GEOMETRIC THEMES IN REPRESENTATION THEORY.

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1. Introduction.

The purpose of this paper is to explore and illustrate how certain geometric ideas may be applied in the context of the theory of representations of groups and algebras. Such applications lead to insights, both algebraic and geometric, as well as suggesting problems. Our discussion will revolve around three themes, the first of which is that of *base change*. The typical application here compares the properties of solution sets of polynomial equations over different domains. E.g., one might ask about relationships among the sets of solutions of the equation $x^2 + y^2 - z^2 = 0$ over the fields $\mathbb{C}, \mathbb{R}, \overline{\mathbb{F}_q}, \mathbb{F}_q$.

Our second theme is *deformation*, particularly of the structure constants of an algebra. A semisimple algebra may be deformed into a non-semisimple algebra in this way. The representation theory of such algebras (e.g. Hecke algebras, quantum groups,...) is often of great interest, and questions about radicals, decomposition numbers and so on may be studied by means of deformation. An example is the theory of cellular algebras [GL1]. The first two themes could loosely be described as "context free geometry".

Much has been written about the " algebrisation of geometry" in the last 50 years. This is the principal tool in the removal of the context from geometry. However in representation theory, there has also been a *geometrisation of algebra*, which is our third theme. A good example of this is the positivity properties of the coefficients of the Kazhdan-Lusztig polynomials, which even for the Hecke algebra of type A_n have no elementary proof and require the question to be put into a geometric context for its solution (see[KL1, KL2] and [DyL]).

Below we shall discuss instances of these themes which arise in the context of studying the representation theory of reductive algebraic groups over various, but particularly finite, fields. In the course of the discussion, we shall also indicate some open problems.

2. Configuration spaces.

These arise in the study of the variety of regular semisimple elements of a Lie algebra (see [L4], [LSe], [Sp1]). For any field K, let

$$M_n(K) = \{(z_1, ..., z_n) \mid z_i \in K, z_i \neq z_j \text{ if } i \neq j\}.$$

The symmetric group Sym_n acts on $M_n(K)$ by permuting coordinates. Denote by $R(\mathrm{Sym}_n)$ the Grothendieck ring of $\mathbb{Q}$-representations of Sym_n. If K is algebraically closed, then $M_n(K)$ is a smooth K-variety and if $\mathcal{D}$ is any functor from such varieties to $\mathbb{Q}$-vector spaces we obtain a representation of Sym_n. Consider

Problem. *Take $K = \mathbb{C}$ (the complex numbers) above. Determine the graded representation of Sym_n on the rational singular cohomology $H^*(M_n(\mathbb{C}))$; i.e., determine $\sum_{i\geq 0} H^i(M_n(\mathbb{C}), \mathbb{Q})t^i$ as an element of the ring $R(Sym_n)[t]$, where t is an indeterminate.*

This is equivalent to the problem of giving, for each element $g \in \mathrm{Sym}_n$, a formula for

$$P_{M_n(\mathbb{C})}(g, t) = \sum_{i\geq 0} \mathrm{trace}(g, H^i(M_n(\mathbb{C}))t^i.$$

This problem may be approached in three different ways (see [L1,L2,L3]). We show here how base change may be used ([L2]) to give an explicit and complete solution to the problem. The space $M_n(\overline{\mathbb{F}_q})$ has a Frobenius endomorphism $F : M_n(\overline{\mathbb{F}_q}) \to M_n(\overline{\mathbb{F}_q})$, given by $F(z_1, ..., z_n) = (z_1^q, ..., z_n^q)$. The Grothendieck fixed point theorem (cf. [D]) may be applied to assert that for any prime ℓ distinct from the characteristic of $\mathbb{F}_q$, and any element $g \in \mathrm{Sym}_n$, we have

$$|M_n(\overline{\mathbb{F}_q})^{gF}| = \sum_{i\geq 0} (-1)^i \mathrm{trace}(gF, H_c^i(M_n(\overline{\mathbb{F}_q}), \overline{\mathbb{Q}_\ell})) \tag{1.1}$$

where the left side is the cardinality of the fixed point set of gF and on the right side one has ℓ-adic cohomology with compact supports (cf. [M]).

To apply this effectively in the current context one has

Theorem (Lehrer [L2], 1992). .

(i) F acts on $H_c^i(M_n(\overline{\mathbb{F}_q}), \overline{\mathbb{Q}_\ell})$ with all eigenvalues equal to q^{i-n} ("purity").

(ii) For almost all characteristics and for any $g \in Sym_n$, we have

$$\mathit{trace}(g, H_c^i(M_n(\overline{\mathbb{F}_q}), \overline{\mathbb{Q}_\ell})) = \mathit{trace}(g, H^{2n-i}(M_n(\mathbb{C})))$$

("comparison").

