Asymptotic Analysis and Special Solutions of a Family of Painlevé-Like Equations

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Introduction

We wish to study the following family of nonlinear ordinary differential equations (ODEs):

$$y''(x) = 2y(x)^3 + x^\mu y(x), \qquad \mu \in \mathbb{Z}^+.$$

In general, y is a map from the complex plane to the complex plane. When $\mu = 1$, we retrieve the traditional second Painlevé equation (P_{II}). We wish to:

- Investigate a physical context which may give rise to such equations.
- Investigate the asymptotic behaviour of solutions.
- Investigate the generalisation of special solutions of P_{II}.

Background on Painlevé II

We have the standard second Painlevé equation:

$$y''(x) = 2y(x)^3 + xy(x).$$

- The equation was derived and studied by Painlevé in a search for ODEs which yield higher transcendental functions as generic solutions (Painlevé transcendents).
- Such transcendental functions are not expressible in terms of classical operations applied to well known mathematical functions. However, such a description may be possible when we consider the independent variable in some asymptotic limit.
- Boutroux studied the first and second Painlevé equations extensively in the limit $|x| \to \infty$. Leading order behaviour was shown to be given by elliptic functions.
- Elliptic functions are bi-periodic, with infinitely many poles distributed throughout the complex plane.
- Boutroux studied special solutions which are asymptotically pole-free in wide sectors of the complex plane, which he called 'tritronquée' (triply-truncated).
- Hastings and McLeod studied a special solution on the real line which is pole-free; this solution is celebrated in mathematical physics as it appears when describing the Tracy-Widom distribution.

Physical Context: Electro-diffusion

Consider a steady flow of charged particles along a single length dimension. We have two species of charged particles (equal and opposite charge) which flow at a rate that is constant with respect to position. It may be shown that the strength of the induced electric field, as a function of position, is governed by a scaled form of P_{II}:

$$\lambda^2 E''(x) = \tfrac{1}{2}\lambda^2 E(x)^3 + (\theta x + B)E(x) + (A_+ - A_-).$$

We have constants $A_\pm, B, \theta$ and λ, which depend on various properties of the system. We may add complexity to the system by modelling particle flux as an n-degree polynomial in x, as opposed to a constant. Then instead of an $xE(x)$ term, we end up with an $x^{n+1}E(x)$ term, thus motivating the study of our broader class of equations (i.e. $\mu = n+1$).

