THE UNIVERSITY OF SYDNEY
STAT2012 STATISTICAL TESTS

Semester 2	**Solution to Tutorial Week 8**	2015

Tutorial questions

1. The number of blocks $r = 5$ and the number of treatments $c = 4$. Summary of data is

$$\begin{aligned} \bar{y}_{i.} &= 4.175 \quad 4.45 \quad 6.275 \quad 6.675 \quad 9.125 \\ \bar{y}_{.j} &= 5.48 \quad 10.16 \quad 2.90 \quad 6.02 \end{aligned}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij} = 122.8, \quad \sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}^2 = 959.42.$$

(a) & (b) The two-way ANOVA tests for treatment and block effects are

1. **Hypothesis:**

$$\begin{aligned} H_0: \ \beta_1 = ... = \beta_c = 0 \quad \text{vs} \quad H_1: \ \text{Not all } \beta_j \text{ are same;} \\ H_0: \ \alpha_1 = ... = \alpha_r = 0 \quad \text{vs} \quad H_1: \ \text{Not all } \alpha_i \text{ are same;} \end{aligned}$$

2. **Test statistic:**

$$\begin{aligned} f_{t0} &= \frac{SST/(c-1)}{SSR/(r-1)(c-1)} = \frac{135.54/3}{6.16/12} = 88.013 \\ f_{b0} &= \frac{SSB/(r-1)}{SSR/(r-1)(c-1)} = \frac{63.728/4}{6.16/12} = 31.036 \end{aligned}$$

where

$$\begin{aligned} CM &= \frac{\left(\sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}\right)^2}{rc} = \frac{122.8^2}{20} = 753.992 \\ SST_o &= \sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}^2 - CM = 959.42 - 753.992 = 205.428 \\ SSB &= c\sum_{i=1}^{r}(\bar{y}_{i.})^2 - CM = 4(4.175^2 + \cdots + 9.125^2) - 753.992 = 63.728 \\ SST &= r\sum_{j=1}^{c}(\bar{y}_{.j})^2 - CM = 5(5.48^2 + 10.16^2 + 2.90^2 + 6.02^2) - 753.992 = 135.54 \\ SSR &= SST_o - SSB - SST = 205.428 - 63.728 - 135.54 = 6.16 \end{aligned}$$

3. **Assumption:** $Y_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma^2)$ and Y_{ij} are independent.

4. **P-value:**

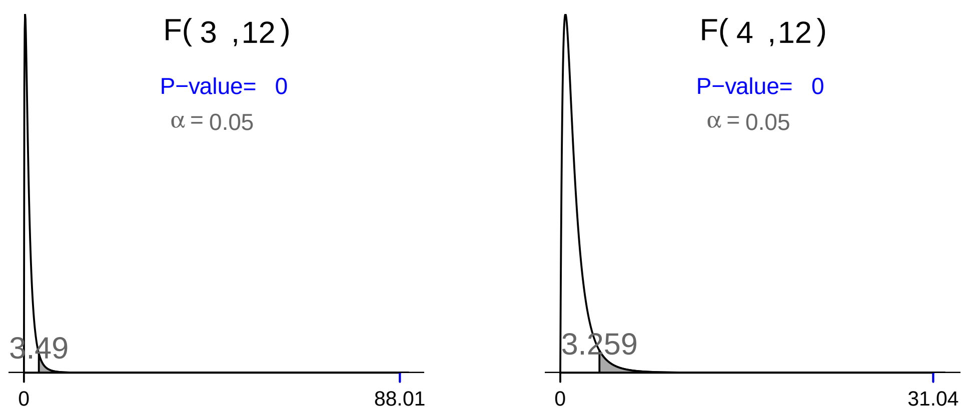
$$\begin{aligned} p\text{-value} &= \Pr(F_{3,12} \geq 88.013) < 0.001 \quad (F_{3,12,0.999} = 10.8), \\ p\text{-value} &= \Pr(F_{4,12} \geq 31.036) < 0.001 \quad (F_{4,12,0.999} = 9.63), \end{aligned}$$

5. **Decision:** Since p-value for treatment effects < 0.05, we reject H_0. There is strong evidence of differences in the yields among the four varieties of barley.

