Normal Form for Completely Integrable Triple-Zero Singularities

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Abstract

The volume-preserving-completely integrable family of triple-zero singularities is introduced. This family is generated by $\mathfrak{sl}_2$ representation associated to the linear part of the system. All such vector fields share a common quadratic invariant. Further, we provide a Poisson structure for the Lie subalgebra from which the second invariant for each vector field can be readily derived. Using $\mathfrak{sl}_2$, the unique normal form for this family is classified.
Keywords: Normal form classification; Triple-zero singularity; $\mathfrak{sl}_2$-Lie algebra representation; Completely integrable vector field.

Algebraic structures

Let

$$\mathbf{N} := \mathbf{x}\frac{\partial}{\partial \mathbf{y}} + 2\mathbf{y}\frac{\partial}{\partial \mathbf{z}},$$

$$\mathbf{M} := \mathbf{z}\frac{\partial}{\partial \mathbf{y}} + 2\mathbf{y}\frac{\partial}{\partial \mathbf{x}}, \quad \text{and } \mathrm{H} := [\mathrm{M}, \mathrm{N}] = \mathbf{MN} - \mathbf{NM} = 2\mathbf{z}\frac{\partial}{\partial \mathbf{z}} - 2\mathbf{x}\frac{\partial}{\partial \mathbf{x}}.$$

The triple $\{\mathbf{M}, \mathbf{N}, \mathrm{H}\}$ generates an $\mathfrak{sl}_2$ Lie algebra, ***i.e.,***

$$[\mathbf{M}, \mathbf{N}] = \mathrm{H}, \qquad [\mathrm{H}, \mathbf{M}] = 2\mathbf{M}, \qquad [\mathrm{H}, \mathbf{N}] = -2\mathbf{N}.$$

We denote $(\mathbf{N}^{\mathbf{n}}\mathbf{f})\,\mathbf{v}$ for the iterative action of N as a differential operator on the scalar function f that is also multiplied with v.

B-Family

We define

$$\mathsf{B}^{\mathbf{l}}_{\mathbf{i,k}} = \frac{(\mathbf{2i-l+1})\mathbf{N}^{\mathbf{l+2}}(\mathbf{z}^{\mathbf{i+1}})\boldsymbol{\Delta}^{\mathbf{k}}}{(\mathbf{i+1})\kappa_{\mathbf{l+2,2i+2}}}\frac{\partial}{\partial \mathbf{x}} + \frac{(\mathbf{i-l})\mathbf{N}^{\mathbf{l+1}}(\mathbf{z}^{\mathbf{i+1}})\boldsymbol{\Delta}^{\mathbf{k}}}{(\mathbf{i+1})\kappa_{\mathbf{l+1,2i+2}}}\frac{\partial}{\partial \mathbf{y}} - \frac{(\mathbf{l+1})\mathrm{N}^{\mathbf{l}}(\mathbf{z}^{\mathbf{i+1}})\boldsymbol{\Delta}^{\mathbf{k}}}{(\mathbf{i+1})\kappa_{\mathbf{l,2i+2}}}\frac{\partial}{\partial \mathbf{z}}.$$

for

$$-\mathbf{1} \leq \mathbf{l} \leq \mathbf{2i+1}, \mathbf{i}, \mathbf{k} \in \mathbb{N}_0 = \mathbb{N} \cup \{\mathbf{0}\}.$$

Hence, $\mathsf{B}^{\mathbf{l}}_{0,0} := -\mathrm{N}$ and $\mathsf{B}^{-1}_{0,0} := -\mathrm{M}$. Now we introduce B as the vector space spanned by all nonlinear vector fields from this family with the nilpotent linear part $\mathsf{B}^{\mathbf{l}}_{0,0}$, ***i.e.,***

$$\mathbf{B} := \mathrm{span}\left\{\mathsf{B}^{\mathbf{1}}_{\mathbf{0,0}} + \sum \mathbf{b}^{\mathbf{l}}_{\mathbf{i,k}}\mathsf{B}^{\mathbf{l}}_{\mathbf{i,k}} \,\middle|\, -\mathbf{1} \leqslant \mathbf{l} \leqslant \mathbf{2i+1}, \mathbf{i} \in \mathbb{N}, \mathbf{k} \in \mathbb{N}_0, \mathbf{b}^{\mathbf{l}}_{\mathbf{i,k}} \in \mathbb{R}\right\}.$$

Poisson algebra structure

Consider the ring of formal power series $\mathbb{R}[[x, y, z]]$ and define a Poisson bracket on the ring's variables by

$$\{\mathbf{x}, \mathbf{y}\} = \mathbf{x}, \quad \{\mathbf{x}, \mathbf{z}\} = \mathbf{2y}, \quad \{\mathbf{y}, \mathbf{z}\} = \mathbf{z}.$$

Since f and g from $\mathbb{R}[[x, y, z]]$ have each a unique representation as formal power series in x, y, and z, the Leibniz rule and bilinearity of the Poisson bracket are sufficient to uniquely determine Poisson structure for all elements in $\mathbb{R}[[\mathbf{x}, \mathbf{y}, \mathbf{z}]]$. Now define

$$\mathfrak{b}^l_{i,k} := -\frac{\mathrm{ad}_x^{l+1}(z^{i+1}\Delta^k)}{(i+1)\kappa_{l+1,2i+2}}, \qquad \text{for } -1 \leq l \leq i+1, \text{ and } i, k \in \mathbb{N}_0,$$

where $\mathrm{ad}_x f := \{x, f\}$ and $\mathrm{ad}_x^n f := \{x, \mathrm{ad}_x^{n-1} f\}$ for $f \in \mathbb{R}[[x, y, z]]$ and $n > 1$.

