INVARIANT THEORY AND EIGENSPACES FOR UNITARY REFLECTION GROUPS

G.I. LEHRER AND J. MICHEL

1. Introduction and statement of results

Let G be a finite group generated by (pseudo)reflections in a complex vector space V of dimension $\ell > 0$. For any linear transformation $A : V \longrightarrow V$ and element $\zeta \in \mathbb{C}$, denote by $V(A, \zeta)$ the ζ-eigenspace of A. It is well known that if S denotes the coordinate ring of V (identified with the symmetric algebra on the dual V^*) the ring S^G of polynomial invariants of G is free; if $f_1, f_2, \ldots, f_\ell$ is a set of homogeneous free generators of S^G, then the degrees $d_i = \deg f_i$ $(i = 1, \ldots, \ell)$ are determined by G, and are called the *invariant degrees* of G.

Denote by $\mathcal{A}$ the set of all reflecting hyperplanes of G, and for each $H \in \mathcal{A}$, fix a corresponding linear form $L_H \in V^*$, and . The polynomial

$$\Pi = \prod_{H \in \mathcal{A}} L_H^{e_H - 1}$$

(where e_H is the order of the (cyclic) isotropy group of H in G) is the unique (up to a non-zero scalar) polynomial f of minimal degree such that $gf = \det gf$ for all $g \in G$. Denote by $\mathcal{H}$ the space of all derivatives of Π. This is the space of G-harmonic polynomials.

If F is the ideal of S generated by the elements of S^G which vanish at $0 \in V$, then S/F realises the regular representation of G, and since $\mathcal{H}$ is the orthogonal complement of F in S with respect to the natural form, the same is true of $\mathcal{H}$. Now $\mathcal{H}$ is clearly graded, and for each i, the graded component $\mathcal{H}_i$ of degree i is a G module. Let q be an indeterminate and write, for any G-module M,

$$P_M(q) = \sum_i \langle \mathcal{H}_i, M \rangle_G q^i = \sum_j q^{m_j(M)}. \tag{1.1}$$

Here $\langle \, , \rangle_G$ denotes the usual intertwining number for complex representations of G. The integers $m_1(M) \leq m_2(M) \leq \cdots \leq m_r(M)$, where $r = \dim M$ since $\mathcal{H}$ realises the regular representation of G, are called the *M-exponents* of G. This terminology is sometimes reserved for irreducible M, but applies generally. There are two noteworthy special cases of this definition. When $M = V$, usually called the reflection representation, the M-exponents are called simply the *exponents* $m_1, \ldots, m_\ell$ of G, so that $m_i = m_i(V)$. When $M = V^*$, the contragredient of V, the corresponding exponents $m_i^* := m_i(V^*)$ $(i = 1, \ldots, \ell)$ are referred to as the *coexponents* of G. It is well known that if the degrees $d_1, \ldots, d_\ell$ are also written in non-decreasing order, then

$$d_i = m_i + 1 \text{ for } i = 1, \ldots, \ell. \tag{1.2}$$

In analogy with (1.2), we define the *codegrees* d_i^* of G by

$$d_i^* = m_i^* - 1 \text{ for } i = 1, \ldots, \ell. \tag{1.3}$$

Consider the following theorems.

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Theorem 1.4. *([LS1]) Let d be any positive integer, and let $\zeta \in \mathbb{C}$ be a primitive d^{th} root of unity. Suppose $g \in G$ is such that $E := V(g, \zeta)$ is maximal among ζ-eigenspaces of elements of G. Define subgroups $N = \{x \in G \mid xE \subset E\}$ and $C = \{x \in G \mid xe = e \text{ for all } e \in E\}$. Then the group N/C acts faithfully as a unitary reflection group on E.*

In the notation of Theorem 1.4, the integer d is said (cf. [Sp]) to be *regular* for G if the maximal eigenspace E contains a vector which is regular for G, i.e., which lies in no reflecting hyperplane. Equivalently, E is not contained in any reflecting hyperplane. By Steinberg's theorem ([St, 1.5]), we have

1.5. In the notation of Theorem 1.4, d is regular for G if and only if $C = \{1\}$.

Theorem 1.6. *([LS2]) Maintain the notation of Theorem 1.4. Let $a(d) = \#\{i \in \{1, \dots, \ell\} \mid d_i \equiv 0(\mod d)\}$ and let $b(d) = \#\{i \in \{1, \dots, \ell\} \mid d_i^* \equiv 0(\mod d)\}$. Then $a(d) \le b(d)$, and we have equality if and only if d is regular for G.*

Theorem 1.4 was proved in [LS1], using intersection multiplicities of hypersurface intersections, and Bezout's theorem. Theorem 1.6 was proved in [LS2], where some case by case analysis was needed, which depended on the Shephard-Todd classification of the irreducible reflection groups.

