Topology ⟷ Combinatorics: Width-type parameters of 3-manifolds

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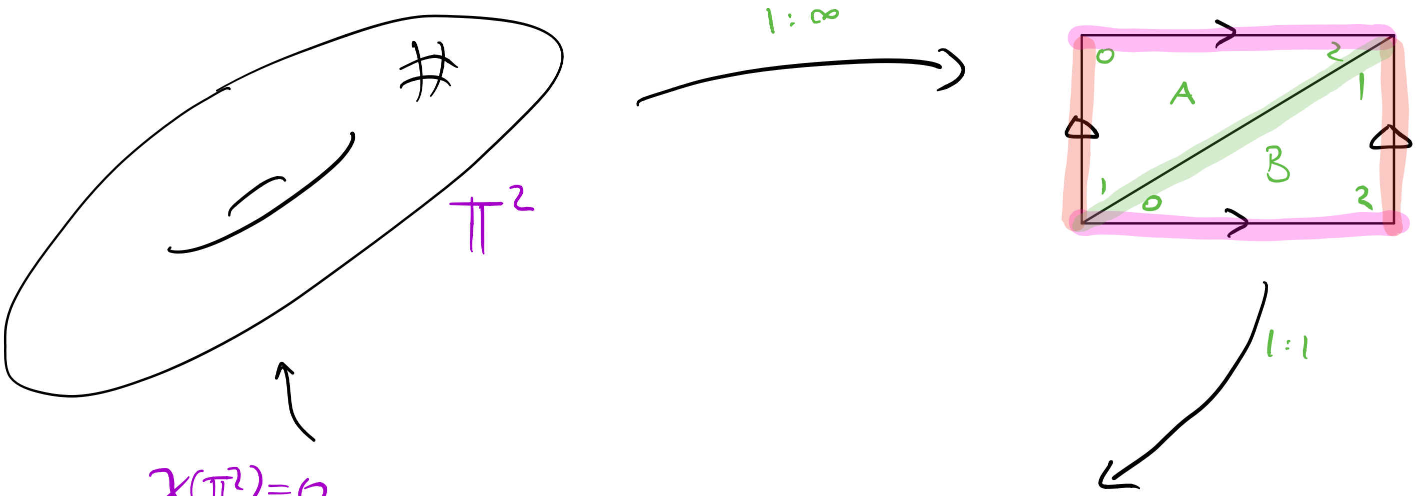


	01	02	12
A	B (12)	B (02)	B (01)
B	A (12)	A (02)	A (01)

$\chi(\mathbb{T}^2) = 0$

$H_*(\mathbb{T}^2) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$

$Or(\mathbb{T}^2) = \mathrm{TRUE}$

Motivation: Make use of theoretical advances in 3-manifold topology in practice.

Given two 3-manifolds M_1 and M_2, do we have

(1) $M_1 \cong M_2$?

(2) $M_1 \ncong M_2$?

heuristics

computationally hard

compute topological invariants

This is "difficult"

Q: Does it have to be difficult?

Observation: If M is given as a

sufficiently nice

"thin"

