8135

THE UNIVERSITY OF SYDNEY

FACULTIES OF ARTS AND SCIENCE

STAT 2012: Statistical Tests (Normal)

Semester 2, 2007 Time allowed: Two hours

Solution to examination

1. One sample:

   (a) (4 marks) t-test: the rejection region on sample mean is

   $$[2]\ \bar{x} \geq k_0 = \mu_0 + t_{\alpha,n-1}\sqrt{\frac{s^2}{n}} = 15 + 1.895\sqrt{\frac{86.286}{8}} = 21.2221.$$

   Since [1] $\bar{x} = 19.5 < 21.2221$, H_0 is accepted.

   (b) (4 marks) z-test: the type II error under H_1 when $\mu = 18$,

   $$\begin{aligned}
   \beta(18) &= \Pr(\text{type II error}) \\
   &= \Pr(\text{Accept } H_0 \mid H_0 \text{ is false and } \mu = 18) \\
   [1] &= \Pr(\bar{x} < 20 |\ \bar{X} \sim \mathcal{N}(18, 80/8)) \\
   [1] &= \Pr\left(z < \frac{20 - 18}{\sqrt{80/8}}\right) \\
   [1] &= \Pr(z < 0.6325) = 0.7357
   \end{aligned}$$

   (c) (5 marks) The sign ranks of $d_i = x_i - 15$ are [1]

   $$-4\ (-4) \quad -3\ (-2.5) \quad 0\ (0) \quad 1\ (1) \quad 3\ (2.5) \quad 5\ (5) \quad 9\ (6) \quad 25\ (7)$$

   $$\begin{aligned}
   [1]\ W^+ &= 1 + 5 + 6 + 7 + 2.5 = 21.5, \\
   [1]\ E(W) &= \frac{n(n+1)}{4} = \frac{7(8)}{4} = 14, \\
   [1]\ Var(W) &= \frac{1}{4}\sum_{i=1}^{7} r_i^2 = \frac{1}{4}(1^2 + (-4)^2 + \ldots + (2.5)^2) = 34.875, \\
   [1]\ p\text{-value} &= \Pr(W^+ \geq 21.5) = \Pr\left(Z > \frac{21.5 - 14}{\sqrt{34.875}}\right) \\
   &= \Pr(Z > 1.27) = 1 - 0.898 = 0.102
   \end{aligned}$$

2. Two samples:

(a) (5 marks) The *2 samples t-test* for the *difference in means of ages of audience who favor programs A and B* is

1. **Hypotheses:** $H_0 : \mu_x = \mu_y$ against $H_1 : \mu_x \neq \mu_y$.
2. **Test statistic:** [2] $t_0 = \dfrac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}}} = \dfrac{18.6667 - 24.7143}{4.6\sqrt{\frac{1}{6} + \frac{1}{7}}} = -2.3631.$

$$
\begin{aligned}
[1]\ s_p^2 &= \frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{(n_x + n_y - 2)} \\
&= \frac{(6-1)7.8667 + (7-1)32.2381}{(6+7-2)} = 4.6^2
\end{aligned}
$$

3. **Assumptions:** $X_i \sim \mathcal{N}(\mu_x, \sigma^2)$ & $Y_i \sim \mathcal{N}(\mu_y, \sigma^2)$. X_i and Y_i are independent. Then $t_0 \sim t_{n_x+n_y-1}$.
4. ***P*-value:** [1] $0.02 < p\text{-value} = 2\Pr(t_{11} < -2.3631) < 0.05$
5. **Decision:** [1] Since p-value is < 0.05, we reject H_0 that the means of ages of audience who favor programs A and B are the same.

(b) (5 marks) The ranks for the combined sample are [1]

X	15	16	18	20	21	22	Y	17	19	23	25	26	30	33
Ranks	1	2	4	6	7	8	Ranks	3	5	9	10	11	12	13

The Wilcoxon rank-sum test for the difference in ages between audience who favor program A and B is

1. **Hypotheses:** $H_0 : \ \mu_x = \mu_y \quad \text{vs} \quad \mu_x \neq \mu_y$.
2. **Test statistic:** [1] $W = 1 + 2 + 4 + 6 + 7 + 8 = 28$
3. **Assumption:** X_i and Y_i follow the same kind of distribution, differ by a shift.
4. ***P*-value:** [1] $2\Pr(W \leq 28) = 2(0.0256) = 0.0512$ (WRS table, $n_1 = 6$, $n_2 = 7$, $w = 28$).
5. **Decision:** [1] Since the p-value is just > 0.05, we accept H_0. There is borderline evidence in the data against H_0 that the means of ages of audience who favor programs A and B are the same.

(c) (3 marks) The 95% confidence interval for the ratio of variances is

$$
\begin{aligned}
& \Big(\frac{s_y^2}{s_x^2}F^{-1}_{0.975,6,5},\ \frac{s_y^2}{s_x^2}F^{-1}_{0.025,6,5}\Big) = \Big(\frac{s_y^2}{s_x^2}/F_{0.975,6,5},\ \frac{s_y^2}{s_x^2}F_{0.975,5,6}\Big) \\
= & \Big(\frac{32.2381}{7.8667}/6.98,\ \frac{32.2381}{7.8667}5.99\Big) = (0.5871,\ 24.5473)
\end{aligned}
$$

The same since CI includes 1.

