Bounds for the genus of a normal surface

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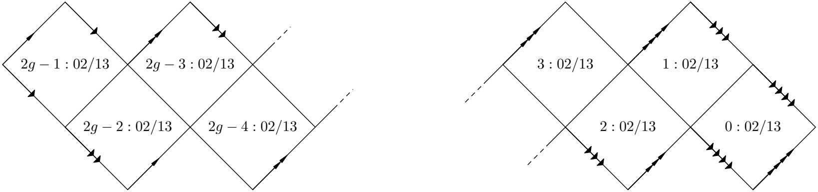
Quadrangulated normal surfaces

Lemma (Bound for quadrilateral surfaces)

Suppose M is a triangulated, compact, orientable 3–manifold, and let S be a closed, connected, orientable, quadrangulated, v-vertex normal surface in M. Then

$$q(S) = 2g(S) + v - 2.$$

Examples



Infinite family of quadrangulated normal surfaces with only two vertices,

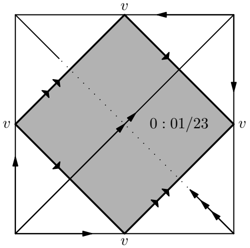
The general orientable case

Theorem (Bound for closed normal surfaces)

Let M be a triangulated, compact, orientable 3–manifold, and S be a closed, connected, orientable normal surface in M. Then

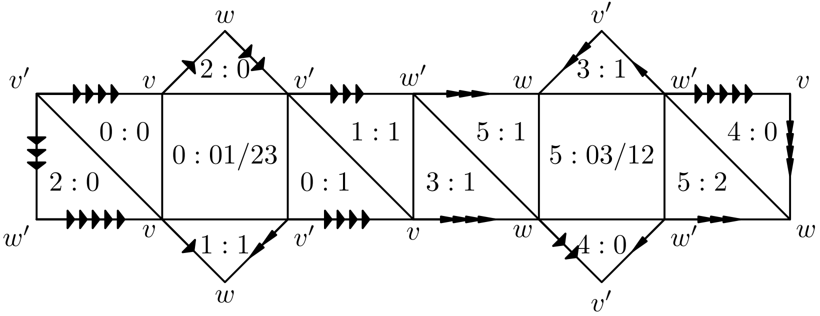
$$3q(S) \geq 2g(S).$$

Examples



A 1-quadrilateral torus (inside the standarrd 1-tetrahedron 3-sphere).

Examples



A 2-quadrilateral genus 3 normal surface (inside a 6-tetrahedra triangulation of the 3-sphere).

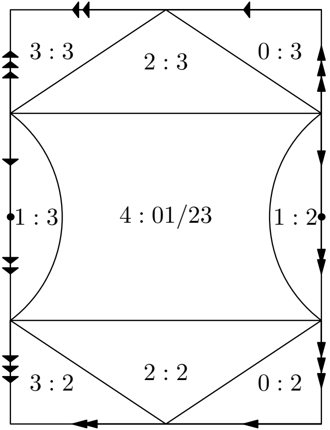
The non-orientable case

Corollary (Bound for non-orientable normal surfaces)

Let M be a triangulated, compact, orientable 3–manifold, and S be a closed, connected non-orientable normal surface in M. Then

$$3(q) \geq g(S) - 1.$$

Example



Non-orientable genus 4 normal surface with only one quadrilateral.

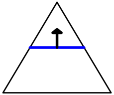
A bound for incompressible normal surfaces

Corollary (Bound for incompressible normal surfaces)

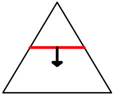
Let M be a triangulated, compact, orientable 3–manifold, and S be a closed, connected, orientable normal surface in M. If S is incompressible, then

