THE UNIVERSITY OF SYDNEY
STAT2012 STATISTICAL TESTS

Semester 2 | **Solution to Tutorial Week 3** | 2015

Tutorial questions

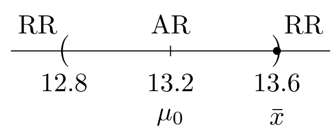
1. We have $n = 6$, $\overline{x} = 13.6$, $\sigma = 0.5$ and $\alpha = 0.05$. The t test is performed to test if the mean amount of sales in a drug store last week is 13.2 thousand dollars.

   (a) **Hypotheses:** H_0: $\mu = 13.2$ vs H_1: $\mu \neq 13.2$

   (b) **Critical value and rejection region (RR):** The critical value is $z_\alpha = 1.96$. The RR for $z_0 = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ is $|z_0| \geq 1.96$, that is, the standardized difference should be at least 1.96 in magnitude in order to be just able to reject H_0.
   The RR (reject H_0) for the sample mean $\bar{x}$ is

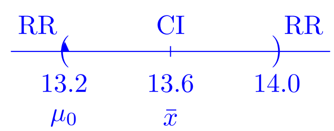
   $$\begin{aligned} &\bar{x} \leq \mu_0 - z_{\alpha/2}\sigma/\sqrt{n} = 13.2 - 1.96 \cdot 0.5/\sqrt{6} = 12.8 \\ \text{or} \quad &\bar{x} \geq \mu_0 + z_{\alpha/2}\sigma/\sqrt{n} = 13.2 + 1.96 \cdot 0.5/\sqrt{6} = 13.6 \end{aligned}$$

   Or the acceptance region (AR) (accept H_0) is $\bar{x} \in (12.8, 13.6)$.



   A 95% CI for μ:

   $$(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = (13.6 - 1.96\frac{0.5}{\sqrt{6}}, 13.6 + 1.96\frac{0.5}{\sqrt{6}}) = (13.2,\ 14.0)$$



   Note that the type I error is $\alpha = 0.05$ under this rejection region.

   (c) **Decision:** Since $\bar{x} = 13.6$ just ≥ 13.6, the RR or $\mu_0 = 13.2$ just $\notin (13.2,\ 14.0)$, the CI, we just reject H_0 and conclude that the sales volume is significiantly larger than the usual sales volume of \$13,200.

   (d) Yes. Result agrees with (c) since p-value=0.05 ≤ 0.05.

   $$p - \text{value} = 2\Pr\left(z \geq \frac{13.6 - 13.2}{0.5/\sqrt{6}}\right) = 2\Pr(z \geq 1.96) = 0.05$$

   (e) **Type I error:** Under H_0,

   $$\begin{aligned} \alpha &= \Pr(\text{type I error}) \\ &= \Pr(\text{Reject } H_0 \mid H_0 \text{ is true}) \\ &= \Pr(\bar{X} \leq 12.8 \text{ or } \bar{X} \geq 13.6 |\ \bar{X} \sim \mathcal{N}(13.2, 0.5^2/6)) \\ &= \Pr\left(z \leq \frac{12.8 - 13.2}{0.5/\sqrt{6}} \text{ or } z \geq \frac{13.6 - 13.2}{0.5/\sqrt{6}}\right) \\ &= \Pr(z \leq -1.96 \text{ or } z \geq 1.96) \\ &= 2(0.025) = 0.05. \end{aligned}$$

