ASYMPTOTIC ANALYSIS OF RICCI FLOW ON $\mathbb{R}^{n+1}$ WITH TYPE-IIB SINGULARITIES

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Abstract. In this paper, we study the precise asymptotics of Type-IIb solutions to Ricci flow on $\mathbb{R}^{n+1}$. In each dimension $n+1 \geq 3$ and for each real number $\lambda > 0$, we construct complete rotationally symmetric solutions to Ricci flow on $\mathbb{R}^{n+1}$ that form in infinite time Type-IIb singularities with the curvature blow-up rate $t^{\lambda-1}$. Near the origin, the blow-ups of such a solution converge uniformly to the Bryant soliton; near spatial infinity, the solution is asymptotically flat at a precise rate depending on λ.

1. Introduction

A one-parameter family of $(n+1)$-dimensional complete smooth Riemannian manifolds $(M, g(t))_{t_0 \leq t < t_1}$ is said to evolve by Hamilton's Ricci flow [21], starting from an initial metric g_0, if $g(t)$ satisfies the equation

$$\partial_t g = -2\,\mathrm{Ric}(g), \qquad g(t_0) = g_0, \tag{1.1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature of the metric.

Let $(M, g(t))$ be a solution to Ricci flow that exists up to a maximal time $T \leq \infty$. If $T < \infty$, then we say the Ricci flow has a finite-time singularity of

- Type-I if $\sup_{M \times [t_0, T)} (T-t)|\mathrm{Rm}(\cdot, t)| < \infty$,
- Type-IIa if $\sup_{M \times [t_0, T)} (T-t)|\mathrm{Rm}(\cdot, t)| = \infty$.

If $T = \infty$, then the infinite-time singularity of this immortal Ricci flow is said to be

- Type-III if $\sup_{M \times [t_0, \infty)} t|\mathrm{Rm}(\cdot, t)| < \infty$,
- Type-IIb if $\sup_{M \times [t_0, \infty)} t|\mathrm{Rm}(\cdot, t)| = \infty$.

If a Ricci flow solution encounters a Type-IIb (or Type-III) singularity, then we also call it a Type-IIb (or Type-III) solution to Ricci flow. Analogous classifications hold for solutions to mean curvature flow (MCF) with $|h(\cdot, t)|^2$,

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the second fundamental form of a hypersurface moving by MCF, replacing $|\operatorname{Rm}(\cdot, t)|$ in the above definitions.

Naturally, we have the following questions (cf. [13, Problem 8.6]): What can be said about the specific blow-up rates of Ricci flow solutions with a Type-IIa or Type-IIb singularity? What about the asymptotic properties of Ricci flow solutions of each type near the singular time T?

In real dimension two, by the work of Hamilton [22] and Chow [11], Ricci flow on S^2 always encounters a Type-I singularity modelled by the round sphere. On the other hand, Daskalopoulos and Hamilton [16] have showed that Ricci flow on $\mathbb{R}^2$ starting from a metric of finite area forms a Type-IIa singularity at the rate $(T-t)^{-2}$. The precise description of the extinction profile of such a solution were later given in [15] and [17]: the solution is modelled by a cigar soliton in an inner region, and has a logarithmic cusp in an outer region.

In real dimension three, Hamilton's seminal work [21] says that Ricci flow of a closed three-manifold with positive Ricci curvature forms a Type-I singularity and shrinks to a round point. This result was later generalised to higher dimensions under other curvature assumptions, e.g., the 2-positive curvature operator by Böhm and Wilking [5]. These Type-I singularities are global in the sense that the volume of the manifold at the singular time T is zero. In comparison, there exist local singularities that form on compact subsets of a manifold and the volume of the manifold remains positive at the singular time T. For example, Type-I nondegenerate neckpinches modelled by the round cylinder have been rigorously constructed on S^{n+1} $(n \geq 2)$ by Angenent and Knopf [1].

In real dimensions $n+1 \geq 3$, Type-IIa singularity was first proved to exist in Ricci flow on S^{n+1} by Gu and Zhu [20]. Concerning the geometric details of such a solution, Garfinkle and Isenberg [19] gave numerical evidence that a degenerate neckpinch in Ricci flow on S^3 is a Type-IIa singularity modelled by the rotationally symmetric Bryant soliton, which was first constructed by Bryant [8] and has been proven by Brendle to be the unique complete non-flat steady gradient Ricci soliton in dimension three under a non-collapsing assumption [6]; see also Brendle's generalisation to higher dimensions [7]. In [3], Angenent, Isenberg and Knopf have constructed on S^{n+1} Ricci flow with Type-IIa singularities modelled on the Bryant soliton with curvature blow-up rate $(T-t)^{-2+2/k}$ for each integer $k \geq 3$. In contrast, Type-IIa singularities to Ricci flow on $\mathbb{R}^{n+1}$ $(n \geq 2)$ with curvature blow-up rates $(T-t)^{-(\lambda+1)}$ for any real number $\lambda \geq 1$ have been constructed by the author in [32]. There are also corresponding results on Type-IIa singularities in MCF by the author and his collaborators [23, 24].

There are several recent results on Ricci flow with Type-IIa singularities. Appleton [4] has showed that Ricci flow on a noncompact four-manifold can develop Type-IIa singularities modelled on the Eguchi-Hanson space. Di Giovanni [18] has proved that asymptotically cylindrical Ricci flow on $\mathbb{R}^{n+1}$ without minimal sphere forms a Type-IIa singularity modelled on the

Bryant soliton after suitable dilations. Stolarski [30] has constructed on certain product manifolds Ricci flows that form Type-IIa singularities with curvature blow-up rates given by arbitrarily large powers of $(T-t)^{-1}$. If we specialise the Ricci flow to Kähler manifolds, then Li, Tian and Zhu have given the first examples of Type-IIa singularities on Fano manifolds [27].

Concerning the Type-IIb singularities in Ricci flow, the simplest example on compact manifolds is a non-flat Ricci-flat Kähler metric on a K3 surface, whose existence follows from Yau's resolution [33] of the Calabi Conjecture; note that this solution is static under Ricci flow. Further results on Kähler-Ricci flows with Type-IIb singularities have been obtained by Tosatti and Zhang [31]. It has been conjectured [12, Conjecture A.38] that Ricci flow on a closed 3-manifolds never forms a Type-IIb singularity .

In general, any steady Ricci soliton is a Type-IIb solution to Ricci flow. Examples of steady Ricci solitons include the cigar soliton on $\mathbb{R}^2$ and the Bryant soliton on $\mathbb{R}^n$ for $n \geq 3$. Both the cigar soliton and the Bryant soliton are rotationally symmetric. Many non-rotationally symmetric steady Ricci solitons have been found recently. Notably, Lai [26] has constructed $\mathbb{Z}_2 \times O(n)$-symmetric, non-rotationally symmetric, steady gradient Ricci solitons on $\mathbb{R}^{n+1}$ for $n \geq 2$. Taking the product of a steady Ricci soliton with $\mathbb{R}^k$ $(k \geq 1)$ produces (somewhat trivially) a Type-IIb solution to Ricci flow. Additional example of Type-IIb solution is found in [9], where Cabezas-Rivas and Wilking constructed an immortal 3-dimensional non-negatively curved complete Ricci flow with unbounded curvature for all time.

In this paper, we are interested in constructing *non-Kähler*, *non-soliton* solutions to Ricci flow with Type-IIb singularities on a complete noncompact manifold and analysing their precise asymptotics as $t \nearrow \infty$.

Throughout this paper, we use C_k $(k \in \mathbb{N})$ to denote a positive constant that depends at most on n or λ, and may change from line to line. The expression "$f \lesssim g$" means $f \leq C_k g$ for some constant C_k; "$A \sim B$"if and only if $A \lesssim B$ and $B \lesssim A$.

Our main result is the following.

Theorem 1.1. *In each dimension $n+1 \geq 3$, for each real number $\lambda > 0$, there exists an open set (in C^2 topology) $\mathcal{G}$ of complete rotationally symmetric metrics, none of which is the Bryant soliton, on $\mathbb{R}^{n+1}$ such that Ricci flow starting at each $g_0 \in \mathcal{G}$ has a unique solution $g(t)$ for $t \in [t_0, \infty)$. The solution $g(t)$ has the following asymptotic properties as $t \nearrow \infty$.*

1. *The singularity is Type-IIb with*
$$\sup_{\mathbb{R}^{n+1}} |\mathrm{Rm}(\cdot, t)| \sim t^{\lambda-1}$$
attained at the origin of $\mathbb{R}^{n+1}$.
2. *If we rescale the solution so that the distance from the origin rescales at the rate $t^{-(1-\lambda)/2}$, then the metric converges uniformly on intervals of order $t^{(1-\lambda)/2}$ to the Bryant soliton.*

(3) *Near spatial infinity, the metric is asymptotically flat, i.e.,* $|\operatorname{Rm}(\cdot, t)| \to 0$*, for all* $t \geq 0$*, with the precise asymptotics at spatial infinity described in Section 3.4.*

In particular, the solution exhibits the asymptotic behaviour of the formal solution described in Section 3.

