Dynamics of Solitons, Breathers, and Rogue Waves in Inhomogeneous Media: Similarity Transformation to Explore Variable-coefficient Nonlinear Model Equations

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Abstract

The coupled nonlinear Schrödinger type equations with varying dispersion and (self-phase & cross-phase modulations and four-wave mixing) nonlinearities, which govern the dynamics of beam propagation inhomogeneous optical media, are exciting objects of study. We discuss similarity transformation, an efficient yet simplest tool to explore such models for bringing out the dynamical characteristics of their underlying nonlinear waves. In this demonstration, we investigate the role of modulated nonlinearities in the dynamics of vector optical solitons, Akhmediev/Ma breathers, and rogue waves with the help of the explicit analytical solution. We highlight the possibility of controlling these nonlinear waves possessing different characteristics.

This poster is based on the articles
(i) J. Phys. A: Math. Theor. 53 (2020) 415701 &
(ii) Phys. Scr. 95 (2020) 095202.

1. Introduction

- ❖The Nonlinear Schrödinger (NLS) equation: pulse/beam propagation in optic fiber & waveguides [1].
- ❖The Gross-Pitaevskii (GP) equation: properties of Bose–Einstein condensates (BECs) at ultra low temperatures [2].
- ❖Generation of different localized waves are possible [1-15].

- ❖*What happens when the medium is inhomogeneous?* [1-9]
- ❖Dynamics of beam propagation in an inhomogeneous Kerr-like nonlinear optical medium.
- ❖Feshbach resonance mechanism with a non-uniform magnetic field in (i) binary & (ii) spinor BECs.
- ❖Solitons in a dispersion managed erbium doped inhomogeneous fiber with gain/loss.
- ❖Dispersion managed solitons in fibers with random birefringence.
- ❖Light bullets in strongly modified by dispersion and nonlinearities.
- ❖Non-autonomous solitons in planar grating waveguides.
- ❖Dissipative managed solitons in graded-index waveguides & optical fiber systems.
- ❖Self-similar optical waves in graded-index layered media.
- ❖The non-autonomous nonlinear waves in 1D NLS type systems and in HD models with modulated dispersion and nonlinearities with gain/loss and higher-order effects.
- ❖Multi-component optical systems such as multi-mode fibers.

2. The Mathematical Model

- ❖The general nonlinear Schrödinger model: Dynamics of beam propagation temporally inhomogeneous optical fiber [1,2,10-13].

- ❖The inhomogeneous coherently coupled nonlinear Schrödinger (ICCNLS) equation consisting of temporally varying (self-phase & cross-phase modulations and four-wave mixing) nonlinearities.

$$i\frac{\partial A_1}{\partial z} + \left(\delta\frac{\partial^2}{\partial x^2} + v(x,z)\right)A_1 + \gamma(z)[\sigma_{11}|A_1|^2 + \sigma_{12}|A_2|^2]A_1 - \delta_1\gamma(z)A_2^2A_1^* = 0, \quad (1a)$$

$$i\frac{\partial A_2}{\partial z} + \left(\delta\frac{\partial^2}{\partial x^2} + v(x,z)\right)A_2 + \gamma(z)[\sigma_{21}|A_1|^2 + \sigma_{22}|A_2|^2]A_2 - \delta_2\gamma(z)A_1^2A_2^* = 0, \quad (1b)$$

- ❖Two optical modes $A_1(z,x)$ and $A_2(z,x)$.
- ❖δ: constant dispersion
- ❖σ_{11} & σ_{22}: Self-phase modulation nonlinearity coefficient
- ❖σ_{12} & σ_{21}: Cross-phase modulation coefficient
- ❖δ_1 & δ_2: Four-wave mixing coefficient
- ❖Refractive index in inhomogeneous media [4-9].
  $n(z,x)=n_0+n_1\, v(z,x) + n_2\gamma(z)I(z,x)$
- ❖Temporally varying nonlinearities $\gamma(z)$.
- ❖Linear variation in refractive index $v(z,x) = F(z)x^2/2$.

