HOMOLOGY STABILITY FOR CLASSICAL REGULAR SEMISIMPLE VARIETIES

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Abstract. We use techniques from homotopy theory, in particular the connection between configuration spaces and iterated loop spaces, to give geometric explanations of stability results for the cohomology of the varieties of regular semisimple elements in the simple complex Lie algebras of classical type A, B or C, as well as in the group $GL_n(\mathbb{C})$. We show that the cohomology spaces of stable versions of these varieties have an algebraic stucture, which identifies them as "free Poisson algebras" with suitable degree shifts. Using this, we are able to give explicit formulae for the corresponding Poincaré series, which lead to power series identities by comparison with earlier work. The cases of type B and C involve ideas from equivariant homotopy theory. Our results may be interpreted in terms of the actions of a Weyl group on its coinvariant algebra (i.e. the coordinate ring of the affine space on which it acts, modulo the invariants of positive degree; this space coincides with the cohomology ring of the flag variety of the associated Lie group) and on the cohomology of its associated complex discriminant variety.

The varieties of regular semisimple elements of a reductive algebraic group and of its Lie algebra figure centrally in several aspects of the representation theory of reductive groups over various fields. Particular instances include the theory of Springer representations of the Weyl group (cf. [Sp], [CG], [LS]) and the notion of "isolated classes", introduced by Lusztig ([Lu]) in connection with his study of intersection complexes on reductive groups. This work is a sequel to [L1], in which closed formulae were given for the Poincaré polynomials of the varieties of regular semisimple elements of the classical complex simple Lie algebras and some of the Lie groups. From these formulae, one could deduce stability results for the Betti numbers as the rank increased. Our purpose

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is to extend and explain these stability results. Specifically, we use constructions from homotopy theory to produce topological spaces whose rational homology coincides, in the stable range, with that of the variety $\mathfrak{G}_{rs}(n)$ (resp. $G_{rs}(n)$) ($n = 1, 2, 3, ...$) of regular semisimple elements in the n^{th} Lie algebra $\mathfrak{G}(n)$ (resp. Lie group $G(n)$) of one of the classical series A_n, B_n or C_n. We shall not discuss the series D_n, which is not quite so straightforward from the present point of view. Our results not only provide a geometric explanation of the stability discovered in [L1], but also give us some new identities of power series, since the homology may now be computed in two different ways. Also, given that these Betti numbers may be expressed in terms of certain graded representations of the Weyl group (cf. [L1, (2.3)]), our results may be interpreted in that context.

The stable spaces we construct resemble $\mathfrak{G}_{rs}(n)$ and $G_{rs}(n)$ in homology, but not in homotopy type. The fundamental groups of $\mathfrak{G}_{rs}(n)$ and $G_{rs}(n)$ are closely related to the Artin braid groups, and become bigger as n increases. The stabilisation holds after we apply Quillen's "plus" construction" to abelianise the fundamental groups. The mere objective of producing a space $\mathfrak{G}_{rs}(\infty)$ such that $H_*(\mathfrak{G}_{rs}(\infty)) = \varinjlim H_*(\mathfrak{G}_{rs}(n))$ could be achieved rather trivially by choosing appropriate maps $\mathfrak{G}_{rs}(n) \to \mathfrak{G}_{rs}(n+1)$ up to homotopy, and forming a "telescope" space; but that would provide no new information and no explanation of the stability.

After reducing the question of stability to that of a space ($X(\mathfrak{G}(n))$ below) closely related to $\mathfrak{G}_{rs}(n)$, our method proceeds by identifying $X(\mathfrak{G}(n))$ in type A with a labelled configuration space, denoted C_n in §2 below. We then exploit the connection between configuration spaces and iterated loop spaces studied in [S1, McS, McD] to show that the homology $H_*(C_n)$ is stably (i.e. as $n \to \infty$) equal to $H_*(\Omega_0)$, where Ω_0 is a connected component of the iterated loop space mentioned above. As part of a more detailed analysis of the situation just described we prove that $\oplus_{n\geq 0} H_*(C_n)$ has the structure of a free Poisson algebra. Our treatment of the cases of the group GL_n and the Lie algebras of type B and C depend on the analysis above. In particular, the stability results for the Lie algebras of type B and C are obtained from those for type A using the results of [S3], in which equivariant versions of the homotopy theoretic results we use are proved. We remark in passing that our work possibly shares some ideas with that of Vassiliev [V].

All homology in this work is taken with coefficients in $\mathbb{Q}$. Thus for any topological space X, $H^i(X)$ will denote $H^i(X, \mathbb{Q})$, and similarly for homology. For any $\mathbb{N}$-graded vector space $M = \oplus_{i\geq 0} M_i$ with finite dimensional graded components, we use the notation $\mathrm{Poin}_M(t) = \sum_{i\geq 0} \dim M_i t^i$. In particular, if X is a topological space with finite dimensional rational homology in each degree, we shall write

$$\mathrm{Poin}_X(t) = \sum_{i\geq 0} \dim H_i(X, \mathbb{Q}) t^i = \sum_{i\geq 0} \dim H^i(X, \mathbb{Q}) t^i \in \mathbb{Q}[[t]].$$

1. Background and notation.

Let G be a complex connected reductive algebraic group. Fix a maximal torus T and Borel subgroup $B \supseteq T$ of G, and write $W = N_G(T)/T$ for the Weyl group. Then

$B = TU$, where U is the unipotent radical of B. Write $\mathfrak{G}, \mathfrak{B}, \mathfrak{T}, \mathfrak{U}$ for the Lie algebras of G, B, T and U respectively. An element $x \in G$ is *regular* (cf. [K], [St]) if its centraliser $C_G(x)$ has minimal dimension (which is equal to r =rank$G$ =dim$T$). Similarly, $X \in \mathfrak{G}$ is regular if $C_G(X) = \{g \in G \mid \mathrm{Ad}\ g(X) = X\}$ has minimal dimension r (here Ad denotes the adjoint action of G on $\mathfrak{G}$). Denote by G_{rs} (resp. $\mathfrak{G}_{rs}$) the set of regular semisimple elements of G (resp $\mathfrak{G}$). Similarly, let T_{rs} (resp. $\mathfrak{T}_{rs}$) denote the set of those elements of T (resp. $\mathfrak{T}$) which are regular in G (resp. $\mathfrak{G}$). The latter are characterised as the set of elements which are not stabilised by any non-trivial element of W. Background concerning these matters may be found in [L1], whose notation we by and large adopt.

It is known that G_{rs} (resp $\mathfrak{G}_{rs}$) is an open dense subvariety of G (resp $\mathfrak{G}$). Moreover in [L1, §2] it was observed that there is a homeomorphism

$$\mathfrak{G}_{rs} \cong (\mathfrak{T}_{rs} \times (G/T))/W$$

which gives rise to the fibration sequence

$$G/T \to \mathfrak{G}_{rs} \to \mathfrak{T}_{rs}/W, \tag{1.1}$$

and also to an isomorphism

$$H^*(\mathfrak{G}_{rs}) \cong H^*(\mathfrak{T}_{rs} \times (G/T))^W \tag{1.2}$$

If we write B_K for the classifying space of a group K, then G/T is the fibre of the natural map $B_T \to B_G$. Now the cohomology ring $H^*(B_T)$ is a polynomial algebra S in dim(T) variables over $\mathbb{Q}$, the variables each having degree 2. Moreover the Weyl group W acts on $H^*(B_T)$ and it is known that $H^*(B_G)$ is the invariant subalgebra S^W, while $H^*(G/T) \cong S/J$, where J is the ideal of S generated by the elements of S^W with no constant term (cf. [L1, (3.2)]). It is also standard that S is a free S^W-module and that we have the following isomorphisms of W-modules:

$$H^*(B_T) \cong H^*(B_G) \otimes H^*(G/T) \cong S^W \otimes (S/J) \tag{1.3}$$

In this note we shall focus principally on the space $X(\mathfrak{G}) = (\mathfrak{T}_{rs} \times B_T)/W$, which is fibred over $\mathfrak{T}_{rs}/W$, with fibre B_T, since W acts freely on the Hausdorff space $\mathcal{T}_{rs}$. Furthermore we have the following diagram, in which the rows and columns are fibration sequences.

$$\begin{tikzcd}
G/T \arrow[r] \arrow[d] & \mathfrak{G}_{rs} \arrow[r] \arrow[d] & \mathfrak{T}_{rs}/W \arrow[d, "\mathrm{id}"'] \\
B_T \arrow[r] \arrow[d] & X(\mathfrak{G}) \arrow[r] \arrow[d] & \mathfrak{T}_{rs}/W \arrow[d] \\
B_G \arrow[r, "\mathrm{id}"] & B_G \arrow[r] & \{\text{point}\}
\end{tikzcd} \tag{1.4}$$

Our purpose in this work is to prove a stability result for the middle vertical sequence

$$\mathfrak{G}_{rs} \to X(\mathfrak{G}) \to B_G \tag{1.5}$$

That is, if $\mathfrak{G}(n)$ is one of the sequences of classical Lie algebras we treat (i.e. $\mathfrak{G}(n)$ is of type A_n, B_n or C_n, $n = 1, 2, 3, ...$), then there are spaces $\mathfrak{G}_{rs}^{\infty}$ and $X^{\infty}(\mathfrak{G})$ whose cohomology respectively coincides with that of $\mathfrak{G}(n)_{rs}$ and $X(\mathfrak{G}(n))$ up to a cetain degree d_n such that $d_n \to \infty$ as $n \to \infty$. We also prove a similar result for the group GL_n.

(1.6) Lemma. *The spectral sequence of the fibration (1.5) collapses, so that we have an isomorphism of S^W-modules*

$$H^*(X(\mathfrak{G})) \cong S^W \otimes H^*(\mathfrak{G}_{rs})$$

Proof. It is clearly sufficient to show that there is a surjection: $H^*(X(\mathfrak{G})) \to H^*(\mathfrak{G}_{rs})$. For this, observe that there is an obvious surjection of graded W-modules:

$$H^*(\mathfrak{T}_{rs}) \otimes S \to H^*(\mathfrak{T}_{rs}) \otimes S/J$$

Since taking fixed points under a finite group is an exact functor in characteristic zero, this induces a surjection

$$H^*(X(\mathfrak{G})) \cong (H^*(\mathfrak{T}_{rs}) \otimes S)^W \to (H^*(\mathfrak{T}_{rs}) \otimes S/J)^W \cong H^*(\mathfrak{G}_{rs})$$

as required. $\square$

Now it is well known that the Poincaré series of S^W is $\prod_i (1 - t^{2d_i})^{-1}$, where the d_i are the invariant degrees of W. These series stabilise in the sense that the series for $W(\mathfrak{G}(n))$ coincides up to degree $2n$ with that of $W(\mathfrak{G}(k))$ for $k \geq n$ because the latter is obtained from the former by adding terms whose degree is at least $2n$. Hence to prove stability results for the varieties $\mathfrak{G}_{rs}$, it suffices to consider the spaces $X(\mathfrak{G})$.

2. Configuration spaces.

The proofs of our principal results depend on the relationship between labelled configuration spaces and loop spaces. In this section we recall some relevant facts. The configuration space formulation applies in the first instance to the general linear groups, so in this section $G(n)$ will always be taken to mean $GL_n(\mathbb{C})$.

If M and P are topological spaces the *labelled configuration space* $C_n(M; P)$ of n-tuples of points of M labelled with points of P means the space of orbits of the symmetric group Sym_n on the space of ordered n-tuples

$$((x_1,\ p_1), \ldots,\ (x_n,\ p_n))$$

with $x_i \in M$, $p_i \in P$, and $x_i \neq x_j$ for $i \neq j$. We shall mainly be interested in the case $M = \mathbb{C}$. The link between our space $X(\mathfrak{G})$ and configuration spaces is given by

(2.0) Remark. *If the group G is taken to be $GL_n(\mathbb{C})$, the space $X(\mathfrak{G})$ of §1 is homeomorphic to $C_n(M;P)$, where $M = \mathbb{C}$ and P is the infinite-dimensional complex projective space $\mathbb{P}(\mathbb{C}^\infty)$.*

To see this, think of Sym_n as the Weyl group W of $GL_n(\mathbb{C})$, and of $\mathbb{C}^n$ as the Lie algebra $\mathfrak{T}$ of the maximal torus consisting of diagonal matrices. Then $\mathfrak{T}_{rs}$ is the space of elements of $\mathbb{C}^n$ with distinct coordinates. Moreover $T \cong (\mathbb{C}^\times)^n$, so that $B_T \cong (B_{\mathbb{C}^\times})^n = P^n$, where $P = B_{\mathbb{C}^\times}$ is the infinite-dimensional complex projective space $\mathbb{P}(\mathbb{C}^\infty)$. Thus

$$C_n(\mathbb{C};P) \cong (\mathfrak{T}_{rs} \times P^n)/W \cong X(\mathfrak{G}),$$

which justifies the remark.

Remark (2.0) makes it possible to bring to bear the machinery of homotopy theory on our questions (cf. [C]). In order to achieve coherence in this work, some of the material which follows will be expository. The reason for this is twofold; firstly, it should be of some use to those readers whose interests are predominantly in representation theory and secondly, it is required for the statement of several of our results, particularly those concerning Poincaré series and the equvariant theory.

There is an attractive explicit description of the rational homology of the configuration spaces $C_n = C_n(\mathbb{R}^d;P)$ for any space P. In particular, it accounts for the stability property. It is in terms of two basic algebraic operations on the homology induced by juxtaposition of configurations. "Juxtaposition" means the map

$$C_n \times C_m \to C_{n+m} \tag{2.1}$$

which puts two labelled configurations side-by-side in $\mathbb{R}^d$, after moving the second in a fixed direction to avoid coincidences:

$$[((x_1,p_1),\dots,(x_n,p_n)),((x_1',p_1'),\dots,(x_m',p_m'))] \longmapsto \\ [(x_1,p_1),\dots,(x_n,p_n),(\tilde{x}_1',p_1'),\dots\tilde{x}_m',p_m')],$$

where

$$\tilde{x}_i' = x_i' + (\Sigma\|x_j\| + \Sigma\|x_k'\|)\,u,$$

for some chosen unit vector $u \in \mathbb{R}^d$.