Theorem 1.1 constructs Type-IIb solutions to Ricci flow on $\mathbb{R}^{n+1}$ for $n \geq 2$ that are not soliton or Kähler. These solutions (and also the Bryant soliton) show that the exponent $(\lambda - 1)$ of the Type-IIb blow-up rate $t^{\lambda-1}$ belongs to a continuum $(-1, \infty)$. The curvature blow-up rates of previous examples of Type-IIb solutions are restricted, e.g. $\lambda = 1$ for steady Ricci solitons. We note that a continuum of curvature blow-up rates has been observed for Type-IIa singularities in Ricci flows on $\mathbb{R}^{n+1}$ [32]. As $t \nearrow \infty$, the Ricci flow solutions constructed in Theorem 1.1 converge uniformly to a non-Euclidean flat metric (cf. Remark 3.3) if $\lambda \in (0, 1)$ and otherwise if $\lambda \geq 1$. As previously mentioned, the Bryant soliton is a Type-IIb solution whose curvature blow-up rate is $t^{\lambda-1}$ with $\lambda = 1$. So we may ask whether or not the Bryant soliton appears as a "phase change" among Type-IIb solutions to Ricci flow when the parameter λ varies across the "critical value" $\lambda = 1$. Lastly, one may compare Theorem 1.1 for Ricci flow with the construction of Type-IIb solutions to MCF in [25].

The proof of Theorem 1.1 uses matched asymptotic analysis and barrier arguments for nonlinear PDE. The same strategy has been implemented for Ricci flow or mean curvature flow with Type-IIa singularities in [23, 24, 32], and Type-IIb MCF solutions in [25]. In Section 2, we recall the set-up for rotationally symmetric Ricci flow on $\mathbb{R}^{n+1}$ and collect some basic facts. In Section 3, we derive approximate (formal) solutions using the method of formal matched asymptotics. In Section 4, we use these approximate solutions to construct the corresponding supersolutions and subsolutions to the rescaled PDE. The supersolutions and subsolutions are ordered and patched together in Section 5 to create barriers to the rescaled PDE; a comparison principle for the subsolutions and supersolutions is also proved there. In Section 6, we complete the proof of Theorem 1.1.

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2. Preliminaries

Let $\mathcal{O}$ denote the origin of $\mathbb{R}^{n+1}$ $(n \geq 2)$. We identify $\mathbb{R}^{n+1} \setminus \mathcal{O}$ with $(0, \infty) \times S^n$ and equip it with the time-dependent warped product metric

$$g = \varphi^2(x, t) dx^2 + \psi^2(x, t) g_{\text{sph}},$$

where $x \in (0, \infty)$ and g_{sph} is the metric of constant sectional curvature one on S^n.

We recall some basic facts about such a metric, cf. [3, Section 2]. The distance s to the origin is

$$s(x,t) := \int_0^x \varphi(y,t)dy.$$

In the s-coordinate, the metric becomes

$$g = ds^2 + \psi^2(s,t)\, g_{\mathrm{sph}}. \tag{2.1}$$

If we extend the metric g to a complete smooth metric, still denoted by g, on $\mathbb{R}^{n+1}$, then ψ necessarily satisfy the boundary conditions

$$\lim_{x\searrow 0}\psi = 0 \quad \text{and} \quad \lim_{x\searrow 0}\psi_s = 1.$$

In this paper, we use the notation $\partial_t|_\cdot$ for taking the time derivative while keeping the quantity "$\cdot$" fixed. Then

$$[\partial_t|_x, \partial_s] = -n\frac{\psi_{ss}}{\psi}\partial_s.$$

In the s-coordinate, the Ricci flow system (1.1) is reduced to the following parabolic PDE for ψ,

$$\partial_t|_x \psi = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi}. \tag{2.2}$$

The function φ, which is suppressed in the s-coordinate, evolves under Ricci flow by

$$\partial_t|_x \log\varphi = n\frac{\psi_{ss}}{\psi}.$$

Let K denote the sectional curvature of a two-plane with one radial and one spherical direction and L the sectional curvature of a two-plane tangential to the sphere $\{x\}\times S^n$. Then

$$K = -\frac{\psi_{ss}}{\psi}, \qquad L = \frac{1-\psi_s^2}{\psi^2}. \tag{2.3}$$

In particular, $|\mathrm{Rm}|^2 = 2nK^2 + n(n-1)L^2$.

Since the metric g is smooth and $\lim_{x\searrow 0}\psi_s = 1$, we must have $\psi_s > 0$ in a neighbourhood of the origin $\mathcal{O}$. So we can use ψ as a new coordinate near the origin to write

$$g = z(\psi,t)^{-1}d\psi^2 + \psi^2 g_{\mathrm{sph}}, \tag{2.4}$$

where $z(\psi,t) := \psi_s^2$. Then the sectional curvatures are rewritten as

$$K = -\frac{z_\psi}{2\psi}, \qquad L = \frac{1-z}{\psi^2}. \tag{2.5}$$

Under Ricci flow, the metric (2.4) evolves by (see [3, Section 2.2])

$$\partial_t|_\psi z = \mathcal{E}_\psi[z], \tag{2.6}$$

where $\mathcal{E}_\psi$ is the purely local quasilinear operator

$$\mathcal{E}_\psi[z] := zz_{\psi\psi} - \frac{1}{2}z_\psi^2 + (n-1-z)\frac{z_\psi}{\psi} + 2(n-1)\frac{(1-z)z}{\psi^2}.$$

The boundary conditions we impose for equation 2.6 are

$$\lim_{\psi\searrow 0} z = 1, \quad \lim_{\psi\nearrow\infty} z = 0.$$

We can split $\mathcal{E}_\psi$ into a linear part and a quadratic part:

$$\mathcal{E}_\psi[z] = \mathcal{L}_\psi[z] + \mathcal{Q}_\psi[z],$$

where

$$\mathcal{L}_\psi[z] := (n-1)\left(\frac{z_\psi}{\psi} + 2\frac{z}{\psi^2}\right), \tag{2.7}$$

$$\mathcal{Q}_\psi[z] := zz_{\psi\psi} - \frac{1}{2}z_\psi^2 - \frac{zz_\psi}{\psi} - 2(n-1)\frac{z^2}{\psi^2}. \tag{2.8}$$

The quadratic part defines a symmetric bilinear operator

$$\hat{\mathcal{Q}}_\psi[z_1, z_2] := \frac{1}{2}\left[z_1(z_2)_{\psi\psi} + z_2(z_1)_{\psi\psi} - (z_1)_\psi(z_2)_\psi\right] \tag{2.9}$$

$$- \frac{z_1(z_2)_\psi + z_2(z_1)_\psi}{2\psi} - 2(n-1)\frac{z_1z_2}{\psi^2}. \tag{2.10}$$

In particular, $\mathcal{Q}_\psi[z] = \hat{\mathcal{Q}}_\psi[z, z]$.

3. Formal solutions

The basic idea behind the construction of the formal solutions, (i.e. approximate solutions) is to analyse the evolution equation (2.6) under various rescalings of ψ and find approximate solutions to the rescaled PDEs. The formal solutions serve as the approximate models which the solutions we discuss here asymptotically approach.

We introduce the following rescaled variables

$$\tau := \log t,$$
$$\sigma := \frac{s}{\sqrt{2(n-1)t}},$$
$$u := \frac{\psi}{\sqrt{2(n-1)t}}.$$

Since we are interested in the asymptotic behaviour of the solution when $t \nearrow \infty$, we can assume $t_0 \geq 1$, and so $\tau_0 = \log t_0 \geq 0$.

In the (u, τ)-coordinates, equation (2.6) becomes the following evolution equation for $z(u, \tau)$.

$$\partial_\tau|_u z = \frac{1}{2(n-1)}\mathcal{E}_u[z] + \frac{1}{2}uz_u, \tag{3.1}$$

where $\tau \in [\log t_0, \infty)$, $u \in (0,\infty)$ and $z \in (0,1]$, and the boundary conditions for (3.1) are $\lim_{u \searrow 0} z = 1$ and $\lim_{u \nearrow \infty} z = 0$. In particular, $z \in (0,1]$ under Ricci flow, which is proved in Lemma 6.2. We seek solutions of equation (3.1) subject to the boundary condition $z(0,\tau) = 1$ and the asymptotic condition $\lim_{u \nearrow \infty} z(u,\tau) = 0$ for all $\tau \geq \tau_0$. In particular, the asymptotic condition that $z \searrow 0$ as $u \nearrow \infty$ is compatible with the consideration of asymptotically flat Riemannian manifolds whose metrics $ds^2 + \psi(s)^2 g_{\text{sph}}$ are defined by $\psi(s)$ with sublinear growth in s, cf. Section 3.4.

3.1. **Formal solution in the exterior region.** The exterior region is expected to be a time-dependent subset in which $u \in (0,\infty)$ and $z \in (0,1)$. Motivated by the asymptotic condition $\lim_{u \nearrow \infty} z(u,\tau) = 0$ for all $\tau \geq \tau_0$, we adopt the following ansatz

$$z = \sum_{m=1}^{\infty} e^{-m\lambda\tau} Z_m(u),$$

where $\lambda > 0$ is a parameter to be specified.

Remark 3.1. This ansatz has been used in constructing Ricci flow solutions with Type-IIa singularities in both the compact setting [3] and the noncompact setting [32].