❖2-CCNLS systems with positive coherent coupling:
$\delta = -1, \sigma_{11} = \sigma_{22} = -1, \sigma_{12} = \sigma_{21} = -2, \delta_1 = \delta_2 = 1,$
$\delta = 1,\ \sigma_{11} = \sigma_{22} = -1, \sigma_{12} = \sigma_{21} = -2, \delta_1 = \delta_2 = 1,$
$\delta = 1,\ \sigma_{11} = \sigma_{22} = 1, \sigma_{12} = \sigma_{21} = 2, \delta_1 = \delta_2 = -1.$

❖2-CCNLS systems with negative coherent coupling:
$\delta = 1,\ \sigma_{11} = \sigma_{22} = 1, \sigma_{12} = \sigma_{21} = 2, \delta_1 = \delta_2 = 1,$
$\delta = -1, \sigma_{11} = \sigma_{22} = -1, \sigma_{12} = \sigma_{21} = -2, \delta_1 = \delta_2 = -1$

❖2-CCNLS system with mixed coherent coupling:
$\delta = 1, \sigma_{11} = -\sigma_{22} = 1, \sigma_{12} = -\sigma_{21} = -2, \delta_1 = -\delta_2 = 1$

❖***Effects of temporally-varying nonlinearities $\gamma(z)$ in the dynamics of solitons, breathers & rogue waves?***

3. Methodology: Similarity Transformation

- ➢A simple mathematical tool with wide applicability and show rich characteristics [1,2,10-13].
- ➢Constant-coefficient CCNLS equations: (Homogeneous System)

$$iQ_{1,Z} + \delta Q_{1,XX} + (\sigma_{11}|Q_1|^2 + \sigma_{12}|Q_2|^2)Q_1 - \delta_1 Q_2^2 Q_1^* = 0, \quad (2a)$$

$$iQ_{2,Z} + \delta Q_{2,XX} + (\sigma_{21}|Q_1|^2 + \sigma_{22}|Q_2|^2)Q_2 - \delta_2 Q_1^2 Q_2^* = 0. \quad (2b)$$

- ➢Similarity Transformation [15]:

$$A_j(x,z) = \rho(z)\, Q_j(X(x,z), Z(z)) \exp[i\zeta(x,z)], \qquad j = 1,2 \quad (3a)$$

- ➢Relationship among parameters:

$$\zeta(x,z) = -\frac{1}{4\delta}\frac{d}{dz}(\ln\gamma)x^2 + \epsilon_1^2\epsilon_2\gamma x - \delta\epsilon_2^2\epsilon_1^4\int\gamma^2 dz, \quad (3b)$$

$$X(x,z) = \epsilon_1\left(\gamma x - 2\delta\epsilon_2\epsilon_1^2\int\gamma^2 dz\right), \quad (3c)$$

$$Z(z) = \epsilon_1^2\int\gamma^2 dz, \quad (3d)$$

$$\rho(z) = \epsilon_1\sqrt{\gamma(z)}, \quad (3e)$$

- ➢With a relation between external potential and nonlinearity coefficient: Riccati equation

$$F(z) = \frac{1}{\delta}\left(\frac{\gamma_z^2}{\gamma^2} - \frac{\gamma_{zz}}{2\gamma}\right). \quad (3f)$$

- ➢***The dynamics of nonlinear waves associated with inhomogeneous model (1) can be explored using the known solutions of associated homogeneous model (2) through similarity transformation (3)!***