"tree-like" dual graph

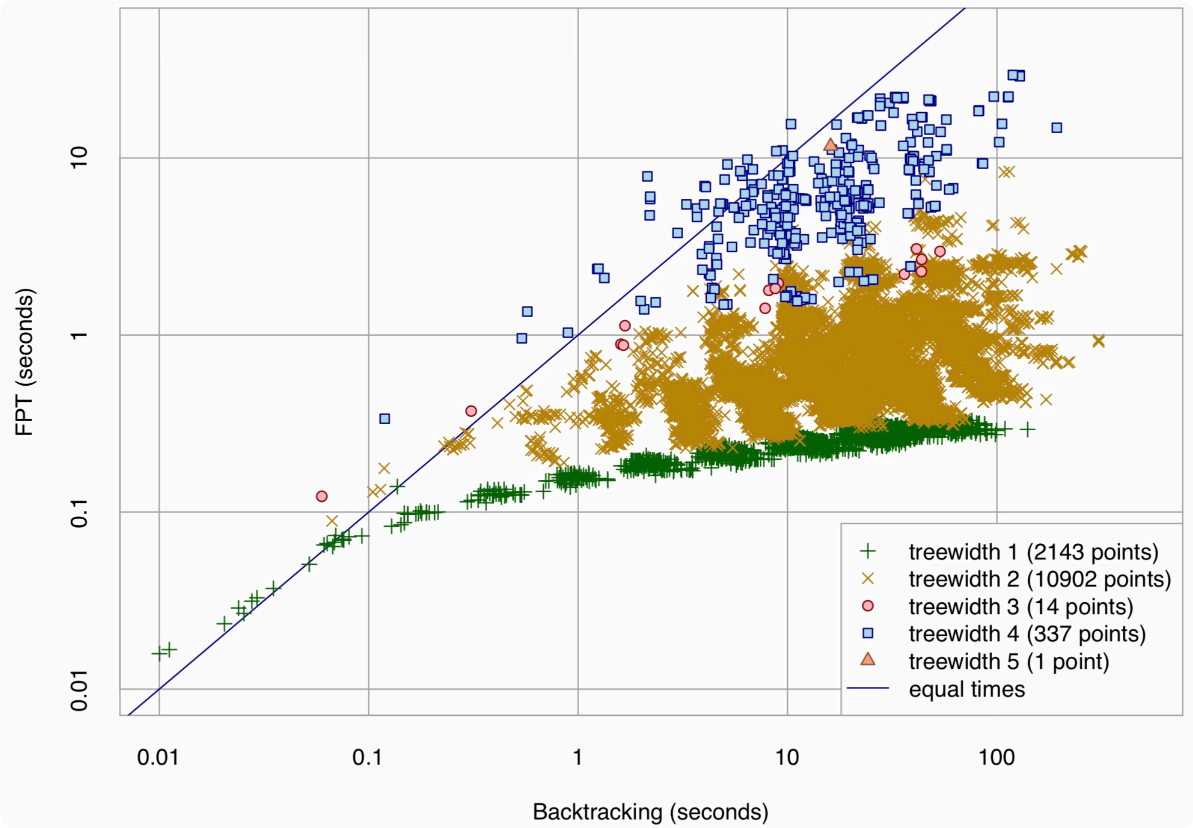
triangulation, then: **No!**

Example thm: (Burton, Maria, S.) T triangulation of 3-manifold M with n tetrahedra and dual graph of treewidth $\leq k$, then Turaev-Viro quantum invariants $TV_r(M)$ can be computed in

$$\boxed{\mathcal{O}\left((r-1)^{6k+6} \cdot k^2 \cdot \log(r) \cdot n \right)}$$

instead of $\mathcal{O}\left((r-1)^{n+1}\right)$

.... and this makes things easier in practice as well...



8 MINS.

Q : Given triangulation T, how to reduce the treewidth of its dual graph ?

sometimes easy

cannot be easy in general

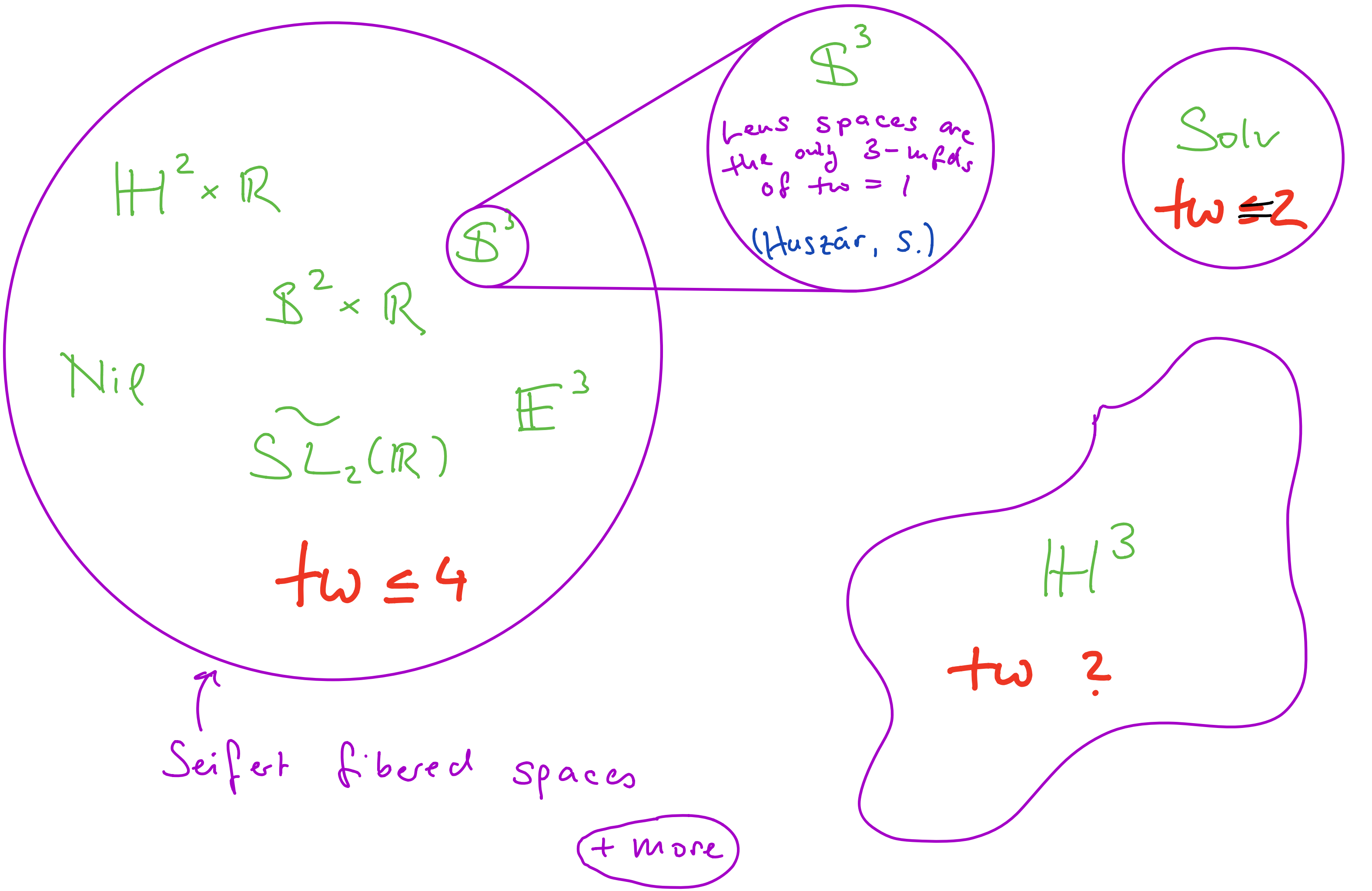
What are the theoretical limitations of this approach?