Taking the Theorem into account in Grothendieck's formula (1.1), we obtain

$$P_{M_n(\mathbb{C})}(g, t) = (-t)^n |M_n(\overline{\mathbb{F}_q})^{gF}|_{q\mapsto -t^{-1}} \tag{1.2}$$

where the right side of (1.2) involves the cardinality of the fixed point set of gF, which is a polynomial in q, with q replaced by $-t^{-1}$.

Examples. 1. Take $g = \mathrm{id}$ in (1.2). Then $|M_n^F| = q(q-1)...(q-(n-1))$, whence the Poincaré polynomial of $M_n(\mathbb{C})$ is $(1+t)(1+2t)...(1+(n-1)t)$, a result originally due to Ar'nold.

2. If $g = (1, 2, ..., n)$ (an n-cycle), then $M_n^{gF} = \{(z_1, ..., z_n) \mid (z_1^q, ..., z_n^q) = (z_2, ..., z_n, z_1)\}$. The cardinality of M_n^{gF} is therefore n times the number of irreducible polynomials of degree n over $\mathbb{F}_q$. It follows easily that $P_{M_n(\mathbb{C})}(g, t) = \sum_{d|n} \mu(\frac{n}{d})(-t)^{n-d}$, where μ is the Möbius function.

With a little more effort, (1.2) provides a complete answer to the original question. The success of the method depended on the theorem concerning purity and comparison. This leads us to

Problem 1. *Prove a general equivariant comparison theorem for ℓ-adic and analytic cohomology of schemes over $\mathbb{Z}$.*

This problem may have connections with the study of purity (see [DiL]).

3. Reductive groups

Denote by $G(K)$ the set of K-points of the group scheme G, which will be assumed to be over $\mathbb{Z}$. For example, if $G = GL_n$, $G(K) = GL_n(K)$. The group $G(\overline{\mathbb{F}_q})$ has a Frobenius endomorphism $F : G(\overline{\mathbb{F}_q}) \to G(\overline{\mathbb{F}_q})$, which corresponds to its $\mathbb{F}_q$-structure, since by assumption, $G(\overline{\mathbb{F}_q})$ is defined over $\mathbb{F}_q$. In case $G = GL_n$, F may be taken to be the map $F : (a_{ij}) \mapsto (a_{ij}^q)$. We write $G^F = G(\overline{\mathbb{F}_q})^F$ for the subgroup of F-fixed points of $G(\overline{\mathbb{F}_q})$. Then $G^F = G(\mathbb{F}_q)$. The complex (or characteristic zero) representation theory of the groups G^F has led to many interesting new areas of research and points of contact with other areas of mathematics. These include Hecke algebras, the Springer representations of Weyl groups and Deligne Lusztig varieties.

These areas have generally arisen in an attempt to discuss the representation theory in a "context free" way. We shall describe in this section some of the connections and show how they lead to interesting problems, particularly in geometry. For the rest of this section, G will denote $G(\overline{\mathbb{F}_q})$, except where otherwise specified.

By a celebrated theorem of Borel and Tits, there is a pair $B_0 \supseteq T_0$, unique up to conjugacy by an element of G^F, such that B_0 is a Borel subgroup of G, T_0 is a maximal torus, and both are F-stable (in general, an F-stable subvariety of G will be called *rational*). In case $G = GL_n$, B_0 and T_0 may be taken to be the subgroups of upper triangular and diagonal matrices respectively.

The Weyl group W enters the story in (at least) three ways:

(i)$W = N_G(T_0)/T_0$, $\cong \mathrm{Sym}_n$ when $G = GL_n$.

(ii) G is the disjoint union $\amalg_{w \in W} B_0 w B_0$ of the Bruhat cells $B_0 w B_0$.

(iii) All maximal tori of G are conjugate in G, by a theorem of Borel. Thus if $\mathcal{T}$ is the variety of all maximal tori, $\mathcal{T} \cong G/N_G(T_0)$. But the rational tori $T \in \mathcal{T}^F$ are not all conjugate under the action of G^F. In fact there is a bijection

$$\mathcal{T}^F/G^F \longrightarrow W/\sim \tag{2.1}$$

(see [L5]), which is defined as follows. If $T \in \mathcal{T}^F$, then $T = gT_0g^{-1}$ for some $g \in G$, and $F(T) = T$ implies that $g^{-1}F(g) \in N_G(T_0)$. Taking $g^{-1}F(g)$ mod T_0 therefore gives an element of W. Taking account of the ambiguity in the choice of g, one obtains the bijection (2.1). We shall speak of a "w-twisted rational torus", understanding that $w \in W$ is only determined up to conjugacy.

