Asymptotic behaviours of q-orthogonal polynomials from a q-Riemann-Hilbert Problem

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November, 2021

Introduction

The theory of orthogonal polynomials is a source of major developments in modern mathematical physics. But, the spectacular outcomes of the theory of orthogonal polynomials with continuous measures are not yet matched by q-orthogonal polynomials, which are orthogonal with respect to a discrete measure supported on the lattice $\{\pm q^k\}_{k\in\mathbb{N}^+}$, for some $0<q<1$. We show that, as was the case for polynomials orthogonal with continuous measures, re-framing q-orthogonal polynomials as solutions to Riemann-Hilbert Problems (RHPs) provides significant insights into their asymptotic behaviour as the degree tends to infinity.

q-difference calculus

q-orthogonal polynomials are most easily studied using q-difference calculus. Some operations and functions which frequently arise in q-difference calculus are defined below.

- The Jackson integral of $f(x)$ from -1 to 1 is defined as

$$\int_{-1}^{1} f(z)d_qz = (1-q)\sum_{k=0}^{\infty}(f(q^k)+f(-q^k))q^k\,.$$

- The Pochhammer symbol $(a;q)_\infty$ is defined as

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)\,.$$

- The function $h(z)$ is defined as

$$h(z) = \sum_{k=-\infty}^{\infty}\frac{2zq^k}{z^2-q^{2k}} = \sum_{k=-\infty}^{\infty}\left(\frac{q^k}{z-q^k}+\frac{q^k}{z+q^k}\right),$$

  and satisfies the q-difference equation $h(qz)=h(z)$.

A sequence of monic q-orthogonal polynomials $\{P_n(z)\}_{n=0}^{\infty}$ is orthogonal with respect to the weight $w(z)$ if

$$\int_{-1}^{1} P_n(z)P_m(z)w(z)d_qz = \gamma_n\delta_{n,m}. \tag{1}$$

Using the RHP setting we find that q-orthogonal polynomials exhibit certain universal asymptotic behaviours.

Asymptotic behaviour of q-orthogonal polynomials

If the orthogonality weight $w(z)$ satisfies
- $w(z)$ is analytic everywhere,
- $w(q^n)=O(q^{2n})$, for large n,

then, for even n, the asymptotic behaviour of the corresponding q-orthogonal polynomials is given by:

$$\begin{aligned}
\lim_{n\to\infty} P_n(t) &= (-1)^{n/2}q^{\frac{n}{2}(\frac{n}{2}-1)}\left(\psi(t)+O(q^n)\right) \text{ for } |t|<q^{-1},\\
\lim_{n\to\infty} P_n(z) &= z^n(z^{-2};q^2)_\infty + O\left(q^{n/2(n/2-1)}\right) \text{ for } |z|>q^{n/2-1},\\
\lim_{n\to\infty}\gamma_n &= q^{n(n-1)/2}\left(2(q^2;q^2)_\infty^2+O(q^n)\right),
\end{aligned}$$

where $t=q^{-n/2}z$, and $\psi(t)$ is an entire function independent of n.

q-RHP

q-orthogonal polynomials naturally arise as entries to the unique matrix solution of the following q-RHP, where the terminology q-RHP is adopted because the RHP is related to the theory of q-difference equations.