   Moreover, since p-value for block effects < 0.05, we reject H_0. There is strong evidence of differences in the yields among the five blocks.

ANOVA table

Source	df	SS	MS	F
Treatments	3	135.54	$\frac{135}{3} = 45.18$	88.013
Blocks	4	63.728	$\frac{63.728}{4} = 15.932$	31.036
Residuals	12	6.16	$\frac{6.16}{12} = 0.5133$	
Total	19	205.428		



(c) If the block effect is dropped from the model assuming the yields under each variety are random sample, it becomes a completely randomized design experiment. Then in this one-way ANOVA model, the SSB will move to SSR resulting in an inflated SSR. Hence

$$\begin{aligned} SSR' &= SSB + SSR = 63.728 + 6.16 = 69.888 \quad \text{with } d.f. = 4 + 12 = 16 \\ f'_{t0} &= \frac{MST}{MSR'} = \frac{135.54/3}{(63.728 + 6.16)/(4 + 12)} = 10.34341 \\ p\text{-value} &= \Pr(F_{3,16} \geq 10.34341) = 0.0005 \end{aligned}$$

which is still significant but the p-value increases showing slightly less evidence.

2. The ranks r_{ij} of the i-th block (type) are given in brackets below:

	Laboratory			
Type	A	B	C	D
I	38.7 (3)	39.2 (4)	34.0 (1.5)	34.0 (1.5)
II	41.5 (4)	39.3 (3)	35.0 (2)	34.8 (1)
III	43.8 (4)	39.7 (3)	39.0 (2)	34.8 (1)
IV	44.5 (4)	41.8 (3)	40.0 (2)	35.4 (1)
V	45.5 (4)	41.8 (2)	43.0 (3)	37.2 (1)
$\sum_{i=1}^{5} r_{ij}$	19	15.0	10.5	5.5
$\sum_{i=1}^{5} r_{ij}^2$	73	47	23.25	6.25

Note $\bar{r} = (4+1)/2 = 2.5$. The Friedman test of the treatment effects is

1. **Hypothesis:**

$$\begin{aligned} H_1 &: \quad \text{No differences among the labs} \quad \text{vs} \\ H_0 &: \quad \text{There are differences among the labs.} \end{aligned}$$

2. **Test statistic:**

$$q_0 \;=\; \frac{SST}{MST'_o} = \frac{20.3}{1.633} = 12.43$$

where

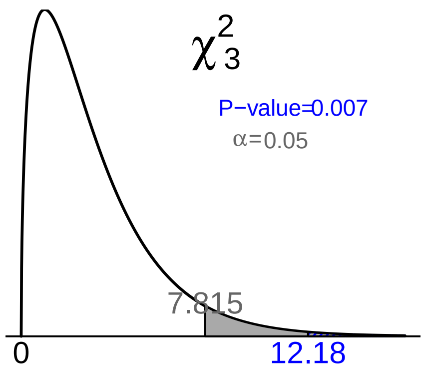
$$\begin{aligned} \bar{r}_{.j} &= 3.8, \quad 3.0, \quad 2.1, \quad 1.1, \\ SST &= r\sum_{j=1}^{c} \bar{r}_{.j}^2 - rc(\bar{r})^2 = 5(3.8^2 + 3^2 + 2.1^2 + 1.1^2) - 5 \times 4 \times 2.5^2 = 20.3 \\ MST'_o &= \frac{1}{r(c-1)}\left(\sum_{i=1}^{r}\sum_{j=1}^{c} r_{ij}^2 - rc\bar{r}^2\right) \\ &= \frac{1}{5(4-1)}(73 + 47 + 23.25 + 6.25 - 5(4)(2.5)^2) = 1.633 \end{aligned}$$

