Collapsibility and 3-sphere recognition

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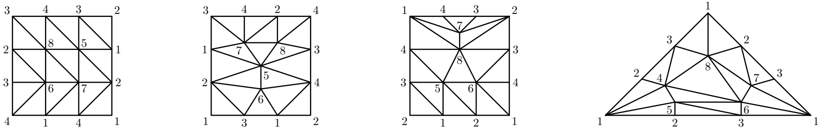
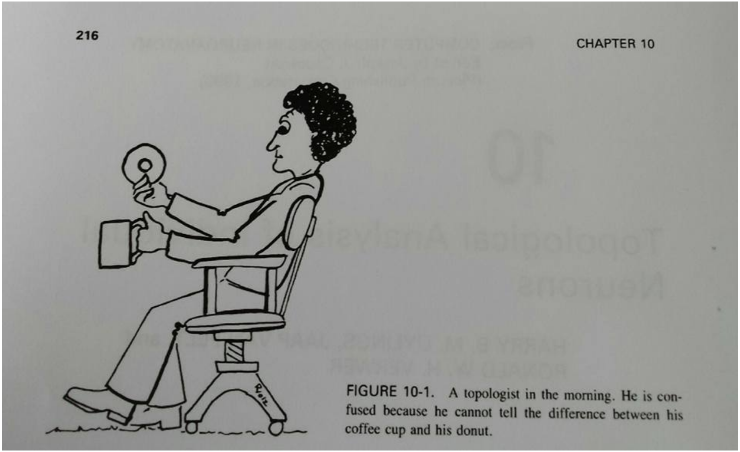


Topology



- (Geometric) Topology is study of manifolds up to continuous deformation
- Important issue: Topological type recognition problems

Topological manifold recognition

- Recognition problem: given two manifolds, are they topologically the same?
- Easy in dimension two. [Brahana 1921]
- *Possible* (in theory) in dimension three. [Haken 1962]
- Undecidable in higher dimensions. [Markov 1958]
- Here: Recognition of the 3-dimensional sphere.

3-sphere recognition: an overview

Input: simplicial triangulations of 3-manifolds, eg.

$$\langle\langle 0,1,2,3\rangle,\langle 0,1,2,4\rangle,\langle 0,1,3,4\rangle,\langle 0,2,3,4\rangle,\langle 1,2,3,4\rangle\rangle$$

Think: boundary of a simplicial 4-polytope.

Deterministic problem solving:

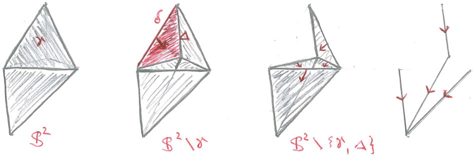
- *Algorithm* to solve 3-sphere recognition. [Rubinstein 1995]
- Simplified and implemented. [Burton 1997-2015]
- Worst case exponential running time.

Heuristic guesses:

- Local modifications, Fundamental group simplifications, Collapsibility.
- Fast but does not solve all instances.

Collapsibility

- C (3-dimensional) simplicial complex.
- Free face: an i-dimensional face δ of C only contained in a single $(i+1)$-dimensional face Δ.
- Elementary collapse: remove δ and Δ from C (homotopy equivalence).
- If there is a sequence of elementary collapses of C to a single vertex, C is called collapsible.
- C collapsible $\Rightarrow$ C contractible.



Idea: Given a 3-manifold triangulation M, remove a tetrahedron γ and check if $M \setminus \gamma$ is collapsible. If yes, $M \cong \mathbb{S}^3$

The problem(s) with collapsibility

- Deciding if M is collapsible is (probably) hard.
- M not collapsible $\not\Rightarrow$ $M \not\cong \mathbb{S}^3$. [Benedetti 2012]

So: why do we care?

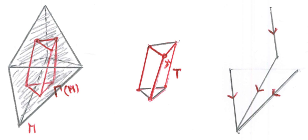
- Algorithmically speaking: it does not matter whether M is difficult to collapse or non-collapsible.
- Possibly work-around: Input M, change $M \mapsto M'$ with better collapsing properties.

- How common are 3-sphere triangulations which are difficult to collapse / non-collapsible (after removing a tetrahedron)?
- Agenda:
  - Define "difficult to collapse"
  - Study emergence of non-collapsible features in small 3-sphere triangulations
  - Essential question: Are there large "clusters" of 3-sphere triangulations which are difficult to collapse.

Collapsing sequences in dimension 3

M 3-manifold triangulation. The dual graph $\Gamma(M)$ of M is the graph whose vertices represent tetrahedra and edges indicate how the tetrahedra are glued along common triangles.

1. Given M, choose a spanning tree $T \subset \Gamma(M)$
2. Remove the "root tetrahedron" γ of M
3. Collapse tetrahedra of $M \setminus \gamma$ along T.
4. Decide collapsibility of the remaining 2-dimensional complex $M_T \subset M$.



Uniform sampling

- M has collapsing probability

$$p = \frac{\#\ \{T : M_T \text{ is collapsible}\}}{\#\ \{\text{sp. trees of } \Gamma(M)\}}$$

- Spanning trees can be sampled uniformly at random in polynomial time. [Guénouche 1983]
- Collapsing probability can be efficiently estimated.

- Study spanning trees T such that S_T is non-collapsible, for S from the census of small (simplicial) 3-sphere triangulations.

Small complexes

Non-collapsing sequences of small triangulations of the 3-sphere.

v	# trig	sample size	non-coll (%)	max (%)
7	3	∞	0.00000	0.00000
8	4	$40,000,000$	0.00109	0.00343
9	50	$2,500,500$	0.02064	0.08000
10	$3,540$	$17,700,000$	0.12429	0.84000
11	1	$\sim 10,000$		$\geq$ 05.43
12	1	$\sim 100,000$		$\geq$ 18.73
13	1	$\sim 100,000$		$\geq$ 30.95
	1	$1,000,000$		16.03 [Lutz 04]
14	1	$\sim 100,000$		$\geq$ 42.53
15	1	$1,000,000$		$\geq$ 97.35
16	1	$1,000,000$		80.61 [BL 13]
...	...	...	...	...
18	1	∞		100 [BL 13]

Thank you

João Paixão and Jonathan Spreer, *Random Collapsibility and 3-Sphere Recognition*. Preprint, 18 pages, 6 figures, 2015. arXiv:1509.07607.