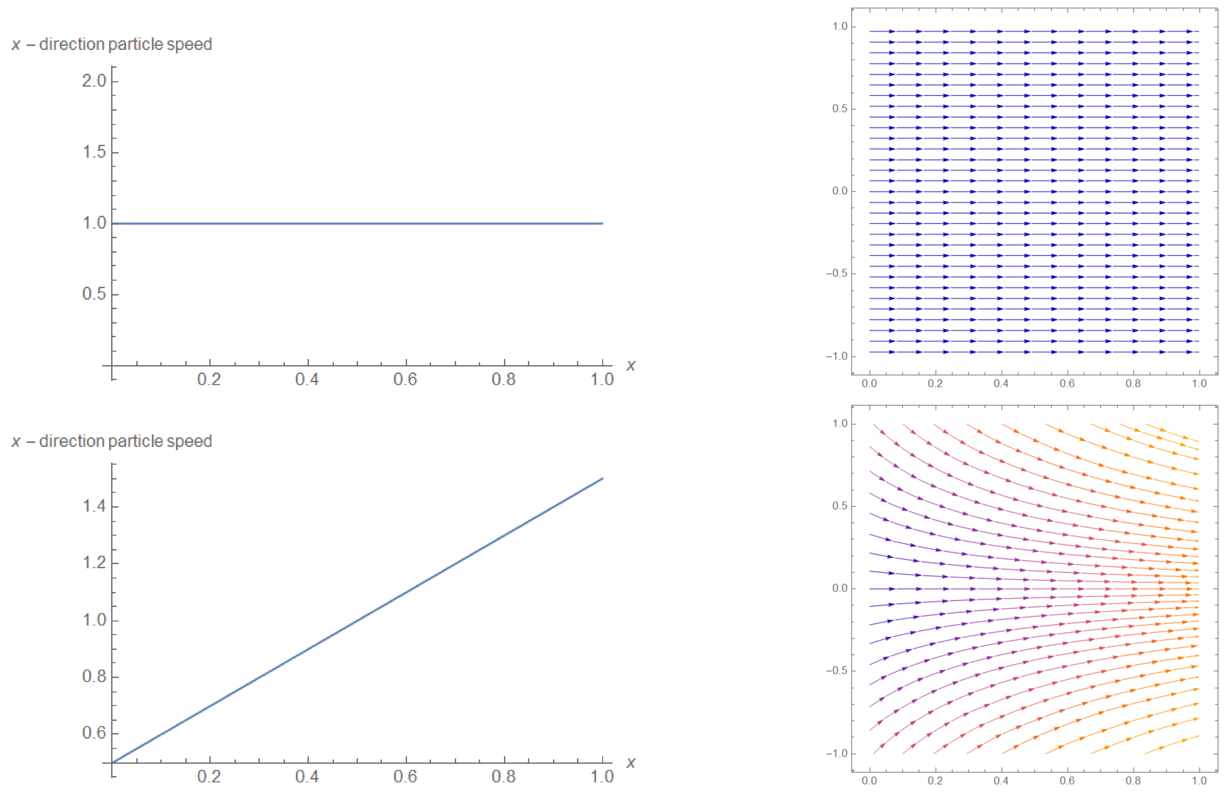

Figure 1: Constant flux with an example of flow in 2D (top row). Linear flux in x with an example of flow in 2D (bottom row).

Asymptotic Analysis

First, we generalise Boutroux's transformation of the dependent and independent variables, and thus derive an ODE which is more amenable to asymptotic analysis:

$$z = \frac{2}{\mu+2}x^{\frac{\mu+2}{2}}, \qquad y(x) = x^{\frac{\mu}{2}}u(z).$$

Using the principle of maximal dominant balance, we have derived a new quantity u, which is of constant order as the independent variable becomes infinitely large. It may be shown that u satisfies:

$$u''(z) = 2u(z)^3 + u(z) - \frac{3\mu}{\mu+2}\frac{u'(z)}{z} - \frac{\mu(\mu-2)}{(\mu+2)^2}\frac{u(z)}{z^2}.$$

Therefore, we have an autonomous general leading order relation: $u'' \sim 2u^3 + u$ as $|z| \to \infty$. This is solved by a Jacobi elliptic function. To obtain leading order algebraic (singularity-free) behaviour, we must assume the sub-maximal dominant balance whereby $2u^3 \sim -u$. Therefore, along with the identically zero solution, we have $u(z) \sim \pm\frac{i}{\sqrt{2}}$; this may be corrected by considering a full asymptotic series in negative powers of z. Meanwhile, we seek an exponential perturbation of the zero solution.

Key Results Concerning Asymptotic Behaviour

We have an asymptotic power-series solution of the form:

$$u(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}, \qquad (a_0 = \pm\tfrac{i}{\sqrt{2}}).$$

In the original variables, we have:

$$y(x) \sim x^{\frac{\mu}{2}} \sum_{n=0}^{\infty} a_n \left(\tfrac{2}{\mu+2}\right)^{-2n} x^{-(\mu+2)n}.$$

Furthermore, we have asymptotically valid exponential behaviour:

$$y(x) \sim c_\pm x^{-\frac{\mu}{4}} \exp[\pm\tfrac{2}{\mu+2}x^{\frac{\mu+2}{2}}].$$

Stokes Phenomenon and Tritronquée Solutions

The exponential behaviour shown above applies when $|x| \to \infty$. However, the exponential exhibits varying behaviour depending on the path taken by x to infinity in the complex plane. Rays in the complex plane for which the argument of the exponential is purely real, and thus the behaviour is purely growing or decaying, we call Stokes lines. On the other hand, rays on which the argument is purely imaginary, and thus the behaviour is oscillatory, we call anti-Stokes lines.

Figure 2: Stokes and anti-Stokes lines in the complex plane when $\mu = 1$ (left) and $\mu = 3$ (right).

The exponential behaviour only maintains validity while it is indeed decaying to zero. Therefore, the exponential above with a given sign is only valid in certain bisectors of the complex plane, bounded by anti-Stokes lines. Hence, we have families of solutions which are 'tronquée'/truncated; singularities are truncated along a special bisector. However, Boutroux studied unique solutions which are 'tritronquée'/triply-truncated, with singularities truncated along three consecutive bisectors.

Key Result Concerning Tritronqué Solutions

We show that there are indeed unique triply-truncated solutions for $\mu > 1$. That is, there are unique solutions which are asymptotically equivalent to a given power series (shown above) in three consecutive special bisectors of the complex plane, and thus singularity free in this wide sector. More specifically, for a given value of μ, we have unique triply-truncated solutions which are asymptotically singularity-free in a sector of angular width $\frac{4\pi}{\mu+2}$.