The Weyl group W acts as a reflection group on the affine space $V = Y(T_0) \otimes_{\mathbb{Z}} \mathbb{C}$, where for any torus T, $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^{\mathrm{rank} T}$, $\mathbb{G}_m$ being the multiplicative group. In the case $G = GL_n$, $W = \mathrm{Sym}_n$ acts on $V \cong \mathbb{C}^n$ by permuting coordinates. Let $S = \mathbb{C}[V]$ be the coordinate ring of V. Then W acts on S contragrediently, preserving degree. Denote by S^W the subring of W-invariant polynomials and J the ideal of S generated by the invariant polynomials with no constant term. The quotient S/J clearly has a degree preserving W-action and by a theorem of Chevalley [C], S/J is a graded version of the regular representation of W. Write

$$P_{S/J}(t) = \sum_{i=0}^{N} (S/J)_i t^i \in R(W)[t] \tag{2.2}$$

where $(S/J)_i$ is the degree i component of S/J, t is an indeterminate and N is the number of positive roots in the root system of G with respect to T.

The Weyl group W acts on $G(\mathbb{C})/T(\mathbb{C})$: $w.gT(\mathbb{C}) = gwT(\mathbb{C})$, for $w \in W, gT(\mathbb{C}) \in G(\mathbb{C})/T(\mathbb{C})$. Hence W acts on the singular cohomology $H^*(G(\mathbb{C})/T(\mathbb{C}), \mathbb{Q})$ and it is classical (cf. [Sp1, L4]) that in the notation of (2.2), we have the following equation in the ring $R(W)[t]$.

$$\sum_{i \geq 0} H^i(G(\mathbb{C})/T(\mathbb{C}))t^i = P_{S/J}(t^2) \tag{2.3}$$

Let $\phi : W \to \mathbb{C}$ be any function which is constant on conjugacy classes. Then ϕ defines a function : $\mathcal{T}^F \to \mathbb{C}$, which we shall also call ϕ, by taking a w-twisted rational torus to $\phi(w)$. The following result may be found in [L5].

Theorem ([L5, (1.10)]). *We have, with the above notation,*

$$\sum_{T \in \mathcal{T}^F} \phi(T) = q^{2N} < P_{S/J}(q^{-1}), \phi >_W$$

where the right hand side indicates the inner product of class functions on W and N is the number of positive roots of G with respect to T.

This result, as well as related ones in [L5], has several consequences which lead to insights and questions in the context of base change.

Corollary.

(i) (Steinberg) The total number of rational maximal tori $|T^F|$ is q^{2N}.

(ii) If $T \in \mathcal{T}^F$ is w-twisted, say that T is positive or negative according to the sign $\epsilon(w)$, where ϵ is the sign character of W. If $\mathcal{T}_+^F$ is the set of positive rational tori, then $|\mathcal{T}_+^F| = \dfrac{(q^{2N} + q^N)}{2}$.

(iii) Let G_{uni} and $\mathfrak{G}_{\text{nil}}$, respectively denote the varieties of unipotent elements of G and nilpotent elements of the Lie algebra $\mathfrak{G}$. Then $|G_{\text{uni}}^F| = |\mathfrak{G}_{\text{nil}}^F| = |\mathcal{T}^F|$.

These results are all special cases of the theorem. They correspond to the cases where ϕ is taken to be $1, \epsilon$ and $|T_w^F|$ respectively.

These and other results lead to

Problem 2. *Compare the varieties $\mathfrak{G}_{\text{nil}}$ and $\mathcal{T}$ generically.*

While the varieties are superficially very different, one being non-singular, the other having interesting singularities, they share many properties, such as the number of rational points over any finite field, and other local properties. One specific issue for investigation may be local cohomology.

Similar considerations involving conjugacy classes and adjoint orbits lead to

Theorem [L5].

(i) Let M_W be the complexified hyperplane complement corresponding to W (for type A, this is the space M_n of the first section). Then $< H^*(M_W, \mathbb{Q}), \epsilon >_W = 0$.

(ii) For any semisimple algebraic group over $\overline{\mathbb{F}_q}$, which is defined over $\mathbb{F}_q$, we have

$$\sum_s \epsilon(s) = (-1)^{\sigma(G)}$$

where the sum is over a set of representatives s of the rational regular semisimple conjugacy classes of G, $\epsilon(s)$ is the sign of the rational torus $Z_G(s)^0$ and $\sigma(G)$ is the semisimple rank of G.

The equivalence of the statements (i) and (ii) of the theorem is an instance of base change. The statement (ii) has been shown ([LT]) to be equivalent to Alperin's conjecture for the finite group G^F.