Note that even with ties,

$$\begin{aligned} q'_0 &= \frac{12r}{c(c+1)}\sum_{j=1}^{c}(\bar{r}_{.j})^2 - 3r(c+1) = \frac{12}{c(c+1)r}\sum_{j=1}^{c}(\sum_{i=1}^{r} r_{ij})^2 - 3r(c+1) \\ &= \frac{12}{4(5)(5)}(19^2 + 15^2 + 10.5^2 + 5.5^2) - 3(5)(5) = 12.18 \end{aligned}$$

is close to 12.43.

3. **Assumption:** No particular assumption for Y_{ij}. We have $q_0 \sim \chi^2_{c-1}$ under H_0.
4. **P-value:** p-value $= \Pr(\chi^2_3 \geq 12.43) < 0.01$ $\quad(\chi_{3,0.99} = 11.345,\ 0.00605$ from R)
5. **Decision:** Since p-value < 0.05, we reject H_0. There is very strong evidence in the data that the smoothness measurements are different across labs.



3. In the case of no ties, the ranks r_{ij} for y_{ij} ranked across row i take the values $j = 1, 2, ..., c$.

   (a) Hence we have

$$\begin{aligned} \sum_{j=1}^{c} r_{ij} &= \sum_{j=1}^{c} j = \frac{1}{2}c(c+1), \\ \sum_{j=1}^{c} r_{ij}^2 &= \sum_{j=1}^{c} j^2 = \frac{1}{6}c(c+1)(2c+1). \end{aligned}$$

Moreover $\bar{r} = \frac{1}{rc}\sum_{i=1}^{r}\sum_{j=1}^{c} r_{ij} = \frac{1}{rc}\sum_{i=1}^{r}\frac{1}{2}c(c+1) = \frac{1}{2}(c+1)$. Then

$$\begin{aligned} SST &= r\sum_{j=1}^{c}(\bar{r}_{\cdot j})^2 - rc(\bar{r})^2 = r\sum_{j=1}^{c}(\bar{r}_{\cdot j})^2 - \frac{1}{4}rc(c+1)^2. \\ MST_o &= \frac{1}{r(c-1)}\left(\sum_{i=1}^{r}\sum_{j=1}^{c} r_{ij}^2 - rc\bar{r}^2\right) \\ &= \frac{1}{6r(c-1)}rc(c+1)(2c+1) - \frac{rc(c+1)^2}{4r(c-1)} \\ &= \frac{2c(c+1)(2c+1) - 3c(c+1)^2}{12(c-1)} \\ &= \frac{(c+1)(4c^2+2c-3c^2-3c)}{12(c-1)} \\ &= \frac{c(c+1)(c-1)}{12(c-1)} = \frac{c(c+1)}{12}. \end{aligned}$$

It follows that

$$
\begin{aligned}
q_0 &= \frac{SST}{MST_o} = \frac{12}{c(c+1)}\left(r\sum_{j=1}^{c}(\bar{r}_{\cdot j})^2 - \frac{1}{4}rc(c+1)^2\right) \\
&= \frac{12r}{c(c+1)}\sum_{j=1}^{c}(\bar{r}_{\cdot j})^2 - \frac{12}{4c(c+1)}rc(c+1)^2 \\
&= \frac{12r}{c(c+1)}\sum_{j=1}^{c}(\bar{r}_{\cdot j})^2 - 3r(c+1)
\end{aligned}
$$