We substitute this ansatz into equation (3.1) and split $\mathcal{E}_u[z]$ into the linear and quadratic parts as given in (2.7) and (2.8), respectively. By comparing the coefficients of $e^{-m\lambda\tau}$ in the resulting equation, we see each Z_m must satisfy the ODE

$$\frac{1}{2}\left(u^{-1} + u\right)\frac{dZ_m}{du} + \left(u^{-2} + m\lambda\right) Z_m = -\frac{1}{2(n-1)} \sum_{i=1}^{m-1} \hat{\mathcal{Q}}_u \left[Z_i, Z_{m-i}\right]. \tag{3.2}$$

When $m = 1$, Z_1 satisfies the linear homogeneous equation

$$\frac{1}{2}\left(u^{-1} + u\right)\frac{dZ_1}{du} + \left(u^{-2} + \lambda\right) Z_1 = 0, \tag{3.3}$$

whose general solution is

$$Z_1(u) = cu^{-2}\left(1 + u^2\right)^{1-\lambda} \tag{3.4}$$

for an arbitrary constant $c \neq 0$.

When $m = 2$, equation (3.2) becomes

$$\frac{1}{2}\left(u^{-1} + u\right)\frac{dZ_2}{du} + \left(u^{-2} + 2\lambda\right) Z_2 = -\frac{1}{2(n-1)} \mathcal{Q}_u \left[Z_1\right], \tag{3.5}$$

where

$$\mathcal{Q}_u\left[Z_1\right] = 2c^2 u^{-6}\left(1 + u^2\right)^{-2\lambda}\left(4 - n(1+u^2)^2 + u^4(1+\lambda)^2 + 2u^2(3+\lambda)\right).$$

The general solution of equation (3.5) is

$$Z_2(u) = \frac{u^{-2}\left(1+u^2\right)^{1-2\lambda}}{n-1} f(u),$$

where

$$f(u) := C_1 - 2c^2\left(\frac{n-4}{u^2} + \frac{1-\lambda^2}{1+u^2}\right) - 2c^2(\lambda-1)\log\left(\frac{u^2}{1+u^2}\right) \tag{3.6}$$

for some arbitrary constant C_1.

Let us now analyse the asymptotics of $e^{-\lambda\tau}Z_1(u) + e^{-2\lambda\tau}Z_2(u)$ as $u \searrow 0$ and $u \nearrow \infty$, respectively. It is straightforward to see that

$$\frac{e^{-2\lambda\tau}Z_2(u)}{e^{-\lambda\tau}Z_1(u)} = \frac{e^{-\lambda\tau}}{c(n-1)}\left(1+u^2\right)^{-\lambda} f(u),$$

where $f(u)$ as defined in (3.6) has the following asymptotics

$$f(u) \sim \begin{cases} c^2u^{-2} + O\left(c^2\log(u^2)\right), & u \searrow 0, \\ C_1 + c^2u^{-2}, & u \nearrow \infty. \end{cases}$$

Therefore, we obtain

$$\frac{e^{-2\lambda\tau}Z_2(u)}{e^{-\lambda\tau}Z_1(u)} = \begin{cases} ce^{-\lambda\tau}\left(u^{-2} + O\left(\log u\right)\right), & u \searrow 0, \\ e^{-\lambda\tau}u^{-2\lambda}\left(C_1 + c^2u^{-2}\right), & u \nearrow \infty. \end{cases}$$

Consequently, we always have

$$\lim_{u\nearrow\infty}\left|\frac{e^{-2\lambda\tau}Z_2(u)}{e^{-\lambda\tau}Z_1(u)}\right| = 0 \tag{3.7}$$

for all $\tau \geq \tau_0$. On the other hand, if $u = e^{-\lambda\tau/2}R$ for some fixed $R > 0$, then

$$\left|\frac{e^{-2\lambda\tau}Z_2(u)}{e^{-\lambda\tau}Z_1(u)}\right| \lesssim c\left(R^{-2} + e^{-\lambda\tau}O\left(\log R + \tau\right)\right), \tag{3.8}$$

which is small for all sufficiently large τ if for a given c we choose R to be large.

Let us label the region where $ue^{\lambda\tau/2} = O(1)$ as the *interior region*. The complement of the interior region is labelled as the *exterior region*. The estimates (3.7) and (3.8) allow us to use the dominant term $e^{-\lambda\tau}Z_1(u)$ as a formal solution (i.e., an approximate solution) in the exterior region, so we define

$$z_{\text{form, ext}} = cu^{-2}\left(1+u^2\right)^{1-\lambda}e^{-\lambda\tau}.$$

3.2. **Formal solution in the interior region.** In the interior region where $ue^{\lambda\tau/2} = O(1)$, we introduce a new variable

$$r := ue^{\lambda\tau/2}.$$

Then in the (r,τ)-coordinates, since

$$\partial_\tau|_r\, z = \partial_\tau|_u\, z - \frac{\lambda}{2}uz_u = \partial_\tau|_u\, z - \frac{\lambda}{2}rz_r,$$

$$\mathcal{E}_u[z] = e^{\lambda\tau}\mathcal{E}_r[z],$$

equation (3.1) for $z(u,\tau)$ becomes the following evolution equation for $z(r,\tau)$:

$$e^{-\lambda\tau}\left\{\partial_\tau|_r\, z + \frac{\lambda-1}{2} r z_r\right\} - \frac{1}{2(n-1)}\mathcal{E}_r[z] = 0. \tag{3.9}$$

Suppose, for the sake of the formal argument, that the term involving $e^{-\lambda\tau}$ is negligible for sufficiently large τ, then this equation is approximated by the equation

$$\mathcal{E}_r\left[\tilde{z}\right] = 0,$$

whose solution, subject to the boundary $\tilde{z}(0) = 1$ and the asymptotic condition $\lim_{r\nearrow\infty} \tilde{z}(r) = 0$, is a Bryant soliton profile function

$$\tilde{z}(r) = \mathfrak{B}\left(Ar\right),$$

where $A > 0$ is a constant whose value will be determined later. The complete smooth metric given by

$$g = \mathfrak{B}^{-1}\left(Ar\right) dr^2 + r^2 g_{\mathrm{sph}}$$

is a scaled version of the Bryant soliton [8].

The function $\mathfrak{B}(r)$ is smooth and strictly monotonically decreasing for all $r > 0$ with the following asymptotics

$$\mathfrak{B}(r) = \begin{cases} 1 - b_2 r^2 + b_3 r^4 + b_4 r^6 + \cdots, & r \searrow 0,\\ c_1 r^{-2} + c_2 r^{-4} + c_3 r^{-6} + \cdots, & r \nearrow \infty, \end{cases} \tag{3.10}$$

where b_k's and c_k's are constants; in particular, $b_2 > 0$, and $b_3 = \frac{n}{n+3} b_2^2$ [2, Appendix B]. In this paper, we normalize $\mathfrak{B}(r)$ by setting $c_1 = 1$. In the interior region, our formal solution is

$$z_{\text{form, int}} = \mathfrak{B}\left(Ar\right).$$

Remark 3.2. If $\lambda = 1$, then $r = ue^{\tau/2} = \psi/\sqrt{2(n-1)}$. In this case, the Bryant soliton $\mathfrak{B}\left(Ar\right) = \mathfrak{B}\left(A\psi/\sqrt{2(n-1)}\right)$ solves equation (2.6) and gives a trivial example of Ricci flow with Type-IIb singularity with the highest curvature blowing up at the rate $O\left(t^{\lambda-1}\right) = O\left(\lambda^0\right) = O(1)$. Our focus in Theorem 1.1 is to construct different solutions for $\lambda \neq 1$.

3.3. **Matching condition.** We now match the formal solutions at the interface of the interior region and the exterior region. If we pick $r = R \gg 1$, then in the interior region, using the asymptotics of $z_{\text{form, int}}$ as $r \nearrow \infty$, we have

$$z_{\text{form, int}} = \mathfrak{B}\left(AR\right) \approx A^{-2}R^{-2}; \tag{3.11}$$

in the exterior region, using the asymptotics of $z_{\text{form, ext}}$ as $u \searrow 0$ and that $u = Re^{-\lambda\tau/2}$, we have

$$z_{\text{form, ext}} = cu^{-2}\left(1+u^2\right)^{1-\lambda} e^{-\lambda\tau} \approx cR^{-2}. \tag{3.12}$$

Equating (3.11) and (3.12), we obtain the matching condition for the formal solution

$$A^{-2} = c. \tag{3.13}$$

The condition 3.13 says that given A and $R > 0$, we can always find such c; equivalently, fixing c and R, then A is determined.