4. Springer representations.

In this section we take G to be the complex algebraic group $G(\mathbb{C})$, $\mathfrak{G} = \text{Lie}(G)$, $\mathcal{B} = G/B_0$ is the flag variety, identified with the set of Borel subgroups of G. For any $A \in \mathfrak{G}$ we shall consider the variety $\mathcal{B}_A = \{B \in \mathcal{B} \mid A \in \text{Lie}(B)\}$, which is sometimes referred to as the "fixed point subvariety" of $\mathcal{B}$ corresponding to A. We shall also need $\mathcal{A}(A)$, which is the group of components of $Z_G(A)$, i.e. $\mathcal{A}(A) = Z_G(A)/Z_G^0(A)$, where $Z_G^0(A)$ is the connected component of the identity in $Z_G(A)$. The varieties $\mathcal{B}_A$ are closed subvarieties of the compact space $\mathcal{B}$, hence are compact. Thus we speak of their singular cohomology $H^i(\mathcal{B}_A)$, which is assumed to have complex (or rational) coefficients. The Springer representations of Weyl groups originally arose in connection with the character theory of the groups G^F, but the theory now has intimate connections with several aspects of representation theory and geometry. The most important facts are summarised in the next statement. If H is any group, we use $\widehat{H}$ to denote its set of irreducible complex representations.

Theorem [Sp1, Sp2]. *With the above notation, there is an action of the Weyl group W on $H^i(\mathcal{B}_A)$ for each i, A which has the following properties.*

(i) If A is nilpotent, $\forall \phi \in \widehat{\mathcal{A}(A)}$, the ϕ-isotypic component $H^{\mathrm{top}}(\mathcal{B}_A)^\phi$ is either 0 or an irreducible W module.

(ii) The non-zero W-modules $H^{\mathrm{top}}(\mathcal{B}_A)^\phi$ ($A \in \mathfrak{G}_{\mathrm{nil}}, \phi \in \widehat{\mathcal{A}(A)}$), form a complete set of irreducible W-modules. Each irreducible module occurs exactly once if the pairs (A, ϕ) are taken modulo simultaneous conjugacy by G.

This leads to the "Springer correspondence": $\widehat{W} \ni \chi \mapsto (A_\chi, \phi_\chi)$, where $A_\chi \in \mathfrak{G}_{\mathrm{nil}}$ and $\phi_\chi \in \widehat{\mathcal{A}(A)}$.

A central question concerning the Springer representations is to describe, for any $A \in \mathfrak{G}_{\mathrm{nil}}$, the polynomial $Q_A(t) = \sum_{i\geq 0} H^{2i}(\mathcal{B}_A)t^i$ as an element of $R(W)[t]$, or better still, as an element of $R(W \times \mathcal{A}(A))[t]$. One motivation (in fact the original one) for this is that if R_w is the generalised character $R_{T_w}^{(G)}(1)$ of G^F which was defined by Deligne and Lusztig ([DL]), T_w being a w-twisted torus, the values of R_w on unipotent elements of G^F may be "generically" described as follows, by the work of Lusztig, Kazhdan, Springer, Spaltenstein, Shoji and others.

The unipotent conjugacy classes of $G^F = G(\mathbb{F}_q)$ are generically parametrised by pairs (A, c), where $A \in \mathfrak{G}_{\mathrm{nil}}/\mathrm{Ad}(G)$ and $c \in \mathcal{A}(A)$. Then

$$R_w(A, c) = Q_A(q)(w, c)$$

, where $Q_A(q)$ is regarded as an element of $R(W \times \mathcal{A}(A))[q]$. This leads naturally to

Problem 3. *For any nilpotent element $A \in \mathfrak{G}$, and character ϕ of $\mathcal{A}(A)$, determine explicitly the polynomials $< Q_A^\phi, \chi >$, where χ is any (irreducible) representation of W and Q_A^ϕ is the ϕ-isotypic component or Q_A. Express the result as a function $f(A, \phi, A_\chi, \phi_\chi)$ in a "context free" way.*

Among the special cases about which something is known is the case $A = 0$. Here $\mathcal{B}_A = \mathcal{B}$, and it is known (see, e.g. [Sp1]) that

$$Q_0(t) = P_{S/J}(t)$$

In this case much (but not all) is known concerning Problem 3. Another case which has been studied is that where $A = A_J$ is a regular nilpotent element in $\mathrm{Lie}(L_J)$, where L_J is the standard Levi subgroup of the standard parabolic subgroup corresponding to the subset J of the simple roots of G. For this case, we have

Theorem. *(Lehrer-Shoji, Spaltenstein) Let ρ_j be the j^{th} exterior power of the reflection representation $\rho = rho_1$ of W. Then*

$$< Q_{A_J}, \rho_j >_W = \sigma_j(q^{b_1(J)}, q^{b_2(J)}, ...)$$

where σ_j denotes the j^{th} elementary symmetric function and the $b_j(J)$ are defined in terms of the topology of the space $M_W(\mathbb{C})$.

