THE UNIVERSITY OF SYDNEY

STAT2012 STATISTICAL TESTS

Semester 2	**Solution to Tutorial Week 4**	2015

Tutorial questions

1. The muscle weight differences, $d_i = y_i - x_i$, are:

$$0.4,\ -0.2,\ 0.5,\ 0.7,\ -0.1,\ 0.5,\ 0.6,\ 0,\ 0.2,\ -0.1$$

Let p_+ be the probability of a positive difference d_i. The *sign test* for the differences in muscular weight between rats injected with biochemical substance and placebo is

1. **Hypotheses:** $H_0 : p_+ = \frac{1}{2}$ against $H_1 : p_+ > \frac{1}{2}$.
   or $H_0 : \mu_d = 0$ against $H_1 : \mu_d > 0$.
2. **Test statistic:** $x = 6$ and $n = 9$. (*ignore 0 difference*).
3. **Assumptions:** D_i are from a symmetric distribution. $X \sim \mathcal{B}(9, 0.5)$ under H_0.
4. P**-value:**

$$\begin{aligned} P(X \geq 6) &= \sum_{i=6}^{9} \binom{9}{i} 0.5^i 0.5^{9-i} \\ &= 1 - P(X \leq 5) = 1 - 0.7461 \text{ (bin. table, } n = 9,\ p = 0.5,\ x = 5) \\ &= 0.2539. \end{aligned}$$

5. **Decision:** Since p-value is > 0.05, we accept H_0 and conclude that the data are consistent with H_0.

2. We have $n = 8$ and max $W^+ = \frac{n(n+1)}{2} = \frac{8\times 9}{2} = 36$. The table of differences $d_i = x_i - 107$ is

d_i	-7	-17	28	1	0	12	20	2	-2
$\lvert d_i \rvert$	7	17	28	1	0	12	20	2	2
Rank r_i	4	6	8	1		5	7	2.5	2.5
Sign r'_i	-4	-6	8	1		5	7	2.5	-2.5

The Wilcoxon sign-rank test for the IQ of the arrested abusers who are 16 or older from the population of interest is

(a) **Hypotheses:** $H_0 : \mu = 107$ against $H_1 : \mu > 107$.

(b) **Test statistic:** $w^+ = 23.5$, $w^- = 12.5$, $w = 12.5$ ($W^+ + W^- = \frac{8\times 9}{2} = 36$).

(c) **Assumption:** X_i follow a symmetric distribution.

(d) P**-value:** Since there are ties, normal approximation is used.

$$\begin{aligned} E(W^+) &= \frac{n(n+1)}{4} = \frac{8(8+1)}{4} = 18 \\ Var(W^+) &= \frac{1}{4}\sum_{i=1}^{8} r_i'^2 = \frac{1}{4}[4^2 + 6^2 + 8^2 + \ldots + 2.5^2] = \frac{203.5}{4} = 50.875 \\ p\text{-value} &= \Pr(W^+ \geq 23.5) = \Pr(W^- \leq 12.5) = \Pr\left(Z < \frac{12.5 - 18}{\sqrt{50.875}}\right) \\ &= \Pr(Z < -0.7711) = 0.220 \end{aligned}$$

(e) **Decision:** Since P-value is > 0.05, we accept H_0. The data is consistent with H_0 that the IQ of the arrested abusers who are 16 or older from the population of interest is 107.

Note that from the WSR table, $\Pr(W^+ \leq 12) = 0.2305$ and $\Pr(W^+ \leq 13) = 0.2734$ and that $\Pr(W^- \leq 12.5) = 0.220$ with ties is not between $\Pr(W^+ \leq 12) = 0.2305$ and $\Pr(W^+ \leq 13) = 0.2734$ without ties because the distribution changes when there are ties.

3. There are $2^3 = 8$ possible signed ranks.

1	2	3	$w^+ = 6$	-1	-2	-3	$w^+ = 0$
-1	2	3	$w^+ = 5$	1	-2	-3	$w^+ = 1$
1	-2	3	$w^+ = 4$	-1	2	-3	$w^+ = 2$
1	2	-3	$w^+ = 3$	-1	-2	3	$w^+ = 3$

Each with prob.$=\frac{1}{8}$ under H_0. Thus

$$\begin{aligned}
\Pr(w^+ \leq 0) &= \frac{1}{8} = 0.125, \quad \Pr(w^+ \leq 5) = \frac{7}{8} = 0.875 \\
\Pr(w^+ \leq 1) &= \frac{2}{8} = 0.250, \quad \Pr(w^+ \leq 4) = \frac{6}{8} = 0.750 \\
\Pr(w^+ \leq 2) &= \frac{3}{8} = 0.375, \quad \Pr(w^+ \leq 3) = \frac{5}{8} = 0.625
\end{aligned}$$

Since the distribution of W^+ is symmetric, the probabilities in the upper area (2nd column above) are dropped from the table. When there are ties or zeros, the ranks will change and hence the WSRD table cannot be applied.

Extra problems

1. (a) $X \sim \mathcal{B}(12, 0.1)$.