3.4. **Features of the formal solution.** Our formal solutions defined in the interior region and the exterior region are valid for all dimensions $n+1 \geq 3$ and give rise to Riemannian metrics on $\mathbb{R}^{n+1}$ as defined in (2.4). In fact, these Riemannian metrics are complete, as will be proven in Lemma 6.1. Since $z_{\text{form, ext}} = e^{-\lambda\tau} cu^{-2}(1+u^2)^{1-\lambda}$, we have at any $\tau < \infty$, as $u \nearrow \infty$, i.e., $\psi = \sqrt{2(n-1)}ue^{\tau/2} \nearrow \infty$, that

$$z = \psi_s^2 \sim e^{-\lambda\tau} cu^{-2\lambda} \sim c\psi^{-2\lambda}, \tag{3.14}$$

$$K = -\frac{z_\psi}{2\psi} \sim -\frac{z_u}{2u} e^{-\tau} \sim \left(tu^2\right)^{-(1+\lambda)} \sim \psi^{-2(1+\lambda)}, \tag{3.15}$$

$$L = \frac{1-z}{\psi^2} = \frac{1-z}{u^2} e^{-\tau} \sim \left(tu^2\right)^{-1} \sim \psi^{-2}. \tag{3.16}$$

Remark 3.3. Conditions (3.15) and (3.16) imply that the metrics given by the barriers (cf. Section 5), and hence the Ricci flow solutions described in Theorem 1.1 (cf. Section 6), are in fact asymptotically flat in the sense that $|\operatorname{Rm}| \to 0$ as one approaches spatial infinity. Recall that an asymptotically conical metric on $\mathbb{R}^{n+1}$ is given by $ds^2 + \alpha^2 s^2 g_{\text{sph}}$, where $\alpha \in (0,1]$ with the case $\alpha = 1$ being the Euclidean metric. For an asymptotically conical metric, $z = \psi_s^2 = \alpha^2 > 0$. As will be shown in Section 6, the solutions in Theorem 1.1 do not satisfy $z \equiv 1$ and therefore are non-Euclidean, the solutions are not asymptotically conical either since $\lim_{s \nearrow \infty} z = 0$. Condition (3.14) implies that the metric $ds^2 + \psi^2 g_{\text{sph}}$ we construct in this paper are *not* asymptotically Euclidean in the sense considered in [14] or [28].

As we move towards the origin $\mathcal{O}$, $z(u) \nearrow 1$ and we enter the interior region where the formal solution $z_{\text{form, int}}$ is a Bryant soliton profile function. At $\mathcal{O}$, we have $K(\mathcal{O}, t) = L(\mathcal{O}, t)$ for all $t \geq t_0$, and the norm of the curvature tensor achieves its maximum value

$$|\operatorname{Rm}(\mathcal{O}, t)| = \sqrt{n(n+1)} L(\mathcal{O}, t) = \frac{\sqrt{n(n+1)}}{2(n-1)} \lim_{r \searrow 0} \frac{1-z}{r^2} e^{(\lambda-1)\tau} = t^{\lambda-1} C,$$

where C is a positive constant depending on n, A, b_2; to be precise, $C = \frac{\sqrt{n(n+1)} A^2 b_2}{2(n-1)}$. Therefore, the formal solution has a Type-IIb singularity if $\lambda > 0$. In particular, the curvature of a Ricci flow solution that asymptotically approaches this formal solution necessarily blows up at the same rate.

4. Subsolutions and supersolutions

Given a parabolic differential operator $\mathcal{P}[v] = \partial_\tau v - \mathcal{D}[v]$ where $\mathcal{D}[\cdot]$ is some second-order elliptic operator, a function v^+ is a *subsolution* of the PDE $\mathcal{P}[v] = 0$ if $\mathcal{P}[v^+] \le 0$ whereas a function v^- is a *supersolution* if $\mathcal{P}[v^-] \ge 0$. If there exist subsolution v^- and supersolution v^+ and in addition, $v^- \le v^+$, then we call v^- a *lower barrier* and v^+ an *upper barrier.*

Suppose the equation $\mathcal{P}[v] = 0$ admits a solution, then the existence of barriers $v^- \le v^+$ implies that there exists a solution v with $v^- \le v \le v^+$. This is the general idea of our argument which will be justified rigorously in this section and next. In this section, we construct subsolutions and supersolutions for equation (3.1) in the interior and the exterior regions. In the next section, we patch them to obtain global upper and lower barriers.

4.1. **Interior region.** Recall equation (3.9) for $z(r,\tau)$. Let us define

$$\mathcal{T}_r[z] := e^{-\lambda\tau}\left\{ \partial_\tau|_r\, z + \frac{\lambda-1}{2} r z_r \right\} - \frac{1}{2(n-1)}\mathcal{E}_r[z] \tag{4.1}$$

so then $z(r,\tau)$ satisfies the equation $\mathcal{T}_r[z] = 0$ in the interior region. The subsolution and supersolution for this equation in the interior region are given in the next lemma.

Lemma 4.1. *For an integer $n \ge 2$, a real number $\lambda > 0$, a constant $A > 0$ and arbitrary constants $a^\pm$, there exist a sufficiently large $\tau_1 < \infty$, a constant $B_1 > 0$ depending only on A, and bounded functions $\beta^\pm(r,\tau) : (0,\infty) \times [\tau_1,\infty) \to \mathbb{R}$ depending on A and $a^\pm$ such that the functions*

$$z^\pm_{int}(r,\tau) := \mathfrak{B}\left(A \left(1 + a^\pm e^{-\lambda\tau/2} \right) r \right) \pm e^{-\lambda\tau}\beta^\pm(r,\tau) \tag{4.2}$$

are supersolution $(+)$ and subsolution $(-)$, respectively, of $\mathcal{T}_r[z] = 0$ in the region $\Omega_{int} := \left\{ 0 \le r \le B_1 e^{\lambda\tau/2} \right\}$ for all $\tau \ge \tau_1$.

Proof. Let us denote $\mathbf{B}^\pm(r,\tau) := \mathfrak{B}\left(A\left(1 + a^\pm e^{-\lambda\tau/2}\right) r\right)$. Then

$$\partial_\tau|_r\, \mathbf{B}^\pm = r\mathbf{B}^\pm_r \frac{-\lambda a^\pm e^{-\lambda\tau/2}}{2(1 + a^\pm e^{-\lambda\tau/2})}.$$

In order for $z^+_{\text{int}} = \mathbf{B}^+(r,\tau) + e^{-\lambda\tau}\beta^+(r,\tau)$ to be a supersolution, we need to show $\mathcal{T}_r\left[z^+_{\text{int}}\right] \ge 0$. Below, for notational clarity, we drop the superscript "+".

Since $\mathbf{B}(r,\tau)$ solves $\mathcal{E}_r[z] = 0$, we obtain

$$\begin{aligned}
\mathcal{T}_r\left[z^+_{\text{int}}\right] =& e^{-\lambda\tau}\left\{ -\frac{\mathcal{L}_r[\beta] + 2\hat{\mathcal{Q}}_r[\mathbf{B},\beta]}{2(n-1)} + \frac{\lambda-1}{2} r\mathbf{B}_r \right\} \\
&+ e^{-3\lambda\tau/2} r\mathbf{B}_r \frac{-\lambda a}{2(1 + a e^{-\lambda\tau/2})} \\
&+ e^{-2\lambda\tau}\left\{ -\lambda\beta + \partial_\tau|_r\,\beta + \frac{\lambda-1}{2} r\beta_r - \frac{\mathcal{Q}_r[\beta]}{2(n-1)} \right\}.
\end{aligned}$$

Set $\hat{A} := 1 + \frac{\lambda-1}{2} = \frac{\lambda+1}{2} > 0$, we define $\beta(r,\tau)$ to be a solution of the equation

$$\mathcal{L}_r[\beta] + 2\hat{\mathfrak{Q}}_r[\mathbf{B},\beta] = 2(n-1)\hat{A}r\mathbf{B}_r. \tag{4.3}$$

Using the definitions of $\mathcal{L}_r$ in (2.7) and $\hat{\mathfrak{Q}}_r$ in (2.8) respectively, equation (4.3) becomes

(4.4)

$$\mathbf{B}\beta_{rr} + \left\{\frac{n-1}{r} - \mathbf{B}_r - \frac{\mathbf{B}}{r}\right\}\beta_r + \left\{\mathbf{B}_{rr} - \frac{\mathbf{B}_r}{r} + 2(n-1)\frac{1-2\mathbf{B}}{r^2}\right\}\beta = 2(n-1)\hat{A}r\mathbf{B}_r.$$

Using the asymptotic expansions of $\mathbf{B}(r,\tau)$ near $r=0$ and $r=\infty$ given in (3.10), we have the following. Near $r=0$, equation (4.4) is approximated by

$$\beta_{rr} + \frac{n-2}{r}\beta_r - \frac{2(n-1)}{r^2}\beta = -C_1 r^2\left(1 + O\left(ae^{-\lambda\tau/2}\right)\right),$$

where $C_1 = 2(n-1)(\gamma+1)b_2A^2$. Near $r=\infty$, equation (4.4) is a perturbation of the following equation

$$\frac{1+O\left(ae^{-\lambda\tau/2}\right)}{(Ar)^2}\beta_{rr} + \frac{n-1}{r}\beta_r + \frac{2(n-1)}{r^2}\beta = -\frac{4(n-1)\hat{A}}{(Ar)^2}\left(1 + O\left(ae^{-\lambda\tau/2}\right)\right).$$

So there exists a solution β to equation (4.3) with the following asymptotics

$$\beta(r,\tau) = \begin{cases} r^2 + O\left(r^4\left(1 + ae^{-\lambda\tau/2}\right)\right), & r \searrow 0, \\ \left(-2\hat{A}/A^2 + o(1)\right)\left(1 + O\left(ae^{-\lambda\tau/2}\right)\right), & r \nearrow \infty. \end{cases} \tag{4.5}$$

