Monodromy Surfaces of q-Painlevé Equations

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Introduction

The classical six Painlevé equations $P_{\mathrm{I}},\ldots,P_{\mathrm{VI}}$, were discovered around 1900 by Paul Painlevé and his colleagues as the canonical nonlinear 2nd order ODEs in the complex domain whose local solutions branch only at a fixed collection of points. For example, any solution of P_{VI},

$$u_{tt}=\left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-t}\right)\frac{u_t^2}{2}-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{u-t}\right)u_t+\frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{u^2}+\frac{\gamma(t-1)}{(u-1)^2}+\frac{\delta t(t-1)}{(u-t)^2}\right),$$

branches only at the points $t=0,1,\infty$ in the complex t-plane.

Each Painlevé equation $P_K: u_{tt}=R_K(u,u_t,t)$, $K=\mathrm{I},\ldots\mathrm{VI}$, is **integrable**: it has an associated linear system

$$Y_z=A_K(z;u,u_t,t)Y,$$

such that, as t moves, the **monodromy data** of the linear system is preserved.
The monodromy data form a complete set of **first integrals** of the corresponding Painlevé equation, which gives rise to a one-to-one correspondence between solutions and monodromy data,

$$\{\textit{solutions of } P_K\}\leftrightarrow\{\textit{monodromy data}\}.$$

Monodromy manifold

The collection of monodromy data

$$M=\{\textit{monodromy data}\}$$

is known as the **monodromy manifold**.
For the classical Painlevé equations, the monodromy manifolds can be identified with explicit **affine cubic surfaces**. For example, the affine cubic surface for Painlevé VI reads

$$\{\rho\in\mathbb{C}^3:\rho_1\rho_2\rho_3+\rho_1^2+\rho_2^2+\rho_3^2+w_1\rho_1+w_2\rho_2+w_3\rho_3+w_4=0\},$$

where the coefficients w_k, $1\leq k\leq 4$, are related to the parameters $\alpha,\beta,\gamma,\delta$. So, for (generic) fixed parameter values, solutions of Painlevé VI are in one-to-one correspondence with points on this cubic surface.

Questions

During the 90ties, many discrete Painlevé equations were derived and ultimately classified by Sakai [7] in 2001. What do their monodromy manifolds look like? Can they be identified with algebraic surfaces?

In [5] we derive an explicit algebraic surface for the monodromy manifold of q-Painlevé VI. The results of this paper are presented in this poster. We also mention the corresponding result in [4] for q-Painlevé IV.

q-Painlevé VI

Let $q\in\mathbb{C}$ with $0<|q|<1$. The q-Painlevé VI equation is given by

$$qP_{\mathrm{VI}}:\begin{cases} f\overline{f} &=\dfrac{(\overline{g}-\kappa_0t)(\overline{g}-\kappa_0^{-1}t)}{(\overline{g}-\kappa_\infty)(\overline{g}-q^{-1}\kappa_\infty^{-1})},\\ g\overline{g} &=\dfrac{(f-\kappa_tt)(f-\kappa_t^{-1}t)}{q(f-\kappa_1)(f-\kappa_1^{-1})},\end{cases}$$

- $f=f(t)$, $\overline{f}=f(qt)$, $g=g(t)$ for t in a domain $T\subseteq\mathbb{C}^*$ with $qT=T$.
- $\kappa=(\kappa_0,\kappa_t,\kappa_1,\kappa_\infty)\in\mathbb{C}^4$ are nonzero complex parameters.

Here, we consider the class of solutions with domain given by a discrete q-spiral,

$$t\in T,\quad T=q^{\mathbb{Z}}t_0=\{\ldots,q^{+2}t_0,q^{+1}t_0,t_0,q^{-1}t_0,q^{-2}t_0,\ldots\},$$

for some $t_0\in\mathbb{C}^*$. We call (f,g) a solution of $qP_{\mathrm{VI}}(\kappa,t_0)$.

We assume **non-resonance** conditions,

$$\kappa_0^2,\kappa_t^2,\kappa_1^2,\kappa_\infty^2\notin q^{\mathbb{Z}},\qquad(\kappa_t\kappa_1)^{\pm1},(\kappa_t/\kappa_1)^{\pm1}\notin t_0q^{\mathbb{Z}}.$$

and **non-splitting** conditions

$$\kappa_0^{\epsilon_0}\kappa_t^{\epsilon_t}\kappa_1^{\epsilon_1}\kappa_\infty^{\epsilon_\infty}\notin q^{\mathbb{Z}},\quad\kappa_0^{\epsilon_0}\kappa_\infty^{\epsilon_\infty}\notin t_0q^{\mathbb{Z}},$$

where $\epsilon_j\in\{\pm1\}$, $j=0,t,1,\infty$.