Main Results

The following items presents geometrical features and representations for the normal form of integrable solenoidal vector fields given by

$$v := -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + \mathrm{v}(x, y, z),$$

where $\mathrm{v} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a vector field without constant and linear part,

$$\mathrm{div}(v) = 0, \qquad \text{and} \qquad v(\Delta) = 0.$$

1. Polynomials $\mathfrak{b}^l_{i,0}$ and Δ are two first integrals for $\mathsf{B}^l_{i,k}$, ***i.e.,***
$$\mathsf{B}^l_{i,k}(\mathfrak{b}^l_{i,0}) = 0, \quad \text{and} \quad \mathsf{B}^l_{i,k}(\Delta) = 0.$$
Indeed for every $v \in B$, $\Psi^{-1}(v) \in \mathfrak{B}$, and Δ are two first integrals for v.
2. A Clebsch potential representation for $\mathsf{B}^l_{i,k}$ is given by
$$\mathsf{B}^l_{i,k} = \Delta^k(\nabla\mathfrak{b}^l_{i,0} \times \nabla\Delta).$$
Which provides an alternative representation for each vector field v in B by using the primary and secondary Clebsch potentials Δ and $\Psi^{-1}(v) \in \mathfrak{B}$.
3. The polynomial functions $\mathfrak{b}^l_{i,0}$ and Δ are two functionally independent first integrals for $\mathsf{B}^l_{i,0}$.
4. The ring of invariants for $\mathsf{B}^{-1}_{i,k}$, $\mathsf{B}^i_{i,k}$, and $\mathsf{B}^{2i+1}_{i,k}$ includes $\langle\Delta, z\rangle$ *(the algebra generated by Δ and z), $\langle\Delta, y\rangle$ and $\langle\Delta, x\rangle$, respectively.*
5. *A formal sum of* B*-terms:*
$$\mathbf{w} := \mathsf{B}^{\mathbf{1}}_{\mathbf{0,0}} + \mathsf{B}^{-\mathbf{1}}_{\mathbf{p,0}} + \sum_{\mathbf{k=p+1}}^{\infty}\sum_{\mathbf{i=0}}^{[\frac{\mathbf{k+p}}{\mathbf{2}}]} \mathbf{b}_{\mathbf{i,k}}\mathsf{B}^{-\mathbf{1}}_{\mathbf{i,k+p-2i}}.$$
6. *The secondary invariant:*
$$\mathbf{w} := -\boldsymbol{\Psi}\Big(\mathbf{x} + \mathbf{z}^{\mathbf{p}} + \sum_{\mathbf{k=p+1}}^{\infty}\sum_{\mathbf{i=0}}^{[\frac{\mathbf{k+p}}{\mathbf{2}}]} \mathbf{b}_{\mathbf{i,k}}\mathbf{z}^{\mathbf{i+1}}(\mathbf{xz} - \mathbf{y}^{\mathbf{2}})^{\mathbf{k+p-2i}}\Big).$$
Here, Ψ is the Lie isomorphism
7. *Vector potential:*
$$\mathbf{w} := \mathbf{curl}\Big((\mathbf{x} + \mathbf{z}^{\mathbf{p+1}} + \sum_{\mathbf{k=p+1}}^{\infty}\sum_{\mathbf{i=0}}^{[\frac{\mathbf{k+p}}{\mathbf{2}}]} \frac{\boldsymbol{\Delta}^{\mathbf{k}}\mathbf{z}^{\mathbf{i+1}}}{\mathbf{i+1}})(\mathbf{z}, -\mathbf{2y}, \mathbf{x})\Big).$$
8. *Functionally independent Clebsch potentials:*
$$\mathbf{w} := \nabla(\mathbf{xz} - \mathbf{y}^{\mathbf{2}}) \times \nabla\Big(\mathbf{x} + \mathbf{z}^{\mathbf{p+1}} + \sum_{\mathbf{k=p+1}}^{\infty}\sum_{\mathbf{i=0}}^{[\frac{\mathbf{k+p}}{\mathbf{2}}]} \frac{\mathbf{b}_{\mathbf{i,k}}\mathbf{z}^{\mathbf{i+1}}(\mathbf{xz} - \mathbf{y}^{\mathbf{2}})^{\mathbf{k+p-2i}}}{\mathbf{i+1}}\Big).$$
9. *Poisson bracket:*
$$\mathbf{w} = \sum_{\mathbf{p=1}}^{\mathbf{3}}\{\mathbf{x}_{\mathbf{p}}, \mathbf{I}(\mathbf{x}, \mathbf{y}, \mathbf{z})\} \cdot \mathbf{e}_{\mathbf{p}},$$
where $\mathbf{I}(x, y, z)$ is the invariant given by
$$\mathbf{I}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x} + \frac{\mathbf{1}}{\mathbf{p+1}}\mathbf{z}^{\mathbf{p+1}} + \sum_{\mathbf{k=p+1}}^{\infty}\sum_{\mathbf{i=0}}^{[\frac{\mathbf{k+p}}{\mathbf{2}}]} \frac{\mathbf{b}_{\mathbf{i,k}}}{\mathbf{i+1}}\mathbf{z}^{\mathbf{i+1}}(\mathbf{xz} - \mathbf{y}^{\mathbf{2}})^{\mathbf{k-2i+p}}.$$
Furthermore, $\{\mathbf{I}(x, y, z), \Delta\} = 0.$

Reference

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