Integrable Systems arising from Separation on S^3

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Geodesic Flow on S^3

Consider the sphere S^3 as embedded in $\mathbb{R}^4$ with Cartesian coordinates $\boldsymbol{Q} = (x_1, x_2, x_3, x_4)$ and momenta $\boldsymbol{P} = (y_1, y_2, y_3, y_4)$. The phase space is T^*S^3 with constraints $\boldsymbol{Q} \cdot \boldsymbol{Q} = 1$ and $\boldsymbol{Q} \cdot \boldsymbol{P} = 0$ and the geodesic Hamiltonian is given by $H = \frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{P}$. This system is known to be superintegrable with a global Hamiltonian S^1-action. The invariants of the corresponding Dirac structure are the six angular momenta $\ell_{ij} = x_i y_j - x_j y_i$. These form a Poisson algebra with two Casimirs $\mathcal{C}_1 = 2H = \sum_{j>i} \ell_{ij}^2$ which is the energy and the Plücker relation $\mathcal{C}_2 = \ell_{12}\ell_{34} - \ell_{13}\ell_{24} + \ell_{14}\ell_{23} = 0$. Performing symplectic reduction gives us the reduced space $S^2 \times S^2$ explicitly described by

$$\mathfrak{C}_1 = \mathcal{C}_1 + 2\mathcal{C}_2 = (\ell_{12} + \ell_{34})^2 + (\ell_{13} - \ell_{24})^2 + (\ell_{14} + \ell_{23})^2 := X_1^2 + X_2^2 + X_3^2 = 1$$
$$\mathfrak{C}_2 = \mathcal{C}_1 + 2\mathcal{C}_2 = (\ell_{12} - \ell_{34})^2 + (\ell_{13} + \ell_{24})^2 + (\ell_{14} - \ell_{23})^2 := Y_1^2 + Y_2^2 + Y_3^2 = 1.$$

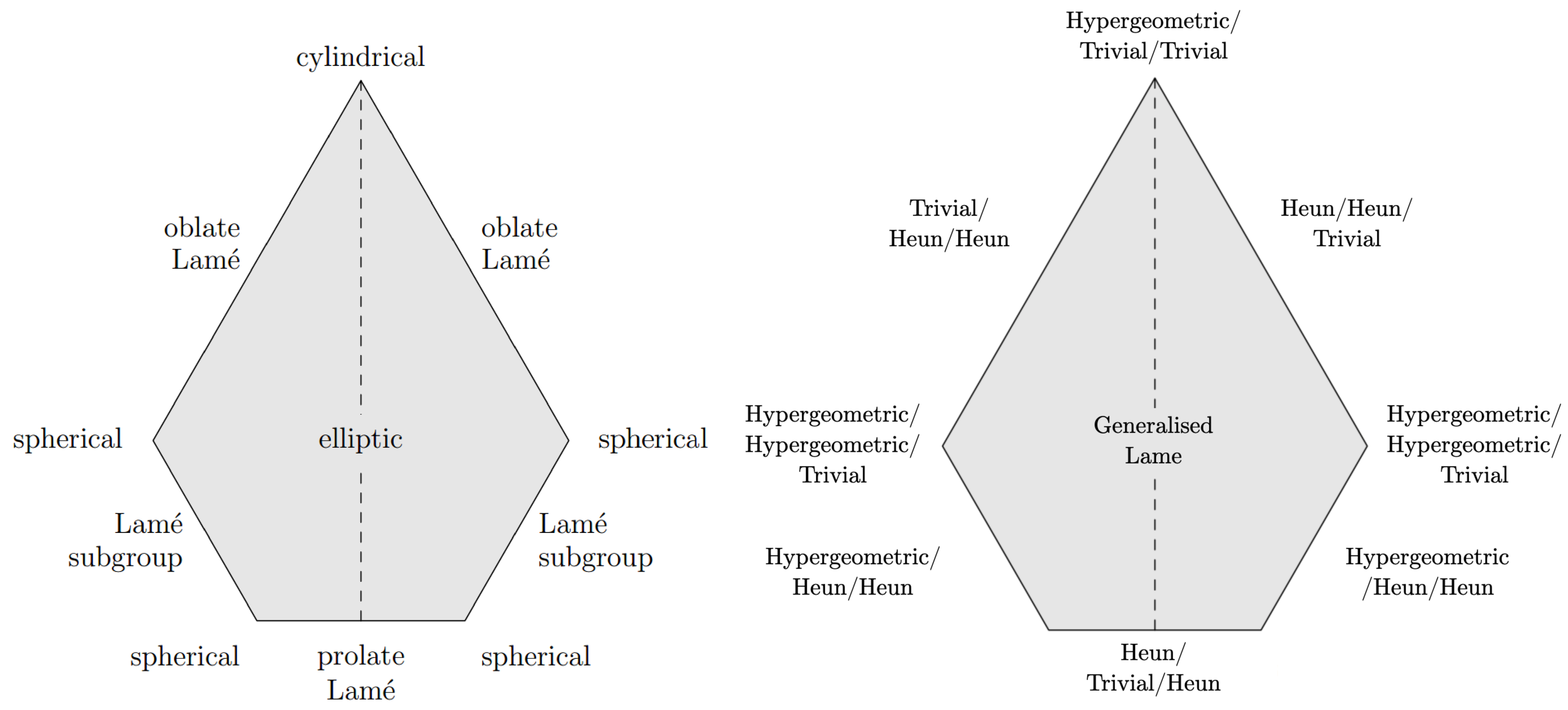
The Poisson algebra in these new coordinates is isomorphic to $\mathfrak{so}(3) \times \mathfrak{so}(3)$.

