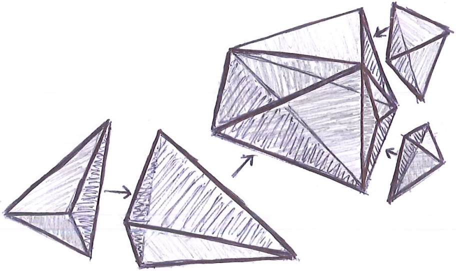
Separation index of graphs and stacked 2-spheres

B. Burton [1] B. Datta [2] N. Singh [2] J. Spreer [1]

[1] The University of Queensland, Brisbane, bab@maths.uq.edu.au, j.spreer@uq.edu.au.
[2] Indian Institute of Science, Bangalore, dattab@math.iisc.ernet.in, nitisin1@in.ibm.com.

UQ, December 8, 2015



Background and motivation

- (Geometric) Topology is study of manifolds (surfaces) up to continuous deformation

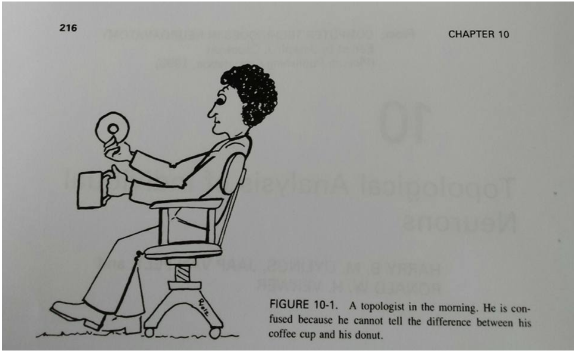

216 CHAPTER 10

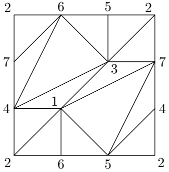
FIGURE 10-1. A topologist in the morning. He is confused because he cannot tell the difference between his coffee cup and his donut.

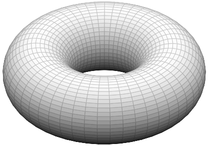
- Here: discrete / combinatorial / computational topology

Background and motivation

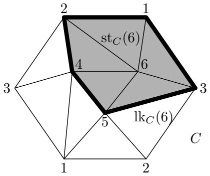
- Represent manifolds (surfaces) as simplicial complexes

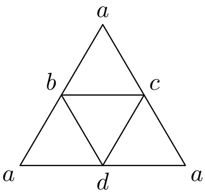
$\langle\langle 1,2,4\rangle, \langle 1,2,6\rangle, \langle 1,3,4\rangle, \langle 1,3,7\rangle,$
$\langle 1,5,6\rangle, \langle 1,5,7\rangle, \langle 2,3,5\rangle, \langle 2,3,7\rangle,$
$\langle 2,4,5\rangle, \langle 2,6,7\rangle, \langle 3,4,6\rangle, \langle 3,5,6\rangle,$
$\langle 4,5,7\rangle, \langle 4,6,7\rangle\rangle$





- A triangulation of a manifold M is a simplicial complex C with underlying set $|C|$ homeomorphic to M.



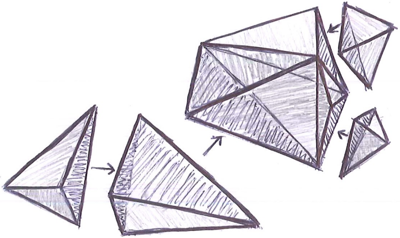


- Here: M 2-sphere $\rightarrow$ planar triangulations.