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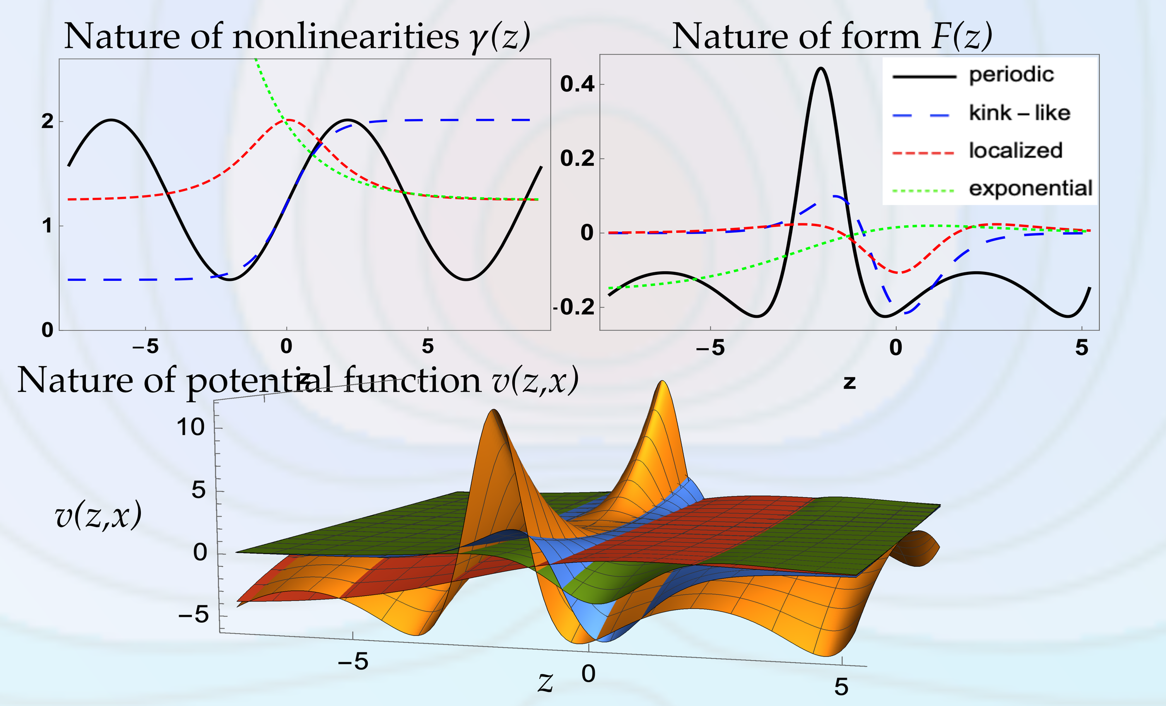
4. Nature of Inhomogeneities

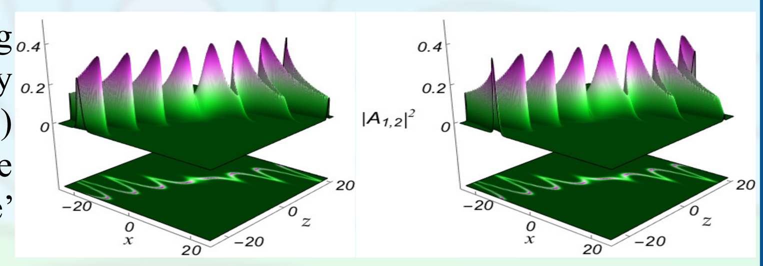
- ➢Different types of nonlinearities are possible based on the nature of medium.
- ➢Periodic modulation: $\gamma(z)=a_1+a_2\sin(a_3z+a_4)$ or $\gamma(z)=a_1+a_2\cos(a_3z+a_4)$
- ➢Localized modulation: $\gamma(z)=a_1+a_2\,\mathrm{sech}^2(a_3z+a_4)$
- ➢Kink-like modulation: $\gamma(z)=a_1+a_2\tanh(a_3z+a_4)$
- ➢Exponential (growth/decay) modulation: $\gamma(z)=a_1+a_2\exp(a_3z+a_4)$

- ➢Inhomogeneities enforce characteristic changes in the dynamics of the associated system (waves propagating through the medium).
- ➢Especially, variation in the Amplitude, velocity (position), interaction nature, etc.