For $J = \emptyset$ (i.e. $A_J = 0$), this is an old result of Solomon. For $j = 1$, see [LSh]. The general case has not been published. Note that in type A, the space $M_W(\mathbb{C})$ is the configuration space of §1 above.

This leads to the fundamental

Problem 4. *Establish direct connections between configuration spaces (like $M_n(\mathbb{C}), M_W$) and Schubert varieties, which are the closures of Bruhat cells B_0wB_0 (see above) in the flag variety of G. This problem should include the case of Kac-Moody groups and quantum groups- i.e., whenever there is a reasonable geometric theory of the flag variety and of the Springer representations.*

The spirit of the Springer construction is an interpolation between the variety $\mathfrak{G}_{rs}$ of regular semisimple elements of $\mathfrak{G}$ and the variety $\mathfrak{G}_{\text{nil}}$. Now $\mathfrak{G}_{rs}$ is related to the configuration spaces M_W (see [L4]) and their homology has a very rich structure of graded Poisson algebra (see [LSe]). This could be a key to Problem 3. In addition, the spaces M_W are related to knot theory, and it would be very interesting to make direct connections between knot invariants and our representation theoretic ideas. One concept which occurs in both contexts of Problem 4 (i.e. in the theory of Schubert varieties and in configuration spaces, via knot theory) is that of a Hecke algebra, which is the subject of our next section.

Another approach to the questions raised in this section is through the study of the space of complex valued functions on a finite Lie algebra, which has a rich structure, including operations of induction and restriction, convolution and Fourier transform (see [Lu2],[L9] and [Sp1]). The study of this space, in particular of the explicit value of certain Fourier transforms, provides a path to some aspects of Problem 3 (see [L10], [Ja]).

5. Hecke algebras

We begin by recalling how Hecke algebras arise in the two contexts of Problem 4. Let $D^b_c(V)$ be the category of bounded complexes of sheaves on a variety V, with constructible cohomology sheaves. If V is acted on by an algebraic group G, $D^b_c(V)^G$ denotes the $G-$ invariant objects in $D^b_c(V)$. Let G be a complex algeraic group and let W be the Weyl group of G. Then there is an associative algebra $H_W(u)$ over the ring $R = \mathbb{Q}[u, u^{-1}]$, which we shall define shorly. This is the Hecke algebra of W, or of G. If $\mathcal{B}$ is the flag variety of G, then there is an associative "convolution product" (see [DyL] and the references there)

$$D^b_c(\mathcal{B} \times \mathcal{B})^G \times D^b_c(\mathcal{B} \times \mathcal{B})^G \longrightarrow D^b_c(\mathcal{B} \times \mathcal{B})^G$$

and a map (cf. [DyL] loc.cit.)

$$D^b_c(\mathcal{B} \times \mathcal{B})^G \longrightarrow H_W(u)$$

which takes convolution of complexes into multiplication in the Hecke algebra. The "Kazhdan-Lusztig basis of $H_W(u)$" consists of the images of the simple objects in $D^b_c(\mathcal{B}\times \mathcal{B})^G$.

On the other hand the algebra $H_W(u)$ has been prominent (at least in the classical types) in the construction of knot invariants, which place it in the context of the spaces $M_W(\mathbb{C})$ (cf. Jones [J1], [J2] or the survey [L6]).

The "Hecke algebras" were originally defined by Iwahori [I] in the following context. Maintaining the notation introduced in §2, when q is any prime power, Iwahori determined the structure of

$$H_W(q) = \mathrm{End}_{\mathbb{C}G^F}(\mathrm{Ind}_{B_0^F}^{G^F}(1)).$$

The reason for the notation on the left side is that the endomorphism algebra on the right turned out to have a $\mathbb{C}$ basis parametrised by W, and multiplication table which may be described with reference only to W, i.e., which is in a sense independent of G. This structure may be described as follows. For any Coxeter system (W, S) with S finite, there is an associated length function $\ell(w)$ for $w \in W$. The algebra $H_W(q)$ has $\mathbb{C}$-basis $\{T_w, w \in W\}$ and multiplication given by

$$T_s T_w = \begin{cases} T_{sw} \text{ if } \ell(sw) > \ell(w) \\ qT_{sw} + (q-1)T_w \text{ if } \ell(sw) < \ell(w) \end{cases} \tag{4.1}$$

for any $s \in S$ and $w \in W$.