Also, the asymptotic expansions

$$-r\mathbf{B}_r = \begin{cases} \left(C_7 r^2 + o\left(r^2\right)\right)\left(1 + O\left(ae^{-\lambda\tau/2}\right)\right), & r \searrow 0, \\ \left(C_8 r^{-2} + o\left(r^{-2}\right)\right)\left(1 + O\left(ae^{-\lambda\tau/2}\right)\right), & r \nearrow \infty, \end{cases}$$

imply that

$$-r\mathbf{B}_r \geq C_9 \min\left\{r^2, r^{-2}\right\}.$$

Then in view of (4.5), we have for $0 < r \leq 1$,

$$\left|-\lambda\beta + \partial_\tau|_r\,\beta + \frac{\lambda-1}{2}r\beta_r - \frac{\mathfrak{Q}_r[\beta]}{2(n-1)}\right| \leq C_{10}r^2,$$

and hence

$$\begin{aligned} \mathfrak{T}_r\left[z^+_{\text{int}}\right] &\geq -e^{-\lambda\tau}r\mathbf{B}_r - e^{-3\lambda\tau/2}C_7r^2 - e^{-2\lambda\tau}C_{10}r^2 \\ &\geq e^{-\lambda\tau}r^2\left(C_9 - e^{-\lambda\tau/2}C_7 - e^{-\lambda\tau}C_{10}\right) \\ &> 0 \end{aligned}$$

for all $\tau \geq \tau_1$ with τ_1 sufficiently large. And for $r \geq 1$ and $\tau \geq \tau_1$,

$$\left|-\lambda\beta + \partial_\tau|_r\,\beta + \frac{\lambda-1}{2}r\beta_r - \frac{\mathfrak{Q}_r[\beta]}{2(n-1)}\right| \leq C_{11},$$

so then

$$\begin{aligned}\mathcal{T}_r\left[z_{\mathrm{int}}^+\right] &\geq -e^{-\lambda\tau} r\mathbf{B}_r - e^{-3\lambda\tau/2}C_8 r^{-2} - e^{-2\lambda\tau}C_{11}\\ &\geq e^{-\lambda\tau}\left(C_9 r^{-2} - e^{-\lambda\tau/2}C_8 r^{-2} - e^{-\lambda\tau}C_{11}\right)\\ &> 0\end{aligned}$$

provided that $r < B_1 e^{\lambda\tau/2}$ with constant $B_1 := \sqrt{C_9/(2C_{11})}$.

Therefore, z_{int}^+ is indeed a supersolution. That z_{int}^- is a subsolution is proved similarly. So the lemma follows. □

Remark 4.2. See the proof of Lemma 4.1, the asymptotics of $\mathbf{B}(r)$ and $\beta(r,\tau)$ as $r \searrow 0$ imply that $\lim_{r\searrow 0} z_{\mathrm{int}}^{\pm}(r,\tau) = 1$ for all $\tau \geq \tau_1$.

4.2. **Exterior region.** Recall equation (3.1) for $z(u,\tau)$. We define

$$\mathcal{F}_u[z] := \partial_\tau|_u\, z - \frac{1}{2(n-1)}\mathcal{E}_u[z] - \frac{1}{2}u z_u, \tag{4.6}$$

$$= \partial_\tau|_u\, z - \frac{1}{2}\left(u^{-1}+u\right) - u^{-2}z - \frac{\mathcal{Q}_u[z]}{2(n-1)}, \tag{4.7}$$

where we used $\mathcal{E}_u[z] = \mathcal{L}_u[z] + \mathcal{Q}_u[z]$, (2.7) and (2.8). In this region, $z(u,\tau)$ satisfies the equation $\mathcal{F}_u[z] = 0$. The next lemma takes care of the subsolution and supersolution for this equation in the exterior region.

From now on, we define $Z(u) := u^{-2}\left(1+u^2\right)^{1-\lambda}$. We note that $Z(u) > 0$ for all $u \in (0,\infty)$.

Lemma 4.3. *For an integer $n \geq 2$, a real number $\lambda > 0$ and constants $c^{\pm} > 0$, there exist function $\zeta : (0,\infty) \to \mathbb{R}$, constants $B_2^{\pm} > 0$, a sufficiently large $\tau_2 < \infty$, and constants $b_*^{\pm}$ depending only on $c^{\pm}$, respectively, such that for any $b \geq b_*^{\pm} > 0$, the functions*

$$z_{ext}^{\pm}(u,\tau) := c^{\pm}e^{-\lambda\tau}Z(u) \pm b^{\pm}e^{-2\lambda\tau}\zeta(u) \tag{4.8}$$

are supersolution $(+)$ and subsolution $(-)$, respectively, of $\mathcal{F}_u[z] = 0$ in the region $\Omega_{ext}^{\pm} := \left\{B_2^{\pm}\sqrt{\frac{b^\pm}{c^\pm}}e^{-\lambda\tau/2} \leq u < \infty\right\}$ and for all $\tau \geq \tau_2$.

Proof. We first prove the lemma for z_{ext}^+. To simplify notation, we omit the superscript "+" in the argument below.

Since $Z(u)$ is a solution of the ODE (3.3), we have

$$\begin{aligned}e^{2\lambda\tau}\mathcal{F}_u\left[z_{\mathrm{ext}}^+\right] = b\left\{-\frac{1}{2}\left(u^{-1}+u\right)\zeta' - \left(u^{-2}+2\lambda\right)\zeta\right\} - \frac{c^2}{2(n-1)}\mathcal{Q}_u[Z]\\ - \frac{bc}{n-1}e^{-\lambda\tau}\hat{\mathcal{Q}}_u[Z,\zeta] - \frac{b^2}{2(n-1)}e^{-2\lambda\tau}\mathcal{Q}_u[\zeta].\end{aligned}$$

Since

$$\mathcal{Q}_u[Z] = 2u^{-6}\left(1+u^2\right)^{-2\lambda} f(u),$$

where

$$f(u) = 4 - n(1+u^2)^2 + u^4(1+\lambda)^2 + 2u^2(3+\lambda),$$

we have for $u \in (0,\infty)$,

$$|f(u)| \leq C_1(1+u^2)^2 \tag{4.9}$$

for some constant C_1 depending only on n and λ.

Let $\zeta : (0,\infty) \to \mathbb{R}$ be a solution of the ODE

$$-\frac{1}{2}\left(u^{-1}+u\right)\zeta' - \left(u^{-2}+2\lambda\right)\zeta = u^{-6}\left(1+u^2\right)^{2-2\lambda}. \tag{4.10}$$

Then we solve this ODE to obtain

$$\zeta(u) := u^{-4}\left(1+u^2\right)^{1-2\lambda}\left(1+C_2u^2\right)$$

for an arbitrary constant C_2. Let us choose $C_2 = 1$, so

$$\zeta(u) := u^{-4}\left(1+u^2\right)^{2-2\lambda}. \tag{4.11}$$

In particular, $\zeta(u) > 0$ for all $u \in (0,\infty)$.

From (4.11), the asymptotics of ζ are

$$\zeta(u) = \begin{cases} u^{-4} + O\left(u^{-2}\right), & u \searrow 0, \\ C_4u^{-4\lambda} + O\left(u^{-2-4\lambda}\right), & u \nearrow \infty. \end{cases}$$

So the following estimates hold. For $B_2e^{-\lambda\tau/2} \leq u < 1$,

$$\left|\hat{\mathfrak{Q}}_u\left[Z,\zeta\right]\right| \leq C_5u^{-8}, \quad \left|\mathfrak{Q}_u\left[\zeta\right]\right| \leq C_6u^{-10}$$

For $1 \leq u < \infty$,

$$\left|\hat{\mathfrak{Q}}_u\left[Z,\zeta\right]\right| \leq C_7u^{-2-6\lambda}, \quad \left|\mathfrak{Q}_u\left[\zeta\right]\right| \leq C_8u^{-2-8\lambda}.$$

Using the definition of ζ and estimate (4.9), we have

$$\begin{aligned} e^{2\lambda\tau}\mathfrak{F}_u\left[z_{\text{ext}}^+\right] &= \left(b - \frac{C_1c^2}{n-1}\right)u^{-6}\left(1+u^2\right)^{2-2\lambda} \\ &\quad - \frac{bc}{n-1}e^{-\lambda\tau}\hat{\mathfrak{Q}}_u[Z,\zeta] - \frac{b^2}{2(n-1)}e^{-2\lambda\tau}\mathfrak{Q}_u\left[\zeta\right] \\ &\geq \frac{b-C_1c^2}{n-1}u^{-6}\left(1+u^2\right)^{2-2\lambda} \\ &\quad - \frac{bc}{n-1}e^{-\lambda\tau}\left|\hat{\mathfrak{Q}}_u[Z,\zeta]\right| - \frac{b^2}{2(n-1)}e^{-2\lambda\tau}\left|\mathfrak{Q}_u[\zeta]\right|. \end{aligned}$$

We choose $b_* = c^2\left(1 + C_1/(n-1)\right)$, then for any $b \geq b_*$, we have the following. For $0 < u \leq 1$, there exists a constant $B_2 > 0$ such that

$$\begin{aligned} e^{2\lambda\tau}\mathfrak{F}_u\left[z_{\text{ext}}^+\right] &\geq C_1u^{-6}\left(c^2 - C_5bcu^{-2}e^{-\lambda\tau} - C_6b^2u^{-4}e^{-2\lambda\tau}\right) \\ &\geq C_1u^{-6}\left(c^2 - C_5c^2B_2^{-2} - C_6c^2B_2^{-2}\right) \end{aligned}$$

$$\begin{aligned} &\geq \frac{C_1}{2}u^{-6}c^2 \\ &> 0 \end{aligned}$$

provided that $u^2e^{\lambda\tau} \geq B_2^2 b/c$, or equivalently,

$$B_2\sqrt{\frac{b}{c}}e^{-\lambda\tau/2} \leq u \leq 1.$$

For $1 \leq u < \infty$, since

$$\left|u^{-6}(1+u^2)^{2-2\lambda}\right| \leq C_1 u^{-2-4\lambda},$$

we have

$$\begin{aligned} e^{2\lambda\tau}\mathcal{F}_u\left[z_{\text{ext}}^+\right] &\geq C_1 u^{-2-4\lambda}\left(c^2 - C_5 c u^{-2\lambda}e^{-\lambda\tau} - C_6 c^2 u^{-4\lambda}e^{-2\lambda\tau}\right) \\ &\geq C_1 u^{-2-4\lambda}\left(c^2 - C_5 bce^{-\lambda\tau} - C_6 b^2 e^{-2\lambda\tau}\right) \\ &> 0 \end{aligned}$$

for all $\tau \geq \tau_2$ with τ_2 sufficiently large.

