EQUIVARIANT POINCARÉ POLYNOMIALS AND COUNTING POINTS OVER FINITE FIELDS.

M. Kisin and G.I. Lehrer

School of Mathematics and Statistics
University of Sydney
N.S.W. 2006, Australia
Email: g.lehrer@maths.usyd.edu.au

Abstract. Suppose a finite group acts as a group of automorphisms of a smooth complex algebraic variety which is defined over a number field. We show how, in certain circumstances, an equivariant comparison theorem in l-adic cohomology may be used to convert the computation of the graded character of the induced action on cohomology into questions about numbers of rational points of varieties over finite fields. This is carried through in three applications: firstly, for the symmetric group acting on the moduli space of n points on a genus zero curve, secondly, for a unitary reflection group acting on the complement of its reflecting hyperplanes and thirdly for the symmetric group action on the space of configurations of points in any smooth variety which satisfies certain strong purity conditions.

Key words and phrases. l-adic cohomology, smooth scheme, group action, comparison.

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§0. Introduction.

Let X be a smooth complex algebraic variety which is defined over a number field K, and suppose that the finite group G acts as a group of K-endomorphisms of X. In this paper we give a general method for calculation of the action of G on the (singular or de Rham) cohomology of X, by reducing X modulo finite primes. The technique applies in particular when X satisfies certain strong purity conditions, and gives rather explicit formulae provided we can compute the zeta function of X modulo these primes, and this can be done in many interesting cases (cf. [L2], [DL]).

To explain the method, we "spread out" X together with its G-action by noting that for some integer $n \geq 1$, they are defined over a finite $\mathbb{Z}[1/n]$-algebra R whose field of fractions is K. Fix a map $R \to \mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_q$ for some prime power q. Let F denote the geometric Frobenius in $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the l-adic cohomology

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$H_c^*(X \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l))$. Since the action of $g \in G$ on this cohomology commutes with that of F, its character can be easily determined in terms of those of F and gF. However the latter two characters can be computed using the Grothendieck-Lefschetz fixed point formula. For gF this works because it is the Frobenius with respect to a "twisted" $\mathbb{F}_q$-structure on X. Finally, we can switch back to the Betti cohomology of X, provided we have the comparison

$$\mathrm{trace}(g, H_c^i((X \otimes \mathbb{C})^{an}, \mathbb{C})) = \mathrm{trace}(g, H_c^i(X \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l)), \tag{A}$$

In §1 we show how (A) may be deduced for almost all characteristics, from some standard facts concerning the comparison of Betti and l-adic cohomology of schemes.

We give three applications of the technique sketched above. First, we give an explicit determination of the action of the symmetric group Sym_n on the cohomology of the moduli space $\mathcal{M}_{0,n}$ of n (ordered) distinct points on a Riemann surface of genus zero (cf. the work of Getzler [G]). In this case our results are a little more explicit than those of [loc. cit.]. In particular we show that the quotient of $\mathcal{M}_{0,n}$ by the symmetric group has the cohomology of a point, and that the alternating representation does not appear in the cohomology of $\mathcal{M}_{0,n}$. We also show that the top cohomology is an extension to Sym_n of the $(n-2)!$ dimensional representation of Sym_{n-1} which arises in the top cohomology of the configuration space of $n-1$ complex points, or in the Lie operad at level $n-1$. We understand that the existence of such an extension was first observed by Kontsevich.

Secondly, for a finite group G, acting by (pseudo)reflections on a finite dimensional complex vector space V, we explain how to compute the action of G on the cohomology of M_G, the complement of the reflecting hyperplanes in V. For real reflection groups, this was done in [L2] by a different method.

Finally, for a variety X which is assumed to satisfy a cohomological purity condition, we determine explicitly the action of the group Sym_n on the cohomology of the configuration space $C_n(X)$ of n distinct ordered points of X in terms of the zeta function of X.

§1 Comparison of l-adic and Betti Cohomology

(1.1) The purpose of this section is to recall some standard facts about comparison of Betti and l-adic cohomology. Suppose that R is a strictly henselian, discrete valuation ring with residue field k. Fix an embedding $R \hookrightarrow \mathbb{C}$. For any finite type R-scheme X, we write $X_k = X \otimes_R k$ and $X(\mathbb{C}) = X_{\mathbb{C}} = X \otimes_R \mathbb{C}$.

Let l be a prime number which is invertible in R. We can consider the l-adic cohomology $H^i(X_k, \bar{\mathbb{Q}}_l)$ with coefficients in an algebraic closure $\bar{\mathbb{Q}}_l$ of $\mathbb{Q}_l$ as well as the cohomology with compact supports $H^i_c(X_k, \bar{\mathbb{Q}}_l)$.

If $X^{an}_{\mathbb{C}}$ denotes the complex analytic space attached to $X_{\mathbb{C}}$, then we also have the analogous Betti cohomologies $H^i(X^{an}_{\mathbb{C}}, \mathbb{C})$, and $H^i_c(X^{an}_{\mathbb{C}}, \mathbb{C})$.

In the sequel we fix once and for all an isomorphism $\bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$.