The map (2.1), which is associative and commutative up to homotopy, induces a bilinear map

$$H_i(C_n) \times H_j(C_m) \to H_{i+j}(C_{n+m})$$

which makes

$$R = H_*(C) = \bigoplus_{n\geq 0} H_*(C_n)$$

into a graded-commutative ring, where C_0 is taken to be the space consisting of a single point. (The grading here is the homological grading, but in fact R is bigraded, the second grading being n.) But more is true, for by permitting the unit vector u to vary on the sphere S^{d-1} in $\mathbb{R}^d$ we can extend (2.1) to a map

$$C_n \times C_m \times S^{d-1} \to C_{n+m}. \tag{2.2}$$

which is a special case of a construction of Boardman and Vogt [Lecture Notes in Math. 347, Springer-Verlag 1973].

If σ is the generator of $H_{d-1}(S^{d-1})$ and $a \in H_i(C_n)$, $b \in H_j(C_m)$, the image of $a \otimes b \otimes \sigma$ in $H_{i+j+s}(C_{n+m})$ will be denoted by $\{a, b\}$. (Here $s = d-1$.) This operation satisfies three algebraic identities reflecting the geometry of the maps (2.2):

$$\begin{aligned} \{b, a\} &= -(-1)^{\delta'_a \delta'_b}\{a, b\}\,, \\ (-1)^{\delta'_a \delta'_b}\{\{a, b\}, c\} + (-1)^{\delta'_b \delta'_c}&\{\{b, c\}, a\} + (-1)^{\delta'_c \delta'_a}\{\{c, a, \}, b\} = 0\,, \\ \{ab, c\} &= (-1)^{\delta_b \delta_c} b\{a, c\} + a\{b, c\}. \end{aligned} \tag{2.3}$$

Here δ_a is the (homology) degree of a, and $\delta'_a = \delta_a + s$.

The relations (2.3) are special cases of [C, Theorem 1.2, p. 215], where the more general case of homology with coefficients in characteristic p is considered.

The first two of these identities tell us that the operation $\{\ ,\ \}$ makes $R[s]$ into a graded Lie algebra, where $R[s]$ is R with its degrees shifted up by s; i.e., if R_i is the homogeneous component of R of degree i, then $\deg_{R[s]} R_i = i + s$ or equivalently, $R[s]_j = R_{j-s}$. If we ignore the signs then the identities (2.3) are precisely those satisfied by the Poisson bracket in the commutative ring of smooth functions on a symplectic manifold. Motivated by this, we shall say that a graded-commutative algebra R with an operation

$$\{\ ,\ \}\ :\ R_i \times R_j \to R_{i+j+s}$$

satisfying (2.3) is a *Poisson algebra of degree s*. We can now state

(2.4) Theorem. *The algebra $R = H_*(C)$ is the free Poisson algebra of degree s generated by $H_*(C_1) = H_*(P)$.*

This theorem is essentially the statement of [C, Theorem 3.2, p. 226], where the more general characteristic p situation is treated. However for the computation of Poincaré series, we shall need the fact that the generating subspace is not weight-homogeneous (see the paragraph preceding (4.12) below).

(2.5) Remark. To make the statement of (2.4) more explicit, we shall make a brief algebraic digression. Let $U = \oplus_{i \geq 0} U_i$ be a graded $\mathbb{Q}$-vector space. Write $\mathcal{T}(U)$ for the free associative (or tensor) algebra on U

$$\mathcal{T}(U) = \bigoplus_{n \geq 0} \otimes^n U \tag{2.5.1}$$

with the (usual) grading induced by that of U. It is well known that the Lie algebra $\mathcal{L}(U) \subset \mathcal{T}(U)$ generated by U inside $\mathcal{T}(U)$ is the free Lie algebra generated by U. The symmetric and exterior algebras on U will be denoted by $\mathcal{S}(U)$ and $\Lambda(U)$ respectively, while $\mathcal{A}(U)$ will denote the free graded-commutative algebra on U. With this notation

$$\mathcal{A}(U) = \bigotimes_{i \geq 0} \mathcal{A}(U_i) \tag{2.5.2}$$

and $\mathcal{A}(U_i) \cong \mathcal{S}(U_i)$ if i is even, while $\mathcal{A}(U_i) \cong \Lambda(U_i)$ if i is odd.

Let $\mathcal{P}_s(U)$ (or $\mathcal{P}(U)$ when the value of s is apparent from the context) denote the free Poisson algebra of degree s (≥ 1) on U. If we write $\mathcal{L}_s(U)$ for $\mathcal{L}(U[s])[-s]$, the "free graded Lie algebra with bracket of degree s", we claim that

$$\mathcal{P}(U) = \mathcal{A}(\mathcal{L}(U[s])[-s]) = \mathcal{A}(\mathcal{L}_s(U)). \tag{2.5.3}$$

To see (2.5.3), it is enough to observe that the bracket on $\mathcal{L}_s(U)$ does indeed induce a unique Poisson bracket (i.e. one which satisfies the identities (2.3)) on $\mathcal{A}(\mathcal{L}_s(U))$. For then it is clear that any linear map from U to a Poisson algebra $\mathcal{P}$ extends uniquely to a homomorphism of Poisson algebras $\mathcal{A}(\mathcal{L}_s(U)) \to \mathcal{P}$.

Let us say that U is *locally finite* if $\dim U_i < \infty$ for each i. Then clearly if U is locally finite, $\mathcal{T}(U)$ and $\mathcal{A}(U)$ are also locally finite if and only if $U_0 = 0$. In fact we have

$$\mathcal{A}(U) = \mathcal{A}(U^\sharp) \otimes \mathcal{S}(U_0) \tag{2.5.4}$$

where $U^\sharp = \oplus_{i>0} U_i$, and all the elements of $\mathcal{S}(U_0)$ have degree zero. Thus if $U_0 = \mathbb{Q}v_0$ has dimension one, $\mathcal{A}(U) = \mathcal{A}(U^\sharp)[v_0]$.

It will be convenient to write $\mathcal{A}^\sharp(U)$ for $\mathcal{A}(U^\sharp)$, and also

$$\mathcal{P}^\sharp(U) = \mathcal{A}^\sharp(\mathcal{L}_s(U)), \tag{2.5.5}$$

so that the free Poisson algebra $\mathcal{P}(U)$ is $\mathcal{P}^\sharp(U) \otimes \mathcal{S}(U_0)$. This completes our algebraic digression.

(2.6) Remark. We have been implicitly assuming that $d \geq 2$. If $d = 1$ and $s = 0$ then $C_n \simeq P^n$, and the algebra $H_*(C)$ is the tensor algebra on $H_*(P)$, which is no longer commutative. Then the bracket $\{\ ,\ \}$ reduces to the graded commutator in $H_*(C)$, and the analogue of Theorem (2.4) is the purely algebraic fact that $H_*(C)$ is the universal enveloping algebra of the free graded Lie algebra on $H_*(P)$ (see (4.2) below).

3. Configuration spaces and iterated loop spaces.

Although as mentioned above, Theorem (2.4) is not new, we shall give a treatment here, in which we deduce it from the relation between labelled configuration spaces and iterated loop spaces, as we shall want to use the loop-space picture in any case.

If P is a topological space we write $S^d(P_+)$ for the d-fold iterated suspension of P_+, where P_+ denotes the disjoint union of P with another point ∞, taken as a base-point. It is convenient to identify $S^d(P_+)$ with the quotient space obtained from $D^d \times P$, where

D^d is the unit disc in $\mathbb{R}^d$, by collapsing the subspace $S^{d-1} \times P$ to a single point, which is again denoted ∞.

The iterated loop space $\Omega^d S^d(P_+)$ can be defined as the space of maps $D^d \to S^d(P_+)$ which take S^{d-1} to ∞, or equally as the space of maps $\mathbb{R}^d \to S^d(P_+)$ which are ∞ outside a compact subspace of $\mathbb{R}^d$. Notice that P is naturally a subspace of $\Omega^d S^d(P_+)$, because of the inclusion

$$P \hookrightarrow \mathrm{Map}\,(D^d \ ; \ D^d \times P).$$

An element of $\Omega^d S^d(P_+)$ has a *degree*, which is the degree of the composite

$$D^d/S^{d-1} \to (D^d \times P)/(S^{d-1} \times P) \to D^d/S^{d-1}\,,$$

where the second map forgets P. Write $\Omega^d S^d_{(n)}(P_+)$ for the elements of degree n.

The crucial thing for us is that there is an inclusion

$$\phi_n \ : \ C_n(\mathbb{R}^d \ ; \ P) \to \Omega^d S^d_{(n)}(P_+)\,.$$

Strictly speaking, ϕ_n is defined on the subspace of $C_n(\mathbb{R}^d; P)$ consisting of configurations $\{(x_i, p_i)\}$ with $\|x_i - x_j\| > 2$ when $i \neq j$, but C_n may be replaced by that subspace, to which it is homotopy equivalent. The definition of $\phi_n(\{(x_i,\ p_i)\}) : \ \mathbb{R}^d \to (D^d \times P)/(S^{d-1} \times P)$ is

$$\begin{aligned}\phi_n(\{(x_i,\ p_i)\})(x) \ &= \ (x - x_i,\ p_i) \in D^d \times P \quad \text{if } \|x - x_i\| \leq 1 \\ &= \ \infty\,, \quad \text{otherwise}\,.\end{aligned}$$

Let us write $\Omega_n = \Omega^d S^d_{(n)}(P_+)$. As an element of Ω_n has "compact support" in $\mathbb{R}^d$ there are "juxtaposition" maps

$$\Omega_n \times \Omega_m \times S^{d-1} \to \Omega_{n+m}$$

analogous to those for C_n, which are obtained by "translating the support" of the second loop. The inclusions $C_n \to \Omega_n$ are compatible with these maps. Thus $H_*(\Omega) = \oplus_{n \in \mathbb{Z}} H_*(\Omega_n)$ is a Poisson algebra of degree s, and

$$\phi \ : \ H_*(C) \to H_*(\Omega)$$

is a homomorphism of Poisson algebras.

The homotopy type of the spaces Ω_n is independent of n, for juxtaposition of any fixed loop in Ω_k gives a homotopy equivalence $\Omega_n \to \Omega_{k+n}$. Thus

(3.1) Proposition. *For any $n \in \mathbb{Z}$, we have*

$$H_*(\Omega) \ = \ H_*(\Omega_0)[v_0, v_0^{-1}],$$

where $v_0 \in H_0(\Omega_1)$ is the class of a point.

The main result linking configuration spaces to loop spaces is

(3.2) Theorem. *The map ϕ_n induces an isomorphism*

$$H_k(C_n) \to H_k(\Omega_n)$$

for $k \le d_n$, where $d_n \to \infty$ as $n \to \infty$.

The remainder of this section will be devoted to a proof of this result, which we could not find in exactly this form in the literature, where usually (cf. e.g., [M], [JB]) the space considered is "reduced". This is the space obtained from $\amalg_n C_n$ by identifying configurations differing only by points labelled with a chosen base-point in the connected space P. The "unreduced" statement (3.2) and particularly the structure theorem (2.4) are distinctly harder to prove, because of the need (cf. §4) for precision about the different connected components C_n and Ω_n.

We shall abuse notation by also using v_0 to denote the generator of $H_0(C_1)$ which corresponds to the v_0 of (3.1) under the map ϕ_1.

With this understanding, we have the following commutative diagram, where the horizontal arrows are (right) multiplication by v_0

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_*(C_{n-1}) & \longrightarrow & H_*(C_n) & \longrightarrow & H_*(C_{n+1}) & \longrightarrow \cdots \\ & \phi_{n-1}\Big\downarrow & & \phi_n\Big\downarrow & & \phi_{n+1}\Big\downarrow & \\ \cdots \longrightarrow & H_*(\Omega_{n-1}) & \longrightarrow & H_*(\Omega_n) & \longrightarrow & H_*(\Omega_{n+1}) & \longrightarrow \cdots \end{array} \qquad (3.3)$$

It is clear from (3.1) that the maps in the lower row of (3.3) are all isomorphisms. The first step in the proof of the theorem (3.2) is to show that the upper horizontal maps are injective, i.e.

(3.4) Proposition. *Write $i_n : H_*(C_n) \to H_*(C_{n+1})$ for the map given by right multiplication by v_0. Then i_n is injective for all $n \ge 1$.*

For the proof of (3.4) we shall require the "transfer maps" $t_{n,m} : H_*(C_n) \to H_*(C_m)$, which are defined as follows for $m \le n$. Denote by $\tilde{C}_n$ the space of (ordered) n-tuples

$$((x_1,\ p_1), \ldots,\ (x_n,\ p_n))$$

with $x_i \in \mathbb{R}^d$, $p_i \in P$, and $x_i \ne x_j$ for $i \ne j$, so that $C_n = \tilde{C}_n/\mathrm{Sym}_n$ is the space of orbits of the symmetric group Sym_n on $\tilde{C}_n$, which is an $n!$-fold covering space of C_n. Define, for $m \le n$, $C_{n,m} := \tilde{C}_n/(\mathrm{Sym}_m \times \mathrm{Sym}_{n-m})$. This is an $\binom{n}{m}$-fold covering of C_n, the fibres corresponding to the various choices of m points from n, and we have an obvious forgetful map $C_{n,m} \to C_m$. The transfer $t_{n,m}$ is defined as the composite

$$H_*(C_n) \xrightarrow{\mathrm{tr}} H_*(C_{n,m}) \to H_*(C_m),$$

where tr is the transfer map of the covering $C_{n,m} \to C_n$ and the second map is induced by the forgetful one referred to above.