It is our purpose in this note to prove some variations on known formulae in the invariant theory of G, and to show how they may be used to give case-free elementary proofs of Theorems 1.4 and 1.6.

2. Invariant theory

We begin by recalling some results of Solomon and Orlik-Solomon, which may be found in [OS].

For any $\mathbb{C}G$-module M, consider the $\mathbb{C}G$-module $S \otimes M^*$, which is canonically isomorphic to $\operatorname{Hom}(M, S)$. To determine the subspace of G-invariant elements, recall that by Chevalley's theorem, $S \cong S^G \otimes \mathcal{H}$ as G-module, whence

$$(S \otimes M*)^G \cong S^G \otimes (\mathcal{H} \otimes M^*)^G, \tag{2.1}$$

which is a free S^G-module of rank $r = \dim M$ (since $\langle \mathcal{H}, M \rangle_G = r$), which is generated over S^G by elements $\{u_i = \sum_{j=1}^r h_{ij} \otimes y_j \mid i = 1, \dots, r\}$, where $\{y_j \mid j = 1, \dots, r\}$ is a $\mathbb{C}$-basis of M^* and the h_{ij} are homogeneous elements of $\mathcal{H}$ whose degrees are the M^*-exponents $m_j(M^*)$ of G. Define $Q_M \in S$ by $Q_M = \det(h_{ij})$. The following properties of Q_M may be found in [OS, pp. 79–82], where Q_M is denoted j_M.

Lemma 2.2. *(i) The polynomial Q_M is uniquely defined by M up to multiplication by a non-zero constant, and has degree $q(M) := \sum_j m_j(M)$.*

(ii) We have $gQ_M = \det_M(g)Q_M$ for any element $g \in G$.

(iii) We have $Q_M = PQ_{\Lambda^r M}$ for some $P \in S^G$.

(iv) If M is a Galois conjugate of the reflection representation V, then $Q_M \in \mathbb{C}^\times Q_{\Lambda^r M}$.

Proposition 2.3. *Let G be a unitary reflection group and let M be a $\mathbb{C}G$ module such that $Q_M \in \mathbb{C}^\times Q_{\Lambda^r M}$, where $r = \dim M$. Let x and y be indeterminates over $\mathbb{C}$. Then*

$$|G|^{-1} \sum_{g \in G} \frac{\det_M(1 + yg)}{\det_V(1 - xg)} = \frac{\prod_{j=1}^r (1 + yx^{m_j(M)})}{\prod_{i=1}^\ell (1 - x^{d_i})} \tag{2.4}$$

Proof. This is essentially [OS, Theorem (3.7)], which is the statement for Galois conjugates of V. The proof of the general case is the same; it uses the obvious bigrading of $(S \otimes \Lambda^* M^*)^G$ and *op. cit. Theorem (3.1)* □

Now let K be the cyclotomic extension of $\mathbb{Q}$ by a primitive $|G|^{\text{th}}$ root of unity. It is proved in [OS, Theorem 2.13] that the Galois conjugates V^σ, where σ is an element of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$, satisfy the conditions of (2.3), and hence that the formula (2.4) holds for $M = V^\sigma$. We shall exploit this to prove the following generalisation of [OS, Theorem 3.3].