Proposition (1.2). *Let X be an open dense subscheme of a proper smooth R-scheme $\bar{X}$, and suppose that the complement $\bar{X}\backslash X$ is a normal crossings divisor relative to R. Then there exist isomorphisms*

$$H^i(X_k, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H^i(X^{an}_{\mathbb{C}}, \mathbb{C}), \tag{1.2.1}$$

and

$$H^i_c(X_k, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H^i_c(X^{an}_{\mathbb{C}}, \mathbb{C}), \tag{1.2.2}$$

These isomorphisms are functorial for maps of smooth R-schemes $f : Y \hookrightarrow X$ which have compactifications $Y \hookrightarrow \bar{Y}$ $X \hookrightarrow \bar{X}$ as above. Note that we do not assume f extends to a map of compactifications.

In particular if G is a group acting on the R-scheme X, then (1.2.1) and (1.2.2) are equivariant with respect to the group action induced on cohomology.

Proof. Let K be the fraction field of R, and denote by $\bar{K}$ the integral closure of K in $\mathbb{C}$. Write $X_{\bar{K}} = X \otimes_R \bar{K}$. Let Λ be any finite ring whose order is invertible in R. Since X is smooth, the complex of nearby cycles $R\Psi\Lambda$ is equal to Λ, and [SGA 7, XIII 2.1.8] yields canonical maps

$$H^i(X_{\bar{K}}, \Lambda) \to H^i(X_k, \Lambda) \tag{1.2.3}$$

and

$$H^i_c(X_k,\Lambda) \to H^i_c(X_{\bar{K}},\Lambda) \tag{1.2.4}$$

This holds even if we do not assume that X has a compactification $\bar{X}$ of the type described above. Assuming the existence of $\bar{X}$, these maps are isomorphisms by [SGA 7, XIII, 2.1.9].

Next by [SGA 4, XVI 1.6] and [SGA 4, XVII 5.2.6] respectively. we have isomorphisms

$$H^i(X_{\bar{K}},\Lambda) \xrightarrow{\sim} H^i(X_{\mathbb{C}},\Lambda) \tag{1.2.5}$$

and

$$H^i_c(X_{\bar{K}},\Lambda) \xrightarrow{\sim} H^i_c(X_{\mathbb{C}},\Lambda) \tag{1.2.6}$$

Finally by [SGA 4, XVI 4.1] and [SGA 4, XVII 5.3.5] respectively, we have isomorphisms

$$H^i(X_{\mathbb{C}},\Lambda) \xrightarrow{\sim} H^i(X^{an}_{\mathbb{C}},\Lambda) \tag{1.2.7}$$

and

$$H^i_c(X_{\mathbb{C}},\Lambda) \xrightarrow{\sim} H^i_c(X^{an}_{\mathbb{C}},\Lambda) \tag{1.2.8}$$

Combining (1.2.3), (1.2.5) and (1.2.7) (resp. (1.2.4), (1.2.6) and (1.2.8)) yields canonical (i.e functorial in $X/\mathrm{Spec}R$) isomorphisms

$$H^i(X_k,\Lambda) \xrightarrow{\sim} H^i(X^{an}_{\mathbb{C}},\Lambda) \tag{1.2.9}$$

and

$$H^i_c(X_k,\Lambda) \xrightarrow{\sim} H^i_c(X^{an}_{\mathbb{C}},\Lambda) \tag{1.2.10}$$

Now let $E \subset \bar{\mathbb{Q}}_l$ be a subfield, which is finite over $\mathbb{Q}_l$, and denote by $\mathcal{O}_E$ its ring of integers. Then setting $\Lambda = \mathcal{O}_E/l^n$ for $n = 1, 2, \ldots$ and applying (1.2.9) we have

$$\begin{aligned} H^i(X_k,\mathcal{O}_E) &= \varprojlim H^i(X_k,\mathcal{O}_E/l^n\mathcal{O}_E) \\ &\xrightarrow{\sim} \varprojlim H^i(X^{an}_{\mathbb{C}},\mathcal{O}_E/l^n\mathcal{O}_E) \xrightarrow{\sim} H^i(X^{an}_{\mathbb{C}},\mathcal{O}_E) \end{aligned} \tag{1.2.11}$$

where the first isomorphism in (1.2.11) is simply the definition of $H^i(X_k, \mathcal{O}_E)$ and the final isomorphism follows, for example, from the universal coefficient theorem. Similarly, we obtain an isomorphism

$$H^i_c(X_k, \mathcal{O}_E) \xrightarrow{\sim} H^i_c(X^{an}_{\mathbb{C}}, \mathcal{O}_E) \tag{1.2.12}$$

Passing to the limit in (1.2.11) and (1.2.12) over all subfields $E \subset \bar{\mathbb{Q}}_l$ which are finite over $\mathbb{Q}_l$, and tensoring by $\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$, we get the analogous isomorphisms but with $\bar{\mathbb{Q}}_l$ in place of $\mathcal{O}_E$. Finally using the chosen isomorphism $\bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ yields the proposition. $\square$

Corollary (1.3). *Let $\mathcal{O}$ be the ring of integers of a number field K and n a positive integer divisible by l. Assume that $\bar{X}$ is a smooth, proper scheme over $\mathcal{O}[1/n]$ and that $X \subset \bar{X}$ is a dense open subscheme, such that $\bar{X}\backslash X$ is a normal crossings divisor relative to $\mathcal{O}[1/n]$. Fix a prime $\mathfrak{p}$ of $\mathcal{O}[1/n]$ and an embedding $\mathcal{O}[1/n] \hookrightarrow \mathbb{C}$. If $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$ and $\bar{k}(\mathfrak{p})$ denotes an algebraic closure of $k(\mathfrak{p})$, then there are canonical isomorphisms*

$$H^i(X_{\bar{k}(\mathfrak{p})}, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H^i(X^{an}_{\mathbb{C}}, \mathbb{C}) \tag{1.3.1}$$

and

$$H^i_c(X_{\bar{k}(\mathfrak{p})}, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H^i_c(X^{an}_{\mathbb{C}}, \mathbb{C}) \tag{1.3.2}$$

In particular, if a group G acts on X, then these isomorphisms are G-equivariant, without assuming that the action of G extends to $\bar{X}$.