(3.5) Lemma. *With the above notation, we have, for $m \leq n$,*

$$t_{n+1,m} \circ i_n = t_{n,m} + i_{m-1} \circ t_{n,m-1} \tag{3.5.1}$$

Proof. We regard C_n as a subspace of C_{n+1}, using the map (2.1) with $m = 1$ and a point of C_1 chosen and fixed. This embedding induces the map $i_n : H_*(C_n) \to H_*(C_{n+1})$. Now consider the restriction to $C_n \subset C_{n+1}$ of the covering $\pi_{n+1,m} : C_{n+1,m} \to C_{n+1}$; this restriction induces the left side of (3.5.1). The (unordered) $(n+1)$-tuple $\tilde{c} \in C_{n+1}$ which is the image of $c \in C_n$ consists of the n points of c together with one other, say p_c ($\in \mathbb{R} \times P$); $\pi^{-1}_{n+1,m}(\tilde{c})$ is in bijection with the set of $\binom{n+1}{m}$ m-subsets of $\tilde{c}$, which evidently divide into the $\binom{n}{m-1}$ which contain p_c and the $\binom{n}{m}$ others. It follows that this restriction is the disjoint union of the coverings $C_{n,m} \to C_n$ and $C_{n,m-1} \to C_n$. This proves (3.5). $\square$

When $m = n$, $t_{n,m}$ is clearly the identity map, so that a particular case of (3.5) is

$$t_{n+1} \circ i_n - i_{n-1} \circ t_n = \mathrm{id}_n \tag{3.5.2}$$

where t_n denotes $t_{n,n-1}$ and id_n is the identity map on $H_*(C_n)$. This relation may be written symbolically

$$[t, i] = 1 \tag{3.5.3}$$

where [,] is the commutator and 1 denotes the identity map. This latter relation encodes (3.5) in full generality if one takes into account the relation

$$t_{n,n-r} = \frac{1}{r!} t_{n-r+1} \circ \ldots \circ t_{n-1} \circ t_n$$

(for $0 \leq r \leq n$), which we shall not require in this work. We shall work below with the symbolic form of the relation; in particular i^k and t^k will respectively denote k iterations of the maps i_n and t_n, so that $i^k = i_n \circ i_{n+1} \circ \ldots \circ i_{n+k-1}$ and the above relation reads $\frac{1}{r!} t^r = t_{n,n-r}$.

The injectivity of the maps i_n will follow from

(3.6) Lemma. *(cf. [S2]) Suppose we have a sequence*

$$\ldots \longrightarrow A_n \xrightarrow{i_n} A_{n+1} \longrightarrow \ldots$$

of $\mathbb{Q}$-vector spaces such that $A_i = 0$ for $i < 0$ and such that there are homomorphisms $t_n : A_n \to A_{n-1}$ (for $0 \leq n$) which satisfy the relation (3.5.2). Define $K_n = \ker t_n$. Then for $n \geq 0$,

(i) $A_n = K_n \oplus i_{n-1}(A_{n-1})$

(ii) $t_{n+1} \circ i_n$ is an automorphism of A_n

Proof. In this proof, which is by induction on $n \geq 0$, we shall use the symbolic notation explained above. The statements are trivial for $n = 0$. Observe that if we know (ii) for $n < N$, then (i) follows for $n = N$, since if $a \in K_N \cap iA_{N-1}$, then $a = ia'$ with $a' \in A_{N-1}$ and $ta = 0$. So $tia' = 0$ whence by (ii) for A_{N-1}, $a' = 0$, so that $a = 0$. Further, given $b \in A_N$, $tb \in A_{N-1}$; so $tb = tib'$ for some $b' \in A_{N-1}$ and $b - ib' \in K_N$. This proves that $A_N = K_N \oplus i(A_{N-1})$.

Now suppose (ii) holds for $n < N$. Then we have (i) for $n \leq N$, whence

$$A_N = K_N \oplus iK_{N-1} \oplus i^2 K_{N-2} \oplus ... \oplus i^N K_0 \qquad (3.6.1)$$

But one deduces easily from (3.5.3) that

$$[t, i^k] = ki^{k-1} \text{ , for } k = 1, 2, 3, ... \qquad (3.6.2)$$

It follows that ti acts on $i^k K_{N-k}$ as multiplication by $k+1$ ($k = 0, 1, 2, ...$). Applying this to (3.6.1) proves (ii) for $n = N$.

□

The next result is clear from the proof of (3.6).

(3.7) Corollary. *With notation as in (3.6), we have, for each $n \geq 0$,*

$$A_n = K_n \oplus iK_{n-1} \oplus i^2 K_{n-2} \oplus ... \oplus i^n K_0$$

We are now in a position to give the

Proof of (3.4). Apply (3.6) with $A_n = H_*(C_n)$ and the maps i_n and t_n as in (3.5). Since ti is an automorphism, i is clearly injective. □

We shall deduce (3.2) with the aid of the following result, which may be found in [S3, Theorem (B)]. It is also alluded to in [S1, p. 213] and [McS] and proved (for the case where P is a point, but the general case is no different) in [McD].

(3.8) Theorem (cf. references above). *The direct limits of the two rows of the diagram (3.3) are isomorphic.*

Proof of (3.2). We clearly have a commutative diagram

$$\begin{array}{ccc}
H_i(C_n) & \xrightarrow{\alpha} & \varinjlim_{m\to\infty} H_i(C_m) \\
\phi_n \downarrow & & \psi \downarrow \\
H_i(\Omega_n) & \xrightarrow{\beta} & \varinjlim_{m\to\infty} H_i(\Omega_m)
\end{array}$$

Now β is an isomorphism by the remarks preceding (3.1) and α is injective by (3.4). But (3.8) asserts that ψ is an isomorphism, from which (3.2) follows. □

It is immediate from (3.8) that

(3.9) Corollary. *The maps ϕ_n induce an isomorphism of Poisson algebras*

$$H_*(C)[v_0^{-1}] \xrightarrow{\simeq} H_*(\Omega)\,.$$

An easy consequence of Theorem (3.2) is

(3.10) Corollary. *Let $\mathfrak{G} = \mathfrak{gl}_n$. For any integer $n \geq 1$, we have*

$$Poin_{\mathfrak{G}_{rs}(n)}(t) \equiv \prod_{j=1}^{\infty}(1-t^{2j}).Poin_{\Omega^2 S^2_{(0)}(P_+)}(t) \ (mod\ t^{d_n})$$

where P is the space $\mathbb{P}(\mathbb{C}^\infty)$ and $d_n \to \infty$ as $n \to \infty$.

It is therefore clear that the power series on the right is the "stable version" of $\mathrm{Poin}_{\mathfrak{G}_{rs}(n)}(t)$, whose existence was proved in [L1].

4. Homology of iterated loop spaces.

To complete the proof of Theorem (2.4) concerning the Poisson structure of $H_*(C)$, we focus in this section on $H_*(\Omega)$, which we know by (3.10) will also suffice to provide the "stable version" of the Poincaré series which we seek. The results (4.4), (4.9) and (4.9)$'$ below may be found in [C, loc. cit.] where the more general situation of characteristic p coefficients is considered. We begin with an arbitrary connected space Z with base point and show that $H_*(\Omega^d SZ)$ has the required Poisson structure. Our application will be to the case where $Z = S^{d-1}P_+$. The first key result which we shall require is

(4.1) Proposition. *([MM]). Let X be a connected H-space. Then*

(i) There is an injection $\imath : \pi_*(X) \otimes \mathbb{Q} \to H_*(X, \mathbb{Q})$ whose image is precisely the set of primitive elements of the (co-commutative) Hopf algebra $H_*(X, \mathbb{Q})$ (recall that x is a primitive element if $\Delta(x) = x \otimes 1 + 1 \otimes x$, Δ being the comultiplication).

(ii) $H_*(X, \mathbb{Q})$ is the universal enveloping algebra of the Lie algebra formed by its primitive elements.

The result (i) may be found in [MM, p. 263], while (ii) is [op. cit., cor 4.18].

(4.1)$'$ Corollary. *If X is homotopy commutative, $H_*(X, \mathbb{Q})$ is the free graded-commutative algebra on its primitive elements.*

The corollary follows from (4.1)(ii) immediately, since in the commutative case, the Lie algebra of primitive elements has trivial Lie product.

The other key result we require is

(4.2) Proposition. *If Z is connected then $H_*(\Omega SZ)$ is the free associative unital algebra on the reduced homology $\tilde{H}_*(Z)$. That is, $H_*(\Omega SZ) \cong \mathcal{T}(\tilde{H}_*(Z))$.*

This result is well known. Its proof is a straightforward consequence of the following result of I. James [J].

Proposition. *If Z is connected, with base point x_0, then the natural map $Z \to \Omega SZ$ induces a homotopy equivalence $M_Z \to \Omega SZ$, where M_Z is the free topological semigroup generated by Z, with the single relation $x_0 = 1$.*

Proof of (4.2). To see how (4.2) follows from James's result, observe that M_Z is filtered by length of word; if $M_Z^{(n)}$ is the subspace consisting of words of length at most n, then $M_Z^{(n)}/M_Z^{(n-1)} \simeq Z \wedge Z \wedge ... \wedge Z$, the n-fold smash product. Hence $H_*(M_Z^{(n)}, M_Z^{(n-1)}) \cong \otimes^n \tilde{H}_*(Z)$, and (4.2) follows easily. $\square$

(4.3) Corollary. *If Z is connected, then*

(i) $\pi_*(\Omega SZ)\otimes\mathbb{Q}$ is the free Lie algebra on $\tilde{H}_*(Z)$.

(ii) $H_*(\Omega SZ) \cong \mathcal{A}(\mathcal{L}(\tilde{H}_*(Z)))$

Proof. Take $X = \Omega SZ$ in (4.1). Then by (4.1)(i), $\pi_*(\Omega SZ)$ may be identified with the set prim $H_*(\Omega SZ)$ of primitive elements of $H_*(\Omega SZ)$. By (4.2), $H_*(\Omega SZ) \cong \mathcal{T}(\tilde{H}_*(Z))$. But for any graded vector space V, the tensor algebra $\mathcal{T}(V)$ is naturally a Hopf algebra, with prim $\mathcal{T}(V) = \mathcal{L}(V)$ (cf. [MM]), so that $\pi_*(\Omega SZ) \cong$ prim $H_*(\Omega SZ) \cong$ prim $\mathcal{T}(\tilde{H}_*(Z)) \cong \mathcal{L}(\tilde{H}_*(Z))$. This is the statement (i), while (ii) is immediate from (4.1)$'$. □

We are now in a position to prove

(4.4) Theorem. *Let P be a connected topological space and let d be an integer greater than 1. For any $n \in \mathbb{Z}$, let Ω_n be the corresponding connected component of the iterated loop space $\Omega^d S^d(P_+)$ (see §2 above). Then in the notation of (2.5), $H_*(\Omega_0)$ is the free graded-commutative algebra on $\mathcal{L}(H_*(P)[s])[-s]^\sharp = \mathcal{L}_s(H_*(P))^\sharp$, where $s = d-1$. That is, $H_*(\Omega_0) \cong \mathcal{P}^\sharp(H_*(P))$.*

Proof. From (4.1), we have $H_*(\Omega_0) \cong \mathcal{A}(\text{prim } H_*(\Omega_0)) \cong \mathcal{A}(\pi_*(\Omega_0)\otimes\mathbb{Q})$. But, recalling that $^\sharp$ indicates that the degree zero component should be deleted, we have

$$\begin{aligned}\pi_*(\Omega_0)\otimes\mathbb{Q} &\cong \pi_*(\Omega^d S^d(P_+))\otimes\mathbb{Q}^\sharp\\ &\cong \pi_*(\Omega S(S^{d-1}P_+))\otimes\mathbb{Q}[-(d-1)]^\sharp\\ &\cong \pi_*(\Omega S(Z))\otimes\mathbb{Q}[-s]^\sharp \text{ where } Z = S^{d-1}P_+ \text{ is connected}\\ &\cong \mathcal{L}(\tilde{H}_*(Z))[-s]^\sharp \text{ by (4.3)(i)}\end{aligned}$$

But $\tilde{H}_*(Z) = \tilde{H}_*(S^{d-1}P_+) \cong H_*(P)[d-1]$. It follows that

$$H_*(\Omega_0) \cong \mathcal{A}^\sharp\left(\mathcal{L}_s(H_*(P))\right),$$

as required.

□

Now (4.4) exhibits $H_*(\Omega_0)$ as a Poisson algebra, with the "Samelson" bracket, induced through the identifications of the proof of (4.4) by the non-commutative ring structure (see (4.2)) on the space $H_*(\Omega SZ)$, where $Z = S^{d-1}P_+$. But we have also introduced a Poisson bracket on $H_*(\Omega_0)$ by means of the juxtaposition maps described in the paragraph preceding (3.1). In order to apply (4.4) to obtain results concerning the stability degree of the homology $H_*(C)$ using the diagram (3.3) or the result (3.11), we need to know that the two brackets coincide. This is addressed in

(4.5) Proposition. *Let Y be a connected space with a base point and let $d > 1$ be an integer. Assume $\pi_i(Y) = 0$ for $i < d$ and that $\pi_d(Y) = \mathbb{Z}$, so that $\Omega^d Y$ has (homotopy equivalent) connected components $\{\Omega_n, n \in \mathbb{Z}\}$. Then there exist natural homomorphisms $T : H_i(\Omega^d Y) \to H_{i+d-1}(\Omega Y)$ (called the transgression homomorphisms) which satisfy* *(i) For each $i \geq 0$ we have commutative diagrams*

$$\begin{array}{ccc} \pi_i(\Omega^d Y) & \xrightarrow{\cong} & \pi_{i+d-1}(\Omega Y) \\ \downarrow & & \downarrow \\ H_i(\Omega^d Y) & \xrightarrow{T} & H_{i+d-1}(\Omega Y) \end{array} \tag{4.5.1}$$

where the vertical maps are the Hurewicz maps.