$$q(S) \geq 2g(S).$$

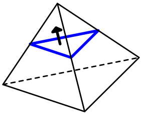
Long and short edges



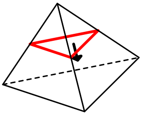
short arc



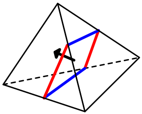
long arc



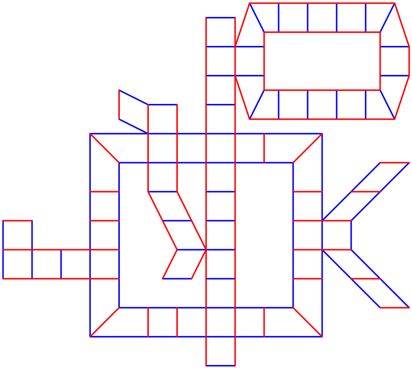
small triangle



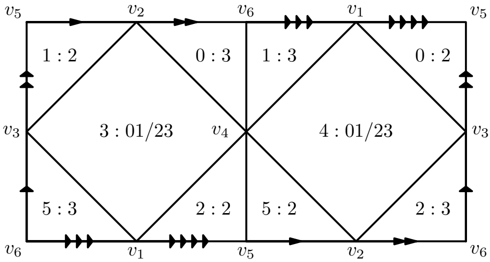
large triangle



quad

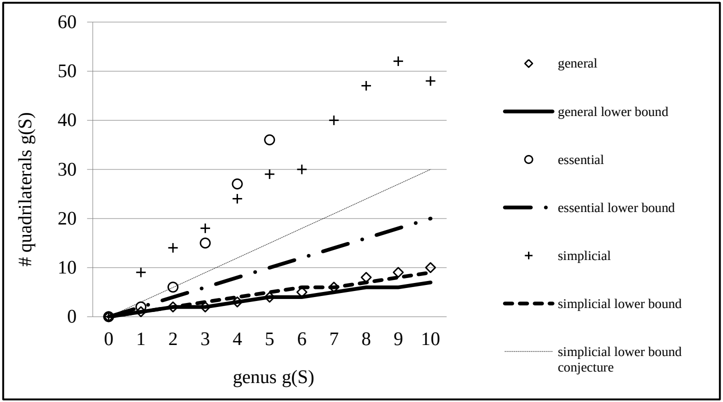


Example



Incompressible torus with only two quadrilaterals (inside a 6 tetrahedra trivial torus bundle).

Summary



60
50
40
30
20
10
0
0 1 2 3 4 5 6 7 8 9 10
quadrilaterals g(S)
genus g(S)
general
general lower bound
essential
essential lower bound
simplicial
simplicial lower bound
simplicial lower bound conjecture

Minimal triangulations of $S_g \times I$

Theorem
Let S_g be a closed, orientable surface of genus at least one, and I be a closed interval. Then every triangulation of $S_g \times I$ has at least $10g - 4$ tetrahedra.

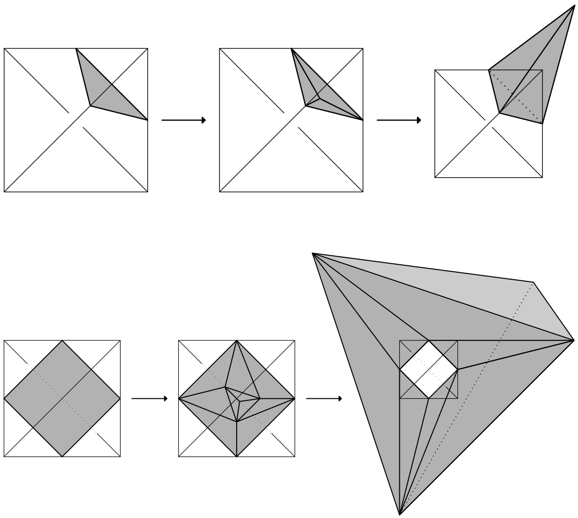
Proposition
There are triangulations of $S_g \times I$, $g \geq 1$ with $10g - 4$ tetrahedra.

Lemma (Bound for splitting surface of product)

Let $M = S_g \times I$, where S_g is a closed, connected, orientable surface, of genus g imbued with a triangulation. Suppose S is a closed, connected, orientable normal surface in M, which separates the two boundary components of M. Then

$$q(S) \geq 2g.$$

The polyhedral realisation problem



The polyhedral realisation problem

Theorem

Let M be a combinatorial 3–manifold and let $S \subset M$ be a closed, orientable normal surface. Then

$$2g(S) < 7f_0(S).$$

Bound for simplicial triangulations

Lemma (Bound for normal surface in simplicial manifold)

Let M be a triangulated, compact, orientable 3–manifold, and S be a closed, connected, orientable normal surface in M. If the triangulation of M is simplicial, then

$$7q(S) \geq 6g(S).$$

Thank you

W. Jaco, J. Johnson, J. Spreer and S. Tillmann, *Bounds for the genus of a normal surface, 2014, in preparation*