Proof. Let R be a strict henselisation of $\mathcal{O}$ at $\mathfrak{p}$. Fix an embedding $R \hookrightarrow \mathbb{C}$ extending $\mathcal{O}[1/n] \hookrightarrow \mathbb{C}$, and an identification of $\bar{k}(\mathfrak{p})$ with the residue field of R. The corollary now follows by applying the Proposition (1.2) to $X \otimes_{\mathcal{O}[1/n]} R$. $\square$

Remark (1.4) The above corollary will be applied to compute the traces of automorphisms of X acting on the Betti cohomology of $X^{an}_{\mathbb{C}}$. For this, the precise ring $\mathcal{O}[1/n]$ over which X is defined is not crucial. In other words, it is usually harmless if we increase the integer n, or equivalently have the statement of Corollary (1.3) only for all but finitely many $\mathfrak{p}$. If we are willing to make this concession, then one has the following result, which obviates the need to check the existence of $\bar{X}$.

Proposition (1.5). *With the notation of (1.3), suppose that X is a smooth $\mathcal{O}[1/n]$ scheme (We do not assume the existence of $\bar{X}$). Then for all but finitely many primes $\mathfrak{p}$ of $\mathcal{O}[1/n]$ we have canonical isomorphisms*

$$H^i(X_{\bar{k}(\mathfrak{p})}, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H^i(X_{\mathbb{C}}^{an}, \mathbb{C}) \tag{1.5.1}$$

and

$$H_c^i(X_{\bar{k}(\mathfrak{p})}, \bar{\mathbb{Q}}_l) \xrightarrow{\sim} H_c^i(X_{\mathbb{C}}^{an}, \mathbb{C}) \tag{1.5.2}$$

As usual, these are equivariant for any group acting on X over $\mathrm{Spec}\mathcal{O}[1/n]$.

Proof. Write $X_\eta = X \otimes_{\mathcal{O}_K} K$. By Nagata's theorem [Na] there exists a proper K-scheme $\bar{X}_\eta$ which admits X_η as a dense open subscheme. By resolution of singularities (see [Hi] or [BM]), we may assume that $\bar{X}_\eta$ is smooth, and that $\bar{X}_\eta \backslash X_\eta$ is a normal crossings divisor. Now a standard limit argument shows that there exists a positive integer m which is divisible by n, a proper $\mathcal{O}[1/m]$-scheme $\bar{X}$ which admits $X_m = X \otimes_{\mathcal{O}[1/n]} \mathcal{O}[1/m]$ as a dense open subset, and such that $\bar{X} \backslash X_m$ is a normal crossings divisor relative to $\mathcal{O}[1/m]$. The result now follows by applying Corollary (1.3) to X_m. $\square$

§2 The symmetric group acting on the moduli space $\mathcal{M}_{0,n}$.

(2.1) In order to apply our method, we shall give a relative, i.e. scheme theoretic, definition of configuration spaces. Fix a map of schemes, $X \to S$ and a positive integer n. We define the configuration space functor $C_n(X)$ from the category Sch$/S$ of S-schemes to sets by putting, for an S-scheme T, $C_n(X)(T)$ equal to the set of ordered n-tuples $(z_1, z_2, \ldots, z_n)$, of pairwise disjoint sections $z_i \in X_T(T)$. Here X_T is the fibre product $X \times_S T$, which we regard both as a scheme over T

and by (common) abuse of notation, as the functor from T-schemes to sets defined by stipulating that for a T-scheme Y, $X_T(Y) = \mathrm{Hom}_T(Y, X_T) = \mathrm{Hom}_S(Y, X)$. The z_i are therefore sections of the morphism $X \times_S T \longrightarrow T$. Two such sections are said to be disjoint if their images in $X \times_S T$ are.

It is easy to see that the functor $C_n(X)$ is representable by an open subset of the fibre product $X \times_S X \times_S \cdots \times_S X$, where there are n terms in the product. We therefore regard this open subscheme as the scheme theoretic version of the configuration space.

(2.2) Consider now the moduli space $\mathcal{M}_{0,n}$ of genus zero curves with n distinct labelled points. This is the functor which assigns to a $\mathbb{Z}$-scheme T the set of ordered n-tuples of disjoint sections $(z_1, \ldots, z_n) \in \mathbb{P}^1_T$ *up to isomorphism.*

We obtain an action of the $\mathbb{Z}$-group scheme $G = PGL_2$ on $C_n(\mathbb{P}^1)$ by letting $g \in G(T) = G_T(T)$ act diagonally on n-tuples in $\mathbb{P}^1_T(T)$. The space $\mathcal{M}_{0,n}$ can be regarded as the quotient of $C_n(\mathbb{P}^1)$ by this action.

$\mathcal{M}_{0,n}$ is represented by a finite type $\mathbb{Z}$-scheme. One way to see this is via the following lemma, which will in any case be needed later. For the rest of this section we assume that $n \geq 3$.