Theorem 2.5. *Let G be a unitary reflection group in V, with basic degrees $\{d_1, \dots, d_\ell\}$. If $\sigma \in \mathrm{Gal}(K/\mathbb{Q}$, denote by $m_i(\sigma)$ the V^σ- exponents of G. Let d be a fixed positive integer and let ζ be a primitive d^{th} root of unity in $\mathbb{C}$. Let $a(d) = \#\{i \mid d \text{ divides } d_i\}$ and let $b_\sigma(d) = \#\{j \mid \zeta^{-\sigma} = \zeta^{m_j(\sigma)}\}$. Then $a(d) \le b_\sigma(d)$, and if we order the d_i and $m_i(\sigma)$ so that $d|d_i$ and $\zeta^{-\sigma} = \zeta^{m_i(\sigma)}$ for $1 \le j \le a(d)$, then, writing $\lambda_1(g), \dots, \lambda_\ell(g)$ for the eigenvalues of $g \in G$ on V, we have the following polynomial identity in $\mathbb{C}[T]$.*

$$\begin{aligned}(2.6)\qquad &\sum_{g\in G} T^{d(g,\zeta)} \prod_{i:\lambda_i(g)\neq\zeta} \frac{1-(\zeta^{-1}\lambda_i(g))^\sigma}{1-\zeta^{-1}\lambda_i(g)} \\ &\qquad = \prod_{1\le j\le a(d)} (T + m_j(\sigma)) \prod_{a(d)+1\le j\le\ell} d_j \frac{1-\zeta^{-\sigma}\zeta^{-m_j(\sigma)}}{1-\zeta^{-d_j}},\end{aligned}$$

where $d(g, \zeta) = \dim V(g, \zeta)$ for $g \in G$.

Proof. We start with the formula (2.4) for $M = V^\sigma$, noting that $\det(g)$ is the product of its eigenvalues.

$$(2.6.1)\qquad |G|^{-1} \sum_{g\in G} \prod_{i=1}^{\ell} \frac{(1 + y\lambda_i(g)^\sigma)}{(1 - x\lambda_i(g))} = \prod_{j=1}^{\ell} \frac{(1 + yx^{m_j(\sigma)})}{(1 - x^{d_j})}.$$

The term corresponding to $g \in G$ on the left side has a zero of order $d(g, \zeta)$ at $x = \zeta^{-1}$ in the denominator. We therefore introduce the change of variable

$$(2.6.2)\qquad y = -\zeta^{-\sigma} - t(x - \zeta^{-1}),$$

where t is an indeterminate over $\mathbb{C}$, which ensures that the same is true of the numerator. Let us compute the value of this term at $x = \zeta^{-1}$. When $\lambda_i(g) \neq \zeta$, then $\frac{(1+y\lambda_i(g)^\sigma)}{(1-x\lambda_i(g))} = \frac{(1-(\zeta^{-1}\lambda_i(g))^\sigma)}{(1-\zeta^{-1}\lambda_i(g))}$. When $\lambda_i(g) = \zeta$, then by differentiation, we see that $\lim_{x\to\zeta^{-1}} \frac{(1+y\lambda_i(g)^\sigma)}{(1-x\lambda_i(g))} = t\zeta^{\sigma-1}$. Hence the left side of (2.6.1) has limit as $x \to \zeta^{-1}$

$$(2.6.3)\qquad |G|^{-1} \sum_{g\in G} (t\zeta^{\sigma-1})^{d(g,\zeta)} \prod_{i:\lambda_i(g)\neq\zeta} \frac{1-(\zeta^{-1}\lambda_i(g))^\sigma}{1-\zeta^{-1}\lambda_i(g)}.$$

But the denominator of the right side of (2.6.1) clearly has a zero of order $a(d)$ at $x = \zeta^{-1}$, whence it follows that the numerator has a zero of multiplicity at least $a(d)$. Moreover it is easily seen that at $x = \zeta^{-1}$, $\frac{\partial}{\partial x}(1 + yx^{m_j(\sigma)}) = -t\zeta^{-m_j(\sigma)} - m_j(\sigma)\zeta^{-(m_j(\sigma)-1)}\zeta^{-\sigma} \neq 0$, so that each factor in the numerator has at most a simple zero at $x = \zeta^{-1}$. But at $x = \zeta^{-1}$, $1 + yx^{m_j(\sigma)} = 1 - \zeta^{-\sigma}\zeta^{-m_j(\sigma)}$, which is zero precisely when

$$(2.6.4)\qquad \zeta^{-\sigma} = \zeta^{m_j(\sigma)}.$$

It follows that the number $b_\sigma(d)$ of indices j for which (2.6.4) holds is at least $a(d)$, which is the first assertion of the Theorem.