Replacing q by an indeterminate u, the relations (4.1) may be thought of as defining an algebra $H_W(u)$ over the ring $R = \mathbb{C}[u, u^{-1}]$, which is known as the generic Hecke algebra. If $\sigma : R \to K$ is any ring homomorphism, the K-algebra $K \otimes_\sigma H_W(u)$ is called a *specialisation* of $H_W(u)$. Such specialisations may be obtained by "putting u equal to a complex number z". Specialisations of $H_W(u)$ are generically semisimple, but the study of non-semisimple specialisations is important for the modular representation theory of the groups G^F and W ([BMM]), for knot theory ([L6, L7]), for connections with Jones invariants ([GL2]) and for several other connections with p-adic and quantum groups.

The algebras $H_W(u)$ fit into the more general context of "cyclotomic Hecke algebras" as follows. The functor $\mathrm{Ind}_{B_0^F}^{G^F} : R(T_0^F) \to R(G^F)$ is a special case of Lusztig induction $R_L^G : R(L^F) \to R(G^F)$, where L is any F-stable Levi subgroup of $G(\overline{\mathbb{F}_q})$. In general R_L^G takes representations to virtual representations, which may be realised on a complex of modules. Investigation of the decomposition of the functors R_L^G leads (cf. [BM]) to "Hecke algebras" associated with centralisers $Z_W(w)$ of certain elements $w \in W$. These centralisers are not in general Coxeter groups, but they are closely related to unitary reflection groups (see [LSp1, LSp2, BM]). Thus one now has, for any unitary (or complex) reflection group W, a "cyclotomic Hecke algebra" $H_W(\mathbf{u}) = H_W(u_1, u_2, ...)$ (see [BMM]) which is defined using the special presentations of the unitary reflection groups which have been determined by Malle et al. All of these algebras are defined ofver rings like $\mathbb{C}[u_1^{\pm 1}, u_2^{\pm 1}, ...]$ and present similar problems concerning their representation theory, particularly at non-semisimple specialisations. The concept of "celular algebra" ([GL1]) is specifically designed to handle non-semisimple specialisations (or "deformations") of algebras which are generically semisimple.

We shall not go into details concerning cellular algebras here; these may be found in [GL1]. Since they provide an approach to several important problems, we list here their main properties. Let A be a finite dimensional cellular algebra over the commutative unital ring R. Then

(i) Any specialisation of A has a cellular structure.

(ii) There is a collection of "cell modules" $\{W_\lambda, \lambda \in \Lambda\}$ for A. Some authors refer to these as standard modules.

(iii) There is an invariant bilinear form ϕ_λ on each cell module W_λ.

(iv) The radical of ϕ_λ coincides with the radical of the module W_λ.

(v) If R is a field, $L_\lambda = W_\lambda / \mathrm{Rad}\phi_\lambda$ is either zero or irreducible.

(vi) The non-zero L_λ form a complete set of irreducible A-modules.

(vii) A is semisimple if and only if each ϕ_λ is non-degenerate.

It is therefore apparent that to study deformations, the existence of a cellular structure is of great assistance. This suggests

Problem 5. *Which of the algebras $H_W(\mathbf{u})$, where W is a unitary reflection group, have a cellular structure? If possible, describe the cellular structure geometrically.*

Currently, cellular structures are known for the cyclotomic Hecke algebras corresponding to the groups $G(r,1,n)$, which include the cases of type A_n, B_n, C_n; in addition types G_2 and F_4 are known to have a cellular structure, as is type D_n when n is odd. In type A_n, the cellular structure is related to the Robinson Schensted correspondence, which has a geometric description in terms of the flag variety (see [St]). It would be desirable to have a geometric description of a cellular structure whenever it exists. Given an algebra A with a cellular structure, it is clear from the properties listed above, that to solve the fundamental problems in its representations theory, viz. to classify and find the dimension of its irreducible modules, it suffices to

(I) Compute the discriminants of the canonical forms ϕ_λ, and

(II) Determine the spaces of homomorphisms between the cell modules.

Observe that $\dim(\mathrm{Hom}_A(W_\mu, W_\lambda))$ is the multiplicity $d_{\mu,\lambda}$ of the irreducible module L_μ in a composition series of W_λ. Since the dimensions of the cell modules W_λ may be assumed known, knowledge of these multiplicities (or decomposition numbers) implies knowledge of the dimensions of the irreducibles.

Problem 6. *Address questions (I) and (II) for those among the $H_W(\mathbf{u})$ which have a cellular structure.*

Note that the decomposition numbers are not understood, even in the case of the Hecke algebra of type A_n, although there are partial results, due to Dipper, James and Mathas.