Lemma (2.3). *There is a canonical isomorphism*

$$H : C_{n-3}(\mathbb{A}^1 - \{0, 1\}) \xrightarrow{\sim} \mathcal{M}_{0,n}.$$

Proof. Let T be a scheme. If $(z_4, \ldots, z_n) \in C_{n-3}(\mathbb{A}^1 - \{0, 1\})(T)$, we define $H(z_4, \ldots, z_n)$ to be the equivalence class of $(0, 1, \infty, z_4, \ldots, z_n)$.

On the other hand consider an n-tuple $(z_1, \ldots, z_n)$ of disjoint sections of $\mathbb{P}^1_T$. There exists a unique element $g \in G(T)$ such that $(gz_1, gz_2, gz_3) = (0, 1, \infty)$. Thus an element of $\mathcal{M}_{0,n}(T)$ is represented by a unique n-tuple of the form $(0, 1, \infty, z_4, \ldots, z_n)$. $\square$

(2.4) For a variety X defined over a finite field $\mathbb{F}_q$, we denote by F the endomorphism of $X \otimes \overline{\mathbb{F}}_q$ obtained by raising local co-ordinates to their q^{th} powers.

There is a natural action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $H^i_c(X \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l)$. We will denote by F the endomorphism of $H^i_c(X \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l)$ induced by the *inverse* of the arithmetic Frobenius ($x \mapsto x^q$) in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Lemma (2.5). *The Frobenius endomorphism F acts on the l-adic cohomology space $H^i_c(\mathcal{M}_{0,n} \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l)$ with all eigenvalues equal to q^{i-n+3}.*

Proof. From Lemma (2.3), we see that $\mathcal{M}_{0,n} \otimes \overline{\mathbb{F}}_q$ is isomorphic to the complement in an affine space of dimension $n-3$ of a finite set of affine linear hyperplanes. The map of (2.3) respects Frobenius, which fixes all the hyperplanes. The result now follows from [L2, Proposition 2.4]. $\square$

Corollary (2.6). *Assume $n \geq 4$ and that t is an indeterminate. For any permutation $w \in \mathrm{Sym}_n$, write*

$$P(w,t) = \sum_i \mathrm{trace}(w, H^i(\mathcal{M}_{0,n}(\mathbb{C})^{an}, \mathbb{C}))t^i$$

for the Poincaré polynomial. Then for infinitely many prime powers q, we have

$$P(w, -q^{-1}) = q^{3-n}|\mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}| \tag{2.6.1}$$

Proof. First observe that since $\mathcal{M}_{0,n}(\mathbb{C})^{an}$ is smooth of (complex) dimension $n-3$, we have

$$H^i_c(\mathcal{M}_{0,n}(\mathbb{C})^{an}, \mathbb{C}) \cong H^{2n-6-i}(\mathcal{M}_{0,n}(\mathbb{C})^{an}, \mathbb{C}).$$

We want to apply Grothendieck's fixed point theorem to wF acting on $\mathcal{M}_{0,n} \otimes \overline{\mathbb{F}}_q$. We can do this once we note that there is a (unique) $\mathbb{F}_q$-scheme $\mathcal{M}^w$ such that $\mathcal{M}^w \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \xrightarrow{\sim} \mathcal{M}_{0,n} \otimes \overline{\mathbb{F}}_q$ and wF becomes the relative Frobenius endomorphism of $\mathcal{M}_{0,n}$ induced by this $\mathbb{F}_q$-structure on $\mathcal{M}_{0,n}$. Thus for almost all q, we have

$$
\begin{aligned}
|\mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}| =& \sum_i (-1)^i \mathrm{trace}(wF, H^i_c(\mathcal{M}_{0,n} \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l))\\
=& \sum_i (-1)^i \mathrm{trace}(w, H^i_c(\mathcal{M}_{0,n} \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_l))q^{i-n+3} \text{ by (2.5)}\\
=& \sum_i (-1)^i \mathrm{trace}(w, H^i_c(\mathcal{M}_{0,n}(\mathbb{C})^{an}, \mathbb{C}))q^{i-n+3} \text{ by (1.3)}\\
=& \sum_i (-q^{-1})^{2n-6-i} \mathrm{trace}(w, H^{2n-6-i}(\mathcal{M}_{0,n}(\mathbb{C})^{an}, \mathbb{C}))q^{n-3},
\end{aligned}
$$

and the result follows. □

The explicit computation of the equivariant Poincaré polynomials of (2.6) is possible because of the next

Proposition (2.7). *Maintain the above notation. Then*

$$
|\mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}| = |C_n(\mathbb{P}^1(\overline{\mathbb{F}}_q))^{wF}||G(\overline{\mathbb{F}}_q)^F|^{-1} = \frac{|C_n(\mathbb{P}^1(\overline{\mathbb{F}}_q))^{wF}|}{q(q^2-1)}
$$