Separable Coordinates

The Hamilton-Jacobi as well as the Schrödinger equation for the geodesic flow on S^3 separates in six orthogonal coordinates. These are classified by the Stasheff polytope shown below. Each of these six orthogonal coordinates gives rise to a classical Liouville integrable system on T^*S^3 when separating the HJ equation. Similarly, we get a quantum integrable system when separating the Schrödinger equation. These systems descend naturally to corresponding classical and quantum integrable systems on the compact symplectic reduced space $S^2 \times S^2$.

The classical first integrals are obtained using the method of Stäckel as are the quantum differential operators. The type of each quantum ODE is shown for each coordinate system in the right figure below. To obtain the joint spectrum we can use either Bargmann quantization or a Frobenius ansatz. The latter gives polynomial eigenfunctions.

It is interesting to note that since all the coordinates can be obtained as limits of the general elliptic coordinates, both the classical and quantum integrable systems corresponding to these coordinates are also degenerations of the most general ellipsoidal system.



Ellipsoidal Coordinates

General ellipsoidal coordinates on S^3 are defined as follows

$$x_i^2 = \frac{\prod_{j=1}^{3}(s_j - e_i)}{\prod_{i \neq j}(e_j - e_i)} \tag{1}$$

where $e_i \le s_i \le e_{i+1}$. The separation constants obtained from Stäckel are $2h = \sum_{j>i} \ell_{ij}^2$, $\eta_1 = \sum_{i<j}(e_n + e_m)\ell_{ij}^2$ and $\eta_2 = \sum_{i<j} e_n e_m \ell_{ij}^2$ where the indices m, n, i, j are all distinct.