The Jimbo-Sakai linear system

The associated Jimbo-Sakai linear system [2] is given by

$$Y(qz)=A(z,t)Y(z),\quad A(z,t)=A_0+zA_1+z^2\begin{pmatrix}\kappa_\infty^{+1} & 0\\ 0 & \kappa_\infty^{-1}\end{pmatrix},$$

where

- $|A(z,t)|=(z-\kappa_t^{+1}t)(z-\kappa_t^{-1}t)(z-\kappa_1^{+1})(z-\kappa_1^{-1})$,
- $A_0=H\begin{pmatrix}\kappa_0^{+1}t & 0\\ 0 & \kappa_0^{-1}t\end{pmatrix}H^{-1}$, for some $H(t)\in GL_2(\mathbb{C})$,

with coordinates $\{f,g,w\}$ defined by

$$A_{12}(z,t)=\kappa_\infty^{-1}w(z-f),$$
$$A_{22}(f,t)=q(f-\kappa_1)(f-\kappa_1^{-1})g.$$

This system is **Fuchsian** and, due to Birkhoff's classical treatment [1] of such Fuchsian systems, the corresponding monodromy is encapsulated by a single connection matrix which relates the following **convergent** series solutions respectively near $z=0$ and $z=\infty$,

$$Y_0(z,t)=z^{\log_q(t)}\Psi_0(z,t)z^{\log_q(\kappa_0)\sigma_3},\qquad\Psi_0(z,t)=H_0(t)+\sum_{n=1}^{\infty}z^nM_n(t),$$

$$Y_\infty(z,t)=z^{\log_q(z)-1}\Psi_\infty(z,t)z^{\log_q(\kappa_\infty)\sigma_3},\qquad\Psi_\infty(z,t)=I+\sum_{n=1}^{\infty}z^{-n}N_n(t),$$

where $\sigma_3=\mathrm{diag}(1,-1)$ is the standard third Pauli spin matrix.
We define the matrix functions

$$P(z,t)=Y_0(z,t)^{-1}Y_\infty(z,t),\quad C(z,t)=\Psi_0(z,t)^{-1}\Psi_\infty(z,t),$$

and call $C(z,t)$ the **connection matrix** or **monodromy** of the linear system.

Substitute a solution (f,g) of qP_{VI} into the linear system and let w satisfy the auxiliary equation

$$\frac{\overline{w}}{w}=\frac{q\kappa_\infty\overline{g}-1}{\overline{g}-\kappa_\infty},$$

then $P(z,t)$ and $C(z,t)$, satisfy

$$P(z,t_m)=P(z,t_0),\quad C(z,t_m)=z^mC(z,t_0),\quad(t_m=q^mt_0),$$

i.e. monodromy is preserved as $t\to qt$ [2].

The monodromy manifold

The connection matrix $C_0(z):=C(z,t_0)$ satisfies

1. $C_0(z)$ is analytic on $\mathbb{C}^*$.
2. $C_0(qz)=z^{-2}\begin{pmatrix}t_0\kappa_0 & 0\\ 0 & t_0\kappa_0^{-1}\end{pmatrix}C_0(z)\begin{pmatrix}\kappa_\infty^{-1} & 0\\ 0 & \kappa_\infty\end{pmatrix}$.
3. $|C_0(z)|=\text{constant}\times\theta_q(\kappa_t^{+1}z/t_0)\theta_q(\kappa_t^{-1}z/t_0)\theta_q(\kappa_1^{+1}z)\theta_q(\kappa_1^{-1}z)$.

We define the **monodromy manifold** $\mathcal{M}(\kappa,t_0)$ as the space of matrices $C_0(z)$ satisfying (1)-(3), quotiented by arbitrary left and right-multiplication by diagonal matrices.

The monodromy manifold $\mathcal{M}(\kappa,t_0)$ was the object of an extensive recent study by Ohyama et al. [6]. Amongst other things, they showed that $\mathcal{M}(\kappa,t_0)$ is an algebraic surface and conjectured that it is smooth. This conjecture follows from the third theorem below.

Results

Theorem [5]

The **monodromy mapping**

$$\{\text{solutions of } qP_{\mathrm{VI}}(\kappa,t_0)\}\to\mathcal{M}(\kappa,t_0),$$

maps the solution space of $qP_{\mathrm{VI}}(\kappa,t_0)$ bijectively onto the monodromy manifold.

For any 2×2 matrix $R\neq0$, with $|R|=0$, define

$$\pi(R)\in\mathbb{CP}^1:\quad R_1=\pi(R)R_2,\quad R=(R_1,R_2).$$

We define **coordinates** $\rho=(\rho_1,\rho_2,\rho_3,\rho_4)$ on the monodromy manifold through

$$\rho_k=\pi(C_0(x_k))\quad(1\leq k\leq4),\quad(x_1,x_2,x_3,x_4):=(\kappa_tt_0,\kappa_t^{-1}t_0,\kappa_1,\kappa_1^{-1}).$$

These coordinates ρ lie in $(\mathbb{CP}^1)^4/\mathbb{C}^*$.