Proof. For $x \in C_n(\mathbb{P}^1(\overline{\mathbb{F}}_q))$, the orbit $G(\overline{\mathbb{F}}_q) \cdot x \in \mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}$ if and only if $wF(x) = g \cdot x$ for some $g \in G(\overline{\mathbb{F}}_q)$. But by Lang's theorem $g = F(h^{-1})h$ for some $h \in G(\overline{\mathbb{F}}_q)$, since G is connected. Thus $h \cdot x \in C_n(\mathbb{P}^1(\overline{\mathbb{F}}_q))^{wF}$. Hence $G(\overline{\mathbb{F}}_q) \cdot x \in \mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}$ if and only if the orbit contains a wF-stable point. But since $G(\overline{\mathbb{F}}_q)$ acts freely on $\mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)$, two wF-stable points of $C_n(\mathbb{P}^1(\overline{\mathbb{F}}_q))$ are in the same $G(\overline{\mathbb{F}}_q)$ orbit if and only if they are transforms of each other under $G(\overline{\mathbb{F}}_q)^F$. □

The explicit values of the polynomials $|\mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}|$ are given by

Lemma (2.8). *Define the polynomials $c_i(q)$ $(i = 1, 2, ...)$ by*

$$
\begin{aligned}
c_1(q) =& q+1\\
c_i(q) =& \sum_{d|i} \mu(\frac{i}{d})q^d \text{ for } i > 1.
\end{aligned}
$$

For positive integers i, m write

$$
C_i^{(m)}(q) = c_i(q)(c_i(q)-i)(c_i(q)-2i)...(c_i(q)-(m-1)i).
$$

If $w \in \mathrm{Sym}_n$ has cycle decomposition (i^{m_i}) (that is, m_i cycles of length i), then

$$|\mathcal{M}_{0,n}(\overline{\mathbb{F}}_q)^{wF}| = \frac{\prod_i C_i^{(m_i)}(q)}{q(q^2-1)}.$$

This is a straightforward computation, given (2.7). Assembling all the above information, we obtain

Theorem (2.9). *The Poincaré polynomial $P(w,t)$ of a permutation w of cycle type (i^{m_i}) acting on the cohomology of the moduli space $\mathcal{M}_{0,n}(\mathbb{C})$ is given by*

$$P(w,t) = \frac{\prod_i Q_i^{(m_i)}(t)}{1-t^2}$$

where, for any pair i, m of positive integers,

$$Q_i^{(m)}(t) = p_i(t)(p_i(t) - i(-t)^i)(p_i(t) - 2i(-t)^i)...(p_i(t) - (m-1)i(-t)^i),$$

and

$$p_i(t) = \begin{cases} 1-t & \text{if} \quad i = 1 \\ \sum_{d|i} \mu(\frac{i}{d})(-t)^{i-d} & \text{if} \quad i > 1 \end{cases}$$

This follows easily from (2.6) and (2.8). As an immediate consequence we recover a result of Getzler [G, 5.5(8)].

Corollary (2.10). *The Poincaré polynomial of $\mathcal{M}_{0,n}(\mathbb{C})$ is $(1+2t)(1+3t)...(1+(n-2)t)$. In particular the dimension of $H^*(\mathcal{M}_{0,n}(\mathbb{C}),\mathbb{C}) = \oplus_i H^i(\mathcal{M}_{0,n}(\mathbb{C}),\mathbb{C})$ is $\frac{(n-1)!}{2}$.*

We shall apply our character formula to deduce properties of the representation of Sym_n on $H^*(\mathcal{M}_{0,n}(\mathbb{C}))$ and on the top cohomology, in analogy with the results of [L1]. To determine character values, the following result will be useful.

Lemma (2.11). *(i) $Q_1^{(m)}(t)$ is divisible by t^2-1 if $m \geq 3$.*

(ii) $Q_2^{(m)}(t)$ is divisible by t^2-1 for $m \geq 2$.

(iii) (i) If $i > 2$, $p_i(t)$ is divisible by $t^2 - 1$ in $\mathbb{Z}[t]$. Write $q_i(t) = \dfrac{p_i(t)}{1-t^2}$. then $q_i(1)$ is a positive integer, and if $\ell_1, \ell_2, ..., \ell_k$ are the distinct prime divisors of i, $q_i(1)$ is explicitly given by the formula

$$q_i(1) = \begin{cases} \frac{1}{2} i \prod_j \left(1 - \frac{1}{\ell_j}\right) & \textit{if } i \not\equiv 2 \textit{(mod 4)} \\ \frac{3}{2} i \prod_j \left(1 - \frac{1}{\ell_j}\right) & \textit{if } i \equiv 2 \textit{(mod 4)} \end{cases}$$

Proof. The assertions (i) and (ii) are easily checked. We turn to the proof of (iii). By the defining property of the Möbius function μ, $p_i(-1) = 0$ for $i > 2$, so $p_i(t)$ is divisible by $t + 1$. But by [L1, Prop. 4.11], we also have $p_i(1) = 0$ for $i > 2$, whence the first assertion of (iii) follows. To prove the explicit formula, first take i to be square free and use de l'Hôpital's rule. The general case follows by substitution. □

Corollary (2.12). *Let χ^* be the character of* Sym_n *on $H^*(\mathcal{M}_{0,n}(\mathbb{C}), \mathbb{C})$, i.e. on the total cohomology space $= \oplus_i H^i(\mathcal{M}_{0,n}(\mathbb{C}), \mathbb{C})$. Then for $w \in \mathrm{Sym}_n$, $\chi^*(w) = 0$ unless w has cycle type*

(i) $w \sim (i^m)$ for some $i \geq 1$, or

(ii) $w \sim (2, i^m)$ for some $i \neq 2$.

For w of type (i) or (ii), the value of $\chi^*(w)$ may be read off from (2.9) and (2.11).