(b) When $c=2$ and with no ties, the ranks are

	$c=1$	$c=2$	Total
$r=1$	r_{11}	$r_{12}=3-r_{11}$	3
$r=2$	r_{21}	$r_{22}=3-r_{21}$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r=r$	r_{r1}	$r_{r2}=3-r_{r1}$	3
Mean	$\bar{r}_{\cdot 1}$	$\bar{r}_{\cdot 2}=3-\bar{r}_{\cdot 1}$	3

where $r_{ij}=1,2$. Hence the Friedman test statistic is

$$
\begin{aligned}
q_0 &= \frac{12r}{c(c+1)}\sum_{j=1}^{2}(\bar{r}_{\cdot j})^2 - 3r(c+1) \\
&= \frac{12r}{2(2+1)}\sum_{j=1}^{2}(\bar{r}_{\cdot j})^2 - 3r(2+1) \\
&= 2r[(\bar{r}_{\cdot 1})^2 + (3-\bar{r}_{\cdot 1})^2 - 4.5] \\
&= 2r[(\bar{r}_{\cdot 1})^2 + 9 - 6\bar{r}_{\cdot 1} + (\bar{r}_{\cdot 1})^2 - 4.5] \\
&= r[4(\bar{r}_{\cdot 1})^2 - 12\bar{r}_{\cdot 1} + 9] = r[2\bar{r}_{\cdot 1} - 3]^2 \\
&= \frac{[2r\bar{r}_{\cdot 1} - 2r - r]^2}{r} = \frac{[2\sum_{i=1}^{r}(r_{i1}-1) - r]^2}{r} = \frac{(2x-r)^2}{r} \\
&= \frac{(x-r/2)^2}{r/4} = z_0^2 \sim \chi_1^2
\end{aligned}
$$

is the standardized test statistic for matched pair since $x=\sum_{i=1}^{r}(r_{i1}-1)$ is the count of signs which are 0,1 for treatment $c=1$, $\sum_{i=1}^{r} r_{i1} = r\bar{r}_{\cdot 1}$ and the number of pairs is $n=r$.

(c) When $c=2$, the number of blocks r is the number of pairs n. The data are

Block/pair	Treatment 1	Treatment 2	Block mean
1	y_{11}	y_{12}	$\frac{1}{2}(y_{11}+y_{12})$
2	y_{21}	y_{22}	$\frac{1}{2}(y_{21}+y_{22})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	y_{n1}	y_{n2}	$\frac{1}{2}(y_{n1}+y_{n2})$
Treatment mean	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	$\frac{1}{2}(\bar{y}_{\cdot 1}+\bar{y}_{\cdot 2})$

Now $\bar{y} = \frac{1}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})$, $r = n$ and the sample size $N = 2n$.

$$\begin{aligned}
SST &= r\sum_{j=1}^{c} \overline{y}_{\cdot j}^2 - N\overline{y}^2 = n(\overline{y}_{\cdot 1}^2 + \overline{y}_{\cdot 2}^2) - 2n\frac{1}{4}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2 \\
&= n(\overline{y}_{\cdot 1}^2 + \overline{y}_{\cdot 2}^2) - \frac{n}{2}(\overline{y}_{\cdot 1}^2 + \overline{y}_{\cdot 2}^2 + 2\bar{y}_{\cdot 1}\bar{y}_{\cdot 2}) \\
&= \frac{n}{2}(\overline{y}_{\cdot 1}^2 + \overline{y}_{\cdot 2}^2 - 2\bar{y}_{\cdot 1}\bar{y}_{\cdot 2}) = \frac{n}{2}(\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2})^2 \\
SST_o &= \sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}^2 - N\overline{y}^2 = \sum_{i=1}^{r} y_{i1}^2 + \sum_{i=1}^{r} y_{i2}^2 - \frac{n}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2 \\
SSB &= c\sum_{i=1}^{r} \overline{y}_{i\cdot}^2 - N\overline{y}^2 = 2\sum_{i=1}^{r} \overline{y}_{i\cdot}^2 - \frac{n}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2 \\
&= \frac{2}{4}\sum_{i=1}^{r}(y_{i1} + y_{i2})^2 - \frac{n}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2 = \frac{1}{2}\sum_{i=1}^{r}(y_{i1}^2 + y_{i2}^2 + 2y_{i1}y_{i2}) - \frac{n}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2 \\
SSR &= SST_o - SST - SSB \\
&= \sum_{i=1}^{r} y_{i1}^2 + \sum_{i=1}^{r} y_{i2}^2 - \frac{n}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2 - \frac{n}{2}(\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2})^2 - \\
&\quad \left[\frac{1}{2}\sum_{i=1}^{r}(y_{i1}^2 + y_{i2}^2 + 2y_{i1}y_{i2}) - \frac{n}{2}(\bar{y}_{\cdot 1} + \bar{y}_{\cdot 2})^2\right] \\
&= \frac{1}{2}\sum_{i=1}^{r}(y_{i1}^2 + y_{i2}^2 - 2y_{i1}y_{i2}) - \frac{n}{2}(\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2})^2 \\
&= \frac{1}{2}\left[\sum_{i=1}^{r}(y_{i1} - y_{i2})^2 - n(\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2})\right] = \frac{(n-1)s_d^2}{2} \\
f_0 &= \frac{SST/(2-1)}{SSR/(n-1)(2-1)} = \frac{\frac{n}{2}(\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2})^2}{\frac{(n-1)s_d^2}{2}/(n-1)} = \frac{(\bar{y}_{\cdot 1} - \bar{y}_{\cdot 2})^2}{s_d^2/n} = t_0^2
\end{aligned}$$