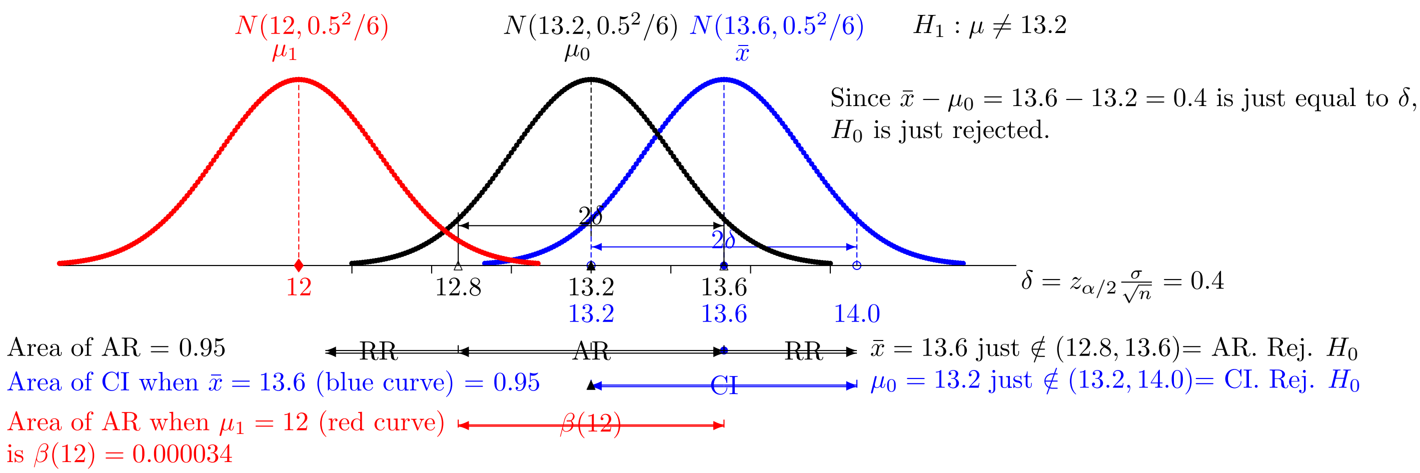
Note that α equals exactly 0.05 because this is the level we set to find the RR $\bar{x} \leq 12.8$ or $\bar{x} \geq 13.6$.

(f) **Type II error:** Under H_1 when $\mu = 12 \neq 13.2$,

$$
\begin{aligned}
\beta(12) &= \Pr(\text{type II error}) \\
&= \Pr(\text{Accept } H_0 \mid H_0 \text{ is false and } \mu = 12) \\
&= \Pr(12.8 < \bar{x} < 13.6 \mid \bar{X} \sim \mathcal{N}(12, 0.5^2/6)) \\
&= \Pr\left(\frac{12.8 - 12}{0.5/\sqrt{6}} < z < \frac{13.6 - 12}{0.5/\sqrt{6}}\right) \\
&= \Pr(3.92 < z < 7.84) \\
&= \Pr(z < 7.84) - \Pr(z < 3.92) \\
&= 1 - 0.999966 = 0.000034.
\end{aligned}
$$

(g) The Power when $\mu = 12$ under H_1 is

$$
\begin{aligned}
\text{Power}(12) &= \Pr(\text{Reject } H_0 \mid H_0 \text{ is false and } \mu = 12) \\
&= 1 - 0.000034 = 0.999966
\end{aligned}
$$



Extra problems

1. (a) (i) Rejection region is

$$
\begin{aligned}
&\bar{x} < \mu_0 - z_{0.95}\sigma/\sqrt{n} = 950 - 1.645 \cdot 200/\sqrt{25} = 884.2 \\
\text{or} \quad &\bar{x} > \mu_0 + z_{0.95}\sigma/\sqrt{n} = 950 + 1.645 \cdot 200/\sqrt{25} = 1015.8
\end{aligned}
$$

(ii) The Type II error is

$$
\begin{aligned}
\beta(1000) &= \Pr(\text{Accept } H_0 \mid H_0 \text{ is false and } \mu = 1000) \\
&= \Pr(884.2 < \bar{x} < 1015.8 \mid \bar{X} \sim \mathcal{N}(1000, 200^2/25)) \\
&= \Pr\left(\frac{884.2 - 1000}{200/\sqrt{25}} < z < \frac{1015.8 - 1000}{200/\sqrt{25}}\right) \\
&= \Pr(-2.89 < z < 0.39) \\
&= 0.6535 - 0.0019 = 0.6516.
\end{aligned}
$$