Proof. It follows from Lemma (2.11) (i) and (iii) and the fact that $p_1(1) = 0$ that $\chi^*(w) = 0$ if (a) w has two cycles of distinct lengths i, j, both > 2 or (b) w has one cycle of length $i > 2$ and either a fixed point or two 2-cycles, or (c) w has two 2-cycles and a fixed point. The elements w of type (i) and (ii) are the ones which remain when those of type (a), (b) or (c) are excluded. □

We shall now prove

Theorem (2.13). *The multiplicity of the trivial character of* Sym_n *in* χ^* *is 1. Thus the quotient* $\widetilde{\mathcal{M}}_{0,n}(\mathbb{C})$ *of* $\mathcal{M}_{0,n}(\mathbb{C})$ *by the symmetric group has the (rational or complex) cohomology of a point.*

Proof. Straightforward application of the character formula (2.9) to the two cases of (2.12) gives the following formula for the multiplicity

$$(\chi^*, 1)_{\mathrm{Sym}_n} = (n!)^{-1} \sum_{w \in \mathrm{Sym}_n} \chi^*(w) :$$

$$\begin{aligned}(\chi^*, 1)_{\mathrm{Sym}_n} =& \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{n} \sum_{im=n, i\geq 3} q_i(1)(-1)^{(i-1)(m-1)} \\ &+ \frac{1}{n-2} \sum_{im=n-2, i\geq 3} q_i(1)(-1)^{(i-1)(m-1)} \\ &+ (-1)^{\frac{n}{2}} \frac{3}{n(n-2)} \end{aligned} \tag{2.13.1}$$

where the last summand $((-1)^{\frac{n}{2}} \dfrac{3}{n(n-2)})$ is omitted if n is odd.

We shall require

Lemma (2.14). *Let* $i = \prod_{j=1}^{k} p_j^{n_j}$ *be a positive integer, where the* p_j *are distinct primes. Define*

$$r(i) = i \prod_j \left(1 - \frac{1}{p_j}\right).$$

Then for any positive integer n, $\sum_{i|n} r(i) = n$

To see (2.14), observe that $r(i)$ is just the Euler ϕ-function evaluated at i, and the assertion is one of its standard properties.

To prove Theorem (2.13), we shall show that the right side of (2.13.1) is less than 2. Since 1_{Sym_n} obviously occurs in $H^0(\mathcal{M}_{0,n}(\mathbb{C}), \mathbb{C})$, this will suffice. For this purpose we use

Lemma (2.15). *(i) If* n *is odd,* $\sum_{i|n, i\geq 3} q_i(1) = \frac{1}{2}(n-1)$.

(ii) If n is even, $\sum_{i|n, i\geq 3} q_i(1) \leq n-2$.

Proof of (2.15). It follows from (2.11)(iii) that $q_i(1) = \frac{1}{2} r(i)$ if $i \not\equiv 2(\text{mod } 4)$, while $q_i(1) = \frac{3}{2} r(i)$ if $i \equiv 2(\text{mod } 4)$. The statement (i) now follows by direct calculation from Lemma (2.14). Suppose now that $n = \prod_{j=1}^{k} p_j^{n_j}$ is even, and that $p_1 = 2$. The divisor i of n satisfies $i \equiv 2(\text{mod } 4)$ if and only if i is of the form $i = 2p_2^{m_2} ... p_k^{m_k}$, and for such i, $r(i) = r(\frac{i}{2})$.

Hence if we write $n_0 = p_2^{n_2} ... p_k^{n_k}$, we have

$$\begin{aligned}
\sum_{i|n, i\geq 3} q_i(1) =& \frac{1}{2} \sum_{i|n, i\geq 3} r(i) + \sum_{i_0|n_0, i_0\geq 3} r(i_0) \\
=& \frac{1}{2}(n-2) + (n_0 - 1) \qquad\qquad (2.15.1) \\
\leq& n-2,
\end{aligned}$$

which proves (ii). □

We now complete the proof of (2.13). Taking absolute values on the right side of (2.13.1), we see that

$$\begin{aligned}
(\chi^*, 1)_{\text{Sym}_n} \leq& \frac{n-1}{n(n-2)} + \frac{n-2}{n} + \frac{n-4}{n-2} \\
=& \frac{2n^2 - 7n + 3}{n(n-2)} \\
<& 2 \quad \text{if} \quad n \geq 3,
\end{aligned}$$

which completes the proof of (2.13) by the above remarks. □

Proposition (2.16). *The alternating character ϵ of Sym_n does not occur in the total cohomology character χ^*.*

Proof. We compare the multiplicity $(\chi^*, \epsilon)_{\text{Sym}_n}$ with $(\chi^*, 1)_{\text{Sym}_n}$. Again, straightforward application of the character formula (2.9) to the two cases of (2.12) gives the following formula for the multiplicity

$$(\chi^*, \epsilon)_{\text{Sym}_n} = (n!)^{-1} \sum_{w \in \text{Sym}_n} \chi^*(w)\epsilon(w) :$$

$$
\begin{aligned}
(\chi^*,\epsilon)_{\mathrm{Sym}_n} =& \frac{1}{2n} - \frac{1}{2n-2} + \frac{1}{n}\sum_{im=n,i\ ge3} q_i(1)(-1)^{(i-1)} \\
& - \frac{1}{n-2}\sum_{im=n-2,i\geq 3} q_i(1)(-1)^{(i-1)} \\
& + \frac{3}{n(n-2)}
\end{aligned} \tag{2.16.1}
$$

where the last summand $(\frac{3}{n(n-2)})$ is omitted if n is odd.