Therefore, z_{ext}^+ is indeed a supersolution. By a similar argument, z_{ext}^- is a subsolution. So the lemma is proven. □

Remark 4.4. See the proof of Lemma 4.3, the asymptotics of $\zeta(u)$ as $u \nearrow \infty$ implies that $\lim_{u\nearrow\infty} z_{\text{ext}}^{\pm}(u,\tau) = 0$ for all $\tau \geq \tau_2$.

5. Upper and lower barriers

According to Lemmata 4.1 and 4.3, the interior region Ω_{int} and the exterior region $\Omega_{\text{ext}}^{\pm}$ overlap for sufficiently large τ. Our goal in this section is to show that the regional supersolutions z_{int}^+ and z_{ext}^+ together with z_{int}^- and z_{ext}^- can be patched together to provide an upper and lower barriers, respectively, for Ricci flow equation (3.1).

In the next two lemmata, we prove in each region the subsolution and supersolution are ordered.

Lemma 5.1. *Let $\beta(r,\tau)$, τ_1 and B_1 be defined as in Lemma 4.1. For $a^- > a^+$, there exists $\tau_3 \geq \tau_1$ such that*

$$z_{int}^{\pm} := \mathfrak{B}\left(A\left(1 + a^{\pm}e^{-\lambda\tau/2}\right)r\right) \pm e^{-\lambda\tau}\beta(r,\tau)$$

satisfy $z_{int}^- < z_{int}^+$ in $\left\{0 \leq r \leq B_1 e^{\lambda\tau/2}\right\}$ for all $\tau \geq \tau_3$.

Proof. Using the asymptotic expansions of $\mathfrak{B}$ (3.10) and β (4.5), we have the following for all sufficiently large $\tau \geq \tau_3$ $ge\tau_1$. Near $r = 0$,

$$\begin{aligned} z_{\text{int}}^+ - z_{\text{int}}^- &= e^{-\lambda\tau/2}\left(2b_2A^2\left(a^- - a^+\right)r^2 + O\left(r^4\right)\right) + e^{-\lambda\tau}\left(-2\hat{A}A^{-2} + o(1)\right) \\ &> 0. \end{aligned}$$

Near $r=\infty$, with $\hat{A}=(\gamma+1)/2$,

$$\begin{aligned} z_{\text{int}}^{+}-z_{\text{int}}^{-} &= e^{-\lambda\tau/2}\left(2A^{-2}\left(a^{-}-a^{+}\right)r^{-2}+O\left(r^{-4}\right)\right)+e^{-\lambda\tau}\left(r^{-2}+o\left(r^{-2}\right)\right) \\ &> 0. \end{aligned}$$

On any bounded interval $c<r<C$, it is straightforward to check that $z_{\text{int}}^{-}<z_{\text{int}}^{+}$. So the lemma is proved. □

Remark 5.2. By choosing $a^{-}>0>a^{+}$, the proof of Lemma 5.1 shows that $z_{\text{int}}^{-}<\mathfrak{B}(Ar)<z_{\text{int}}^{+}$ in $\left\{0\le r\le B_1e^{\lambda\tau/2}\right\}$ for all $\tau\ge\tau_3$.

Lemma 5.3. *Let $B_2^{\pm}$, $b^{\pm}$, $c^{\pm}$, τ_2 be from Lemma 4.3 and define* $R_2 := \max\left\{B_2^{+}\sqrt{\frac{b^{+}}{c^{+}}},\ B_2^{-}\sqrt{\frac{b^{-}}{c^{-}}}\right\}$. *If $c^{+}\ge c^{-}$, then there exists $\tau_4\ge\tau_2$ such that*

$$z_{ext}^{\pm} := c^{\pm}e^{-\lambda\tau}Z(u)\pm b^{\pm}e^{-2\lambda\tau}\zeta(u)$$

satisfy $z_{ext}^{-}<z_{ext}^{+}$ in $\{R_2e^{-\lambda\tau/2}<u<\infty\}$ for all $\tau\ge\tau_4$.

Proof. Using the definitions of Z and ζ, and choosing $C_2\ge 0$, and recall that $c^{+}\ge c^{-}$ implies $b^{+}>b^{-}$, we have

$$\begin{aligned} e^{\lambda\tau}\left(z_{\text{ext}}^{+}-z_{\text{ext}}^{-}\right) &= \left(c^{+}-c^{-}\right)Z(u)+e^{-\lambda\tau}\left(b^{+}+b^{-}\right)\zeta(u) \\ &= \left(c^{+}-c^{-}\right)\frac{(1+u^2)^{1+\lambda}}{u^2}+e^{-\lambda\tau}\frac{\left(b^{+}+b^{-}\right)\left(1+(n-1)C_2u^2\right)}{(n-1)u^4\left(1+u^2\right)^{2\lambda-1}} \\ &> 0 \end{aligned}$$

for all $u\in(0,\infty)$ for all sufficiently large $\tau\ge\tau_4\ge\tau_2$. So the lemma is proved. □

To patch the supersolution in the interior region with that in the exterior region, we state and prove a patching lemma for z_{int}^{+} and z_{ext}^{+}. We omit the patching lemma for z_{int}^{-} and z_{ext}^{-}, since its statement and proof are analogous. To shorten the notation, we write a^{+}, b^{+}, c^{+} as a, b, c.

Remark 5.4. By choosing $c^{-}\le c\le c^{+}$, the proof of Lemma 5.3 shows that $z_{\text{ext}}^{-}<e^{-\lambda\tau}cZ(u)<z_{\text{ext}}^{+}$ in $\{R_2e^{-\lambda\tau/2}<u<\infty\}$ for all $\tau\ge\tau_4$.

Lemma 5.5. *Let τ_3 be from Lemma 5.1 and τ_4 from Lemma 5.3. Let $R_D := D\sqrt{b/c}$ where $D>0$ is arbitrary. Suppose A and c satisfy the following inequality*

$$\left(1+\frac{3}{8}D^{-2}\right)c<A^{-2}<\left(1+\frac{1}{2}D^{-2}\right)c. \tag{5.1}$$

Then there exists $\tau_5\ge\max\{\tau_3,\tau_4\}$ sufficiently large such that

$$\left(z_{int}^{+}-z_{ext}^{+}\right)(R_D,\tau)<0,\quad \left(z_{int}^{+}-z_{ext}^{+}\right)(2R_D,\tau)>0$$

for all $\tau\ge\tau_5$.

Proof. At the interface of interior and exterior regions, we have the following for $\tau \geq \tau_5$. In the interior region, we have as $r \nearrow \infty$ that

$$\begin{aligned} z^+_{\text{int}} &= \mathbf{B}(r,\tau) + e^{-\lambda\tau}\beta(r,\tau) \\ &= \left(A^{-2}r^{-2} + c_2A^{-4}r^{-4} + O\left(r^{-6}\right)\right)\left(1 + O\left(ae^{-\lambda\tau/2}\right)\right) + O\left(e^{-\lambda\tau}\right). \end{aligned}$$

In the exterior region, we have as $u = re^{-\lambda\tau/2} \searrow 0$ that

$$\begin{aligned} z^+_{\text{ext}} &= e^{-\lambda\tau}\left(cu^{-2} + O(1)\right) + e^{-2\lambda\tau}\left(bu^{-4} + O\left(u^{-2}\right)\right) \\ &= cr^{-2} + br^{-4} + O\left(e^{-\lambda\tau}r^{-2}\right) \end{aligned}$$

So on bounded r-interval, we have

$$r^2\left(z^+_{\text{int}} - z^+_{\text{ext}}\right) = \left(A^{-2} - c\right) + \left(c_2A^{-4} - b + O\left(r^{-2}\right)\right)r^{-2} + O\left(e^{-\lambda\tau/2}\right).$$

Let us choose a constant $\hat{C}$ so large that for

$$b \geq \hat{C}A^{-4} \quad \text{and} \quad b \geq \hat{C}\sqrt{c},$$

we have

$$\left|\frac{c_2c}{bA^4} + O\left(\frac{c^2}{b^2}\right)\right| \leq \frac{c}{2}.$$