Generalised Hastings-McLeod Solution

We define the generalised Hastings-McLeod solution via a boundary value problem on the real line. We require that as x goes to positive infinity, $y(x)$ is exponentially decaying to zero. On the other hand, when x goes to negative infinity, $y(x)$ is asymptotic to the previously discussed power series:

$$y(x) \sim kx^{-\frac{\mu}{4}}\exp[-\tfrac{2}{\mu+2}x^{\frac{\mu+2}{2}}], \qquad x \to +\infty,$$

$$y(x) \sim \sqrt{-\tfrac{x^\mu}{2}}, \qquad x \to -\infty.$$

Lemma: Integral Equation

By assuming the condition at positive infinity, we may derive the following integral equation:

$$y_k(x) = kf(x) + 2\int_x^\infty (f(x)g(t) - g(x)f(t))y_k(t)^3 dt,$$

where

$$f(x) \sim x^{-\frac{\mu}{4}}\exp[-\tfrac{2}{\mu+2}x^{\frac{\mu+2}{2}}], \qquad g(x) \sim x^{-\frac{\mu}{4}}\exp[+\tfrac{2}{\mu+2}x^{\frac{\mu+2}{2}}], \qquad x \to +\infty.$$

This shows us that for any $k \in (0,\infty)$, there is a unique solution satisfying the right boundary condition. Furthermore, at any fixed x, we see that $y_k(x)$ is continuous in k.

We find that the solutions $\{y_k; k \in (0,\infty)\}$ may be categorised into two groups:

- Type 1: The solution $y_k(x)$ blows up at some finite point as x decreases.
- Type 2: The solution $y_k(x)$ is bounded on the real line, becoming oscillatory left of some negative x.

The Hastings-McLeod type solution may then be thought of as a 'boundary' between Type 1 and Type 2. We can show the existence of this solution by showing that Type 1 and 2 solutions, respectively, correspond to disjoint open intervals of k-values. This would imply that there exists at least one value of k giving a solution which does not become singular and is not bounded. From previous asymptotic analysis, we know that such a solution must be asymptotic to the given power series as $x \to -\infty$.

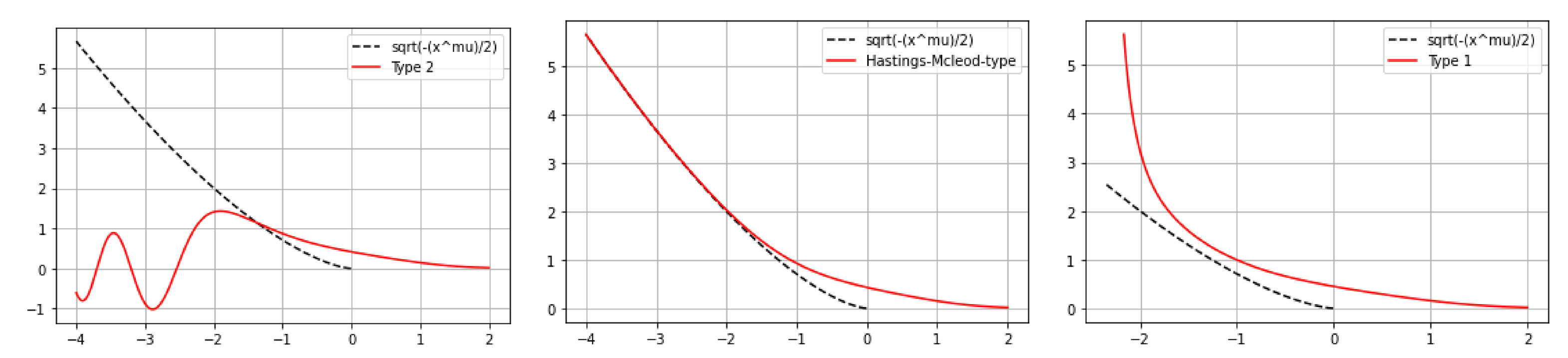

Figure 3: Example of Type 1 (right), Type 2 (left), and the generalised Hastings-McLeod solution (middle), when $\mu = 3$.

Key Result Concerning the Hastings McLeod Solution

We find that when $\mu > 1$ is even, we only have Type 1 solutions, i.e. solutions that become singular at some finite point on the real line. However, when $\mu > 1$ is odd, we find that there is a unique generalised Hastings-Mcleod solution, as defined above.

References

1. Painlevé, P. (1900). Mémoire sur les équations différentielles dont l'intégrale générale est uniforme. *Bulletin de la Société Mathématique de France, 28*, 201-261.
2. Boutroux, P. (1913). Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre. In *Annales Scientifiques de l'École Normale Supérieure* (Vol. 30, pp. 255-375).
3. Hastings, S. P., & Mcleod, J. B. (1980). A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Archive for Rational Mechanics and Analysis, 73*(1), 31-51.
4. Bass, L. (1964). Electrical structures of interfaces in steady electrolysis. *Transactions of the Faraday Society, 60*, 1656-1663.