If n is odd, each summand of the right side of (2.13.1) is positive, while some of those in (2.16.1) are negative, whence $(\chi^*,\epsilon)_{\mathrm{Sym}_n}$ is an integer satisfying $0 \leq (\chi^*,\epsilon)_{\mathrm{Sym}_n} < (\chi^*,1)_{\mathrm{Sym}_n} = 1$, whence $(\chi^*,\epsilon)_{\mathrm{Sym}_n} = 0$.

Suppose n is even. We shall use

Lemma (2.17). *If n is even,* $\sum_{i|n,i\geq 3} q_i(1)(-1)^i = \frac{1}{2}(n-2)$.

Proof of (2.17). A similar argument to that of (2.15) shows that in the notation of (2.15.1),

$$\sum_{i|n,i\geq 3,i \text{ odd}} q_i(1) = \frac{1}{2}(n_0 - 1).$$

Combining this with (2.15.1) immediately yields the Lemma. $\square$

Two simple applications of (2.17) now make the explicit evaluation of the right side of (2.16.1) possible, confirming that $(\chi^*,\epsilon)_{\mathrm{Sym}_n} = 0$ as stated. $\square$

We conclude this section with a description of the top cohomology (cf. [G, 3.10, 3.12).

Proposition 2.18. *Let χ^{top} be the character of Sym_n on $H^{n-3}(\mathcal{M}_{0,n}(\mathbb{C}),\mathbb{C})$, the top cohomology. Then*

(i) For $w \in \mathrm{Sym}_n$, $\chi^{top}(w) = 0$ unless w has cycle type (i^m) (any i) or $(1,i^m)$ $(i>1)$. Thus χ^{top} is supported on the elements of Sym_n which are regular in the sense of Springer [Sp].

(ii) The restriction of χ^{top} to Sym_{n-1} is the character induced from a faithful character of the cyclic group generated by a Coxeter element of Sym_{n-1}, tensored with the sign character.

(iii) The restriction of χ^{top} to Sym_{n-2} *is the regular character of* Sym_{n-2}.

Proof. (i) It is evident that $\mathrm{degree}(Q_i^{(m)}(t)) \leq mi - 1$, except when $i = m = 1$. Since $n = \sum_i m_i i$, (i) follows immediately.

(ii) The value of χ^{top} on elements of the form $(1, i^m)$ may be read off from (2.9): it is $(-1)^{(i-1)(m-1)}\mu(i)i^{m-1}(m-1)!$. This proves the assertion (cf. [L2, §5]).

(iii) This is trivial from (i), which shows that the character of the restriction is supported on the unit element of Sym_{n-2}. □

§3 Unitary Reflection Groups.

A (pseudo)reflection in $V = \mathbb{C}^n$ is a semisimple linear transformation which fixes a hyperplane pointwise. A unitary reflection group is a finite group of linear transformations of $\mathbb{C}^n$ which is generated by reflections. Let G be such a group, let $\mathcal{A}$ be the set of fixed hyperplanes of the reflections in G and let $M_G = V \setminus \cup_{H \in \mathcal{A}} H$ be the corresponding hyperplane complement. The action of G (or more generally, the normaliser $N_{GL(V)}(G)$) on $H^*(M_G, \mathbb{C})$ is important for several representation theoretic applications (cf. [L3, L4, LSe]). In [L2] it was shown how this action could be computed using equivariant comparison of l-adic and Betti cohomology when G is real. In [L3],[BL] it is shown how the computation of the equivariant Poincaré polynomials

$$P_G(g, t) = \sum_{i \geq 0} \mathrm{trace}(g, H^i(M_G, \mathbb{C}))t^i \quad (g \in N_{GL(V)}(G)) \tag{3.1.1}$$

may be reduced to a small number of special cases. Here we generalise the results of [L2] to the unitary case. The generalisation provides a straightforward way of resolving the special cases.

Any unitary reflection group is defined over a cyclotomic ring $R = \mathbb{Z}[\zeta]$, where ζ is a root of unity. This means that there is a finite free R-module V_R, equipped with a G-action, which induces the G-action on V via some embedding $R \hookrightarrow \mathbb{C}$. It is easy to see that after replacing R by $R[1/m]$ for some integer m, we may define a subcheme of the affine R-scheme associated to V_R, by removing the fixed R-hyperplanes of the reflections in G. We again denote this scheme by M_G.

For a (height 1) prime $\mathfrak{p}$ of R we denote by $k(\mathfrak{p})$ its residue field, and by $\bar{k}(\mathfrak{p})$ an algebraic closure of $k(\mathfrak{p})$.

Theorem (3.2). *Let G be any finite unitary reflection group acting on $V = \mathbb{C}^n$, let R be as in the preceding paragraph and assume that $g \in N_{GL(V)}(G)$ is defined over R. Then for almost all primes $\mathfrak{p}$ of R the Poincaré polynomial $P_G(g,t)$ (see (3.1.1)) satisfies*

$$P_G(g, -q^{-1}) = q^{-n}|M_G(\bar{k}(\mathfrak{p}))^{gF}| \tag{3.2.1}$$

where $q = |k(\mathfrak{p})|$, and F denotes the q-power Frobenius.