We therefore assume numbering chosen so that $\zeta^{d_j} = 1$ and $\zeta^{-\sigma} = \zeta^{m_j(\sigma)}$ for $1 \le j \le a(d)$ and evaluate the right side of (2.6.1) at $x = \zeta^{-1}$ in the same way as the left side, obtaining

$$(2.6.5)\qquad \prod_{1\le j\le a(d)} d_j^{-1}(t\zeta^{\sigma-1} + m_j(\sigma)) \prod_{a(d)+1\le j\le\ell} \frac{1-\zeta^{-\sigma}\zeta^{-m_j(\sigma)}}{1-\zeta^{-d_j}}.$$

The Theorem now follows immediately by comparing (2.6.3) and (2.6.5), taking into account that $|G| = d_1 \dots d_\ell$ and replacing $t\zeta^{\sigma-1}$ by T. □

We next state four corollaries of Theorem 2.5, the first two of which are well known; we believe the others are new.

Corollary 2.7. *(*[OS, Theorem 3.3]*) With notation as in the Theorem above, we have*

$$\sum_{g\in G} T^{d(g,1)} \prod_{i:\lambda_i(g)\neq 1} \frac{1-\lambda_i(g)^\sigma}{1-\lambda_i(g)} = \prod_{j=1}^{\ell}(T+m_j(\sigma)). \tag{2.7.1}$$

This is simply the case $d = 1$ (whence $\zeta = 1$) of the Theorem.

Corollary 2.8. *(*[PW]*) We have*

$$\sum_{g\in G} T^{d(g,\zeta)} = \prod_{1\leq j\leq a(d)} (T+m_j) \prod_{j>a(d)} d_j = \prod_{1\leq j\leq a(d)} (T+d_j-1) \prod_{j>a(d)} d_j. \tag{2.8.1}$$

This is the case $\sigma = \mathrm{Id}$ of the Theorem. Note that here the integers $m_j(\sigma) = m_j = d_j - 1$ are the usual exponents.

We believe the next result is new.

Corollary 2.9. *Maintaining the above notation, we have*

$$\begin{aligned}(-1)^\ell \sum_{g\in G} \det(g^{-1})(-T)^{d(g,\zeta)} &= \prod_{1\leq j\leq a(d)} (T+m_j^*) \prod_{a(d)+1\leq j\leq \ell} d_j \frac{1-\zeta^{1-m_j^*}}{1-\zeta^{-d_j}} \\ &= \prod_{1\leq j\leq a(d)} (T+d_j^*+1) \prod_{a(d)+1\leq j\leq \ell} d_j \frac{1-\zeta^{-d_j^*}}{1-\zeta^{-d_j}},\end{aligned} \tag{2.10}$$

where the integers $m_j^* = d_j^* + 1$ are the coexponents of G (1.3).

Proof. Take σ to be complex conjugation in (2.6). Then by definition, $m_j(\sigma) = m_j^*$ and σ acts as inversion on roots of unity. Taking into account the facts that for any $\lambda(\neq 1) \in \mathbb{C}$, $\frac{1-\lambda^{-1}}{1-\lambda} = -\lambda^{-1}$, and that for $g \in G$, $\det(g) = \zeta^{d(g,\zeta)} \prod_{i:\lambda_i(g)\neq\zeta} \lambda_i(g)$, the formula (2.10) now follows directly from (2.6). □

There are of course many other special cases which might be explored. We give one other.

Corollary 2.11. *With the above notation, let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ and write $b_\sigma(2)$ for the number of generalised exponents $m_j(\sigma)$ which are odd. Recall that $a(2)$ is the number of even degrees of G. Then $a(2) \leq b_\sigma(2)$ and we have*

$$\begin{aligned}&\sum_{g\in G} T^{d(g,-1)} \prod_{i:\lambda_i(g)\neq -1} \frac{1+\lambda_i(g)^\sigma}{1+\lambda_i(g)} \\ &\qquad = \prod_{1\leq j\leq a(2)} (T+m_j(\sigma)) \prod_{a(2)+1\leq j\leq \ell} d_j \frac{1+(-1)^{m_j(\sigma)}}{2}.\end{aligned} \tag{2.11.1}$$

This is obtained from (2.6) by putting $d = 2$ and hence $\zeta = -1$.