3. Two-way data without replicate:

The number of blocks $r = 5$ and the number of treatments $c = 3$.

(a) (6 marks) [3]

$$
\begin{aligned}
CM &= \frac{1}{rc}\Big(\sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}\Big)^2 = \frac{192^2}{15} = 2457.6 \\
SST_o &= \sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}^2 - CM = 2696 - 2457.6 = 238.4 \\
SSB &= \frac{1}{c}\sum_{i=1}^{r}\Big(\sum_{j=1}^{2} y_{i\cdot}\Big)^2 - CM = \frac{1}{3}(31^2 + 38^2 + \ldots + 44^2) - 2457.6 = 43.06667 \\
SST &= \frac{1}{r}\sum_{j=1}^{c}\Big(\sum_{i=1}^{r} y_{\cdot j}\Big)^2 - CM = \frac{1}{5}(81^2 + 64^2 + 47^2) - 2457.6 = 115.6 \\
SSR &= SST_o - SSB - SST = 238.4 - 43.067 - 115.6 = 79.733
\end{aligned}
$$

[3] The ANOVA table for two-way data without replicate is

ANOVA table

Source	df	SS	MS	F
Treatments (program)	2	115.6	$\frac{115.6}{2} = 57.8$	$\frac{57.8}{9.967} = 5.799$
Blocks (weight)	4	43.067	$\frac{43.067}{4} = 10.767$	$\frac{10.767}{9.967} = 1.080$
Residuals	8	79.733	$\frac{79.733}{8} = 9.967$	
Total	14	238.4		

(b) (4 marks) The two-way ANOVA tests for treatment and block effects are

1. **Hypothesis:** [1] $H_0:\ \beta_1 = \beta_2 = \beta_3 = 0$ vs $H_1:$ Not all β_j are the same
2. **Test statistic:** [1]

$$
f_{t0} = \frac{SST/(c-1)}{SSR/(r-1)(c-1)} = \frac{115.6/2}{79.733/8} = 5.799
$$

3. **Assumption:** $Y_{ij} \sim \mathcal{N}(\mu + \alpha_i + \beta_j, \sigma^2)$ and Y_{ij} are independent.

4. ***P*-value:** [1]

$$0.025 < p\text{-value} \;=\; \Pr(F_{2,8} \geq 5.799) < 0.05 \quad (F_{2,8,0.95} = 4.46,\; F_{2,8,0.975} = 6.06).$$

5. **Decision:** [1] Since p-value for treatment effects < 0.05, we reject H_0. There is strong evidence of difference in means across the three weight-reduction programs.

(c) (5 marks) The ranks r_{ij} of the i-th block(type) are given in brackets below: [2]

	Laboratory		
Type	A	B	C
I	11 (2)	12 (3)	8 (1)
II	21 (3)	8 (1)	9 (2)
III	15 (2)	16 (3)	13 (1)
IV	16 (3)	13 (2)	6 (1)
V	18 (3)	15 (2)	11 (1)
$\sum_{i=1}^{5} r_{ij}$	13	11	6

Note $\bar{r} = (3+1)/2 = 2$. The Friedman test of the treatment effects is

1. **Hypothesis:**

$$H_1: \quad \text{No differences across programs} \quad \text{vs}$$
$$H_0: \quad \text{There are differences across programs.}$$

2. **Test statistic:** [1] Without ties,

$$\begin{aligned} q_0 &= \frac{12r}{c(c+1)} \sum_{j=1}^{c} (\bar{r}_{\cdot j})^2 - 3r(c+1) = \frac{12}{c(c+1)r} \sum_{j=1}^{c} \Big(\sum_{i=1}^{r} r_{ij} \Big)^2 - 3r(c+1) \\ &= \frac{12}{3(4)(5)} (13^2 + 11^2 + 6^2) - 3(5)(4) = 5.2 \end{aligned}$$

3. **Assumption:** No particular assumption for Y_{ij}. We have $q_0 \sim \chi^2_{c-1}$ under H_0.

4. **P-value:** [1] $0.05 < p\text{-value} = \Pr(\chi_2^2 \geq 5.2) < 0.1$

$$(\chi_{2,0.9} = 4.605,\ \chi_{2,0.95} = 5.991)$$

5. **Decision:** [1] Since p-value > 0.05, we accept H_0. The data is consistent with H_0 that the means across programs are the same.