q-RHP

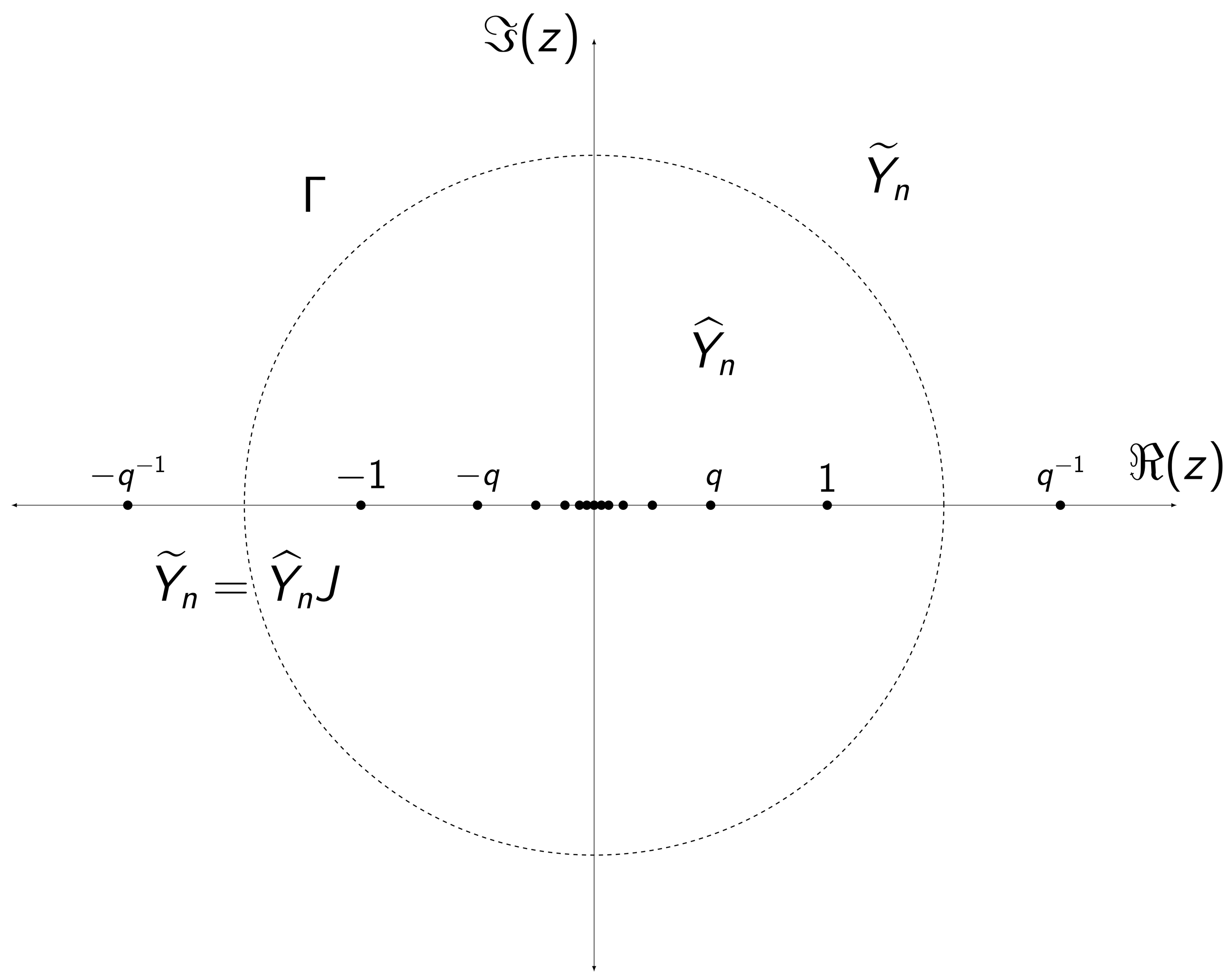
- *Analyticity:* $Y_n(z)$ is analytic on $\mathbb{C}\setminus\Gamma$, where Γ is depicted in Figure 1.
- *Jump condition:* Let $\widetilde{Y}_n(z)$ be $Y_n(z)$ restricted ext(Γ) and $\widehat{Y}_n(z)$ be $Y_n(z)$ restricted to int(Γ)

$$\widetilde{Y}_n(s) = \widehat{Y}_n(s)\begin{pmatrix}1 & w(s)h(s)\\ 0 & 1\end{pmatrix},\ s\in\Gamma.$$

- *Asymptotic decay:* $Y_n(z)$ satisfies

$$Y_n(z)\begin{pmatrix}z^{-n} & 0\\ 0 & z^n\end{pmatrix} = I + O\left(\frac{1}{|z|}\right),\ \text{as } |z|\to\infty.$$

Figure 1: q-RHP satisfied by q-orthogonal polynomials.