Then at $r = R_D$,

$$\begin{aligned} R_D^2\left(z^+_{\text{int}} - z^+_{\text{ext}}\right) &= (A^{-2} - c) + \left[\frac{c_2c}{bA^4} + O\left(\frac{c^2}{b^2}\right) - c\right]D^{-2} + O(\tau e^{-\lambda\tau}) \\ &\leq A^{-2} - \left(1 + \frac{1}{2}D^{-2}\right)c + O\left(e^{-\lambda\tau/2}\right), \end{aligned}$$

and at $r = 2R_D$,

$$\begin{aligned} 4R_D^2\left(z^+_{\text{int}} - z^+_{\text{ext}}\right) &= (A^{-2} - c) + \left[\frac{c_2c}{bA^4} + O\left(\frac{c^2}{b^2}\right) - c\right]\frac{D^{-2}}{4} + O(\tau e^{-\lambda\tau}) \\ &\geq A^{-2} - \left(1 + \frac{3}{8}D^{-2}\right)c + O\left(e^{-\lambda\tau/2}\right). \end{aligned}$$

Now choose A and c according to (5.1), then the lemma follows for $\tau \geq \tau_5$. □

For fixed $\lambda \neq 1$ and constants $A, b^\pm, c^\pm$ chosen so far, we define the upper barrier z^+ for equation (3.1) by

$$z^+ := \begin{cases} z^+_{\text{int}}, & \text{if} \quad 0 < u \leq R_De^{-\lambda\tau/2}, \\ \min\left\{z^+_{\text{int}},\ z^+_{\text{ext}}\right\}, & \text{if} \quad R_De^{-\lambda\tau/2} \leq u \leq 2R_De^{-\lambda\tau/2}, \\ z^+_{\text{ext}}, & \text{if} \quad 2e^{-\lambda\tau/2}R_D \leq u < \infty. \end{cases} \tag{5.2}$$

The lower barrier $z^- = z^-(u,\tau)$ for equation (3.1) is defined analogously using z^-_{int} and z^-_{ext}; in particular, $z^- := \max\{z^-_{\text{int}}, z^-_{\text{ext}}\}$ for $R_De^{-\lambda\tau/2} \leq u \leq$

$2R_D e^{-\lambda\tau/2}$. By remarks 5.2 and 5.4, we see that z^+ stays strictly above the formal solution and z^- strictly below the formal solution.

Lemmata 5.1–5.5 together with Remarks 4.2 and 4.4 imply the following proposition.

Proposition 5.6. *There exist a sufficiently large $\tau_0 < \infty$ and positive continuous, piecewise smooth functions $z^\pm = z^\pm(u,\tau)$ defined for $0 < u < \infty$ and $\tau \geq \tau_0$ such that the following hold.*

(B1) *$z^\pm$ are upper (+) and lower (−) barriers to equation* (3.1)*, respectively.*
(B2) *$z^- < z^+$; near $u = 0$, $z^\pm = z^\pm_{int}$; as $u \nearrow \infty$, $z^\pm = z^\pm_{ext}$.*
(B3) *At any $\tau \in [\tau_0, \infty)$, we have*

$$\lim_{u\searrow 0} z^- = \lim_{u\searrow 0} z^+ = 1, \quad \lim_{u\nearrow\infty} z^- = \lim_{u\nearrow\infty} z^+ = 0.$$

Remark 5.7. By construction, where z^+ (or z^-) is not smooth, the corner is concave (or convex).

We end this section with a comparison principle for the equation (3.1).

Proposition 5.8. *Let $\bar\tau \in [\tau_0, \infty)$ be arbitrary. Let $z^\pm$ be two non-negative sub-(−) and super- (+) solutions of equation* (3.1) *respectively. Moreover, assume*

(C1) *$z^-(u,\tau_0) < z^+(u,\tau_0)$ for $0 < u < \infty$;*
(C2) *$z^-(0,\tau) \leq z^+(0,\tau)$, and $\lim_{u\nearrow\infty}(z^-(u,\tau) - z^+(u,\tau)) \leq 0$ for all $\tau \in [\tau_0, \bar\tau]$.*

Then $z^-(u,\tau) \leq z^+(u,\tau)$ in $[0,\infty)\times[\tau_0,\bar\tau]$.

Remark 5.9. In this proposition, we assume $z^\pm$ are smooth. The result also holds for the continuous, piecewise smooth barriers $z^\pm$ constructed earlier, see Remark 5.7. When applying the comparison principle, we will only evaluate $z^\pm$ at "points of first contact with a given smooth function" which are necessarily smooth points of $z^\pm$ for each $\tau \geq \tau_0$.

Proof of Proposition 5.8. By (C1) and (C2), for any given $\varepsilon > 0$, there exists $R = R(\varepsilon)$ such that $z^+ > z^-$ on $[R,\infty)\times[\tau_0,\bar\tau]$ and $(z^+ - z^-)(R) > \epsilon$.

Define

$$w := e^{-\mu\tau}\left(z^+ - z^-\right) + \varepsilon,$$

where $\mu > 0$ is to be chosen. Then $w > 0$ on the parabolic boundary of the evolution by assumptions (C1) and (C2). We claim that $w > 0$ in $(R,\infty)\times[\tau_0,\bar\tau]$. Suppose the contrary, then there must be an interior point u_* and a first time τ_* such that $w(u_*,\tau_*) = 0$ and $w_\tau(u_*,\tau_*) \leq 0$. Moreover, at (u_*,τ_*), we have

$$z^+ = z^- - \varepsilon e^{-\mu\tau}, \quad z^+_u = z^-_u, \quad z^+_{uu} \geq z^-_{uu}.$$

Then at (u_*,τ_*),

$$0 \geq e^{\mu\tau_*}\,\partial|_\tau\, w_\tau$$

$$\begin{aligned}
&= \left(z^+_\tau - z^-_\tau\right) - \mu\left(z^+ - z^-\right) \\
&\geq \left(z^+ - z^-\right)\left(u^{-2} - \mu\right) + \frac{\mathcal{Q}_u[z^+] - \mathcal{Q}_u[z^-]}{2(n-1)} \\
&= \left(z^- - z^+\right)\left\{\mu + \frac{(z^+_u/u) - z^+_{uu}}{2(n-1)} + \frac{z^+ + z^- - 1}{u^2}\right\} + z^-\left(z^+_{uu} - z^-_{uu}\right) \\
&\geq \varepsilon e^{-\mu\tau_*}\left\{\mu - \left.\frac{\mathcal{Q}_u[z^+] - \mathcal{Q}_u[z^-]}{2(n-1)}\right|_{(u_*,\tau_*)} - \frac{1}{u_*^2}\right\} \\
&= \varepsilon e^{-\mu\tau_*}\left\{\mu - (\text{bounded term independent of } \mu)\right\}
\end{aligned}$$

Since $\epsilon > 0$ is fixed, we choose μ sufficiently large, then at (u_*, τ_*) we have

$$0 \geq \partial|_\tau \, w > 0,$$

which is a contradiction. Hence, the claim is true. In the proof of the claim, μ may depend on ζ^+, ζ^- and $\bar{\tau}$, but not on $\epsilon > 0$. Therefore, letting $\epsilon \to 0$, the proposition follows. □

6. Proof of Theorem 1.1

For any solution z of equation (3.1) we have the following.

Lemma 6.1. *Suppose* $0 < z \leq z^+$. *If* $\lambda > 0$, *then* z *determines a* complete *rotationally symmetric metric* $g := z^{-1} d\psi^2 + \psi^2 g_{sph}$ *on* $\mathbb{R}^{n+1}$.

Proof. By definition g is rotationally symmetric. To see that g is a complete metric, it suffices to show that any radial geodesic γ starting from the origin has infinite length in the s-coordinate. The length of γ in s-coordinate is a function of u and τ given by

$$\frac{e^{-\tau/2}}{\sqrt{2(n-1)}} s(u,\tau) = \sigma(u) = \int_0^u \frac{d\sigma}{d\hat{u}} d\hat{u}.$$

Since $z = \psi_s^2 = 2\,(n-1)\, u_\sigma^2$, and $0 < z \leq z^+$ by hypothesis, we have

$$\sigma(u) \geq \int_{u_0}^u \frac{1}{\sqrt{z}} d\hat{u} \geq \int_{u_0}^u \frac{1}{\sqrt{z^+}} d\hat{u}.$$

Recall that

$$z^+_{\text{ext}} = e^{-\lambda\tau} c u^{-2}\left(1+u^2\right)^{1-\lambda} + e^{-2\lambda\tau} b u^{-4}\left(1+u^2\right)^{2-2\lambda}$$

So for u_0 and τ_0 sufficiently large, $z^+ = z^+_{\text{ext}}$ in $[u_0, 1) \times [\tau_0, \infty)$ with

$$z^+_{\text{ext}} \lesssim e^{-\lambda\tau} u^{-2\lambda}.$$

It follows that

$$\frac{e^{-\tau/2}}{\sqrt{2(n-1)}} s(u,\tau) \gtrsim \int_{u_0}^u \frac{1}{\sqrt{z^+}} d\hat{u} = \int_{u_0}^u \frac{1}{\sqrt{z^+_{\text{ext}}}} d\hat{u} \gtrsim e^{\lambda\tau/2} \int_{u_0}^u \hat{u}^\lambda d\hat{u} = u^{1+\lambda} - u_0^{1+\lambda}.$$

Therefore, for each $\tau \geq \tau_0$, $\lim\limits_{u \nearrow \infty} s(u,\tau) = \infty$, whence the lemma follows. □

Since $z = \psi_s^2$, where s is the arclength from the origin, and we are working with complete metrics on $\mathbb{R}^{n+1}$, we have $\psi_s > 0$. In particular, choosing ψ such that $\psi_s > 0$. As explained in [18, 28], the condition $\psi_s > 0$ can be interpreted as the absence of minimal sphere in the manifold. Also, our formal solution and barriers all satisfy $\lim_{s\searrow 0+} \psi_s = 1$ and $\lim_{s\nearrow\infty} \psi_s = 0$. The following lemma bounds ψ_s along Ricci flow. In particular, minimal hyperspheres cannot appear along the Ricci flow solution if none existed at the initial time.