Extra problems

1. The number of blocks $r = 5$ and the number of treatments $c = 4$. Summary of data is

$$\begin{aligned} \bar{y}_{i.} &= 51 \quad 59.25 \quad 63.25 \quad 24 \quad 44 \\ \bar{y}_{.j} &= 47.4 \quad 54.2 \quad 39.4 \quad 52.2 \end{aligned}$$

$$\sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij} = 966, \quad \sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}^2 = 52426.$$

The two-way ANOVA tests for treatment and block effects are

1. **Hypotheses:**

$$\begin{aligned} H_0:\ \beta_1 = \ldots = \beta_c = 0 \quad &\text{vs} \quad H_1:\ \text{Not all } \beta_j \text{ are same;} \\ H_0:\ \alpha_1 = \ldots = \alpha_r = 0 \quad &\text{vs} \quad H_1:\ \text{Not all } \alpha_i \text{ are same;} \end{aligned}$$

2. **Test statistic:**

$$\begin{aligned} f_{t0} &= \frac{SST/(c-1)}{SSR/(r-1)(c-1)} = \frac{650.2/3}{1279.3/12} = 2.033 \\ f_{b0} &= \frac{SSB/(r-1)}{SSR/(r-1)(c-1)} = \frac{3838.7/4}{1279.3/12} = 9.002 \end{aligned}$$

where

$$\begin{aligned} CM &= \frac{\left(\sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}\right)^2}{rc} = \frac{966^2}{20} = 46657.8 \\ SST_o &= \sum_{i=1}^{r}\sum_{j=1}^{c} y_{ij}^2 - CM = 52426 - 46657.8 = 5768.2 \\ SSB &= c\sum_{i=1}^{r}(\bar{y}_{i.})^2 - CM = 4(51^2 + \cdots + 44^2) - 46657.8 = 3838.7 \\ SST &= r\sum_{j=1}^{c}(\bar{y}_{.j})^2 - CM = 5(47.4^2 + 54.2^2 + 39.4^2 + 52.2^2) - 46657.8 = 650.2 \\ SSR &= SST_o - SSB - SST = 5768.2 - 3838.7 - 650.2 = 1279.3 \end{aligned}$$

3. **Assumption:** $Y_{ij} \sim \mathcal{N}(\mu_{ij}, \sigma^2)$ and Y_{ij} are independent.