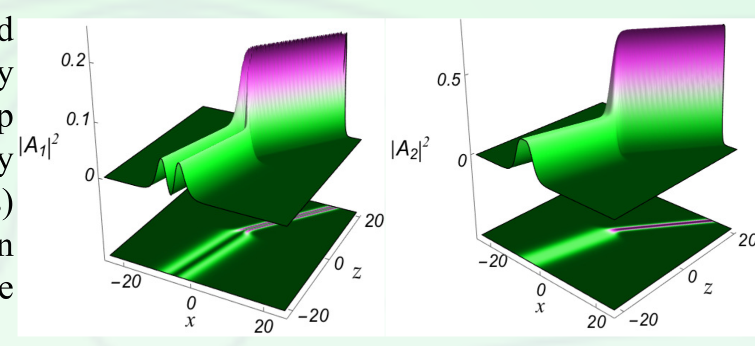


5. Modulated Vector Optical Solitons

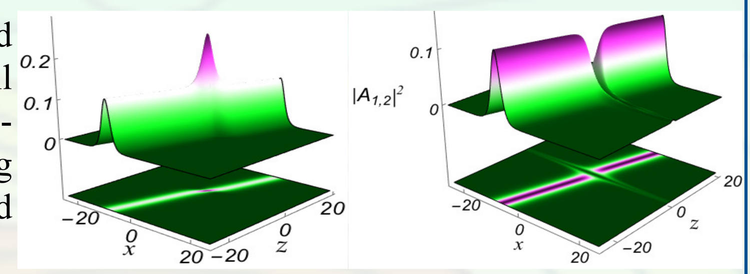
❖ Periodically oscillating inhomogeneous incoherently coupled solitons (IICSs) having single-hump profile with 'sine' and 'cosine' nonlinearities [15].



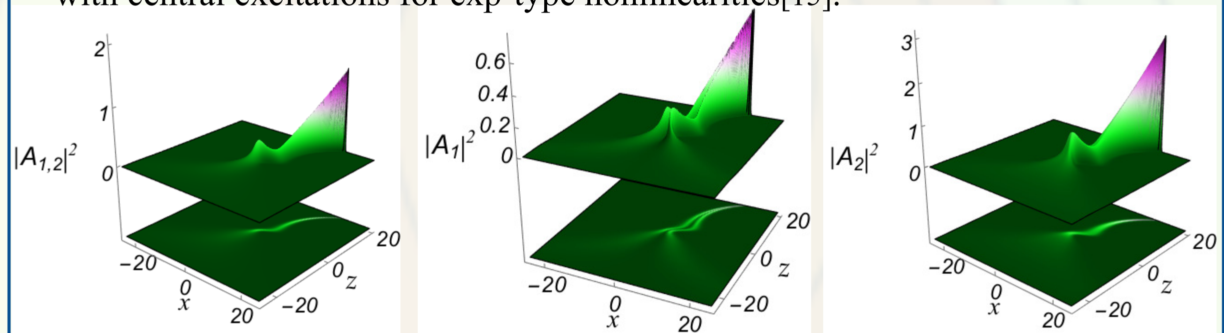
❖ Amplification accompanied by compression in intensity of double-hump & flat-top inhomogeneous coherently coupled solitons (ICCSs) with deflection in direction (velocity) due to kink-line nonlinearity.



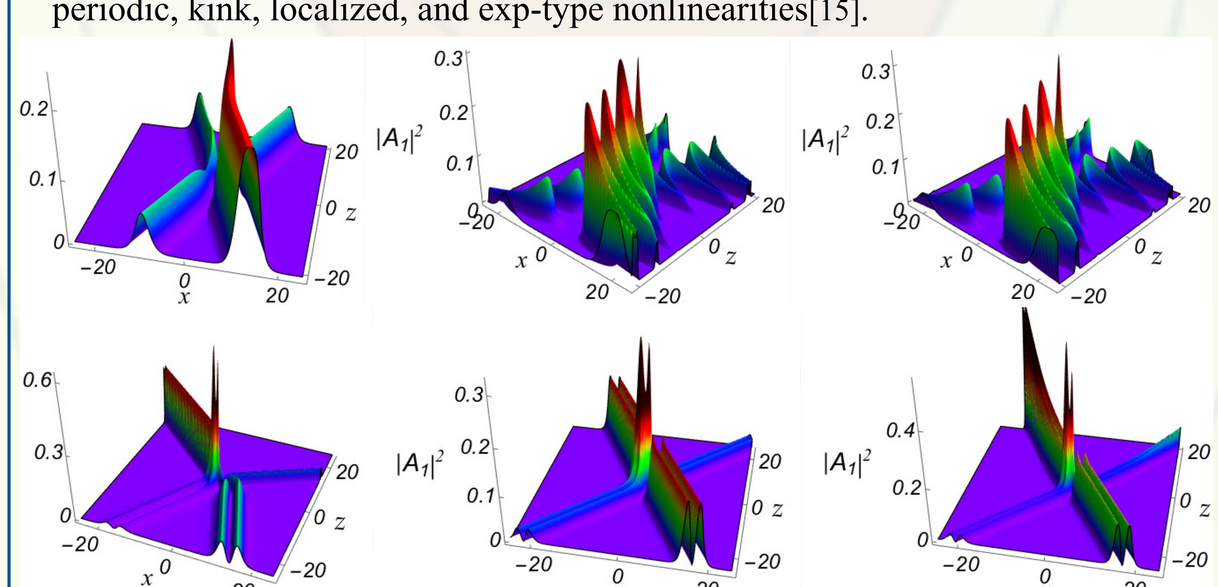
❖ Tunneling through a localized barrier and cross-over a well in the intensity of single-hump ICCSs with narrowing & broadening due to localized sech-type nonlinearities.