   (b) The probability of less than 2 defective items on a particular day is

$$\begin{aligned}
P(X < 2) &= \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^{1} \binom{12}{i} 0.1^i 0.9^{12-i} \\
&= 12(0.0314) + 0.28243 = 0.659 \quad \text{(or from R)}
\end{aligned}$$

   (c) The binomial test on the proportion of defective items produced after the new work practices is

   1. **Hypotheses:** $H_0: \ p = 0.1$ against $H_1: \ p < 0.1$
   2. **Test statistic:** $T = X = 11$.
   3. **Assumption:** Independent trials with constant probability of success. Then $X \sim \mathcal{B}(200, 0.1)$ under H_0 or $X \sim \mathcal{N}(200(0.1), 200(0.1)(0.9))$ approximately using normal approximation by CLT.
   4. ***P*-value:**

$$\begin{aligned}
p\text{-value} &= \Pr(X \leq 11) = \Pr\left(Z < \frac{11 + 0.5 - 200 \cdot 0.1}{\sqrt{200 \cdot 0.1 \cdot 0.9}}\right) \\
&= \Pr(Z < -2.0035) = 0.02256
\end{aligned}$$

5. **Decision:** Since p-value is < 0.05, we reject H_0. There is strong evidence in the data against H_0. The defective rate should be less than 10%.

Note that the exact probability is $\Pr(X \leq 11) = 0.01678953$ from R.

2. The binomial test on the proportion of success of an experiment is

   (a) **Hypotheses:** $H_0: \ p = 0.6$ against $H_1: \ p > 0.6$

   (b) **Test statistic:** $T = X = 40$.

   (c) **Assumption:** Independent trials with constant probability of success. Then $X \sim \mathcal{B}(50, 0.6)$ under H_0 or $X \sim \mathcal{N}(50(0.6), 50(0.6)(0.4))$ approximately using CLT.

   (d) P**-value:**

$$
\begin{aligned}
p\text{-value} &= \Pr(X \geq 40) = \Pr\left(Z > \frac{40 - 0.5 - 50 \cdot 0.6}{\sqrt{50 \cdot 0.6 \cdot 0.4}}\right) \\
&= \Pr(Z > 2.742414) = 0.003049471
\end{aligned}
$$

   (e) **Decision:** Since p-value is < 0.05, we reject H_0. There is strong evidence in the data against H_0. The proportion of success of an experiment is more than 60%.

3. The differences are

   `2 -6 -4 4 1 2 6 4 -3 2`

   The sign test for the difference in typing speeds for secretaries on two different brands of computer keyboards is

   1. **Hypotheses:** $H_0 : p_+ = \frac{1}{2}$ against $H_1 : p_+ \neq \frac{1}{2}$.

   2. **Test statistic:** $x = 7$, $n = 10$.

   3. **Assumption:** X_i are independent binomial trials. Hence $X \sim \mathcal{B}(10, 0.5)$ under H_0.

   4. P**-value:**

$$
\begin{aligned}
2P(X \geq 7) &= 2\sum_{i=7}^{10} \binom{10}{i} 0.5^i 0.5^{10-i} \\
&= 2 \cdot 0.5^{10} \left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right] \\
&= 2 \cdot 0.000977[10 \cdot 9 \cdot 8/(3 \cdot 2) + 10(9)/2 + 10 + 1] \\
&= 2 \cdot 0.1719 = 0.3438 \quad \text{(or from binomial table } n = 10,\ p = 0.5,\ x = 3)
\end{aligned}
$$

   5. **Decision:** Since P-value is > 0.05, we accept H_0 and conclude that there is no difference in the typing speeds for secretaries on two different brands of computer keyboards.

4. The table of differences is

d_i	2	-6	-4	4	1	2	6	4	-3	2
$\|d_i\|$	2	6	4	4	1	2	6	4	3	2
Rank r_i	3	9.5	7	7	1	3	9.5	7	5	3
Sign r_i	3	-9.5	-7	7	1	3	9.5	7	-5	3

The Wilcoxon sign-rank test for the differences in FVC before and after treatment is

1. **Hypotheses:** $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.
2. **Test statistic:** $w^+ = 33.5$, $w^- = 21.5$, $w = 21.5$.
3. **Assumption:** X_i follow a symmetric distribution.
4. P**-value:**

$$
\begin{aligned}
E(W^+) &= \frac{n(n+1)}{4} = \frac{10(10+1)}{4} = 27.5 \\
Var(W^+) &= \frac{1}{4}\sum_{i=1}^{10} r_i^2 = \frac{380.5}{4} = 95.125 \\
p\text{-value} &= 2\Pr(W \leq 21.5) \\
&\approx 2\Pr\left(Z < \frac{w - E(W^+)}{\sqrt{Var(W^+)}}\right) = 2\Pr\left(Z < \frac{21.5 - 27.5}{\sqrt{95.125}}\right) \\
&= 2\Pr(Z < -0.6152) = 2(0.2692) = 0.5384
\end{aligned}
$$

5. **Decision:** Since P-value is > 0.05, we accept H_0. The data is consistent with H_0 that there is no difference in the typing speeds for secretaries on two different brands of computer keyboards. .