Stacked balls and spheres

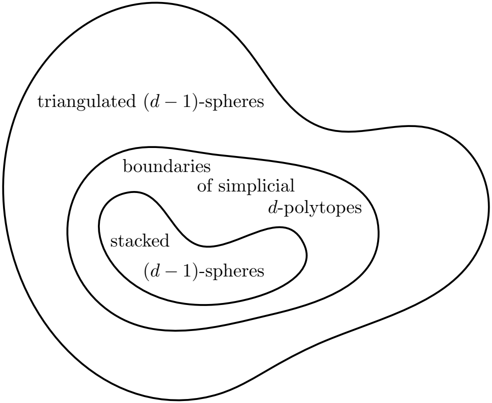
A stacked d-ball is defined recursively:

- A d-simplex is a stacked d-ball.
- A simplicial complex obtained from a stacked d-ball B by gluing a d-simplex along a $(d-1)$-boundary face of B is a stacked d-ball.



A stacked $(d-1)$-sphere is the boundary complex of a stacked d-ball.

Stacked balls and spheres



- convex embeddings easy to obtain
- important role in triangulated manifolds

More background and motivation

- Triangulations of d-manifolds, $d > 3$, with $(d-1)$-dimensional stacked spheres as vertex links allow efficient representations of
  - handle decompositions
  - Morse functions
  - topological features[1].
- Local condition $\Rightarrow$ global properties
- Strong ties between combinatorics and topology
- Proof techniques fundamentally fail for dimension $d = 3$

[1] See work by Bagchi, Effenberger, Kühnel

The case $d > 3$

Stacked balls / spheres are characterised by their f-vector[2].

Theorem (Kalai 1987)

Let S be a triangulated $(d-1)$-sphere, $d > 3$. Then S has at least as many i-dimensional faces $i \leq (d-1)$ as a stacked $(d-1)$-sphere with equality if and only if S is stacked.

[2] The i-th entry of the f-vector of a simplicial complex denotes its number of i-dimensional faces

The case $d = 3$

Let S be a triangulation of the 2-sphere with n vertices, e edges and t triangles.

- $2e = 3t$ (every edge is contained in exactly two triangles)
- $n - e + t = 2$ (Euler 1752)
- $\Rightarrow f(S) = (n, 3n - 6, 2n - 4)$

In particular: stacked spheres can not be characterised by their f-vectors.

The separation index σ

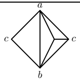
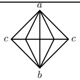
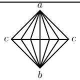
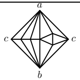
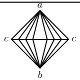
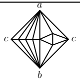
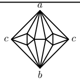
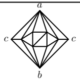
Definition

G graph on n vertices, $G[A]$ subgraph induced by $A \subset V(G)$, $q(G[A])$ # connected components of $G[A]$. The separation index $\sigma(G)$ of G is defined to be

$$\sigma(G) := \sum_{A \subseteq V(G)} \frac{q(G[A]) - 1}{\binom{n}{|A|}}.$$

- $\sigma(G) \geq -1$ with equality iff $G = K_n$ $(q(\emptyset) = 0)$
- For S 2-sphere triangulation we write $\sigma(S) = \sigma(\mathrm{skel}_1(S)) = \sigma(G_S)$

See also Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, 1977.

Triangulation S	$\sigma(S)$	Triangulation S	$\sigma(S)$
	-1		$-4/5$
	$-8/35$		$-2/7$
	$2/9$		$8/63$
	$2/21$		$1/21$

How difficult is it to compute $\sigma(S)$, S n-vertex 2-sphere?

- Naive algorithm: $O(2^n)$.
- Fixed parameter tractable algorithm: $O(nc^k)$, k treewidth of $G_S \Rightarrow O(nc^{\sqrt{n}})$.

Results

Theorem (Burton, Datta, Singh, S. 2014)

Let S be an n-vertex triangulated 2-sphere. Then

$$\sigma(S) \leq \frac{(n-8)(n+1)}{20},$$

where equality occurs if and only if S is a stacked sphere.

Corollary

"Slight variations of the topological results about the efficiency of triangulations of manifolds with stacked vertex links hold in dimension $d = 3$."

Thank you

- B. Burton, B. Datta, N. Singh, J. Spreer. Separation index of graphs and stacked 2-spheres. *J. Combin. Theory (A)*, 136:184-197, 2015.
- B. Burton, B. Datta, J. Spreer. *Flag 2-spheres and a lower bound for the Separation index*, 2015. In preparation.