3. Eigenspaces

In this section we show how the results above may be applied to questions concerning maximal eigenspaces of elements of G. We begin with the

Proof of Theorem 1.4. We use the notation of the statement (1.4). It follows from [Sp, Theorem 3.4] (whose proof is elementary) that the maximal ζ-eigenspaces $V(g,\zeta)$ all have dimension $a(d)$ and are conjugate under the action of G. We compute the coefficient of $T^{a(d)}$ on both sides of (2.8.1), obtaining the equality

$$\frac{|G||C|}{|N|} = \prod_{j:d\nmid d_j} d_j,$$

the left side being the number $|C|$ of elements $g \in G$ such that $V(g,\zeta) = E$ multiplied by the number $\frac{|G|}{|N|}$ of distinct conjugates of E. Since $|G| = \prod_{j=1}^{\ell} d_j$, it follows that $|\frac{N}{C}| = \prod_{j:d|d_j} d_j$, which by [Sp, Proposition 3.5(ii)] proves Theorem 1.4.

The next result includes Theorem 1.6.

Theorem 3.1. *(i) Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. In the notation of Theorem 2.6, if d is regular for G, then $b_\sigma(d) = a(d)$.*

(ii) When σ is complex conjugation, so that $b_\sigma(d) = b(d)$, then the converse of (i) holds. That is, d is regular for G if and only if $a(d) = b(d)$.

Proof. Maintain the notation of the proof of Theorem 1.4 above. Clearly the set $\{g \in G \mid V(g,\zeta) = E\}$ is a coset g_0C of the pointwise stabiliser $C = G_E$ of E in G. Hence if $C = 1$, which is the regular case, there is a unique element g with $V(g,\zeta) = E$. If we write g_h for the corresponding element when E is replaced by a conjugate hE $(h \in G)$ of E, we have clearly, that $g_h = hgh^{-1}$, so that in particular, all the elements g_h have the same eigenvalues.

Let us compare the coefficients of $T^{a(d)}$ in both sides of (2.6) when d is regular. We obtain

$$\prod_{i:\lambda_i(g)\neq\zeta} \frac{1-(\zeta^{-1}\lambda_i(g))^\sigma}{1-\zeta^{-1}\lambda_i(g)} = \prod_{a(d)+1\le j\le\ell} \frac{1-\zeta^{-\sigma}\zeta^{-m_j(\sigma)}}{1-\zeta^{-d_j}}. \tag{3.1.1}$$

Now the right side is zero if and only if $a(d) \lneq b_\sigma(d)$, while the left side is clearly non-zero, which proves (i).

To prove (ii), assume that d is not regular for G. Then by the observations above, the coefficient of $T^{a(d)}$ in the left side of (2.9) has a factor $\sum_{h\in C}\det^{-1}(h)$ which is zero if C is a non-trivial reflection subgroup of G. Hence the right side of (2.9) is zero, so that $\zeta^{1-m_j^*} = 1$ for some $j > a(d)$. Hence $b(d) \gneq a(d)$. $\square$

Note that the relation (3.1.1) holds generally whenever d is regular for G, $\sigma \in \mathrm{Gal}(K/\mathbb{Q}$ and g is a d- regular element of G. The particular case when σ is complex conjugation yields

Corollary 3.2. *Let g be a d-regular element of the unitary reflection group G and let ζ be a primitive d^{th} root of unity. Then*

$$\det(g) = \prod_{a(d)+1\le j\le\ell} \frac{1-\zeta^{-d_j}}{\zeta^{1-m_j^*}-1} = \zeta^{-\sum_{j=1}^{\ell} m_j}.$$

The first equality is immediate from (2.9), while the second follows from [Sp, Theorem 4.2(v)]. The second equality may also be written

$$\prod_{j>a(d)} (1-\zeta^{d_j^*}) = \zeta^{\ell-a(d)} \prod_{j>a(d)} (1-\zeta^{d_j}).$$

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School of Mathematics and Statistics, University of Sydney, N.S.W. 2006, Australia, Fax: +61 2 9351 4534

E-mail address: `gusl@maths.usyd.edu.au`

LAMFA, Université de Picardie-Jules Verne 33, Rue Saint-Leu 80039 Amiens Cedex France
and
Institut de Mathématiques, Université Paris VII, 175, rue du Chevaleret 75013 Paris France

E-mail address: `jmichel@math.jussieu.fr`

URL: `www.math.jussieu.fr/~jmichel`