(ii) Let $\epsilon : H_*(\Omega^d Y) \to \mathbb{Q}$ be the augmentation map (which is zero except in degree 0). Then

$$T(xy) = T(x)\epsilon(y) + \epsilon(x)T(y) \tag{4.5.2}$$

(iii) For $a, b \in H_*(\Omega^d Y)$, we have

$$T\{a, b\}_{d-1} = [Ta, Tb] \tag{4.5.3}$$

where $\{\ ,\ \}_{d-1}$ is the Poisson bracket on $H_*(\Omega^d Y)$ and $[\ ,\ ]$ is the graded (Samelson) bracket on $H_*(\Omega Y)$

Since this statement involves nothing more than standard constructions in algebraic topology and is well known to experts, we shall here simply point out that the homomorphisms T are defined as compositions, induced by

$$H_i(\Omega^d Y) \to H_{i+d-1}(S^{d-1}\Omega^d Y) \to H_{i+d-1}(\Omega Y),$$

where the first map is the suspension homomorphism and the second is induced by the adjunction map $S^k\Omega^k(X) = S^k \wedge \Omega^k(X) \to X$, defined for any space X by $(\sigma, \phi) \mapsto \phi(\sigma) \in X$, ($\sigma \in S^k$, $\phi \in \Omega^k(X) = \mathrm{Map}(S^k, X)$), i.e. "evaluation of loops" (here $\wedge$ denotes the smash product of spaces).

From these maps we obtain the commutative diagrams (4.5.1).

(4.6) Lemma. *With notation as in (4.5), we have*

(i) T induces isomorphisms

$$prim\ H_i(\Omega_0) \to prim\ H_{i+d-1}(\Omega Y)$$

for $i \geq 1$.

(ii) Write g_0 for the generator of $\pi_d(Y)$ which corresponds to the component Ω_1. Let $v_0 \in H_0(\Omega_1)$ be the image of g_0 under the Hurewicz map $\pi_0(\Omega^d Y) \to H_0(\Omega^d Y)$. Define the subspace $U \subset H_*(\Omega_0) \oplus H_*(\Omega_1)$ by

$$U = prim\ H_*(\Omega_0) \oplus \mathbb{Q}v_0$$

Then T induces an isomorphism $: U \to prim\ H_*(\Omega Y)[-(d-1)]$.

Proof. Since ΩY is a connected H-space, we see from (4.1)(i) that the Hurewicz maps induce isomorphisms $\pi_j(\Omega Y)\otimes\mathbb{Q} \to$ prim $H_j(\Omega Y)$ for all j. Moreover for $i > 0$ we also have isomorphisms $\pi_i(\Omega^d Y)\otimes\mathbb{Q} \to \text{prim } H_i(\Omega_0)$. Taking the diagrams (4.5.1) into account, this implies (i).

When $i = 0$, the diagram (4.5.1) becomes

$$\begin{array}{ccc} \mathbb{Z} \cong < g_0 > = \pi_0(\Omega^d Y) & \xrightarrow{\cong} & \pi_{d-1}(\Omega Y) \cong \mathbb{Z} \\ \big\downarrow{\scriptstyle g_0 \mapsto v_0} & & \big\downarrow \\ \mathbb{Q}[v_0, v_0^{-1}] \cong H_0(\Omega^d Y) & \xrightarrow{T} & H_{d-1}(\Omega Y) \cong \mathbb{Q} \end{array} \tag{4.6.4}$$

where $\mathbb{Q}[v_0, v_0^{-1}]$ may be thought of as the group ring of the fundamental group $\mathbb{Z} \cong < g_0 >$ and $T : \mathbb{Q}[v_0, v_0^{-1}] \to \mathbb{Q}$ is the derivation which maps v_0^k to k.

It follows that T maps $\mathbb{Q}v_0$ isomorphically to prim $H_{d-1}(\Omega Y)$. Combining this fact with (i), we see that T maps $U = \text{prim } H_*(\Omega_0)\oplus\mathbb{Q}v_0$ isomorphically to $\oplus_{i\geq 0}\text{prim } H_{i+d-1}(\Omega Y)$ $= \text{prim } H_*(\Omega Y)[-(d-1)]$.

□

(4.7) Corollary. *In the notation above, we have*
(i) $\oplus_{n\geq 0} H_*(\Omega_n) \cong \mathcal{A}(U)$.
(ii) $H_*(\Omega^d Y) \cong H_*(\Omega_0)[v_0, v_0^{-1}] \cong \mathcal{A}(U)[v_0^{-1}]$.

Proof. We have

$$\begin{aligned} \mathcal{A}(U) &\cong \mathcal{A}(\text{prim } H_*(\Omega_0) \oplus \mathbb{Q}v_0) \\ &\cong \mathcal{A}(\text{prim } H_*(\Omega_0))[v_0] \\ &\cong H_*(\Omega_0)[v_0] \\ &\cong \bigoplus_{n\geq 0} H_*(\Omega_n) \end{aligned}$$

which proves (i). Part (ii) is now clear.

□

Henceforth we take $Z = S^{d-1}P_+$ and $Y = SZ = S^d P_+$. Thus $\Omega Y = \Omega SZ$.

(4.8) Corollary. *The Poisson structure on $H_*(\Omega_0)$ arising from (4.4) coincides with that which comes from the juxtaposition maps of §3.*

Proof. By (4.6)(i), the transgression T induces the isomorphism

$$\begin{aligned} \text{prim } H_*(\Omega_0) &\xrightarrow{\sim} (\text{prim } H_*(\Omega SZ)[-(d-1)])^\sharp \\ &\cong \mathcal{L}(\tilde{H}_*(Z))[-(d-1)]^\sharp \end{aligned}$$

of the proof of (4.4) (recall from (2.5) that the symbol $^\sharp$ indicates that the degree zero component should be discarded). Moreover by (4.5)(iii), T maps the Poisson bracket in $H_*(\Omega_0)$ to the Lie product in $\mathcal{L}(\tilde{H}_*(Z))[-(d-1)]^\sharp$. This proves (4.8). □

(4.9) Corollary. *The isomorphism*

$$T : U \to prim\ H_*(\Omega SZ)[-(d-1)] \cong \mathcal{L}(H_*(P)[d-1])[-(d-1)]$$

of (4.6)(ii) extends to an isomorphism of Poisson algebras

$$\tilde{T} : \bigoplus_{n\geq 0} H_*(\Omega_n) \to \mathcal{P}(H_*(P)) \cong \mathcal{A}(\mathcal{L}_s(H_*(P)))$$

where $s = d-1$ and $\mathcal{P}$ denotes the free Poisson algebra of degree s.

Proof. From (4.7) and (4.8), there is an isomorphism of Poisson algebras

$$\rho : \bigoplus_{n\geq 0} H_*(\Omega_n) \xrightarrow{\sim} \mathcal{A}(U).$$

But from (4.6)(ii), T provides an isomorphism

$$\mu : U \xrightarrow{\sim} \text{prim } H_*(\Omega Y)[-(d-1)] \cong \mathcal{L}(H_*(P)[d-1])[-(d-1)],$$

where $Y = SZ = S^d P_+$, the latter identification coming from (4.3). The isomorphism μ clearly extends to $\nu : \mathcal{A}(U) \to \mathcal{P}(H_*(P))$. The required isomorphism $\tilde{T}$ is now obtained by composing ρ with ν. $\square$

The next statement is now clear.

(4.9)′ Corollary. *In the notation of (4.9), $\tilde{T}$ induces an isomorphism of Poisson algebras $H_*(\Omega^d S^d P_+) \to \mathcal{P}(H_*(P))[v_0^{-1}]$.*

We now turn to the diagram (3.3) and the proof of Theorem (2.4). We know from (4.3) that prim $H_*(\Omega Y) \cong \mathcal{L}(\tilde{H}_*(Z))$, which contains $\tilde{H}_*(Z) = H_*(P)[d-1]$. Hence there is a natural inclusion $H_*(P) \hookrightarrow H_*(\Omega Y)[-(d-1)]$. The purpose of the next result is to show that this inclusion is precisely the map which is induced, after composition with $T : H_*(\Omega^d Y) \to H_*(\Omega Y)[-(d-1)]$, by the map $P \to \Omega^d Y = \Omega^d S^d P_+$, which is, up to homotopy, the map ϕ_1 of the diagram (3.3).

(4.10) Proposition. *Let $i : P \to \Omega^d S^d P_+$ be the natural inclusion. The composition $H_*(P) \xrightarrow{i_*} H_*(\Omega^d S^d P_+) \xrightarrow{T} H_*(\Omega Y)[-s]$ coincides with the inclusion $H_*(P) \to \mathcal{L}(H_*(P)[s])[-s] = prim\ H_*(\Omega Y)[-s]$, where $s = d-1$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H_*(P) \arrow[r, "\sigma_1"] \arrow[d, "i_*"] & H_*(Z = S^s P_+)[-s] \arrow[d, "S^s i_*"] & \\
H_*(\Omega^d S^d P_+) \arrow[r, "\sigma_2"] & H_*(S^s \Omega^d SZ)[-s] \arrow[r, "\alpha"] & H_*(\Omega SZ)[-s]
\end{tikzcd} \tag{4.10.1}$$

where σ_1 and σ_2 are the suspension isomorphisms and α is induced by the adjunction map.

Since the transgression $T = \alpha \circ \sigma_2$, the first map of the Proposition is $\alpha \circ \sigma_2 \circ i_*$. We show that the second is $\alpha \circ S^{d-1} i_* \circ \sigma_1$, which is the same map by commutativity.

Now $\alpha \circ S^{d-1} i_* : \tilde{H}_*(Z) \to H_*(\Omega S Z)$ is induced in homology by the composite

$$Z = S^{d-1} P_+ \xrightarrow{S^{d-1} i} S^{d-1}\, \Omega^d S Z \longrightarrow \Omega S Z$$

where the second map is the adjunction. But this map is up to homotopy just the map which sends Z to the words of length ≤ 1 in M_Z (see the proof of (4.2)). The result now follows from the realisation of $H_*(\Omega S Z)$ as the tensor algebra on $H_*(P)[d-1]$ which is described in the proof of (4.2) above. □

(4.10)$'$ Corollary. *Maintain the notation of (4.10). The image $i_*(H_*(P))$ coincides with $v_0 U_1 \oplus \mathbb{Q} v_0$, where U_1 is the subspace of $H_*(\Omega_0)$ which corresponds to $H_*(P)^\sharp \subset \mathcal{P}(H_*(P))$ under the isomorphism $\tilde{T}$ of (4.9).*

Proof. It follows from (4.10) that T maps $i_*(H_*(P))$ isomorphically to $H_*(P) \subset \mathcal{P}(H_*(P))$. But by (4.5)(iii), if a and b are elements of $H_*(\Omega^d S^d P_+)$ which have degree greater than zero, then $T(ab) = 0$. Hence by (4.9), $i_*(H_*(P))$ is contained in $\sum_{k \geq 0} v_0^k \text{prim } H_*(\Omega_0) \oplus v_0^k \mathbb{Q}$. Since $i_*(H_*(P)) \subset H_*(\Omega_1)$, it follows that $i_*(H_*(P)) \subset v_0 \text{prim } H_*(\Omega_0) \oplus v_0 \mathbb{Q}$. But T maps the latter subspace isomorphically to prim $H_*(\Omega S Z)[-s]$, since for any element a of degree greater than zero, (4.5)(iii) shows that $T(a v_0) = T(a)$. This implies the result. □

Proof of Theorem (2.4). The isomorphism $H_*(P) \xrightarrow{\sim} H_*(C_1)$ induces a homomorphism of Poisson algebras

$$\eta : \mathcal{P}(H_*(P)) \to H_*(C) = \bigoplus_{n \geq 0} H_*(C_n)$$

We shall show that η is an isomorphism. Recall from §3 that the maps ϕ_n $n = 1, 2, 3, \ldots$ define a homomorphism of Poisson algebras

$$\phi : H_*(C) = \bigoplus_{n \geq 0} H_*(C_n) \to \bigoplus_{n \in \mathbb{Z}} H_*(\Omega_n) = H_*(\Omega)$$

Moreover (3.9) asserts that the induced map $\phi : H_*(C)[v_0^{-1}] \to H_*(\Omega)$ is an isomorphism of Poisson algebras. To make use of this, extend η to a homomorphism of Poisson algebras $\mathcal{P}(H_*(P))[v_0^{-1}] \to H_*(C)[v_0^{-1}]$, also denoted by η. Since multiplication by v_0 is injective in $H_*(C)$ by (3.4), it will suffice to prove that this η is an isomorphism.

Consider the composition $\gamma = \tilde{T} \circ \phi \circ \eta : \mathcal{P}(H_*(P))[v_0^{-1}] \to \mathcal{P}(H_*(P))[v_0^{-1}]$. Since ϕ and $\tilde{T}$ are isomorphisms, it will suffice to show that γ is an isomorphism. Define the subspace $V_2 = v_0^{-1} H_*(P)^\sharp \oplus \mathbb{Q} v_0$ of $\mathcal{P}(H_*(P))[v_0^{-1}]$ (the $^\sharp$ indicates that the degree zero term should be deleted). Clearly $\mathcal{P}(H_*(P))[v_0^{-1}] = \mathcal{P}(V_2)[v_0^{-1}]$. Moreover it follows from (4.10)$'$ that V_2 is mapped by γ isomorphically (in the sense of graded vector spaces) to a subspace V_1 of $\mathcal{P}(H_*(P))[v_0^{-1}]$ which satisfies $\mathcal{P}(H_*(P))[v_0^{-1}] = \mathcal{P}(V_1)[v_0^{-1}]$. Hence γ is an isomorphism and the result follows. □

We now focus on the special case which is of particular interest to us, viz. when $P = \mathbb{P}(\mathbb{C}^\infty)$. The following result is well known.

(4.11) Lemma. *Let $P = \mathbb{P}(\mathbb{C}^\infty)$. Then the Poincaré series of $H_*(P)$ is* $\dfrac{1}{1-t^2}$.