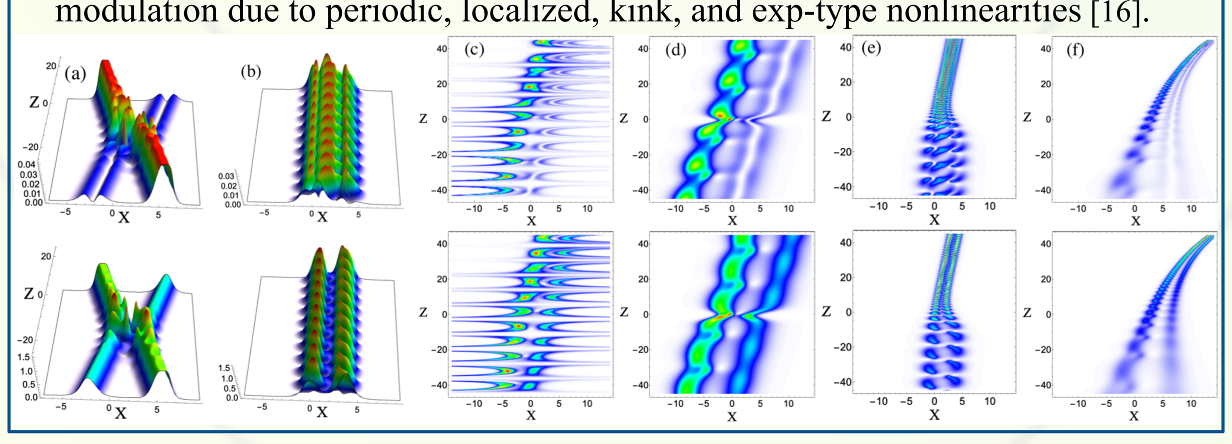
❖ Exponential growth of single-hump IICSs and double-hump & flat-top ICCSs with central excitations for exp-type nonlinearities[15].



❖ Transition of energy (intensity) switching collision of ICCS with IICSs with periodic, kink, localized, and exp-type nonlinearities[15].

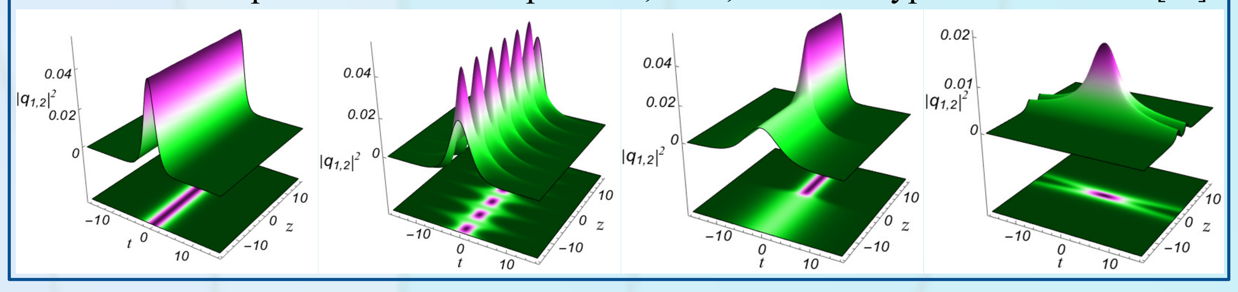


❖ Bound soliton molecule formation through a transition from collision and their modulation due to periodic, localized, kink, and exp-type nonlinearities [16].