$$tw(M) = \min_{T \text{ triang. of } M} \{ tw(\text{dualgraph}(T)) \}$$

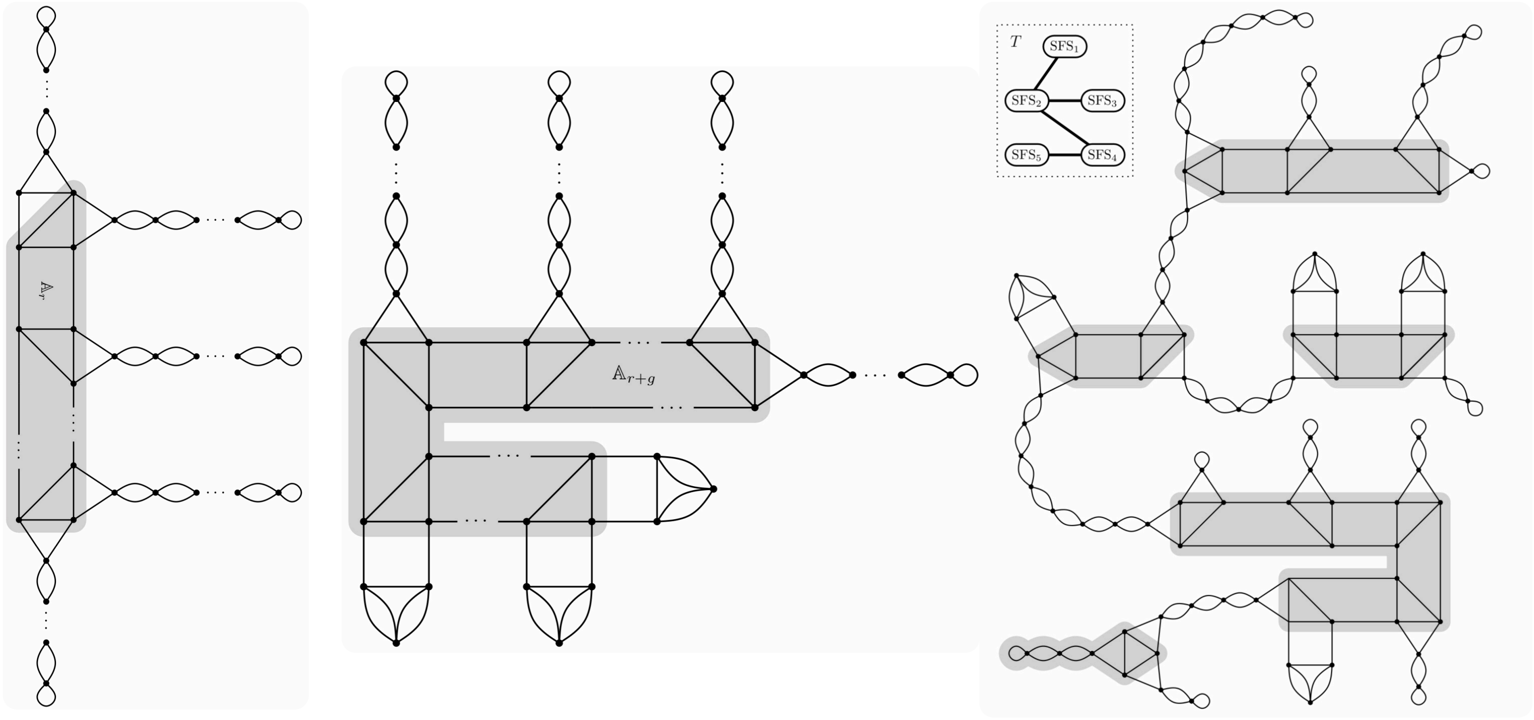
Q' : Given 3-manifold M, what is tw(M)?