Next, let us recall that multiplication in the Hecke algebra of a Coxeter group satisfies remarkable positivity properties (see [DyL] and the references there). These are understood geometrically in some cases, but are valid in a wider range of cases (see, e.g., [Dy]). In addition, there is a large body of evidence concerning "fake and generic degrees" and "Howlett-Lehrer theory" for "unipotent representations" associated with cyclotomic Hecke algebras as well as further elaborate structure (see Spetses I [BMM]). Therefore one could ask for a more direct connection between the reflection group W, the Hecke algebra $H_W(\mathbf{u})$ and the Green functions (called Q_A above). Ideas such as those in [So] may be relevant here. A simple explicit question is

Problem 7. *For any unitary reflection group W, find an algebraic variety $\mathcal{V}(W)$ such that $H^*(\mathcal{V}(W)) \cong S/J$ as graded W-module, where degrees are possibly inflated (as in the Lie group case, where $\mathcal{V}(W)$ may be taken to be G/T).*

More generally, one might ask for abstract geometric realisations of the graded modules Q_A of §3, which is related to Problem 3 above. By abstract, we understand constructions which depend only on the reflection data. This theme and the deformation theory of $H_W(\mathbf{u})$ are combined in the next problem, which is not specific.

Problem 8. *Give a general setting for interpolation between semisimple and nilpotent objects, such as elements of an algebraic group, varieties, algebras,...*

An example of this is the very construction of the Springer representations given in [KL3] (see also [Sh]). One takes a local system over the variety of regular semisimple elements, extends to a perverse sheaf on the whole Lie algebra (this is a complex of sheaves) and restricts to the nilpotent variety.

6. Braids and Hecke algebras.

The theme of §2, viz. configuration spaces, arises also in the consideration of Hecke algebras and their connections with knot invariants. Suppose W is a finite Coxeter group acting as a reflection group in $\mathbb{C}^n$. We have referrred above to the space M_W, which is obtained from $\mathbb{C}^n$ by removing the reflecting hyperplanes of W. Now the Coxeter presentation of W is of the form

$$W =< s \in S; s^2 = 1, sts... = tst... \text{for } s, t \in S >,$$

where there are m_{st} terms on each side of the second relation.

The relations of the second type are called the braid relations for W. It is well known that the group

$$B_W =< s \in S; \text{ braid relations } >$$

is the fundamental group of the space $X_W = M_W/W$ of orbits of W on M_W. It is the "generalised Artin braid group" corresponding to W, the name having its origin in the case of type A_n, where B_W is the original $n+1$-string braid group. Similar statements to those above apply even when W is taken to be a finite group generated by complex (pseudo)-reflections in $\mathbb{C}^n$ (see [BMM]), although the definition of the groups B_W is more complicated. It is straightforward that there is a group homomorphism

$$\phi_W : B_W \longrightarrow H_W(\mathbf{u})$$

given by $s \in S \mapsto T_s$.

Problem 9. *Is the homomorphism ϕ_W injective?*

Solution of this problem would have important implications for the theory of invariants of oriented links. It is open even for Hecke algebras of type A_n for small n, e.g. $n = 4$.

Finally, we observe that although we have spoken principally about finite dimensional algebrtas, the ideas we have mentioned have a wider range of applicability. For example in [GL2], the concept of cellular algebras is successfully applied to give a complete description of the irreducible representations of the affine Temperley-Lieb algebras. A by-product of this discussion was an explicit formula for the "augmentation" or "Jones" idempotent of the finite dimensional Temperley-Lieb algebras, which is important in knot invariant theory, in the case when the parameter is a root of unity. We therefore conclude with

Problem 10. *Generalise the theory of cellular algebras to*
(i) infinite dimensional associative algebras (e.g. Schur algebras), and
(ii) non-associative algebras, e.g. Lie algebras, Hopf algebras.

References

[BT] Armand Borel and Jacques Tits, Groupes réductifs, Inst. Hautes Études Sci. Publ. Math. No. 27 1965, 55–150.

[BM] M. Broué and G. Malle, Zyklotomische Heckealgebren, Représentations unipotentes génriques et blocs des groupes réductifs finis. Astérisque No. 212 , (1993), 119–189.

[BMM] M. Broué, G. Malle and J. Michel, Towards Spetses I, to appear, Transf. Groups, 2000.

[C] Claude Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math. 77 (1955), 778–782.

[D] P. Deligne, Cohomologie Étale (SGA $4\frac{1}{2}$), Springer L.N.M. 509 , Springer-Verlag, Berlin, 1977

[DL] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields. Ann. of Math. (2) 103 (1976), 103–161.

[DiL] A. Dimca and G.I. Lehrer, Purity and equivariant weight polynomials, Aust. Math. Soc. Lecture Series 9 (1996), 161-182.

