Parameterised complexity of problems in computational geometry and topology

Ben Burton, *Jonathan Spreer*

University of Queensland

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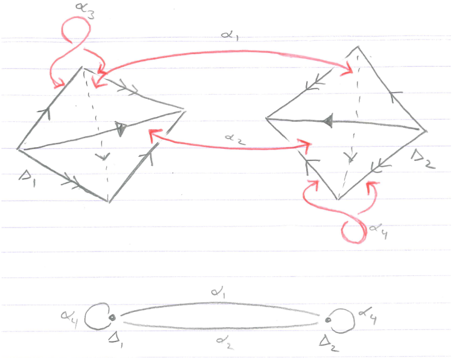
Motivation

- Computational Geometry and Topology – solving topological problems with the computer.
- Situation in dimension 3:
  - Most things are solvable (3-sphere recognition, unknot recognition, homeomorphism problem)
  - Problems don't have polynomial time deterministic algorithms, or even are NP-complete.
  - In many cases computations are feasible – sometimes even extremely fast(?)
  - How can we explain fast heuristics?
- Situation in higher dimensions:
  - Topological questions become undecidable.
  - Heuristics is all we (can) have.

Triangulations

- Generalised triangulation: collection of d-simplices glued together along their $(d-1)$-faces.
- More general than simplicial complexes but related.



Triangulation $\mathcal{T}$ of $S^2 \times S^1$ with its face pairing graph $\Gamma(\mathcal{T})$.

Classical complexity theory

- Decision problems: Algorithmic problems which can only be answered by "yes" or "no".
  Example: *"Is the input triangulation a 3-sphere"*
- A decision problem for which every "yes"-answer can be verified easily (in polynomial time) lies in **NP**.
  Example: *"Can we choose k nodes of a graph such that all arcs touch at least one of these k vertices?"*
- If a problem is at least as hard as the hardest problem in **NP** it is said to be **NP**-hard.
- If an **NP**-hard problem is in **NP** it is called **NP**-complete.

NP-complete problems are not expected to be polynomial time solvable (unless $P = NP$).

Parameterised complexity theory

- Let f be an (**NP**-complete) problem with input set A, a parameter of f is a function $k : A \to \mathbb{N}$.
- The pair (f, k) is called a parameterised (decision) problem. Example: *"Is there a Morse function of the input triangulation with $\leq k$ critical points"*
- f is called fixed parameter tractable with respect to k, if solving f is polynomial in the input size $|a|$ for fixed k, more precisely, if $f(a) \in O(g(k(a))|a|^{O(1)})$, where $g : \mathbb{N} \to \mathbb{N}$ arbitrary.

Exactly what we want:

- $|a|$ = size of triangulation
- k = topological complexity of triangulation

Parameterised complexity for triangulations

What are candidates for good parameters of a triangulation $\mathcal{T}$?

- Most successful candidate so far: tree-width of the face pairing graph $\Gamma(\mathcal{T})$.
  - The tree-width measures how tree-like a graph is.
  - Example: trees have tree-width 1, complete graphs K_n have tree-width $n-1$.
  - Computing tree-width is easy when the tree-width is small.
  - Many (face pairing graphs of) triangulations have small tree-width.
  - Allows constructive proofs yielding fast dynamic programming algorithms.
- Other graph properties of $\Gamma(\mathcal{T})$.
- Number of critical points of a Morse function.

Taut angle structures

A taut angle structure on a 3-dimensional triangulation $\mathcal{T}$ is a combinatorial structure which relates to the existing of a complete hyperbolic structure.

Theorem (Burton and S., 2012)

Deciding whether or not $\mathcal{T}$ admits a taut angle structure is **NP**-*complete.*

Theorem (Burton and S., 2012)

If $\Gamma(\mathcal{T})$ has treewidth $\leq k$, then we can decide whether or not $\mathcal{T}$ admits a taut angle structure in $O(n \cdot k \cdot 3^{7k})$ time, in particular the problem is fixed parameter tractable.

Discrete Morse theory

The number of critical points of a discrete Morse function on a triangulation $\mathcal{T}$ gives an upper bound on the topological complexity of the triangulation.

Theorem (Joswig and Pfetsch, 2006)

Finding a discrete Morse function with the minimum number of critical points of $\mathcal{T}$ is **NP***-hard.*

Theorem (Burton, Lewiner, Paixao and S., 2013)

Deciding whether $\mathcal{T}$ has a discrete Morse function with $\leq k$ critical points remains hard even if k is small (the problem is $W[P]$-complete).

Theorem (Burton, Lewiner, Paixao and S., 2013)

Deciding whether $\mathcal{T}$ has a discrete Morse function with $\leq k$ critical points is fixed parameter tractible in the tree-width of $\Gamma(\mathcal{T})$.

A meta theorem for triangulations

Theorem (Burton and Downey, 2013)

For every monadic second order logic sentence ϕ on a d-manifold triangulation $\mathcal{T}$, testing $\phi \models \mathcal{T}$ is fixed parameter tractable in the tree-width of $\Gamma(\mathcal{T})$.

In other words: every problem on triangulations which can be expressed in monadic second order logic is fixed parameter tractable in the tree-width of the face pairing graph of the input triangulation.

Final remarks

Future research:

- What about unknot recognition / 3-sphere recognition?
- Tree-width approach difficult.

While we cannot handle high topological complexity we might be able to trick inefficient triangulations.

Thank you

B.A. Burton and J. Spreer, The complexity of detecting taut angle structures on triangulations, *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) SIAM*, 2013, pp. 168–183.

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B.A. Burton, R.G. Downey, *Courcelle's theorem for triangulations*. In preparation, 2013.