The unique solution to the q-RHP is given by

$$Y_n(z) = \begin{bmatrix} P_n(z) & \oint_\Gamma \frac{P_n(s)w(s)h(s)}{2\pi i(z-s)}ds\\ \gamma_{n-1}^{-1}P_{n-1}(z) & \oint_\Gamma\frac{P_{n-1}(s)w(s)h(s)}{2\pi i(z-s)\gamma_{n-1}}ds\end{bmatrix}, \tag{2}$$

where $\{P_n(z)\}_{n=0}^{\infty}$ satisfies the orthogonality relation given by Equation (1).

The key step in realising that $Y_n(z)$ is a solution to the q-RHP is applying Cauchy's residue theorem to observe that

$$\oint_\Gamma \frac{P_n(s)w(s)h(s)}{2\pi i(z-s)}ds = \int_{-1}^{1}\frac{P_n(s)w(s)}{z-s}d_qs.$$

By expressing $1/(z-s)$ as a geometric sum it follows $Y_n(z)$ satisfies the asymptotic decay condition of the q-RHP and is thus the unique solution.

Asymptotics

By making a series of transformations to Equation (2) we can obtain a solution, $W_n(z)$, which converges to a limit $\mathcal{W}$ as $n\to\infty$. In order to achieve this we need to introduce the function

$$f(z) = (z^{-2};q^2)_\infty.$$

A quick summary of the transformations is as follows:

1. Define,

$$W_n(z) = \begin{cases} Y_n(z)\begin{bmatrix} f(z)^{-1}z^{-n} & 0\\ 0 & f(z)z^n\end{bmatrix}, & \text{for } z\in \text{ext}(\Gamma),\\ Y_n(z), \text{ for } z\in\text{int}(\Gamma).\end{cases}$$

2. Deform the jump contour Γ such that it is shrunk by a factor of $q^{n/2}$, i.e. if it was the unit circle it would now be the the circle radius $q^{n/2}$.
3. Change the coordinate system to $t=zq^{-n/2}$, Γ should now be invariant in this coordinate system.

These transformations result in a solution $W_n(t)$ which approaches $\mathcal{W}(t)$ as $n\to\infty$, where $\mathcal{W}(t)$ is the unique solution to the following RHP:

Model RHP

- $\mathcal{W}(t)$ is meromorphic in $\mathbb{C}\setminus\Gamma$, with simple poles at $t=\pm q^{-k}$ for $k\in\mathbb{N}_1$.
- $\widetilde{\mathcal{W}}(t)$ and $\widehat{\mathcal{W}}(t)$ satisfy

$$\widetilde{\mathcal{W}}(t) = \widehat{\mathcal{W}}(t)\begin{bmatrix} g(t)^{-1} & g(t)h(t)\\ 0 & g(t)\end{bmatrix},\ t\in\Gamma,$$

- $\mathcal{W}(t)$ satisfies

$$\mathcal{W}(t) = I + O\left(\frac{1}{|t|}\right),\ \text{as } |t|\to\infty.$$

- The residue at the poles $t=\pm q^{-k}$ for $k\in\mathbb{N}_1$ is given by

$$\text{Res}(\mathcal{W}(\pm q^{-k})) = \lim_{t\to\pm q^{-k}}\mathcal{W}(t)\begin{bmatrix}0 & 0\\ (t\mp q^{-k})g(t)^{-2}h(t)^{-1} & 0\end{bmatrix}.$$

Using techniques from q-difference calculus one can derive the explicit solution $\mathcal{W}(t)$ which satisfies the above RHP. Furthermore, each entry can be expressed in terms of a power series either about $t=0$ or $t=\infty$. Retracing the transformations $Y_n(z)\to W_n(t)$ enables one to determine the asymptotic behaviour of q-orthogonal polynomials.

Acknowledgements

This work was supported the Australian Government Research Training Program and by the University of Sydney Postgraduate Research Supplementary Scholarship in Integrable Systems.

References

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