The above results may be used to make more explicit the connection between $H_*(\Omega)$ and $H_*(C)$ in this case. We shall use the following notation. We work in the Poisson algebra

$$\Gamma = \bigoplus_{n \geq 0} H_*(\Omega_n) \cong \mathcal{A}(U) \cong \mathcal{P}(H_*(P))$$

where U is as in (4.9), $s = 1$ and $P = \mathbb{P}(\mathbb{C}^\infty)$. Note that Γ is bigraded: if $x \in H_i(\Omega_n)$, we say x has degree i and weight n and write $\delta(x) = i$ and $\omega(x) = n$. Now by (4.11), $H_*(P)$ has a basis $\{v_k\}_{k=0,1,2,...}$, with $v_k \in H_{2k}(P)$. Then the free Lie algebra $\mathcal{L}(H_*(P)[1])$ has a $\mathbb{Q}$-basis consisting of elements $\{v_I\}_I$, over sequences $I = i_1, i_2, ..., i_r$ of elements of $\{0, 1, 2, ...\}$, where $v_I = \{v_{i_1}, \{v_{i_2}, ...\{v_{i_{r-1}}, v_{i_r}\}...\}\}$. In the space $U \cong \mathcal{L}(H_*(P)[1])[-1]$, this element v_I has degree $2(i_1 + i_2 + ... + i_{r-1} + i_r) + r - 1$. The weight $\omega(v_k)$ is 0 if $k > 0$ and 1 if $k = 0$; then $\omega(v_I) = \omega(v_{i_1}) + \omega(v_{i_2}) + ... + \omega(v_{i_r})$. The algebra $\Gamma = \mathcal{A}(U)$ has a $\mathbb{Q}$-basis consisting of the set of words in the v_I (which commute with each other up to sign).

For any sequence I as above, write $|I| = r$ and $\|I\| = \sum_j i_j$. Then for any word $v = v_{I_1}...v_{I_k}$, we have $\delta(v) = \sum_j (2\|I_j\| + |I_j|) - k$ and $\omega(v) = \sum_j \omega(v_{I_j})$. Now (4.10)$'$ asserts that $i_*(H_*(P))$ has $\mathbb{Q}$-basis $\{v_0, v_0v_1, v_0v_2, v_0v_3, ...\}$ and Theorem (2.4) identifies $H_*(C) = \oplus_{n \geq 0} H_*(C_n)$ as the subspace of Γ generated as Poisson algebra by $i_*(H_*(P))$. Write $w_0 = v_0$ and $w_i = v_0 v_i$ for $i \geq 1$. Then included in (2.4) is the assertion that $H_*(C)$ is the free Poisson algebra of degree 1 on the w_i. Thus $H_*(C)$ has basis consisting of words $w = w_{I_1}...w_{I_k}$, in complete analogy with the above discusssion for the v's. However note that since $\omega(w_i) = 1$ for any i, $\omega(w) = \sum_j |I_j|$, while $\delta(w) = \sum_j (2\|I_j\| + |I_j|) - k$ as before. In particular, $H_*(C_n)$ is identified as the subspace of Γ spanned by the words $w = w_{I_1}...w_{I_k}$ of weight n.

This enables us to sharpen Theorem (3.2) as follows.

(4.12) Theorem. *In the notation of §§2,3, let $P = \mathbb{P}(\mathbb{C}^\infty)$ and take $d = 2$. Then we have isomorphisms*

$$H_k(C_n) \to H_k(\Omega_n)$$

for $k \leq n$.

Proof. Maintaining the notation of the above discussion, $H_*(C_n)$ is identified with the subspace of $\Gamma = \oplus_{n \geq 0} H_*(\Omega_n)$ which is spanned additively by the words $w = w_{I_1} w_{I_2} ... w_{I_s}$ such that $|I_1| + |I_2| + ... + |I_s| = n$. We shall prove

(4.12.1) *Every element of $H_*(C_n)$ which has degree less than n is in $v_0 H_*(C_{n-1})$.*

Our Theorem follows from (4.12.1) because of Theorem (3.2), taking into account that the maps $H_*(C_n) \to H_*(C_{n+1})$ in (3.3) above are multiplication by v_0. Explicitly, if $k < n \leq n'$, then

$$\begin{aligned} H_k(C_{n'}) &= v_0 H_k(C_{n'-1}) = v_0^2 H_k(C_{n'-2}) = ... \\ &= v_0^{n'-n} H_k(C_n) = v_0^{n'-n+1} H_k(C_{n-1}) \cong H_k(\Omega_0) \end{aligned} \tag{4.12.2}$$

From (4.12.2) it follows that for $k \leq n-1$, $H_k(C_{n-1}) \cong H_k(\Omega_0)$, which is our statement for $n-1$.

Now to prove (4.12.1), take a basis element $w = w_{I_1} w_{I_2} ... w_{I_s}$ of $H_*(C_n)$ as above. Then $\omega(w) = |I_1| + |I_2| + ... + |I_s| = n$. We seek an element w which is not divisible by v_0, such that $\delta(w) = \sum_j (2\|I_j\| + |I_j|) - k = 2\sum_j \|I_j\| + n - k$ is minimal. First observe that if no sequence I_j consists of 0's alone, then $\|I_j\| \geq 1$ for each j, so that $\delta(w) \geq n + k \geq n + 1$. Next, note that $\{v_0, v_0\}^2 = 0$ by graded commutativity and $\{v_0, \{v_0, v_0\}\} = 0$ by the Jacobi identity (the second relation (2.3)). Hence if any I_j consists only of 0's, $|I_j| = 2$ (since w is not divisible by v_0) and there is at most one such I_j. It follows that for each sequence I_j except possibly one ($= (0,0)$), we have $|I_j| \geq 1$. Hence $\delta(w) \geq 2(k-1) + n - k = k + n - 2$. Hence if $n \geq 3$ (so $k \geq 2$) $\delta(w) \geq n$ as required. The cases $n \leq 2$ are trivial. □

Note that Theorem (4.12) may also be deduced from [C, Theorem 3.1, p.225] using arguments well known to homotopy theorists.

(4.13) Remark. The argument of the proof just given shows that there is up to sign a unique basis element of degree n which is not divisible by v_0. This element is

$$\{w_0, w_0\}\{w_0, \{w_0, \{w_0, ..., \{w_0, w_1\}...\}\}\} = \{v_0, v_0\}\{v_0, \{v_0, \{v_0, ..., \{v_0, v_0 v_1\}...\}\}\} \tag{4.13.1}$$

where the number of w_0's in the first expression is $n-1$.

One may check using the relations (2.3) that

$$\{v_0, \{v_0, v_0 v_1\}\} = \{v_0, \{v_0, \{v_0, v_1\}\}\}$$

from which it follows that the element of (4.13.1) may be written, if $n \geq 6$ as

$$\{v_0, v_0\}\{v_0, \{v_0, \{v_0, ..., \{v_0, \{v_0, v_1\}...\}\}\},$$

where the number of v_0's is n (recall that $\omega(v_i) = 0$ if $i > 0$).

From the above proof one obtains a little more.

(4.14) Corollary (cf. [L1, §6], (8.6) below). *If $P = \mathbb{P}(\mathbb{C}^\infty)$, then $H_k(C_n(\mathbb{C}, P))$ is independent of n for $k \leq n$. Moreover we have, for any $n \geq 3$,*

$$dim\, H_n(C_n(\mathbb{C}, P)) = dim\, H_n(C_{n-1}(\mathbb{C}, P)) + 1$$

Proof. The first statement is obvious. For the second one uses the remark (4.13) that the basis element (4.13.1) is the only one in $H_n(C_n)$ which is not in $v_0 H_n(C_{n-1})$. The restriction $n \geq 3$ is necessary since the element of (4.13.1) only exists if $n - 1 \geq 2$. The statement is false for $n = 2$. □

We now have the following sharpening of (3.10), which confirms conjecture (6.3) of [L1].

(4.15) Corollary. *Let $V = H_*(P)[1]$, where $P = \mathbb{P}(\mathbb{C}^\infty)$, so that $Poin_V(t) = \dfrac{t}{1-t^2}$. Write $r(t) = Poin_{\mathcal{A}^\sharp(\mathcal{L}(V)[-1])}(t)$ (see (1.5)) and define the power series*

$$\psi(t) = \prod_{j=1}^{\infty}(1-t^{2j})r(t).$$

If $\mathfrak{G}_{rs}(n) = \mathfrak{gl}_n(\mathbb{C})$, then $\psi(t) - Poin_{\mathfrak{G}_{rs}(n)}(t) = t^{n+1}$ + higher terms.

5. The group case.

In this section we discuss the case of the variety G_{rs}, where $G = GL_n(\mathbb{C})$. The situation is similar to the Lie algebra case and we recall the background briefly. We have

$$G_{rs} \cong (T_{rs} \times (G/T))/W$$

and there is a commutative diagram analogous to (1.4)

$$\begin{array}{ccccc}
G/T & \longrightarrow & G_{rs} & \longrightarrow & T_{rs}/W \\
\downarrow & & \downarrow & & \downarrow \mathrm{id} \\
B_T & \longrightarrow & X(G) & \longrightarrow & T_{rs}/W \\
\downarrow & & \downarrow & & \downarrow \\
B_G & \xrightarrow{\mathrm{id}} & B_G & \longrightarrow & \{\text{point}\}
\end{array} \tag{5.1}$$

where $X(G) = (T_{rs} \times B_T)/W$.

In analogy with (1.5), we have

$$H^*(X(G)) \cong S^W \otimes H^*(G_{rs})$$

from which it follows that

$$\mathrm{Poin}_{G_{rs}}(t) = \prod_i (1-t^{2d_i})\mathrm{Poin}_{X(G)}(t)$$

Thus we focus on the space $X(G)$, which may be identified (cf. §2 above) with the labelled configuration space $C_n(\mathbb{C}^\times; P)$, where $P = \mathbb{P}(\mathbb{C}^\infty)$. Our principal tool in this case is that when X is any open connected subspace of $\mathbb{C}$, there is a map

$$\phi_{X,n} : C_n(X; P) \to \mathrm{Map}_n(X^+; S^2(P_+)) \tag{5.2}$$

where X^+ denotes the one point compactification of X and Map_n is the space of base point preserving maps of degree n. The definition of these maps is similar to that of the ϕ_n of §3, which of course are a special case. In analogy with the earlier case where $X = \mathbb{C}$, the space $M_n = \mathrm{Map}_n(X^+; S^2(P_+))$ is independent of n in the sense that it is one of the connected components of $M = \mathrm{Map}(X^+; S^2(P_+)) = \amalg_{n\in\mathbb{Z}} M_n$ and all of these components are homotopy equivalent. The key result we need is

(5.3) Proposition. *([McD, Theorem 1]) The maps $\phi_{X,n}$ of (5.2) induce isomorphisms in homology up to degree d_n where $d_n \to \infty$ as $n \to \infty$.*

Thus to understand the stable homology of $X(G) = C_n(\mathbb{C}^\times; P)$, we investigate the homology of M_0. By (4.1), we have

$$H_*(M_0) \cong \mathcal{A}(\text{prim } H_*(M_0)) \cong \mathcal{A}(\pi_*(M_0)\otimes\mathbb{Q}) \tag{5.4}$$

To study $\pi_*(M)$, we use the fact that since $X \mapsto X^+$ defines a contravariant functor on the category of open subspaces with inclusions, ϕ_X above is covariant in X. In particular if $X = \mathbb{C}^\times$, the inclusion $X \hookrightarrow \mathbb{C}$ induces $\mathbb{C}^+ = S^2 \longrightarrow S^2/S^0 = (\mathbb{C}^\times)^+$, where S^0 is thought of as $\{0, \infty\} \subset \mathbb{C} \cup \{\infty\} = S^2$.

The cofibration sequence $S^0 \to S^2 \to S^2/S^0$ induces, for any space Y with base point, a fibration sequence

$$\text{Map}(S^2/S^0; Y) \to \text{Map}(S^2; Y) \to \text{Map}(S^0; Y) = Y.$$

This yields a long exact sequence

$$... \to \pi_{i+1}(Y) \to \pi_i(M) \to \pi_i(\Omega^2 Y) \to \pi_i(Y) \to \pi_{i-1}(M) \to ... \tag{5.5}$$

which breaks up into short exact sequences

$$0 \to \pi_{i+1}(Y) \to \pi_i(M) \to \pi_i(\Omega^2 Y) = \pi_{i+2}(Y) \to 0$$

which split because the homotopy equivalence $S^2/S^0 \simeq S^2 \vee S^1$ gives us a right inverse to the natural map $M \to \Omega^2 Y$. Thus we have

(5.6) Proposition. *With the above notation, we have*

$$\pi_i(M) \cong \pi_{i+1}(Y) \oplus \pi_{i+2}(Y).$$

We shall apply this with $Y = S^2(P_+)$. Our purpose is to determine the graded vector space $\pi_*(M_0)\otimes\mathbb{Q}$, with a view to applying (5.4). We shall use the fact that $\pi_*(M_0)\otimes\mathbb{Q} = (\pi_*(M)\otimes\mathbb{Q})^\sharp$.

(5.7) Theorem. *Let M be the space of maps from $\mathbb{C}^\times$ to $S^2(P_+)$. Then we have an isomorphism of graded vector spaces*

$$\pi_*(M)\otimes\mathbb{Q} \cong \mathcal{L}(H_*(P)[1]) \oplus \mathcal{L}(H_*(P)[1])[-1]$$

Proof. Taking $Y = S^2(P_+)$ in (5.6), we obtain

$$\pi_*(M)\otimes\mathbb{Q} \cong \pi_*(S^2 P_+)\otimes\mathbb{Q}[-1] \oplus \pi_*(S^2 P_+)\otimes\mathbb{Q}[-2] \tag{5.7.1}$$

But $\pi_*(S^2 P_+)\otimes\mathbb{Q} \cong \pi_*(\Omega^2 S^2 P_+)\otimes\mathbb{Q}[2]$. Taking this into account, (5.7.1) becomes

$$\pi_*(M)\otimes\mathbb{Q} \cong \pi_*(\Omega^2 S^2 P_+)\otimes\mathbb{Q}[1] \oplus \pi_*(\Omega^2 S^2 P_+)\otimes\mathbb{Q} \tag{5.7.2}$$

Now $\pi_0(\Omega^2 S^2 P_+) \cong \mathbb{Z}$. This, together with (4.6) and (4.8) shows that

$$\pi_*(\Omega^2 S^2 P_+)\otimes\mathbb{Q} \cong \mathcal{L}(H_*(P)[1])[-1] \tag{5.7.3}$$

Substituting into (5.7.2), we obtain (5.7). $\square$

Deleting the components of degree zero from both sides of (5.7), we obtain

(5.8) Corollary. *Let M_0 be the space of maps of degree zero from $\mathbb{C}^\times$ to $S^2(P_+)$. Then as graded vector space, we have*

$$\pi_*(M_0)\otimes\mathbb{Q} \cong \mathcal{L}(H_*(P)[1]) \oplus \mathcal{L}(H_*(P)[1])[-1]^\sharp$$

Combining the statements above we obtain

(5.9) Theorem. *Let $G = GL_n(\mathbb{C})$ and let $X(G)$ be the space defined in (5.1). Then the space M_0 of base point preserving maps of degree zero from S^2/S^0 to $S^2(P_+)$, where $P = \mathbb{P}(\mathbb{C}^\infty)$, has homology which coincides with the stable part of $H_*(X(G))$. This homology is given, as graded vector space, by*

$$H_*(M_0) \cong \mathcal{T}(H_*(P)[1]) \otimes H_*(\Omega_0)$$

where Ω_0 is as in §§2,3 and $\mathcal{T}$ denotes the tensor algebra.