(b) (i) Rejection region is

$$\begin{aligned} &\bar{x} < \mu_0 - z_{0.95}\sigma/\sqrt{n} = 950 - 1.645 \cdot 200/\sqrt{40} = 897.99 \\ \text{or} \quad &\bar{x} > \mu_0 + z_{0.95}\sigma/\sqrt{n} = 950 + 1.645 \cdot 200/\sqrt{40} = 1002.02 \end{aligned}$$

(ii) The Type II error is

$$\begin{aligned} \beta(1000) &= \Pr\left(\text{Accept } H_0 \mid H_0 \text{ is false and } \mu = 1000\right) \\ &= \Pr(897.99 < \bar{x} < 1002.02 |\ \bar{X} \sim \mathcal{N}(1000, 200^2/40)) \\ &= \Pr\left(\frac{897.99 - 1000}{200/\sqrt{40}} < z < \frac{1002.02 - 1000}{200/\sqrt{40}}\right) \\ &= \Pr(-3.2260 < z < 0.0637) \\ &= 0.5254 - 0.0006 = 0.5248. \end{aligned}$$

β decreases as n increases.

(c) (i) Rejection region is

$$\begin{aligned} &\bar{x} < \mu_0 - z_{0.975}\sigma/\sqrt{n} = 950 - 1.96 \cdot 200/\sqrt{25} = 871.6015 \\ \text{or} \quad &\bar{x} > \mu_0 + z_{0.975}\sigma/\sqrt{n} = 950 + 1.96 \cdot 200/\sqrt{25} = 1028.399 \end{aligned}$$

(ii) The Type II error is

$$\begin{aligned} \beta(1000) &= \Pr\left(\text{Accept } H_0 \mid H_0 \text{ is false and } \mu = 1000\right) \\ &= \Pr(871.6015 < \bar{x} < 1028.399 |\ \bar{X} \sim \mathcal{N}(1000, 200^2/25)) \\ &= \Pr\left(\frac{871.6015 - 1000}{200/\sqrt{25}} < z < \frac{1028.399 - 1000}{200/\sqrt{25}}\right) \\ &= \Pr(-3.210 < z < 0.710) \\ &= 0.7611 - 0.0007 = 0.7605. \end{aligned}$$

β increases as α decreases.

(d) The power when $\mu = 1000$ is

$$\begin{aligned} \text{Power}(1000) &= \Pr\left(\text{Reject } H_0 \mid H_0 \text{ is false and } \mu = 1000\right) \\ &= 1 - \beta(1000) = 1 - 0.6516 = 0.3465 \end{aligned}$$

Hence the probability of rejecting $H_0 : \mu = 950$ when in fact $\mu = 1000$ is 0.3465 .

2. The hypotheses are $H_0 : \mu = 2.5$ vs $H_1 : \mu > 2.5$. We have the true variance $\sigma = 1.8$, $n = 250$ and $\bar{x} = 2.76$. The test statistic is

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{2.76 - 2.5}{1.8/\sqrt{250}} = 2.284$$

The p-value is

$$\Pr(Z > 2.284) = 1 - 0.9888 = 0.0112$$

The rejection region for sample mean $\bar{x}$ is:

$$\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \geq z_{0.95} \quad \Rightarrow \quad \bar{x} \geq \mu_0 + z_{0.95}\frac{\sigma}{\sqrt{n}} = 2.5 + 1.645 \times \frac{1.8}{\sqrt{250}} = 2.68727$$

Hence the type II error is

$$\begin{aligned}\beta(3) &= \Pr(\text{accept } H_0 | \, H_0 \text{ is false and } \mu = 3) \\ &= \Pr\left(\bar{x} < 2.68727 | \, \bar{x} \sim N\left(3, \frac{1.8^2}{250}\right)\right) \\ &= \Pr\left(Z < \frac{2.68727 - 3}{1.8/\sqrt{250}}\right) \\ &= \Pr(Z < -2.747052) = 0.003006677\end{aligned}$$

3. The rejection region is

$$\bar{x} < \mu_0 - t_{n-1}(\alpha)s/\sqrt{n} = 50 - 1.711 \cdot 4/\sqrt{25} = 48.6312$$

since the sample mean is 45 which is less than 48.6312, reject H_0.