Proof. This is entirely analogous to the proof of (2.6) above. It follows from [L2, Prop. 2.4] that F acts on $H_c^i(M_G \otimes_R \bar{k}(\mathfrak{p}), \overline{\mathbb{Q}}_l)$ with all eigenvalues equal to q^{i-n}. A calculation like that in the proof of (2.6) now yields the result. $\square$

The method implied by (3.2) has been applied to obtain some equivariant Poincaré polynomials in the case of the cyclotomic reflection groups in §8 of [BL].

§4 Application to Configuration Spaces.

Let Z be a scheme of the type in (1.5) above, over $R = \mathcal{O}[1/n]$ and suppose $\dim(Z) = n$. For $\mathfrak{p}$ a height 1 prime of R, set $q = |k(\mathfrak{p})|$. Assume that for all but finitely many $\mathfrak{p}$, the q-power Frobenius (denoted F) acts on $H_c^i(Z_{\bar{k}(\mathfrak{p})}, \bar{\mathbb{Q}}_l)$ with all eigenvalues equal to q^{i-n} i.e., in the language of [DL], that Z is minimally pure (mp). Examples of such varieties include hyperplane complements, algebraic tori, complements of kernels of characters of tori, toric varieties etc (see [DL]). The symmetric group Sym_n acts by permutation of coordinates on the configuration R-scheme $C_n(Z)$. It is then easy to compute the action of Sym_n on the cohomology of the analytic configuration space $C_n(Z_{\mathbb{C}})^{an}$ using the above method.

For this purpose, define, for any positive integer m,

$$Z_m^o = Z(\overline{\mathbb{F}}_q)^{F^m} \setminus \bigcup_{k|m, k\neq m} Z(\overline{\mathbb{F}}_q)^{F^k} \tag{4.1.1}$$

and further define polynomials $a_m(t)$ and $b_m(t) \in \mathbb{Z}[t]$ by

$$\begin{aligned} a_m(q) =& |Z(\overline{\mathbb{F}}_q)^{F^m}| \\ b_m(q) =& |Z_m^o|. \end{aligned} \tag{4.1.2}$$

That these are polynomials follows from our assumption on the eigenvalues of F acting on the cohomology of Z, together with Grothendieck's fixed point formula. Clearly we have

Lemma (4.1). *The polynomials of (4.1.2) satisfy the relations*

$$a_m(t) = \sum_{d|m} b_d(t) \quad , \textit{ and} \tag{4.1.3}$$

$$b_m(t) = \sum_{d|m} \mu(\frac{m}{d}) a_d(t). \tag{4.1.4}$$

Let X_n be the R-scheme $C_n(Z)$, and let $w \in \mathrm{Sym}_n$ have cycle type (i^{m_i}). The next result is proved by a simple computation of exactly the same nature as that of §2 above.

Lemma (4.2). *We have*

$$|X_n(\overline{\mathbb{F}}_q)^{wF}| = \prod_i B_i^{(m_i)}(q),$$

where $B_i^{(m)}(q) = b_i(q)(b_i(q) - i)(b_i(q) - 2i)...(b_i(q) - (m-1)i)$.

This will be used in the same way as above.

Proposition (4.3). *Let $X_n(\mathbb{C})$ be the analytic space associated to $C_n(Z)$. Then for almost all primes $\mathfrak{p}$ of R, and any element $w \sim (i^{m_i}) \in \mathrm{Sym}_n$, the equivariant Poincaré polynomial*

$$P_{X_n}(w, t) = \sum_i \mathrm{trace}(w, H^i(X_n(\mathbb{C}), \mathbb{C})t^i$$

satisfies

$$P_{X_n}(w,-q^{-1}) = q^{-n}|X_n(\bar{k}(\mathfrak{p}))^{wF}| \tag{4.3.1}$$

The proof of (4.3) is again analogous to that of (2.6). Together, (4.1),(4.2) and (4.3) give an explicit expression for the graded character of Sym_n on $H^*(X_n(\mathbb{C}),\mathbb{C})$, given only the number of rational points $|Z(\bar{k}(\mathfrak{p}))^{F^m}|$ for each m, or equivalently the zeta function of Z.

Example (4.4). Following Manin ([M]) we define, for Z as above, the generating function

$$\phi_Z(t,z) = \sum_{n\geq 1} P_{X_n}(1,t)\frac{z^n}{n!} \in \mathbb{Q}[t][[z]].$$

Then in the above notation, we have

$$\phi_Z(t,z) = (1-tz)^{b_1(-t^{-1})}. \tag{4.4.1}$$

More generally, if $x_1, x_2, ...$ is a set of commuting independent variables (cf [G, 5.1]) we define, for any class function λ_n on Sym_n the "characteristic"

$$ch_n(\lambda_n) = (n!)^{-1} \sum_{w\in \mathrm{Sym}_n} \lambda_n(w) x_1^{m_1} x_2^{m_2} ...$$

where w has cycle type (i^{m_i}). For a sequence $\Lambda = (\lambda_n)$ of class functions, define

$$ch(\Lambda) = \sum_{n\geq 1} ch_n(\lambda_n) z^n.$$

If we take $\lambda_n(w) = P_{X_n}(w,t)$, we obtain the following formula for $ch(\Lambda)$:

$$ch(\Lambda) = \prod_{i\geq 1} (1-tx_i z)^{i^{-1}b_i(-t^{-1})}. \tag{4.4.2}$$

where $b_i(t)$ is the polynomial defined in (4.1).

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