Proof. Everything has been proved, except the statement about $H_*(M_0)$. For this, we have

$$\begin{aligned}
H_*(M_0) &\cong \mathcal{A}(\pi_*(M_0)\otimes\mathbb{Q}) \text{ by (5.4)}\\
&\cong \mathcal{A}\{\mathcal{L}(H_*(P)[1]) \oplus (\mathcal{L}(H_*(P)[1])[-1])^\sharp\} \text{ by (5.8)}\\
&\cong \mathcal{A}(\mathcal{L}(H_*(P)[1])) \otimes \mathcal{A}((\mathcal{L}(H_*(P)[1])[-1])^\sharp)\\
&\cong \mathcal{T}(H_*(P)[1]) \otimes \mathcal{A}((\mathcal{L}(H_*(P)[1])[-1])^\sharp) \text{ see (7.0)}\\
&\cong \mathcal{T}(H_*(P)[1]) \otimes H_*(\Omega_0) \text{ by (4.4)}
\end{aligned}$$

as required. □

(5.10) Corollary. *In the notation of (4.15), define the power series*

$$\begin{aligned}
\Psi(t) &= \frac{1-t^2}{1-t-t^2}\psi(t)\\
&= \frac{1-t^2}{1-t-t^2}\prod_{j=1}^{\infty}(1-t^{2j})r(t)
\end{aligned}$$

where $r(t) = Poin_{\mathcal{A}^\sharp(\mathcal{L}(V)[-1])}(t)$ and $V = H_*(P)[1]$.

Then $Poin_{G_{rs}}(t) \equiv \Psi(t) \bmod t^{d_n}$, where $d_n \to \infty$ as $n \to \infty$.

This is an immediate corollary of (5.9), taking into account that if $\mathrm{Poin}_V(t) = p(t)$, then $\mathrm{Poin}_{\mathcal{T}(V)}(t) = \dfrac{1}{1-p(t)}$ for any graded vector space V and that if $V = H_*(P)[1]$, then $\mathrm{Poin}_V(t) = \dfrac{t}{1-t^2}$.

6. The orthogonal and symplectic cases- equivariant theory.

In this section we take G to be $Sp_{2n}(\mathbb{C})$ and $\mathfrak{G} = \text{Lie } G$. Then as in (1.6), we have

$$\text{Poin}_{\mathfrak{G}_{rs}}(t) = \prod_{1}^{n}(1-t^{4i})\text{Poin}_{X(\mathfrak{G})}(t)$$

where $X(\mathfrak{G}) = (\mathfrak{T}_{rs} \times B_T)/W$ as in §1.

Our strategy will be to identify $X(\mathfrak{G})$ as the space of fixed points of an involution acting on a configuration space and then to use the equivariant theory developed in [S3]. The Weyl group W of $Sp_{2n}(\mathbb{C})$ acts on $\mathfrak{T} \cong \mathbb{C}^n$ by permuting n-tuples and changing any number of their signs. Thus $\mathfrak{T}_{rs} \times B_T$ may be thought of as the space of ordered $n-$tuples $((x_1,p_1),...,(x_n,p_n))$ of pairs with $x_i \in \mathbb{C}^\times$, $p_i \in P$ $(= \mathbb{P}(\mathbb{C}^\infty))$ and $x_i \neq \pm x_j$ if $i \neq j$. The Weyl group acts by permuting the n pairs (x_i,p_i) and by changing any pair (x_i,p_i) to $(-x_i,\bar{p}_i)$, where $\bar{p}_i$ denotes complex conjugation. Hence the space $(\mathfrak{T}_{rs} \times B_T)/W$ of orbits may be identified with the (unordered) set of $2n$ pairs $\{(x_i,p_i),(-x_i,\bar{p}_i); i=1,...,n\}$. The following assertion is now clear.

(6.1). *If $\mathfrak{G} = Lie\ Sp_{2n}(\mathbb{C})$ as above, the space $X(\mathfrak{G})$ may be identified with the space of fixed points of the involution θ of the configuration space $C_{2n}(\mathbb{C}^\times, P)$, where*

$$\theta(\{(x_1,p_1),(x_2,p_2),...,(x_{2n},p_{2n})\}) = \{(-x_1,\bar{p}_1),(-x_2,\bar{p}_2),...,(-x_{2n},\bar{p}_{2n})\} \qquad (6.1.1)$$

Note that there is a similar involution on configurations of an odd number of points, but the fixed point space is then empty.

(6.2) Remark. In this section, several spaces will have involutory homeomorphisms related to the θ of (6.1.1). We shall take some care to explain what these are, but denote them all by θ, since they are closely related. In particular, the action $x \mapsto -x$ of θ on $\mathbb{C}^\times$ induces actions on $\mathbb{C}$ and $S^2 = \mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$ which fix 0 and ∞; similarly, θ acts (compatibly) on $(\mathbb{C}^\times)^+ = S^2/\{0,\infty\}$. Further, θ acts on $S^2(P_+) \cong \dfrac{D^2 \times P}{S^1 \times P}$ via $\theta(z,p) = (-z,\bar{p})$, if $z \in D^2, p \in P$. Since θ fixes the base points, there is a corresponding action of θ on $\text{Map}((\mathbb{C}^\times)^+, S^2(P_+))$, given by $\omega \mapsto^\theta \omega$, where

$${}^\theta\omega(x) = \theta(\omega(\theta x)),$$

for $\omega \in \text{Map}((\mathbb{C}^\times)^+, S^2(P_+)), x \in (\mathbb{C}^\times)^+$. For reference below, we record here the well known fact that the involution $p \mapsto \bar{p}$ of P induces in homology the map which acts on $H_{2k}(P)$ as multiplication by $(-1)^k$, which we also denote by θ.

The main result we require here is an equivariant generalisation of (5.3), namely

(6.3) Theorem.
(i) The maps $\phi_{\mathbb{C}^\times,2n}$ of (5.2) respect the action of θ, i.e. $\phi_{\mathbb{C}^\times,2n} \circ \theta = \theta \circ \phi_{\mathbb{C}^\times,2n}$.
(ii) The maps $\phi_{\mathbb{C}^\times,2n}$ induce homology isomorphisms for the subspaces of fixed points:

$$H_i(C_{2n}(\mathbb{C}^\times,P)^\theta) \to H_i(Map_{2n}((\mathbb{C}^\times)^+, S^2(P_+))^\theta)$$

for $0 \le i \le e_n$*, where* $e_n \to \infty$ *as* $n \to \infty$*.*

The first statement is clear from the above descriptions of θ, while the second statement follows from [S3, Theorem B].

We shall see below (6.10) that M_0^θ is connected. It follows that in the notation of §5, M_0^θ is a stable version of $X(\mathfrak{G})$, in the sense that

$$H_i(X(\mathfrak{G}(n))) \cong H_i(M_0^\theta) \text{ for } 0 \le i \le e_n$$

and $e_n \to \infty$ as $n \to \infty$.

It therefore remains only to compute $H_*(M_0^\theta)$. But M_0^θ is again a connected H-space, since θ respects the product in M, up to homotopy. Hence

$$H_*(M_0^\theta) \cong \mathcal{A}(\text{prim } H_*(M_0^\theta)) \cong \mathcal{A}(\pi_*(M_0^\theta) \otimes \mathbb{Q}) \tag{6.4}$$

and we are reduced to investigating $\pi_*(M_0^\theta)\otimes\mathbb{Q}$. For this, observe that, if we regard $\mathbb{C}^\times$ as a union of concentric circles, $(\mathbb{C}^\times)^+ \cong \dfrac{\mathbb{C} \cup \{\infty\}}{\{0,\infty\}}$ may be identified with $\dfrac{I \times S^1}{\{0,\infty\} \times S^1}$, where I is thought of as the closed interval $[0,\infty] \subset \mathbb{C}^+$. This identification is θ-equivariant if θ acts on $\dfrac{I \times S^1}{\{0,\infty\} \times S^1}$ via its action on S^1 (regarded as the unit circle in $\mathbb{C}$) alone, i.e. θ fixes I pointwise and acts antipodally on S^1.

For the moment let Y be any path connected space with an involution θ (we shall soon take $Y = S^2P_+$). Then this identification leads to an obvious θ-equivariant homeomorphism

$$\text{Map}((\mathbb{C}^\times)^+, Y) \xrightarrow{\cong} \Omega\text{Map}'(S^1, Y) \cong \text{Map}(S^1, \text{Map}'(\frac{I}{\{0,\infty\}}, Y))$$

where Map$'$ indicates maps which are not necessarily base point preserving. Taking into account the action of θ on the right side, this gives

(6.5) Lemma. *For any space* Y *with an involution* θ*, let* $M = Map((\mathbb{C}^\times)^+, Y)$ *and* $M' = Map'(S^1, Y)$*. Then the homeomorphism above induces a homeomorphism*

$$M^\theta \xrightarrow{\cong} \Omega((M')^\theta)$$

If we fix a point of S^1 and take $\epsilon : (M')^\theta \to Y$ to be evaluation at that point, the fibre $\epsilon^{-1}(y)$ ($y \in Y$) consists of the paths from y to $\theta(y)$ (by θ-invariance, which gives $\omega(-z) = \theta(\omega(z))$ for any $z \in S^1$ and $\omega \in (M')^\theta$), a space which is homotopy equivalent to the space of loops in Y which are based at $y \in Y$, i.e. to ΩY. It follows that we have a fibration

$$\Omega Y \to (M')^\theta \to Y$$

from which we obtain the long exact sequence

$$... \to \pi_{i+1}(Y) \xrightarrow{\partial} \pi_i(\Omega Y) \cong \pi_{i+1}(Y) \to \pi_i((M')^\theta) \to \pi_i(Y) \xrightarrow{\partial} \pi_{i-1}(\Omega Y) \cong \pi_i(Y) \to ... \tag{6.6}$$

(6.7) Proposition. *With the above notation, we have, for any path connected space Y, and for any i,*

$$\pi_i((M')^\theta)\otimes\mathbb{Q} \cong \pi_i(Y)^\theta\otimes\mathbb{Q} \oplus \pi_{i+1}(Y)^\theta\otimes\mathbb{Q}$$

where θ acts on the homotopy groups on the right via the action induced by the action of θ on Y.

Proof. From the general definition of the homotopy long exact sequence (6.6), it is clear that the connecting maps ∂ are equal to $\theta - \mathrm{id}$. It follows that the middle part of (6.6) may be written as a short exact sequence

$$0 \to \frac{\pi_{i+1}(Y)}{(\theta - \mathrm{id})\pi_{i+1}(Y)} \to \pi_i((M')^\theta) \to (\pi_i(Y))^\theta \to 0$$

Since $\frac{\pi_{i+1}(Y)}{(\theta - \mathrm{id})\pi_{i+1}(Y)}\otimes\mathbb{Q} \cong (\pi_{i+1}(Y))^\theta\otimes\mathbb{Q}$, the result follows. $\square$

We now take $Y = S^2(P_+)$ and begin with an observation concerning the action of θ.

(6.8) Lemma. *The map $\theta : Y \to Y$ is homotopic to the map $\theta' : Y \to Y$ which is functorially induced on Y by suspension of θ acting on P.*

Proof. Clearly $Y = \dfrac{D^2 \times P}{S^1 \times P} = \dfrac{S^2 \times P}{\{\infty\} \times P}$ and θ acts on $S^2 \cong \mathbb{C} \cup \{\infty\}$ via $z \mapsto -z$. But this action on S^2 is homotopic to the identity, the map $(t, z) \mapsto e^{\pi i t} z$ ($t \in [0, 1], z \in S^2$) providing a homotopy through maps which fix 0 and ∞. $\square$

(6.9) Theorem. *We have*

$$\pi_*(M^\theta)\otimes\mathbb{Q} \cong \mathcal{L}(H_*(P)[1])^\theta \oplus (\mathcal{L}(H_*(P)[1])[-1])^\theta$$

where θ acts on M in the manner described in (6.2) and on the right hand side via its action on $H_*(P)$, extended functorially.

Proof. It follows from (6.5) that

$$\begin{aligned}\pi_*(M^\theta)\otimes\mathbb{Q} &\cong \pi_*((M')^\theta)\otimes\mathbb{Q}[-1] \\ &\cong \pi_*(Y)^\theta\otimes\mathbb{Q}[-1] \oplus \pi_*(Y)^\theta\otimes\mathbb{Q}[-2] \text{ by (6.7)}\end{aligned} \tag{6.9.1}$$

where by (6.8), the action of θ on the right is induced by its action on P.