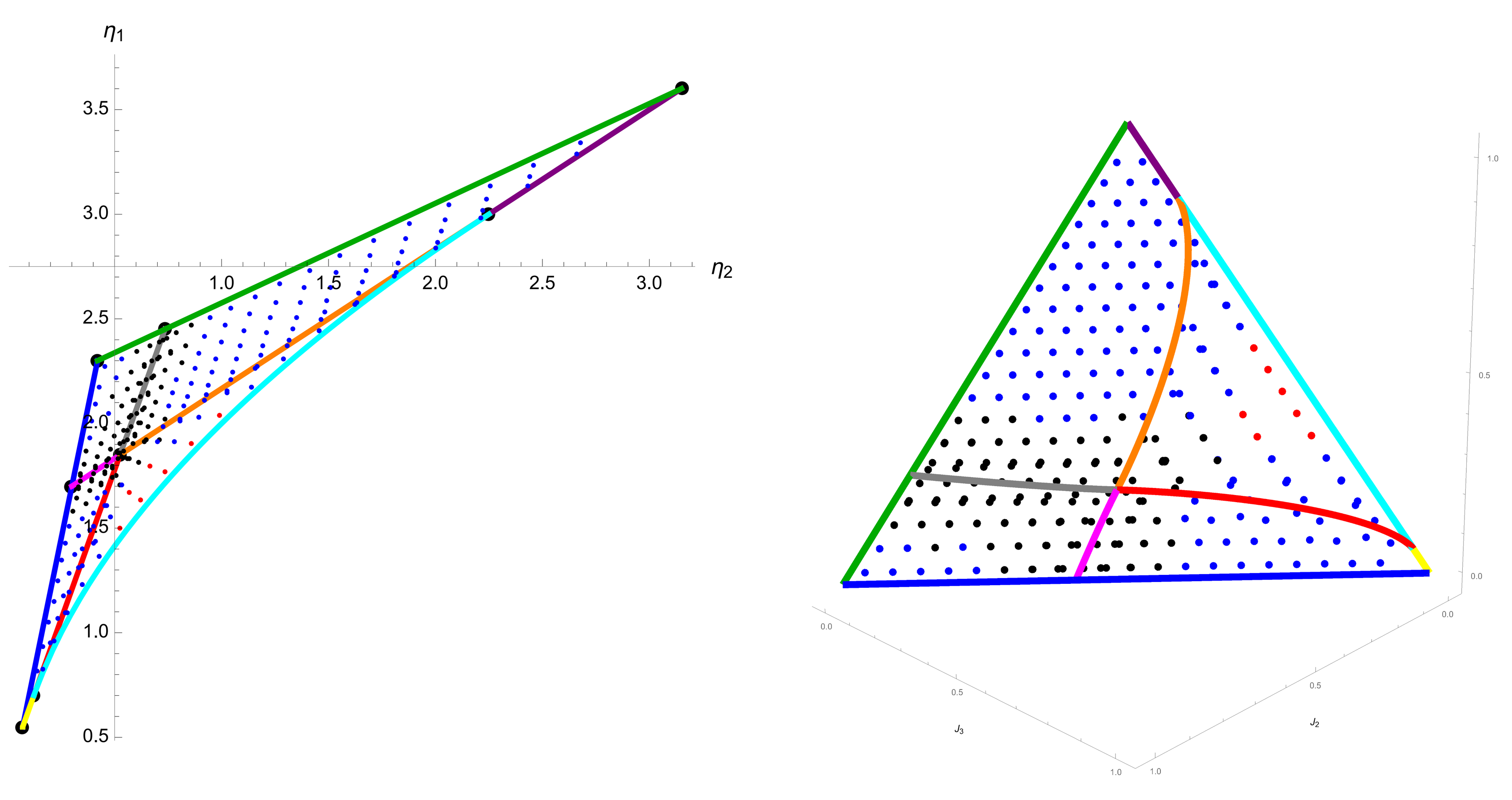
The separated momenta and clasical actions are given by

$$p_i^2 = -\frac{2hs_i^2 - \eta_1 s_i + \eta_2}{4\prod_{k=1}^{4}(s_i - e_k)} := \frac{-R(s_i)}{4A(s_i)} \qquad I_i = \frac{1}{2\pi}\int_{r_1}^{r_2}\sqrt{-\frac{R(s_i)}{A(s_i)}}\,ds_i$$

for $i \in \{1,2,3\}$, with r_1 and r_2 being the 2 roots of $R(z)$. We can show that:

Theorem 1: *The image of the action map is a convex polytope (an equilateral triangle in this case) whose boundary is defined by $I_1 + I_2 + I_3 = \sqrt{2h}$. This is true for all six coordnate systems.*

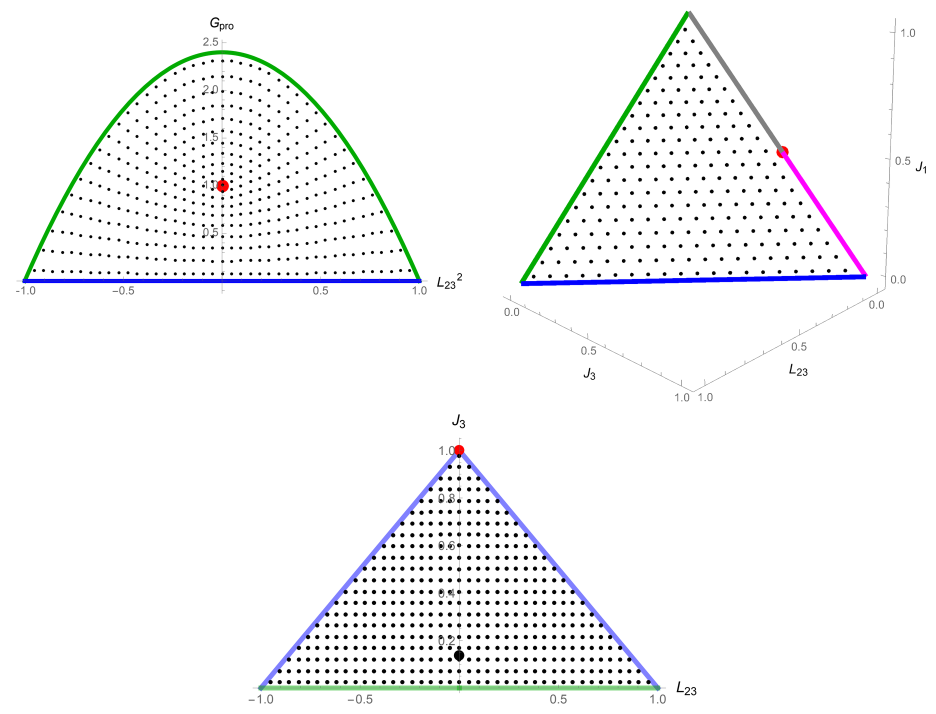
Below are the momentum and action maps with the quantum spectrum for the ellipsoidal coordinates.