Lemma 6.2. *Suppose that the initial metric g_0 satisfies $0 < \psi_s \leq 1$, then $0 < \psi_s \leq 1$ for as long as the solution to Ricci flow exists.*

Proof. Denoting $v = \psi_s$, then by [1, Equation (16)] the evolution of v is

$$v_t = v_{ss} + \frac{n-2}{\psi} v v_s + \frac{n-1}{\psi^2}(1 - v^2)v.$$

By the maximum principle, $v \leq 1$; by [18, Lemma3.1], $v > 0$. Thus, the lemma is proved. □

Remark 6.3. The condition $\psi_s > 0$ is interpreted as the absence of minimal sphere in the manifold, cf. [18, 28].

We now prove the main results in this paper.

Proof of Theorem 1.1. Let $n + 1 \geq 3$ and fix $\lambda > 0$. Let τ_5 be given in Lemma 5.5. We pick $\tau_0 \geq \tau_5$ so that all results in Sections 4 and 5 apply. Note that $t_0 = e_0^{\tau}$.

Let $z^+(u, \tau)$ and $z^-(u, \tau)$ be given in Section 5. Then at $\tau = \tau_0$, we have $0 < z^-(u, \tau_0) < z^+(u, \tau_0)$ for all $u \in (0, \infty)$. We define an initial data z_0 between $z^+(u, \tau_0)$ and $z^-(u, \tau_0)$ as follows.

(1) On $[MR_D e^{-\lambda\tau_0/2}, \infty)$, where $M \geq 2$ is a constant to be specified, we define $\hat{z}_0(u) = cu^{-2}(1 + u^2)^{1-\lambda}$ where $c \in [c^-, c^+]$. By Remark 5.4, $z^- < \hat{z}_0(u) < z^+$ on $[MR_D e^{-\lambda\tau_0/2}, \infty)$. Using

$$K = -\frac{z_\psi}{2\psi} = -\frac{z_u}{2u}\frac{e^{-\tau}}{2(n-1)} = -\frac{z_r}{2r}\frac{e^{(\lambda-1)\tau}}{2(n-1)},$$

$$L = \frac{1-z}{\psi^2} = \frac{1-z}{u^2}\frac{e^{-\tau}}{2(n-1)} = \frac{1-z}{r^2}\frac{e^{(\lambda-1)\tau}}{2(n-1)}.$$

we have at $\tau = \tau_0$ and on $(MR_D e^{-\lambda\tau_0/2}, \infty)$,

$$2(n-1)e^{\tau_0}(L - K) = u^{-2} + O(u^{-4}) > 0$$

if we choose $M \geq 2$ large enough. Also recall (3.15) and (3.16), both K and L decay to zero as $u \nearrow \infty$.

(2) Recall the asymptotic expansion of $\mathbf{B}(r)$ as $r \searrow 0$ from (3.10). Then there exists $R_* > 0$ such that for $r \in [0, R_*]$,

$$\mathbf{B}(Ar) = 1 - b_2 A^2 r^2 + \frac{n}{n+3} b_2^2 A^4 r^4 + O(r^6).$$

On $[0, R_* e^{-\lambda\tau_0/2}]$, we define $\hat{z}_0(u) = \mathbf{B}(Aue^{\lambda\tau_0/2})$. At $\tau = \tau_0$ and on $[0, R_* e^{-\lambda\tau_0/2}]$, we have $z^- < \hat{z}_0 < z^+$ by Remark 5.2, we also have

$$2(n-1)e^{(1-\lambda)\tau_0}(L-K) = \frac{n}{n+3}b_2^2 A^4 r^2 + O(r^4) > 0$$

if we choose R_* to be small enough.

(3) On $[R_* e^{-\lambda\tau_0/2}, MR_D e^{-\lambda\tau_0/2}]$, we connect $\hat{z}_0$ in (1) with $\hat{z}_0$ in (2) by a piecewise linear continuous function strictly between z^- and z^+. This is possible since $z^- < z+$ for all $u \in (0,\infty)$. We have $L > 0$ and $K = 0$ where $\hat{z}_0$ is linear, so in this region $L - K > 0$ except at finitely many points where $\hat{z}_0$ has a corner.

(4) By (1)–(3), we have a continuous, piecewise smooth function $\hat{z}_0$ defined on $[0,\infty)$ such that at $\tau = \tau_0$, $z^- < \hat{z}_0 < z^+$ everywhere, and $L - K > 0$ for all $u \in (0,\infty)$ except at finitely many points. We can then smooth out $\hat{z}_0$ at to get a smooth z_0 for which $z^- < z_0 < z^+$ and $L - K > 0$ for all $u \in (0,\infty)$ at $\tau = \tau_0$.

By Lemma 6.1, z_0 determines a complete rotationally symmetric metric g_0 on $\mathbb{R}^{n+1}$. It is straightforward to check that g_0 has bounded sectional curvatures everywhere, and K and L decay to zero at spatial infinity. Since the sectional curvatures depend smoothly on the metric, there is a neighbourhood of g_0 in the C^2 topology corresponding to an open set of z's around z_0, all of which lie between z^- and z^+, satisfy $L - K > 0$ everywhere, and determine complete rotationally symmetric metrics with bounded curvatures.

There exists a unique solution $g(t)$ to Ricci flow starting from g_0 [10, 29]. We choose $\psi_s > 0$ initially, so $(\mathbb{R}^{n+1}, g_0)$ does not contain any minimal sphere. By construction, the sectional curvatures of g_0 decay to zero at spatial infinity. Thus, $g(t)$ is immortal [18, Theorem 1.2].

The profile $z(u,\tau)$ of $g(t)$ is the unique solution of equation (3.1) for $0 < u < \infty$ and $\tau \geq \tau_0$, with boundary condition $z(0,\tau) = 1$ and asymptotic condition $\lim_{u \nearrow \infty} z(u,\tau) = 0$, and initial data $z(u,\tau_0) = z_0$. By the comparison principle in Proposition 5.8, we have $0 < z^-(u,\tau) \leq z(u,\tau) \leq z^+(u,\tau)$ for all $\tau \geq \tau_0$. In particular, $z(u,\tau)$ defines a complete, rotationally symmetric, smooth metric $g(t)$ on $\mathbb{R}^{n+1}$ by Lemma 6.1.

As $t = e^\tau \nearrow \infty$, the asymptotic behaviour of the solution agrees with that of the barriers, and hence with that of the formal solution. In particular, the sectional curvatures of $K(t)$ and $L(t)$ of $g(t)$ at the origin $\mathcal{O}$ are

$$K(t)|_{\mathcal{O}} = L(t)|_{\mathcal{O}} \sim t^{\lambda-1}.$$

If we define $\alpha = \psi^2(L-K)$, then $\alpha \geq 0$ along the Ricci flow [18, Lemma 2.3]. Moreover, we have

$$\partial_s L = \partial_s\left(\frac{1-\psi_s^2}{\psi^2}\right) = -2\frac{\psi_s}{\psi^3}\alpha \leq 0,$$

where we used $\psi_s > 0$ by Lemma 6.2. Hence, along the flow,

$$|\mathrm{Rm}|^2(\cdot,t) = 2nK^2 + n(n-1)L^2 \le n(n+1)L^2 \le n(n+1)\left. L^2\right|_{\mathcal{O}} = \left.|\mathrm{Rm}|^2\right|_{\mathcal{O}}(t),$$

which implies that $\sup_{\mathbb{R}^{n+1}} |\mathrm{Rm}(\cdot,t)|$ is attained at the origin. Now part (1) of Theorem 1.1 is proved.

Since $z^- \le z(u,\tau) \le z^+$ for any $\tau < \infty$, and the solution $z(u,\tau)$ exhibits the asymptotic behaviour of $z^{\pm}$. Near the origin, $z(u,\tau)$ converges uniformly to the Bryant soliton profile function for $0 < u < R_D e^{-\lambda\tau}$. Near spatial infinity, i.e., as $u \nearrow \infty$, $z(u,\tau) \searrow 0$ at a rate depending on λ as is given in (3.14), and so the sectional curvatures K and L are asymptotically flat according to (3.15) and (3.16), respectively. Thus, $g(t)$ has the asymptotic behaviour described in parts (2) and (3) of Theorem 1.1.

Therefore, Theorem 1.1 is proved. □

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