Tight triangulations of 3-manifolds

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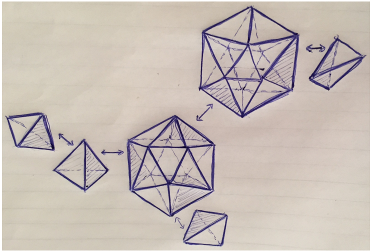


WDWC?

- Convexity is a very important condition in many mathematical results.
- Limited use in geometry and topology: many objects cannot be convex.
- Intuitive notion: many objects look more convex than others.

WDWC?

- A particular embedding of a topological space into some Euclidean space $\mathbb{E}^d$ is said to be tight, if it is "as convex as possible" given its topological constraints.
- More precisely: $M \subset E^d$, compact, connected is called tight with respect to a field $\mathbb{F}$ if for every open or closed half space $h \subset E^d$ the induced homomorphism

$$H_\star(h \cap M, \mathbb{F}) \hookrightarrow H_\star(M, \mathbb{F})$$

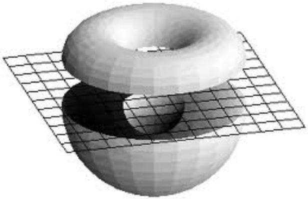
is injective.

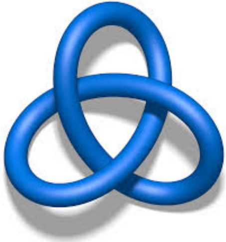
An embedding is tight if it is $\mathbb{F}$-tight for at least one field $\mathbb{F}$.

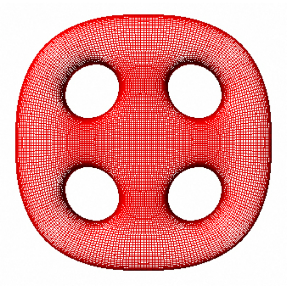
WDWC?

- Generalises convexity.
- Tightness minimises total absolut curvature (Alexandrov, 1938; Milnor, Chern and Lashof, Kuiper, 1950's).
- Compatible with Morse theory (Kühnel, 1995, more in around 4 minutes).

Examples







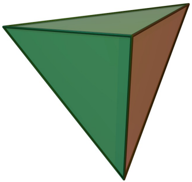
Simplicial (polyhedral) setting

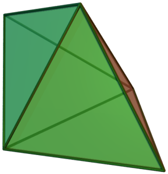
Intrinsic version of tightness (no embedding):

- C connected abstract simplicial complex with vertex set $V(C)$, $\mathbb{F}$ field, then C is tight with respect to $\mathbb{F}$ if for all subsets $W \subset V(C)$,

$$H_\star(C[W], \mathbb{F}) \hookrightarrow H_\star(C, \mathbb{F})$$

is injective ($C[W]$ is the subcomplex of C induced by W).





Relation to Morse theory

- Tightness for abstract simplicial complexes: tight with respect to all embeddings.
- Kühnel (1995): An abstract simplicial complex is tight iff all of its Morse functions* are perfect.[1]
- Relaxation of tightness: find one perfect Morse function.
- Naturally, the latter problem can be efficiently verified ($\in$ **NP**) and the former can be efficiently disproved ($\in$ **co-NP**).
- Finding one perfect Morse function is known to be difficult ($\in$ **NP-hard**)*.[2]

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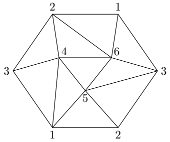
[1] W. Kühnel. *Tight polyhedral submanifolds and tight triangulations*, Springer, 1995

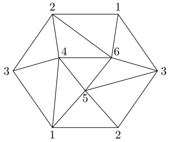
[2] Joswig, Pfetsch. *Computing optimal Morse matchings*, 2006

Relation to Morse theory



- Deciding tightness is easy in dimension two.
- In higher dimensions known algorithms all have exponential running time.
- In dimension three classical tricks to establish tightness fail.

Relation to Morse theory



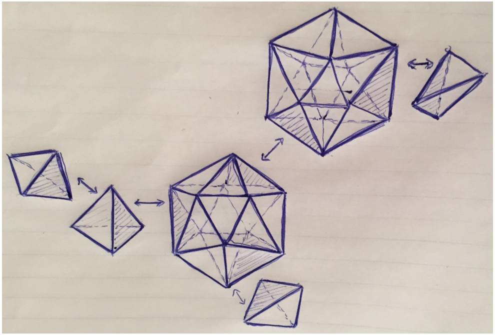
- Deciding tightness is easy in dimension two.
- In higher dimensions known algorithms all have exponential running time.
- In dimension three classical tricks to establish tightness fail.

Sounds familiar?

The main result

Theorem (Bagchi, Datta, S. 2014)

All vertex links of a tight 3-manifold triangulation M are connected sums of the boundary complexes of the tetrahedron and (possibly) the icosahedron.



A connected sum of primitive 2-sphere triangulations.

The main result

Lemma

Let M be a tight 3-manifold triangulation, let $S = \mathrm{lk}_M(v)$ be a vertex link and let $C \subset S$ be an induced cycle in S, then $M[V(C) \cup \{v\}]$ is a tight surface.

$\Rightarrow$ Length of C must be $\not\equiv 1(3)$ (cannot be 4).

Lemma

Let S be a primitive 2-sphere triangulation without cycles of length $\equiv 1(3)$, then S cannot have vertices of degree greater than five.

$\Rightarrow$ The only primitive 2-spheres in vertex links of tight 3-manifold triangulations are  and .

The consequences

"Corollary"

Let M be a 3-manifold with $\beta_1(M, \mathbb{F}_2) < 189$ *admitting a tight triangulation, then M is homeomorphic to one of the following manifolds*

$$S^3, \quad (S^1 \times S^2)^{\#k}, \quad (S^1 \underline{\times} S^2)^{\#k},$$

for values $k \in \{1, 12, 19, 21, 30, 63, 78, 82, 99, 154, 177, 183\}$.

"Corollary"

Deciding tightness for triangulations of 3-manifolds is in **P**.

Thank you

- B. Bagchi, B. Datta, J. Spreer. *Tight triangulations of closed 3-manifolds*. 19 pages, 1 figure.
  arXiv:1412.0412[math.GT]
- B. Bagchi, B. Burton, B. Datta, N. Singh, J. Spreer. *Efficient algorithms to decide tightness* . 18 pages, 3 figures.
  arXiv:1412.1547[cs.CG]
- F. Effenberger, J. Spreer. *simpcomp - A GAP package for simplicial complexes*, Version 2.0.0, 2014,
  http://code.google.com/p/simpcomp/