4. P**-value:**

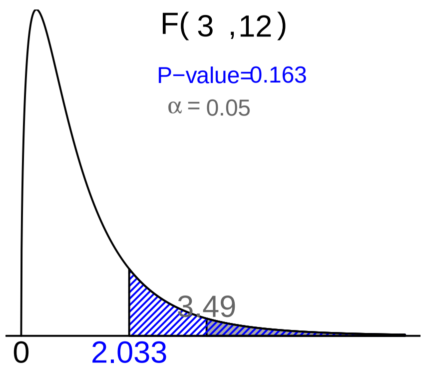
$$\begin{aligned} p\text{-value} &= \Pr(F_{3,12} \geq 2.033) > 0.1 \quad (F_{3,12,0.900} = 2.61;\ 0.1630 \text{ from R}), \\ p\text{-value} &= \Pr(F_{4,12} \geq 9.002) < 0.005 \quad (F_{4,12,0.995} = 6.52;\ 0.0013 \text{ from R}) \end{aligned}$$

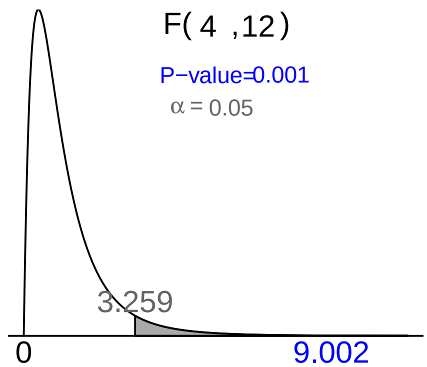
5. **Decision:** Since p-value for treatment effects > 0.05, we accept H_0. The data is consistent with the null hypothesis that the bacteria counts across the four methods are the same.

   Moreover, since p-value for block effects < 0.05, we reject H_0. There is strong evidence of differences in the bacteria counts among the five types of food.

ANOVA table

Source	df	SS	MS	F
Treatments (Method)	3	650.2	$\frac{650.2}{3} = 216.733$	$\frac{216.733}{106.608} = 2.033$
Blocks (Food)	4	3838.7	$\frac{3838.7}{4} = 959.675$	$\frac{216.733}{106.608} = 9.002$
Residuals	12	1279.3	$\frac{1279.3}{12} = 106.608$	
Total	19	5768.2		





2. The ranks r_{ij} of the i-th block(type) are given in brackets below:

	Bacteria count			
	Method 1	Method 2	Method 3	Method 4
Beef	47 (2)	53 (3)	36 (1)	68 (4)
Chicken	53 (2)	61 (3)	48 (1)	75 (4)
Turkey	68 (3)	85 (4)	55 (2)	45 (1)
Eggs	25 (3)	24 (2)	20 (1)	27 (4)
Milk	44 (2)	48 (4)	38 (1)	46 (3)
$\sum_{i=1}^{5} r_{ij}$	12	16	6	16
$\sum_{i=1}^{5} r_{ij}^2$	30	54	8	58

Note $\bar{r} = (4+1)/2 = 2.5$. The Friedman test of the treatment effects is

1. **Hypothesis:**

H_1 : No differences in bacteria counts among methods vs

H_0 : There are differences in bacteria counts among methods.

2. **Test statistic:** There are no ties. Hence

$$q_0 = \frac{12r}{c(c+1)} \sum_{j=1}^{c} (\bar{r}_{.j})^2 - 3r(c+1)$$

$$= \frac{12}{4(5)(5)}(12^2 + 16^2 + 6^2 + 16^2) - 3(5)(5) = 8.04$$

3. **Assumption:** No particular assumption for Y_{ij}. We have $q_0 \sim \chi^2_{c-1}$ under H_0.
4. P**-value:** $p\text{-value} = \Pr(\chi^2_3 \geq 8.04) < 0.05 \quad (\chi_{3,0.95} = 7.815,\ 0.00605 \text{ from R}))$.
5. **Decision:** Since p-value < 0.05, we reject H_0. There is strong evidence in the data that bacteria counts across methods are different.