But the isomorphism $\pi_*(S^2P_+)\otimes\mathbb{Q} \cong \pi_*(\Omega^2S^2P_+)\otimes\mathbb{Q}[2]$ is θ-equivariant and therefore induces an isomorphism of the spaces of θ-fixed points. Moreover, because the isomorphism $\pi_*(\Omega^2S^2P_+)\otimes\mathbb{Q} \cong \mathcal{L}(H_*(P)[1])[-1]$ of (5.7.3) is equivariant with respect to the θ-action of the statement, we have

$$\begin{aligned}\pi_*(Y)^\theta\otimes\mathbb{Q} &\cong \pi_*(\Omega^2S^2P_+)^\theta[2]\otimes\mathbb{Q} \\ &\cong (\mathcal{L}(H_*(P)[1])[1])^\theta\end{aligned}$$

This, together with (6.9.1), implies the theorem. $\square$

(6.10) Lemma. *The space M_0^θ is connected. It is one of the connected components of M^θ, which are all homotopy equivalent.*

Proof. We shall show that $\pi_0(M^\theta) \cong \mathbb{Z}$ and that the natural map $\pi_0(M^\theta) \to \pi_0(M)$ is injective. The result will follow from this, for otherwise all the components of M^θ would be contained in the M_0, which is manifestly not the case.

Observe that since $Y = S^2P_+$ is connected, the end of the long exact sequence (6.6) reads as follows

$$\pi_2(Y) \xrightarrow{\theta-\mathrm{id}} \pi_2(Y) \to \pi_1((M')^\theta) = \pi_0(M^\theta) \to \pi_0(Y) = 0 \qquad (6.10.1)$$

But θ acts as the identity map on $\pi_2(Y) = \pi_2(S^2) = \pi_0(\Omega^2 S^2 P_+) = \mathbb{Z}$, so that $\pi_0(M^\theta) \cong \mathbb{Z} \cong \pi_0(M)$ is immediate from (6.10.1). It follows that $\pi_0(M^\theta) \to \pi_0(M)$ is injective, and (6.10) follows. $\square$

The next statement is now a consequence of (6.9) and (6.10).

(6.11) Corollary. *Let M_0 etc. be as above. Then*

$$\pi_*(M_0^\theta) \otimes \mathbb{Q} \cong \mathcal{L}(H_*(P)[1])^\theta \oplus (\mathcal{L}(H_*(P)[1])[-1]^\sharp)^\theta$$

.

Combining (6.11) and (6.4), we obtain

(6.12) Theorem. *Let $\mathfrak{G}$ be the Lie algebra of a complex connected reductive Lie group of type B_n or C_n. Let $X(\mathfrak{G})$ be the space described in (6.1). Let M_0 be the space of maps defined in (5.9) and let θ be the involution of M_0 defined in (6.2). Then the space M_0^θ has homology which coincides with the stable part of $H_*(X(\mathfrak{G}))$. If we write P for the space $\mathbb{P}(\mathbb{C}^\infty)$, this homology is given, as graded vector space by*

$$H_*(M_0^\theta) \cong \mathcal{A}\left(\mathcal{L}(H_*(P)[1])^\theta \oplus (\mathcal{L}(H_*(P)[1])[-1]^\sharp)^\theta\right)$$

where the action of θ on the right hand side is induced functorially by its action on the graded vector space $H_*(P)$, which is multiplication by $(-1)^k$ in degree $2k$.

We may express this in terms of Poincaré series as follows.

(6.13) Corollary. *Maintain the notation of (6.12). Let U be the graded vector space $U = \mathcal{L}(H_*(P)[1])^\theta \oplus (\mathcal{L}(H_*(P)[1])[-1]^\sharp)^\theta$. Define the power series $s(t)$ by*

$$s(t) = Poin_{\mathcal{A}(U)}(t) = Poin_{\mathcal{A}(\mathcal{L}(H_*(P)[1])^\theta)}(t) Poin_{\mathcal{A}((\mathcal{L}(H_*(P)[1])[-1]^\sharp)^\theta)}(t).$$

Then $Poin_{\mathfrak{G}_{rs}}(t) \equiv \prod_{j=1}^{\infty}(1-t^{4j})s(t)$ *(mod t^{e_n}), where $e_n \to \infty$ as $n \to \infty$.*

7. Computations of the Poincaré series.

Let $V = \bigoplus_{i\geq 0} V_i$ be a graded $\mathbb{Q}$-vector space with Poincaré series $p(t)$. Whenever we refer to the Poincaré series of a space, we assume that it is locally finite, i.e. has finite dimensional graded components. As above, $\mathcal{T}(V), \mathcal{L}(V)$ and $\mathcal{A}(V)$ respectively denote the tensor algebra, free Lie algebra and free graded-commutative algebra on V. Computations in the same spirit as those below may be found in Serre [Se, p. LA 4.6]. It is elementary that if $p(t) = \sum_{k=0}^{\infty} p_k t^k$,

$$\begin{aligned} \mathrm{Poin}_{\mathcal{T}(V)}(t) =& (1 - p(t))^{-1} \\ \mathrm{Poin}_{\mathcal{A}(V)}(t) =& \prod_{k\geq 1} (1 - (-t)^k)^{(-1)^{k-1} p_k} \end{aligned} \tag{7.0}$$

where the first relation follows from (2.5.1) and the second follows from (2.5.2). Note that $\mathcal{A}(V)$ is not locally finite unless $V_0 = 0$, which is assumed for the second relation and henceforth.

Now the Poincaré-Birkhoff-Witt theorem asserts (cf [R, Ch 1]) that $\mathcal{T}(V) = \mathcal{A}(\mathcal{L}(V))$ is the free graded-commutative algebra on $\mathcal{L}(V)$. Hence if we write

$$\mathrm{Poin}_{\mathcal{L}(V)}(t) = q(t) = \sum_{k\geq 1} q_k t^k$$

then

$$\mathrm{Poin}_{\mathcal{T}(V)}(t) = (1 - p(t))^{-1} = \prod_{k\geq 1} (1 - (-t)^k)^{(-1)^{k-1} q_k} \tag{7.1}$$

More generally, let θ be a graded involution of V and write $p(\theta, t) = \sum_{i\geq 0} \mathrm{trace}(\theta, V_i) t^i$, so that $p(t) = p(\mathrm{id}, t)$. Then θ acts on $\mathcal{T}(V)$ and $\mathcal{L}(V)$; write $q_k^{\pm}$ for the respective dimensions of the ± 1 eigenspaces of θ on the degree k component L_k of $\mathcal{L}(V)$. The same analysis as above then yields, if we write $\mathrm{Poin}_M(\theta, t) = \sum_{i\geq 0} \mathrm{trace}(\theta, M_i) t^i$ where M is any graded module on which θ acts,

$$\begin{aligned} \mathrm{Poin}_{\mathcal{T}(V)}(\theta, t) &= \prod_{k\geq 1} \left(1 - (-t)^k\right)^{(-1)^{k-1} q_k^+} \left(1 + (-t)^k\right)^{(-1)^{k-1} q_k^-} \\ &= (1 - p(\theta, t))^{-1} \end{aligned} \tag{7.2}$$

Now write

$$\begin{aligned} -\log(1 - p(t)) &= \sum_{n\geq 1} \alpha_n \frac{(-t)^n}{n}, \\ -\log(1 - p(\theta, t)) &= \sum_{n\geq 1} \beta_n \frac{(-t)^n}{n} \end{aligned} \tag{7.3}$$

Then differentiating (7.3), we see that

$$\begin{aligned}\sum_{n\geq 1}\alpha_n t^n &= \frac{-tp'(-t)}{1-p(-t)}\\ \sum_{n\geq 1}\beta_n t^n &= \frac{-tp'(\theta,-t)}{1-p(\theta,-t)}\end{aligned} \tag{7.3$'$}$$

where $f'(t)$ denotes the derivative of $f(t)$ with respect to t.

Taking logarithms in (7.1) and (7.2), we obtain

$$\begin{aligned}\alpha_n &= \sum_{k|n}(-1)^k kq_k\\ \beta_n &= \sum_{k|n}(-1)^k\left\{kq_k^+ + (-1)^{\frac{n}{k}}kq_k^-\right\}\end{aligned} \tag{7.4}$$

Now the relations (7.4) may be inverted using the usual Möbius formula together with the relation

$$\sum_{k|n}(-1)^k\mu(\frac{n}{k}) = \begin{cases} -1 \text{ if } n=1 \\ 0 \text{ unless } n=1 \text{ or } 2 \\ 2 \text{ if } n=2\end{cases}$$

This results in the relations

$$q_m = q_m^+ + q_m^- = \frac{(-1)^m}{m}\sum_{k|m}\mu(\frac{m}{k})\alpha_k \text{ for } m\geq 1 \tag{7.5}$$

and

$$\frac{(-1)^m}{m}\sum_{k|m}\mu(\frac{m}{k})\beta_k = \begin{cases} q_m^+ - q_m^- & \text{if } m \text{ is odd}\\ q_m^+ - q_m^- + (-1)^{\frac{m}{2}}q_{\frac{m}{2}}^- & \text{if } m \text{ is even}\end{cases} \tag{7.6}$$

We write Q_m for the expression on the left side of (7.6)

The relations (7.3), (7.5) and (7.6) may now be used to deduce explicit values for the $q_k^{\pm}$.

We shall require the q_k explicitly when $V = H_*(P)[1]$, where $P = \mathbb{P}(\mathbb{C}^\infty) = B_{GL_1} = B_{S^1}$. In this case it is known (see (4.11)) that

$$p(t) = \frac{t}{1-t^2}$$

which leads to (cf. (7.3$'$))

$$\sum_{n\geq 1}\alpha_n t^n = \frac{-t(1+t^2)}{(1-t^2)(1+t-t^2)} \tag{7.7}$$

Moreover for the orthogonal and symplectic cases, we need to consider the involution θ which acts on V_{2k+1} as multiplication by $(-1)^k$ (see (6.2)). Thus we have

$$p(\theta,t)=\frac{t}{1+t^2}$$

which implies (again cf. (7.3′))

$$\begin{aligned}\sum_{n\geq 1}\beta_n t^n &= \frac{-t(1-t^2)}{(1+t^2)(1+t+t^2)}\\ &= \frac{-t(1-t^2)(1-t)(1+t^3)(1-t^2+t^4)}{1-t^{12}}\\ &= \frac{-t+t^2+2t^3-3t^4-t^5+4t^6-t^7-3t^8+2t^9+t^{10}-t^{11}}{1-t^{12}}\end{aligned} \tag{7.8}$$

(7.9) Proposition. *With the above notation, when θ and V are as in (7.8), then Q_m $\left(=\dfrac{(-1)^m}{m}\sum_{k|m}\mu(\dfrac{m}{k})\beta_k\right)=0$ for $m>4$, while for $m=1,2,3,4$, the respective values of Q_m are $1,1,-1,-1$.*

Proof. It is clear from (7.8) that β_n depends only on the residue class of n mod 12. However closer inspection of (7.8) reveals that in fact β_n depends only on the greatest common divisor of n and 12. Expressed in this form, the values of β_n are given in the following table.

$$\begin{array}{ccrrrrrr} \gcd(n,12) & : & 1 & 2 & 3 & 4 & 6 & 12\\ \beta_n & : & -1 & 1 & 2 & -3 & 4 & 0\end{array} \tag{7.9.1}$$

If $m=2^a3^bp_1^{m_1}...p_l^{m_l}$ where the p_i are primes distinct from 2 and 3, then any divisor k of m is of the form $k=2^{a'}3^{b'}p_1^{k_1}...p_l^{k_l}$ where the indices are less than or equal to those of m. Now $\gcd(k,12)$ depends only on a' and b'. Moreover $\mu(\dfrac{m}{k})=0$ unless each of the indices of m and k differ by at most 1. It follows that if the sum for Q_m is expressed as a sum of components each of which corresponds to a fixed pair (a',b'), then each component has a factor $\sum_{S\subseteq\{1,\ldots,l\}}(-1)^{|S|}$, which is zero if $l\neq 0$. Hence $Q_m=0$ unless $m=2^a3^b$. For m of this type, one needs to check 12 cases, which are defined by the values of a $(=0,1,2$ or $>2)$ and b $(=0,1$ or $>1)$. An explicit calculation using the values in (7.9.1) then yields the statement.

□

(7.10) Corollary. *Let V and θ be as in (7.8). Then for $m>4$, we have*

(1) $q_m^+=q_m^-$ *if m is odd*
(2) $q_m^+=q_m^- -(-1)^{\frac{m}{2}}q^-_{\frac{m}{2}}$ *if m is even*

This is immediate from (7.6) and (7.9). Moreover a simple induction using (7.10) and (7.5) shows that the $q_m^{\pm}$ may be computed from the q_m using the following formulae.

(7.11) Corollary. *Let V and θ be as in (7.8).*

(1) *If $m = 2^a m_1$, where m_1 is odd and not equal to 1, then*

$$q_m^+ = \frac{1}{2}q_m - \frac{1}{4}q_{\frac{m}{2}} - \frac{1}{8}q_{\frac{m}{4}} - \dots - \frac{1}{2^a}q_{\frac{m}{2^{a-1}}} + \frac{1}{2^{a+1}}q_{\frac{m}{2^a}}$$

and

$$q_m^- = \frac{1}{2}q_m + \frac{1}{4}q_{\frac{m}{2}} + \frac{1}{8}q_{\frac{m}{4}} + \dots + \frac{1}{2^a}q_{\frac{m}{2^{a-1}}} - \frac{1}{2^{a+1}}q_{\frac{m}{2^a}}$$

(2) *If $m = 2^a$, then q_m^+ is respectively $1, 1, 0$ when $a = 0, 1, 2$. When $a \geq 3$ we have*

$$q_{2^a}^+ = \frac{1}{2}q_{2^a} - \frac{1}{4}q_{2^{a-1}} - \dots - \frac{1}{2^{a-2}}q_8 - \frac{1}{2^{a-2}}$$

and

$$q_{2^a}^- = \frac{1}{2}q_{2^a} + \frac{1}{4}q_{2^{a-1}} + \dots + \frac{1}{2^{a-2}}q_8 + \frac{1}{2^{a-2}}$$

We conclude this section by giving explicitly the applications of the above computations to the situations we have considered above.