[Dy] M.J. Dyer, On some generalisations of the Kazhdan-Lusztig polynomials for "universal" Coxeter systems, J. Algebra 116 (1988), 353–371

[DyL] M.J. Dyer and G.I. Lehrer, On positivity in Hecke algebras, Geom. Ded. 25 (1990), 115-125.

[GL1] J.J. Graham and G.I. Lehrer, Cellular algebras, Inventiones Math. 123 (1996), 1-34

[GL2] J.J. Graham and G.I. Lehrer, The representation theory of affine Temperley-Lieb algebras, L'Enseignment Math. 44 (1998), 173–218.

[I] Nagayoshi Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field. J. Fac. Sci. Univ. Tokyo Sect. I 10 (1964), 215–236.

[J1] V.F.R. Jones, A polynomial invariant for knots via von Neumann algebras, Bull. Amer. Math. Soc. 12 (1985), 103–111

[J2] V.F.R. Jones, Hecke algebra representations of braid groups and link polynomials. Ann. of Math. (2) 126 (1987), 335–388.

[Ja] D.G.A. Jackson, Values of the Fourier transforms of Green functions at regular elements, preprint, University of Sydney 1998, to appear.

[KL1] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979), 165–184.

[KL2] D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, Geometry of the Laplace operator (Proc. Sympos. Pure Math. XXXVI, Amer. Math. Soc., Providence, R.I., 1980, Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 185–203.

[KL3] D. Kazhdan and G. Lusztig, A topological approach to Springer's representations. Adv. in Math. 38 (1980), 222–228

[L1] G.I. Lehrer, On the Poincaré series associated with Coxeter group actions on complements of hyperplanes, J. London Math. Soc. (2) 35, (1987), 275-294.

[L6] G.I. Lehrer, A survey of Hecke algebras and the Artin braid groups, Contemp. Math. 78 (1988), 365-385.

[L7] G.I. Lehrer, Group representations, geometry and topology, Springer Lecture Notes in Mathematics 1456 (1990) 14-31.

[L5] G.I. Lehrer, Rational tori, semisimple orbits and the topology of hyperplane complements, Commentarii Math. Helv 67 (1992), 226-251.

[L2] G.I. Lehrer, On the ℓ-adic cohomology of hyperplane complements, Bull. Lond. Math. Soc. 24 (1992), 76-82.

[L9] G.I. Lehrer, The space of invariant functions on a Lie algebra, Trans. Amer. Math. Soc. 348 (1996), 31-50.

[L3] G.I. Lehrer, Poincaré series for unitary reflection groups, Inventiones Math. 120 (1995) 411-425.

[L4] G.I. Lehrer, The cohomology of the regular semisimple variety, J.Algebra 199 (1998), 666-689.

[L10] G.I. Lehrer, Fourier transforms, nilpotent orbits, Hall polynomials and Green functions, Finite reductive Groups (Luminy 1994), 291-309, Progr. Math. 141 Birkhäuser, Boston MA 1997.

[Lu1] G. Lusztig, Green functions and character sheaves, Ann. of Math. (2) 131 (1990), 355–408.

[Lu2] G. Lusztig, Fourier transforms on a semisimple Lie algebra over F_q, Algebraic groups Utrecht 1986, Lecture Notes in Math. Springer, Berlin-New York, 1271,(1987), 177–188.

[LSe] G.I. Lehrer and G.B. Segal, Iterated loop spaces and homology stability for regular semisimple varieties, to appear, Math. Zeit. 1999.

[LSh] G.I. Lehrer and T. Shoji, On flag varieties, hyperplane complements and the Springer representations of Weyl groups, J. Aust. Math. Soc. (Series A) (1990), 449-485.

[LSp1] G.I. Lehrer and T. A. Springer, Intersection multiplicities and reflection subquotients of unitary reflection groups I, in *Geometric Group Theory Down Under: Proceeedings of a special year in Geometric Group Theory, Canberra, Australia 1996*, edited by Cossey et al., de Gruyter 1999, 181–194.

[LSp2] G.I. Lehrer and T.A. Springer, Reflection subquotients of unitary reflection groups, to appear, Canadian J. Math., 1999.

[LT] G.I. Lehrer and J. Thévenaz, Sur la conjecture d'Alperin pour les groupes réductifs finis, Comptes Rendus Acad. Sci. Par. t.315(1993), Sér. 1, 1347-1351

[M] J.S. Milne, Étale cohomology. Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980.

[So] W. Soergel, Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, J. Amer. Math. Soc. 3 (1990), 421–445.

[Sp1] T.A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups. Invent. Math. 36 (1976), 173–207.

[Sp2] T.A. Springer, A construction of representations of Weyl groups, Invent. Math.

44 (1978), 279–293.

[St] R. Steinberg, On the desingularization of the unipotent variety, Invent. Math. 36 (1976), 209–224.