4. Regression analysis:

(a) (4 marks) Given

$$\sum_{i=1}^{n} x_i = 99, \quad \sum_{i=1}^{n} y_i = 373, \quad n = 8,$$
$$\sum_{i=1}^{n} x_i^2 = 1275, \quad \sum_{i=1}^{n} y_i^2 = 17849, \quad \sum_{i=1}^{n} x_i y_i = 4761,$$

$$\begin{aligned}
[0.5]\ S_{xx} &= \sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n} = 1275 - \frac{99^2}{8} = 49.875, \\
[0.5]\ S_{xy} &= \sum_{i=1}^{n} x_i y_i - \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n} = 4761 - \frac{(99)(373)}{8} = 145.125, \\
[1]\ \hat{\beta} &= \frac{S_{xy}}{S_{xx}} = \frac{145.125}{49.875} = 2.909774, \\
[1]\ \hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} = \frac{373}{8} - 2.909774\frac{99}{8} = 10.61654.
\end{aligned}$$

Hence the fitted least squares line is

$$[1]\ \hat{y} = \hat{\alpha} + \hat{\beta}\, x = 10.61654 + 2.909774x.$$

(b) (6 marks) The test for the regression model in (a) is

1. **Hypotheses:** H_0: $\beta = 0$ vs $H_1:\ \beta \neq 0$.
2. **Test statistic:** [1] $f_0 = \dfrac{SSR_e}{SSR/(n-2)} = \dfrac{422.281}{35.59/6} = 71.18298$, where

$$\begin{aligned}
S_{yy} &= \sum_{i=1}^{n} y_i^2 - \frac{(\sum_{i=1}^{n} y_i)^2}{n} = 17849 - \frac{373^2}{8} = 457.875, \\
[0.5]\ SSR_e &= \frac{S_{xy}^2}{S_{xx}} = \frac{145.125^2}{49.875} = 422.281 \\
[0.5]\ SSR &= S_{yy} - \frac{S_{xy}^2}{S_{xx}} = 457.875 - 422.281 = 35.59
\end{aligned}$$

3. **Assumption:** $Y_i \sim (\alpha + \beta x_i, \sigma^2)$. Y_i are independent.

4. **P-value:** [1] p-value $= \Pr(F_{1,6} > 71.18298) < 0.001$ $(F_{1,6,0.999} = 35.5)$

5. **Decision:** [1] Since p-value < 0.05, we reject H_0. There are strong evidence in the data of a linear relationship between income (Y \$1000s) and education (X in years).

(c) (5 marks) Predicted income for $x_0 = 10$:

$$\begin{aligned}
[1]\ \hat{y}|\, x_0 = 10 &= \hat{\alpha} + \hat{\beta} x_0 = 10.61654 + 2.909774(10) = 39.71429(\$000). \\
[2]\ \text{s.e.}(\hat{y}|\, x_0 = 10) &= \sqrt{s^2\left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)} \\
&= \sqrt{\frac{35.59}{6}\left(1 + \frac{1}{8} + \frac{(10 - 99/8)^2}{49.875}\right)} = 2.710127.
\end{aligned}$$

The 95% Prediction Interval for the income of a person who has 10 years of education:

$$\left[(\hat{\alpha} + \hat{\beta} x_0) - t_{\alpha/2,n-2}\ s_e\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}},\ (\hat{\alpha} + \hat{\beta} x_0) + t_{\alpha/2,n-2}\ s_e\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}\right]$$

$$\begin{aligned}
[1] &= (39.71429 - 2.447 \times 2.710127,\ 39.71429 + 2.447 \times 2.710127) \\
[1] &= (33.08284,\ 46.34573).
\end{aligned}$$

5. (a) To test for independence:

(i) (6 marks) The expected frequencies:

		Very interesting ($j = 1$)	Fairly interesting ($j = 1$)	Not interesting ($j = 1$)	Marginal prob. $p_{i\cdot}$
Male	n_{ij}	70	41	9	0.6
($i = 1$)	$\hat{n}_{ij} = np_{i\cdot}p_{\cdot j}$	63	45	12	120
	$\frac{(n_{ij} - np_{i\cdot}p_{\cdot j})^2}{np_{i\cdot}p_{\cdot j}}$	0.7778	0.3556	0.7500	
Female	n_{ij}	35	34	11	0.4
($i = 2$)	$\hat{n}_{ij} = np_{i\cdot}p_{\cdot j}$	42	30	8	80
	$\frac{(n_{ij} - np_{i\cdot}p_{\cdot j})^2}{np_{i\cdot}p_{\cdot j}}$	1.1667	0.5333	1.1250	
	Marginal prob. $p_{\cdot j}$	0.525	0.375	0.1	
	$\sum_i \hat{n}_{ij}$	105	75	20	200
	$\sum_{i,j} \frac{(n_{ij} - np_{i\cdot}p_{\cdot j})^2}{np_{i\cdot}p_{\cdot j}}$				4.7083

(ii) (3 marks) The Chi-square test is

1. **Hypothesis:** H_0: Factor i & factor j are independent vs H_1: They are dependent
2. **Test statistic:** [1] $\chi_0^2 = \sum_{i,j} \frac{(n_{ij} - np_{i\cdot}p_{\cdot j})^2}{np_{i\cdot}p_{\cdot j}} = 4.7083$
3. **P-value:** [1] $0.05 < \Pr(\chi_2^2 > 4.7083) < 0.1 \quad (\chi_{2,0.9} = 4.605, \ \chi_{2,0.95} = 5.991)$
4. **Conclusion:** [1] Do not reject H_0. The data are consistent with the null hypothesis that the two factors are independent.