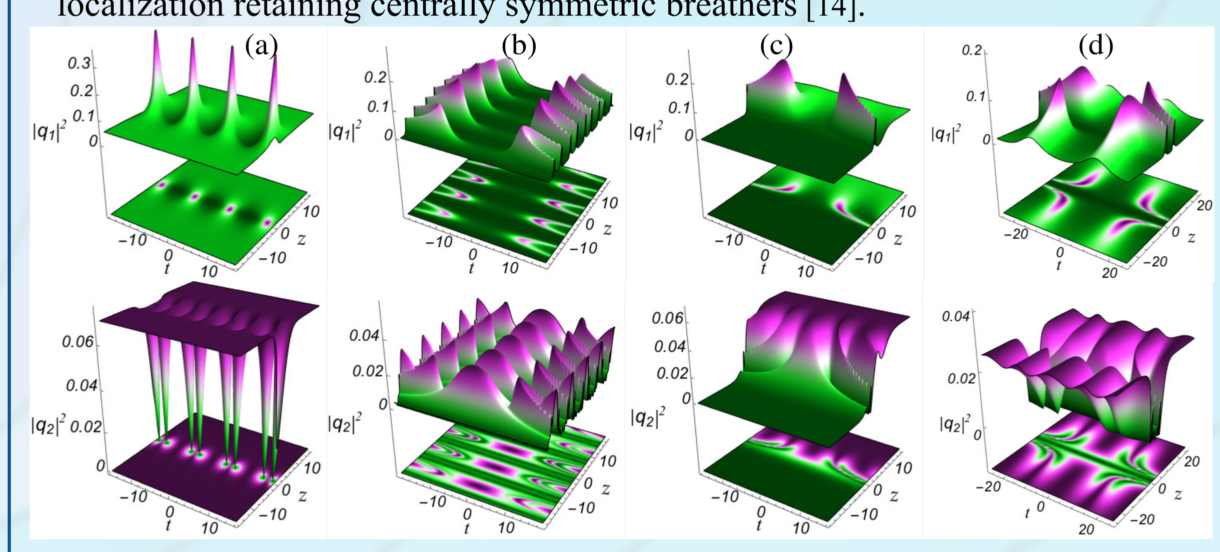
6. Modulated Optical Solitons

- ❖ $A_1(z,x) \to q_1(z,t); A_2(z,x) \to q_2(z,t)$
- ❖ Transition of stable bright optical soliton to a breather, amplified soliton, and localized lump formation due to periodic, kink, and sech-type nonlinearities [14].