Degenerate Systems

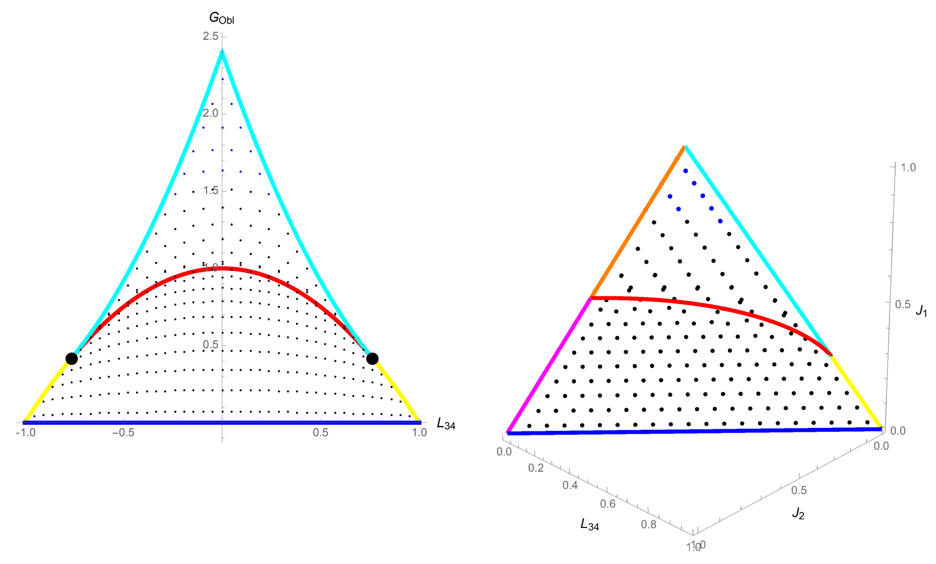
1. Prolate $e_2 = e_3$

Prolate coordinates give a semi-toric system with the angular momentum ℓ_{23} as one of the integrals.



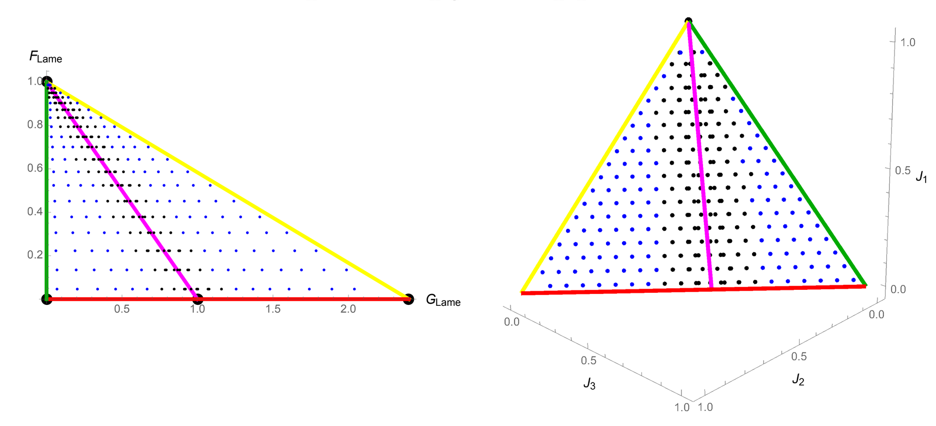
2. Oblate $e_3 = e_4$

The oblate Lamé coordinates also have angular momentum ℓ_{23} as a first integral. However, this system is not semi-toric since there are hyperbolic and degenerate points.



3. Lamé

Lamé reduction coordinates are extensions of coordinates on S^2 onto S^3. This is essentially equivalent to setting 3 of the semi-major axes e_i to be equal. It is interesting to note that this system is superintegrable with $F = \ell_{12}^2 + \ell_{13}^2 + \ell_{14}^2$ as the Hamiltonian.



4. Spherical/Cylindrical

The cylindrical coordinates are a further degeneration of the oblate coordinates with both $e_1 = e_2$ and $e_3 = e_4$. The system is toric with 2 global S^1 actions generated by ℓ_{14} and ℓ_{23}. Spherical cordinates are a further degeneration of Lamé coordinates.