Theorem [5]

The coordinates ρ yield an embedding of the monodromy surface into $(\mathbb{CP}^1)^4/\mathbb{C}^*$. The image of this embedding is given by the surface $\mathcal{S}^*(\kappa,t_0)$ in $(\mathbb{CP}^1)^4/\mathbb{C}^*$, defined as the zero set

$$T_{12}\rho_1\rho_2+T_{13}\rho_1\rho_3+T_{14}\rho_1\rho_4+T_{23}\rho_2\rho_3+T_{24}\rho_2\rho_4+T_{34}\rho_3\rho_4=0,\qquad(1)$$

where

$$T_{12}=\theta_q\left(\kappa_t^2,\kappa_1^2\right)\theta_q\left(\kappa_0\kappa_\infty^{-1}t_0,\kappa_0^{-1}\kappa_\infty^{-1}t_0\right)\kappa_\infty^2,$$
$$T_{34}=\theta_q\left(\kappa_t^2,\kappa_1^2\right)\theta_q\left(\kappa_0\kappa_\infty t_0,\kappa_0^{-1}\kappa_\infty t_0\right),$$
$$T_{13}=-\theta_q\left(\kappa_t\kappa_1^{-1}t_0,\kappa_t^{-1}\kappa_1t_0\right)\theta_q\left(\kappa_t\kappa_1\kappa_0^{-1}\kappa_\infty^{-1},\kappa_0\kappa_t\kappa_1\kappa_\infty^{-1}\right)\kappa_\infty^2,$$
$$T_{24}=-\theta_q\left(\kappa_t\kappa_1^{-1}t_0,\kappa_t^{-1}\kappa_1t_0\right)\theta_q\left(\kappa_0\kappa_t\kappa_1\kappa_\infty,\kappa_t\kappa_1\kappa_\infty\kappa_0^{-1}\right),$$
$$T_{23}=\theta_q\left(\kappa_t\kappa_1t_0,\kappa_t^{-1}\kappa_1^{-1}t_0\right)\theta_q\left(\kappa_t\kappa_\infty\kappa_0^{-1}\kappa_1^{-1},\kappa_0\kappa_t\kappa_\infty\kappa_1^{-1}\right)\kappa_1^2,$$
$$T_{14}=\theta_q\left(\kappa_t\kappa_1t_0,\kappa_t^{-1}\kappa_1^{-1}t_0\right)\theta_q\left(\kappa_1\kappa_\infty\kappa_0^{-1}\kappa_t^{-1},\kappa_0\kappa_1\kappa_\infty\kappa_t^{-1}\right)\kappa_t^2,$$

minus a closed curve $\mathcal{X}$ defined by the intersection of (1) as κ_0 varies over $\mathbb{C}^*$.
In particular,

$$\mathcal{M}(\kappa,t_0)\to\mathcal{S}^*(\kappa,t_0),\ \textit{where }[C_0]\mapsto[\rho],$$

is a bijection.

Corollary

The general solution of $qP_{\mathrm{VI}}(\kappa,t_0)$ can be parametrised by

$$f(t)=f(t;\kappa,t_0,\mathrm{p}),$$
$$g(t)=g(t;\kappa,t_0,\mathrm{p}),$$

with p varying in $\mathcal{S}^*(\kappa,t_0)$.

Theorem [5]

The monodromy manifold $\mathcal{M}(\kappa,t_0)$, or equivalently the surface $\mathcal{S}^*(\kappa,t_0)$, is a smooth and affine algebraic surface.

q-Painlevé IV [4]

Joshi and Nakazono [3] derived a linear system for q-Painlevé IV which is Fuchsian. Similarly to the above, we identified the corresponding monodromy manifold with an algebraic surface in $(\mathbb{CP}^1)^3$, defined through

$$\begin{aligned}&+\theta_q(+a_0,+a_1,+a_2)\left(\theta_q(t_0)\rho_1\rho_2\rho_3-\theta_q(-t_0)\right)\\&-\theta_q(-a_0,+a_1,-a_2)\left(\theta_q(t_0)\rho_1-\theta_q(-t_0)\rho_2\rho_3\right)\\&+\theta_q(+a_0,-a_1,-a_2)\left(\theta_q(t_0)\rho_2-\theta_q(-t_0)\rho_1\rho_3\right)\\&-\theta_q(-a_0,-a_1,+a_2)\left(\theta_q(t_0)\rho_3-\theta_q(-t_0)\rho_1\rho_2\right)=0,\end{aligned}$$

minus a closed curve (defined by intersection over t_0).

References

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