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Superintegrable Hamiltonian Systems on Conformal Manifolds

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Basic definitions

- (M^n, g) (pseudo-)Riemannian manifold, Hamiltonian $H = g_{\mathbf{x}}(\mathbf{p},\mathbf{p}) + V(\mathbf{x})$.
- A (second-order) conformally superintegrable system is a Hamiltonian H together with $2n-1$ functionally independent (conformal) integrals $F = C^{ij}(x)p_ip_j + W(x)$ on the phase-space, i.e.

$$\dot{F} = \{H, F\} = \omega\, H\,. \qquad (1)$$

  for some $\omega = \omega^i p_i$ linear in momenta $\mathbf{p}$.
- If $\omega = 0$ then the system is a proper, i.e. superintegrable in the usual sense.
- Equation (1) is equivalent to the following:
  1. The condition that the C_{ij} are components of a conformal Killing tensor.
  2. The integrability condition for W

$$0 = ddW = dCdV + dV \wedge \omega + V d\omega\,,$$

  which is called Bertrand-Darboux condition.

Overview

To date, non-degenerate conformally superintegrable systems are classified in dimensions 2 and 3. We find

- A framework for an algebraic-geometric classification of (conformally) superintegrable systems, particularly in dimensions $n \geq 3$.
- A definition of superintegrability on conformal manifolds ('c-superintegrability'), and conformally invariant conditions for superintegrability.
- New obstructions in dimension $n \geq 4$ (see blue box).
- Properly superintegrable systems on constant curvature spaces correspond to eigenfunctions of the Laplacian. The eigenvalue is given by the curvature of the manifold.
- Abundant systems only exist on conformally flat manifolds.
- The generic system cannot be realised as a flat properly superintegrable system (known in 2D,3D, new in $n \geq 4$).

Conformal transformations

Stäckel transformations aka coupling constant metamorphosis (CCM) are a well-known equivalence relation on properly superintegrable systems.

Classical formulation:
Let $H_\alpha = g(\mathbf{p},\mathbf{p}) + V + \alpha U$ admit the integral $F_\alpha = C(\mathbf{p},\mathbf{p}) + W_\alpha$.
Then $\tilde{H}_c = \frac{H+c}{U}$ admits the integral

$$\tilde{F}_c = F(\tilde{H}_c).$$

$\Rightarrow$ Equivalence relation on superintegrable systems.

- If $V = 0$, we obtain the classical Maupertuis-Jacobi transformation (going back to the 1700s).
- In conformally superintegrable systems, we work with integrals of the form

$$F = C^{ij}(\mathbf{x})p_ip_j + W(\mathbf{x})$$

  such that C_{ij} are components of a trace-free conformal Killing tensor.
- Such conformal integrals are invariant under conformal transformations. The Hamiltonian transforms as a density of conformal weight -2.

Non-degeneracy

means the existence of a $(n+2)$-parameter family of Hamiltonians

$$H(\mathbf{x},\mathbf{p}) = g(\mathbf{p},\mathbf{p}) + \sum_k c_k V^{(k)}$$

compatible with the same conformal integrals.

- The integrability conditions for the potential are trivially satisfied.
- Non-degenerate systems are classified to date in dimension 2 and 3.
- In 2D: degenerate systems are restrictions of abundant systems.

Associated conformal Killing tensors

- The kinetic parts of the conformal integrals $F = C(\mathbf{p},\mathbf{p}) + W$ are conformal Killing tensor fields,

$$C_{(ij,k)} = g_{(ij}\omega_{k)}.$$

- We say that a conformally superintegrable system is irreducible if its associated Killing tensors are irreducible. Such systems satisfy

$$V_{,ij} = T_{ij}{}^k V_{,k} + \tau_{ij}\,V + \frac{1}{n}g_{ij}\,\Delta V.$$

  All known non-degenerate systems are irreducible.
- For non-degenerate systems, covariant derivatives of trace-free conformal Killing tensors associated to non-degenerate systems satisfy

$$C_{ij,k} = P_{ijkmn}C_{mn}$$

  for a certain tensor P. *Such a relation does not exist for generic conformal Killing tensors.*

Obstruction equation

An abundant superintegrable system is described by a trace-free totally symmetric tensor field S_{ijk} and a conformal scale function σ (i.e. a tensor density of weight 1). The structure tensor S_{ijk} is conformally equivariant (of weight 2), and unique for non-degenerate systems.

- The underlying metric has to be conformally flat.
- A c-superintegrable system corresponds to a solution of

$$(\Psi_{ija}\Psi^a{}_{kl} - \Psi_{ika}\Psi^a{}_{jl})_\circ = 0$$

  where $\Psi_{ijk} = S_{ijk}(x_0)$ is a harmonic cubic form in n variables,

$$\Psi_{ijk}p^ip^jp^k.$$

  Here $\circ$ denotes trace-freeness in all pairs of free indices.

Abundant systems

Abundant systems admit $\frac{n(n+1)}{2}$ linearly independent integrals $F^{(\alpha)}$ compatible with a non-degenerate family of Hamiltonians.

- All known non-degenerate systems are abundant.
- The integrability conditions for the associated trace-free conformal Killing tensors of abundant systems are trivially satisfied.
- Functional independence of $2n-1$ conformal integrals is guaranteed for almost every potential from the non-degenerate family.

Examples

- Isotropic harmonic oscillator
  The 'zero' system: $H = c_1(\mathbf{x}-\mathbf{c})^2 + c_0$.
  Structure tensor: $S_{ijk} = 0$. Conformal scale: $\sigma = 1$.
- 'Generic' system (G is the round metric on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$)

$$H = G(\mathbf{p},\mathbf{p}) + \sum_{k=1}^{n+1}\frac{c_k}{y_k^2} + c_0$$

  Structure tensor: $S = \mathring{\nabla}^3(y_k^2\ln(y_k))$. Conformal scale: $\sigma = \left(\prod_{k=1}^{n+1} y_k\right)^{-\frac{1}{5}}$.
  Here $\mathring{\nabla}^3$ is the trace-free part of the symmetrization of the third covariant derivative.
- Conformal equivalence The Hamiltonians

$$H = p_x^2 + p_zp_{\bar{z}} + c_4x + c_3(4x^2 + z\bar{z}) + \frac{c_2}{z^2} + \frac{c_1\bar{z}}{z^3} + c_0$$

  and $\tilde{H} = z^2H$ admit conformally equivalent non-degenerate properly superintegrable systems.
  $\Rightarrow$ They share the same structure tensor $S = -\frac{3}{2}\mathring{\nabla}^3(z\bar{z}\,\ln(z))$.
  Here ∇ is the Levi-Civita connection of $g = dx^2 + dzd\bar{z}$.
- Geodesically equivalent systems

$$H = \frac{p_1p_2}{x+y^2} + \frac{c_1}{x+y^2} + \frac{c_2\,y}{x+y^2} + \frac{c_3\,y(y^2-3x)}{x+y^2} + c_4$$

  admits a non-degenerate system. Its metric admits a projective symmetry (preserving unparametrised geodesics).
  H induces a non-degenerate system for the metric

$$g = \frac{x+y^2}{(3x-y^2)^6}\left(9(x+y^2)dx^2 - 4y(9x+y^2)dxdy + 12x(x+y^2)dy^2\right)$$

  which has the same unparametrised geodesics. The systems are both projectively and conformally equivalent [Vol2020].

References & Acknowledgments

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