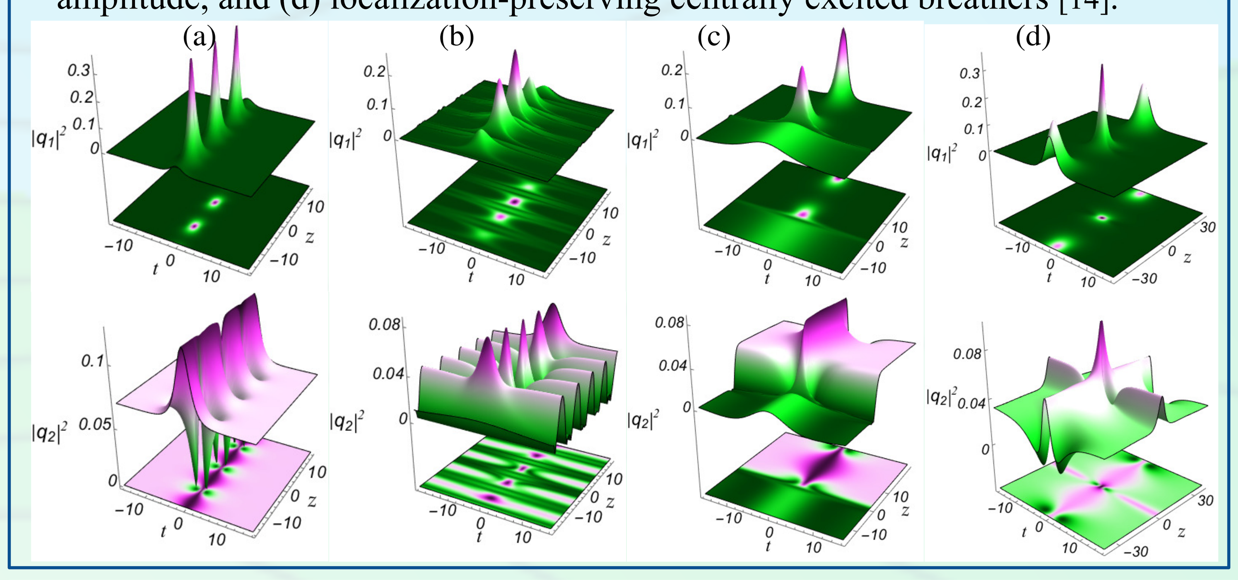


7. Modulated Optical Breathers

❖ Dynamics of bright-dark Akhmediev breathers with (a) constant nonlinearity and their transformation due to (b) periodic, (c) kink-like, and (d) localized well-type nonlinearities revealing (b) localization-broken doubly-periodic breathers, (c) escalated background amplitude time-periodic breathers, and (d) localization retaining centrally symmetric breathers [14].

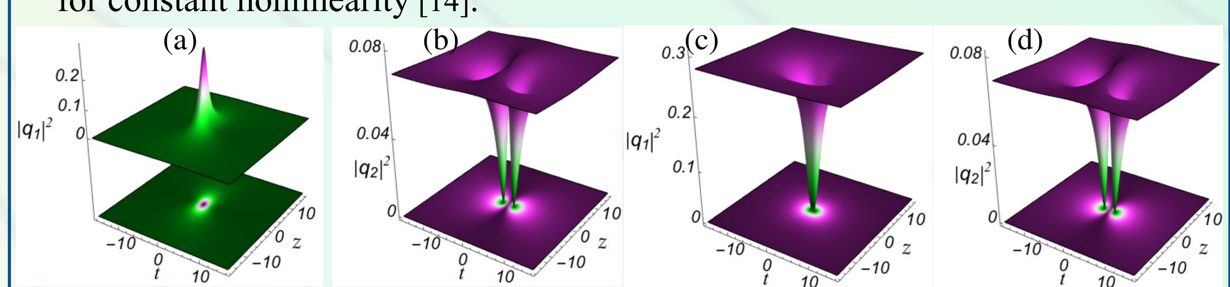


❖ Dynamics of Kuznetsov-Ma breathers with (a) constant nonlinearity and their modulation due to (b) periodic, (c) kink-like, and (d) well-type nonlinearities depicting (b) localization-broken, (c) localization-sustaining escalated amplitude, and (d) localization-preserving centrally excited breathers [14].