(7.12) Theorem. *Let $V = H_*(P)[1]$, where $P = \mathbb{P}(\mathbb{C}^\infty)$. Define the sequences $(\alpha_n, n \geq 1)$ and $(\beta_n, n \geq 1)$ of integers by*

$$\sum_{n\geq 1} \alpha_n t^n = \frac{-t(1+t^2)}{(1-t^2)(1+t-t^2)}.$$

and

$$\sum_{n\geq 1} \beta_n t^n = \frac{-t(1-t^2)}{(1+t^2)(1+t+t^2)}.$$

For $m \geq 1$, let q_m and q_m^+ be given by the formulae (7.5) and (7.11) respectively. Then $q_m, q_m^+ \in \mathbb{Z}_{\geq 0}$ for all m and the power series $r(t)$ and $s(t)$ of (4.15), (5.10) and (6.13) are given by

$$r(t) = Poin_{\mathcal{A}^\sharp(\mathcal{L}(V)[-1])}(t) = \prod_{k\geq 1}(1-(-t)^k)^{(-1)^{k-1}q_{k+1}}$$

and

$$s(t) = Poin_{\mathcal{A}(\mathcal{L}(V)^\theta)}(t) Poin_{\mathcal{A}^\sharp(\mathcal{L}(V)[-1]^\theta)}(t) = \prod_{k\geq 1}(1-(-t)^k)^{(-1)^{k-1}(q_k^+ + q_{k+1}^+)}$$

Proof. It follows from (7.4),(7.5),(7.7) and (7.11) that

$$\mathrm{Poin}_{\mathcal{L}(V)}(t) = \sum_{k\geq 1} q_k t^k.$$

and that

$$\mathrm{Poin}_{\mathcal{L}(V)^\theta}(t) = \sum_{k\geq 1} q_k^+ t^k.$$

The statements are now immediate from (7.0) and the definitions of $r(t)$ and $s(t)$. $\square$

We conclude by giving some explicit values of the integers above.

The first fifty of the q_m $(m = 1, 2, ...)$ are given by

$1, 1, 1, 1, 2, 3, 4, 5, 8, 13, 18, 25, 40, 62, 90, 135, 210, 324, 492, 750,$
$1164, 1809, 2786, 4305, 6710, 10460, 16264, 25350, 39650, 62057, 97108, 152145,$
$238818, 375165, 589520, 927200, 1459960, 2300346, 3626200, 5720274, 9030450,$
$14264242, 22542396, 35644500, 56393760, 89264833, 141358274, 223955235,$
$354975428, 562878415, 892893050$

Similarly, the first fifty of the q_m^+ $(m = 1, 2, ...)$ are given by

$1, 1, 0, 0, 1, 2, 2, 2, 4, 7, 9, 12, 20, 32, 45, 66, 105, 164, 246, 372, 582,$
$909, 1393, 2146, 3355, 5240, 8132, 12660, 19825, 31051, 48554, 76038, 119409,$
$187635, 294760, 463520, 729980, 1150296, 1813100, 2859948, 4515225, 7132412,$
$11271198, 17821800, 28196880, 44633113, 70679137, 111976538, 177487714,$
$281440885, 446446525$

For the various Poincaré series $\sum_{i=0}^{\infty} a_i t^i$, we list the coefficients in the order $a_0, a_1, a_2, ...$.

The Poincaré series of (4.15), i.e. the stable Poincaré series for the Lie algebra of type A has coefficients

$1, 1, 0, 1, 2, 4, 6, 9, 17, 30, 47, 75, 131, 221, 358, 589,$
$987, 1640, 2695, 4435, 7346, 12153, 19993, 32886, 54232, 89349,$
$146880, 241416, 396995, 652509, 1071496,$
$1758884, 2887206, 4737820, 7770521, 12740561, 20886071, 34229894,$
$56079853, 91853755, 150417991, 246264599, 403083757, 659619347, 1079213964,$
$1765368168, 2887185977, 4720983277, 7718168486, 12615970769, 20618279891,$
32274001177

The Poincaré series of (5.10), i.e. the stable Poincaré series for the Lie group of type A has coefficients

$1, 2, 2, 4, 8, 15, 27, 47, 85, 153, 268, 466, 818, 1430, 2475, 4273, 7377,$
$12701, 21786, 37282, 63719, 108719, 185085, 314537, 533861, 904861,$
$1531370, 2588298, 4369783, 7369174, 12413458, 20889007, 35118175, 58986118,$
$98987608, 165976467, 278079625, 465545425, 778818832, 1301988118,$
$2175145088, 3631544050, 6059354904, 10104253702, 16839738813, 28049741336,$
$46697452162, 77702808607, 129231243278, 214829039377, 356960394060$

The Poincaré series of (6.13), i.e. the stable Poincaré series for the Lie algebra of type B or C has coefficients

$1, 2, 2, 2, 2, 5, 12, 20, 29, 45, 75, 127, 213, 349, 568, 928, 1520, 2495,$
$4094, 6699, 10950, 17908, 29286, 47864, 78196, 127722, 208562, 340463, 555623,$
$906581, 1478943, 2412061, 3933043, 6412032, 10451715, 17033326, 27754758,$
$45217744, 73657343, 119966288, 195362519, 318101973, 517889138, 843050542,$
$1372204935, 2233240667, 3634168102, 5913284785, 9620742031, 15651144529,$
$25459088704, 40254595697$

8. The unstable case and power series identities.

Theorem (2.4) may be applied directly to obtain the Poincaré series of the configuration space C_n. In this section we do this, show how to use the resulting computation to give an alternative proof of the stability results of §4 and deduce some identities of power series by comparing with the results of [L1]. As was pointed out in §2, (and also in [C, p. 239]), $H_*(C)$ is in fact bigraded and here we do no more than exploit the fact that our proof of Theorem (2.4) determines the bigrading explicitly. As in §4, we refer to the two gradings as "degree" and "weight", these referring respectively to i and n in $H_i(C_n)$. Recall that we take C_0 to be a point.

(8.1) Definition. With notation as in §2, write $\beta_{i,n} = \dim H_i(C_n)$ and define the power series

$$P(t,x) = \sum_{i\geq 0, n\geq 0} \beta_{i,n} t^i x^n \in \mathbb{Q}[[t,x]][t^{-1}]$$

Now Theorem (2.4) asserts that $H_*(C) \cong \mathcal{A}(\mathcal{L}(H_*(P)[1])[-1])$, where the shift refers to the (homology) grading by i. Moreover since $H_*(P)$ is concentrated in weight 1, the (bigraded) Poincaré series of $H_*(P)[1]$ is $\dfrac{xt}{1-t^2}$. This enables us to carry through an analysis entirely analogous to that in §7, to obtain the following explicit expression for $P(t,x)$

(8.2) Theorem. *Define sequences of integers $\alpha_{k,n}$ and $q_{k,n}$ for $k = 1, 2, 3, ...$ and $n = 1, 2, 3, ...$ by*

$$\sum_{k\geq 0, n\geq 1} \alpha_{k,n} t^k x^n = \frac{-tx(1+t^2)}{(1-t^2)(1+xt-t^2)} \tag{8.2.1}$$

and

$$q_{k,n} = \frac{(-1)^k}{k} \sum_{d|\gcd\,(k,n)} \mu(d) \alpha_{\frac{k}{d},\frac{n}{d}}. \tag{8.2.2}$$

Then

$$P(t,x) = \prod_{k\geq 0, n\geq 1} (1-(-t)^k x^n)^{(-1)^{k-1} q_{k+1,n}}$$

To investigate stability we require

(8.3) Lemma. *In the notation of (8.2), we have*

(i) $q_{k,n} = 0$ *if* $k < n$

(ii) $q_{n,n} = 0$ *unless* $n = 1$ *or* 2, *in which case* $q_{n,n} = 1$.

(iii) For any n, $q_{n+1,n} = 0$.

(iv) For any n, $q_{n+2,n} = 1$

(v) For any n, $q_{n+3,n} = 0$

Proof. It is clear immediately from (8.2.1) that $\alpha_{k,n} = 0$ if $n > k$ and (i) follows. A closer inspection of (8.2.1) yields that for any $n \geq 1$, $\alpha_{n,n} = (-1)^n$, $\alpha_{n+2,n} = (-1)^n(n+2)$ and that $\alpha_{n+1,n} = \alpha_{n+3,n} = 0$. Straightforward computations using these facts together with (8.2.2), produce the stated equations. □

These values of the $q_{k,n}$ may be substituted into the expression in Theorem (8.2) to give the following formula for $P(t,x)$.

(8.4) Corollary. *We have*

$$P(t,x) = (1-x)^{-1}(1+tx^2)\prod_{n\geq 1}(1-(-t)^{n+1}x^n)^{(-1)^n}\prod_{1\leq n\leq k-3}(1-(-t)^k x^n)^{(-1)^{k-1}q_{k+1,n}}$$

The next result, which is a consequence of the above computations, relates the stable and unstable homology of the spaces C_n. It sharpens (3.10) and confirms and generalises conjecture (6.3) of [L1] (cf. (4.14) above).

(8.5) Theorem. *Let* C_n *be the configuration space above (it is the space* $X(\mathfrak{G})$ *for* $\mathfrak{G} = \mathfrak{gl}_n$*). For any* $n \geq 1$ *we have*

(i) For $0 \leq i \leq N$, *we have*

$$dim\ H_i(C_N) = dim\ H_i(C_i).$$

(ii) For $i \geq 3$, *we have*

$$dim\ H_i(C_{i-1}) + 1 = dim\ H_i(C_i).$$

(iii) For $i \geq 4$, *we have*

$$dim\ H_i(C_{i-2}) + 2 + \nu(i-3) = dim\ H_i(C_i).$$

where ν *is the function on positive integers defined by*

$$\nu(n) = \begin{cases} k \text{ if } n = 2k+1 \\ 2\ell - 1 \text{ if } n = 4\ell \\ 2\ell + 1 \text{ if } n = 4\ell + 2 \end{cases}$$

Proof. Write

$$1 + \sum_{1\leq n<k}\gamma_{k,n}t^k x^n = \prod_{n\geq 1}(1-(-t)^{n+1}x^n)^{(-1)^n}\prod_{1\leq n\leq k-3}(1-(-t)^k x^n)^{(-1)^{k-1}q_{k+1,n}} \tag{8.5.1}$$

and

$$1+tx^2+\sum_{3\le n\le k}\xi_{k,n}t^kx^n=(1+tx^2)(1+\sum_{1\le n<k}\gamma_{k,n}t^kx^n) \qquad (8.5.2.)$$

Then it follows from (8.4) that for any pair i,N of integers, dim $H_i(C_N)$ is the coefficient of t^ix^N in

$$(1+x+x^2+...)(1+tx^2+\sum_{3\le n\le k}\xi_{k,n}t^kx^n)=P(t,x). \qquad (8.5.3)$$

The statement (i) is trivial for $i=0$. For $i>0$, it follows from (8.5.3) that dim $H_1(C_N)=1$, dim $H_2(C_N)=0$ for any N, and that for $i\ge 3$,

$$\dim H_i(C_N)=\xi_{i,3}+\xi_{i,4}+...+\xi_{i,N}, \qquad (8.5.4)$$

where $\xi_{k,n}$ is taken to be 0 if $n>k$. When $i\le N$, the right hand side of (8.5.4) is clearly equal to dim $H_i(C_i)$, which proves (i).

For (ii), observe that by (8.5.4), dim $H_i(C_i)-$ dim $H_i(C_{i-1})=\xi_{i,i}$, which by (8.5.2) is equal to $\gamma_{i-1,i-2}$. But it is apparent from (8.5.1) that $\gamma_{n+1,n}=1$ for any $n\ge 1$, which proves (ii).

Applying (8.5.4) again, we see that for $i\ge 3$, dim $H_i(C_i)-$ dim $H_i(C_{i-2})=\xi_{i,i}+\xi_{i,i-1}$. We already know that $\xi_{i,i}=1$. Moreover for $i\ge 4$, $\xi_{i,i-1}=\gamma_{i,i-1}+\gamma_{i-1,i-3}$. Now a calculation using the explicit formula (8.5.1) shows that $\gamma_{n+1,n}=1$ for $n\ge 1$ and that $\gamma_{n+2,n}=\nu(n)$, where the function ν is as defined in the statement of the theorem. The result follows. □

Theorem (8.5) has the following refinement of (4.15) as an immediate consequence.

(8.5)′ Corollary. *With notation as in (4.15) and (8.5), we have*

$$\psi(t)-Poin_{\mathfrak{G}_{rs}}(t)=t^{n+1}+\nu(n-1)t^{n+2}+\textit{higher terms}$$

.

We conclude with a power series identity which arises from a comparison between the expression (8.2) for $P(t,x)$ and the expression obtained in [L1] using a different method.

For $i\ge 1$, define polynomials $q_i(t)\in\mathbb{C}[[t,x]][t^{-1}]$ by

$$q_i(t)=\frac{1}{i}\sum_{d|i}\mu(\frac{i}{d})(-t)^{-d}$$

where $\mu(j)$ is the arithmetical Möbius function.

(8.6) Theorem. *With the above notation, define* $\Gamma(t,x)\in\mathbb{C}[[t,x]]$ *by*

$$\Gamma(t,x)=\prod_{i\ge 1}\left(1+\frac{(-tx)^i}{1-t^{2i}}\right)^{q_i(t)} \qquad (8.6.1)$$

Then $\Gamma(t,x)=P(t,x)$.

More explicitly, in the notation of (8.2), we have

$$\prod_{i\geq 1}\left(1+\frac{(-tx)^i}{1-t^{2i}}\right)^{q_i(t)}=\prod_{k\geq 0,n\geq 1}(1-(-t)^k x^n)^{(-1)^{k-1}q_{k+1,n}}.$$

Theorem (8.6) follows immediately from (8.2) and [L1, proof of (6.1)]. Similar power series identities may be written down for the other cases we have considered.

Finally we remark that by [L1, §6],

$$\Gamma(t,x)=(1+t)^{-1}\prod_{i\geq 1}(1-t^{2i}+(-tx)^i)^{q_i(t)},$$

which leads to a slightly different version of the identity (8.6).

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