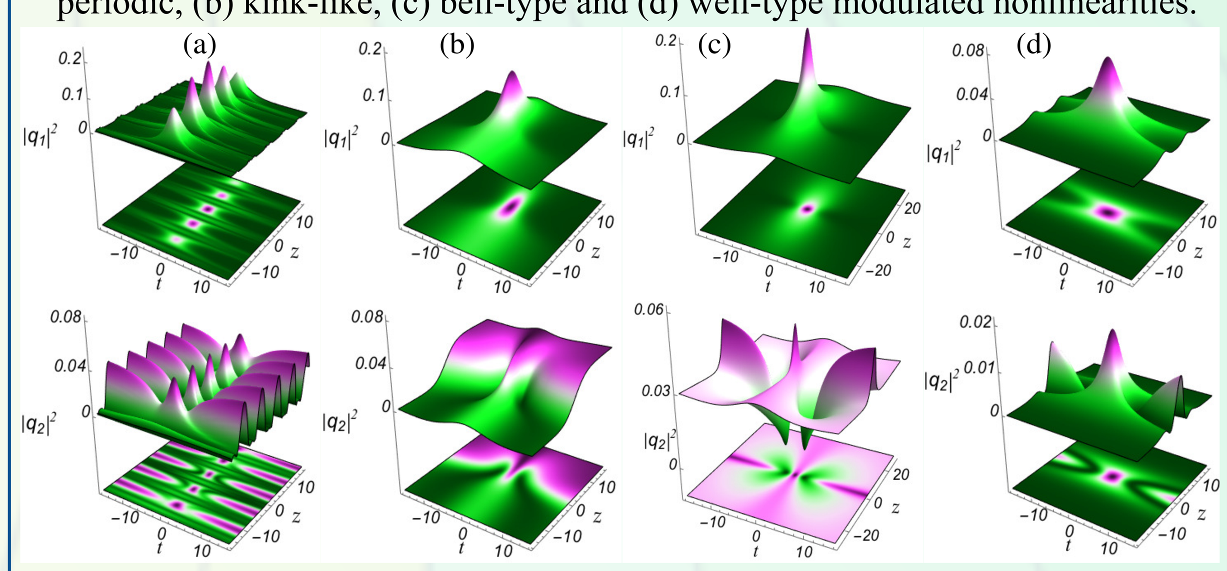


8. Modulated Rogue Waves

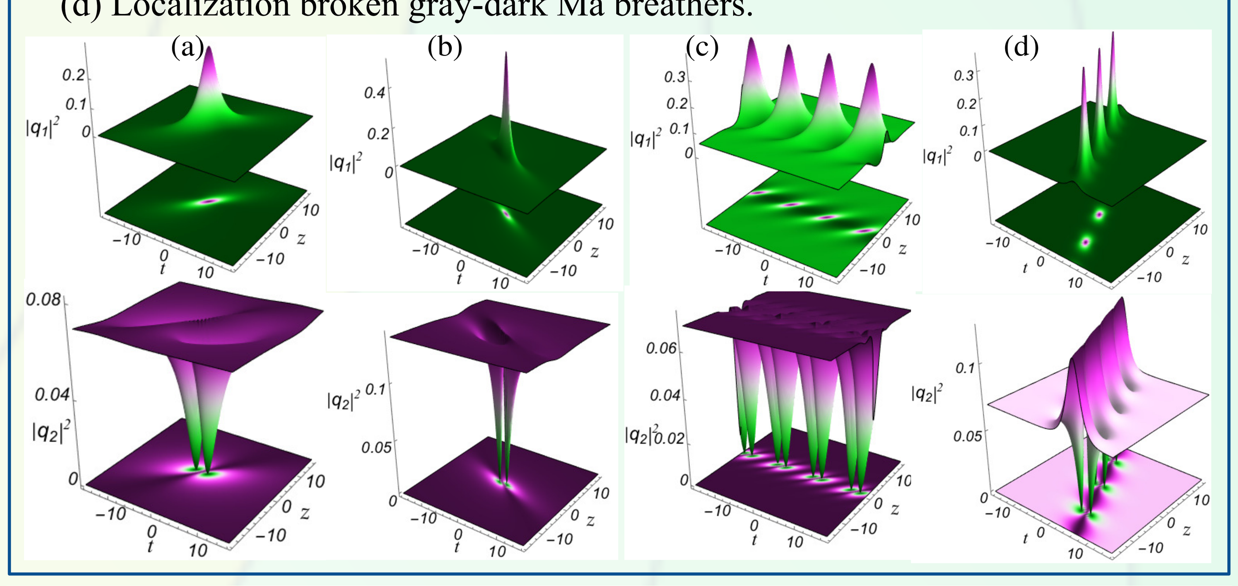
❖ Dynamics of (a-b) bright-dark rogue waves and (c-d) dark-dark rogue waves for constant nonlinearity [14].



❖ Transition of bright-dark rogue waves to (a) A-shaped & M-shaped periodic wave trains, (b) localized amplification with compression, (c) tunneling through a barrier, (d) exciton with side band formation due to modulated (a) periodic, (b) kink-like, (c) bell-type and (d) well-type modulated nonlinearities.



❖ Role of the similarity (ϵ_1 and ϵ_2) parameters: (a-b) Rotation (changing the inclination) of single-hump bright and double-well dark rogue waves. (c) Localization preserving gray-dark Akhmediev breathers. (d) Localization broken gray-dark Ma breathers.



Summary & Conclusion

- ➢ Nonlinear waves in inhomogeneous media shows interesting dynamics.
- ➢ Possible to understand and manipulate the behaviour of waves.
- ➢ Studies on nonlinear waves in inhomogeneous media is continue to be worth exploring due to their occurrence in various contexts and phenomena.
- ➢ Due to the availability of limited analytical tools, identification of new methods needs further investigation.

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