Answer to Q' helps answering initial question.

Building blocks of 3-manifolds: Geometric 3-manifolds



Triangulations of Seifert Fibered Spaces with tw = 2



Q: Are there 3-manifolds of arbitrarily high treewidth?

Theorem (Huszár, S., Wagner): Yes

naive mathematician

There exist 3-manifolds of arbitrarily high tree-width.

This theorem connects combinatorial properties of triangulations with topological properties of their underlying manifolds.

just like $\boxed{v-e+f=2}$

↳ What is the underlying link?

$M \to$ handle decompositions

thin position

1

triangulations

2

dualgraph(T)

treewidth & congestion

20 MINS.

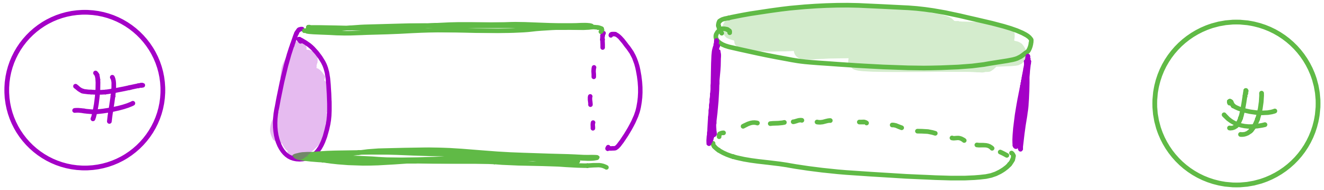
HANDLE DECOMPOSITIONS

k-handle:

- $\mathbb{B}^d = \mathbb{B}^{d-k} \times \mathbb{B}^k$
- $\partial\left(\mathbb{B}^{d-k} \times \mathbb{B}^k\right) = \left(\mathbb{S}^{d-k-1} \times \mathbb{B}^k\right) \cup \left(\mathbb{B}^{d-k} \times \mathbb{S}^{k-1}\right)$

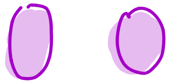
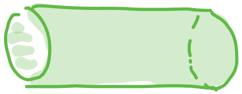
d = 3:

0-handles　　1-handles　　2-handles　　3-handles



$$\mathbb{B}^3 \times \mathbb{B}^0 \cong \mathbb{B}^2 \times \mathbb{B}^1 \cong \mathbb{B}^1 \times \mathbb{B}^2 \cong \mathbb{B}^0 \times \mathbb{B}^3$$

$$\mathbb{B}^2 \times \mathbb{B}^1 \;\big\downarrow \partial$$

$\left(\mathbb{B}^2 \times \mathbb{S}^0\right) \cup \left(\mathbb{S}^1 \times \mathbb{B}^1\right) =$  $\cup$ 

HANDLE DECOMPOSITIONS

Handle attachment:

dim : d

(1) Take an existing d-dim. manifold M

(2) Take a k-handle H_k

(3) Identify a region $R \cong \mathbb{B}^{d-k} \times \mathbb{S}^{k-1}$ on ∂M

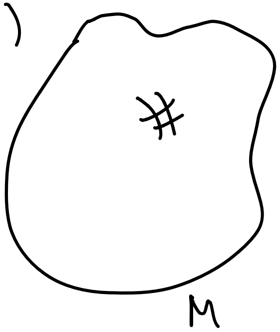
(4) Glue H_k to M along R

attachment map

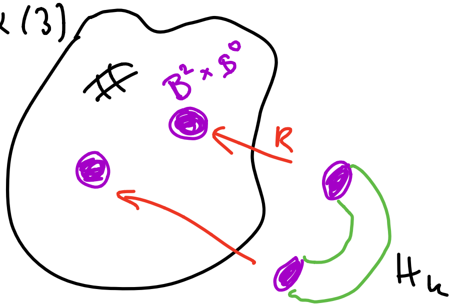
$N = M \cup_R H_k$

Example: $d = 3$; $k = 1$

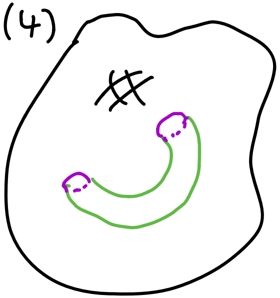
(1)



(2) & (3)



(4)



HANDLE DECOMPOSITIONS

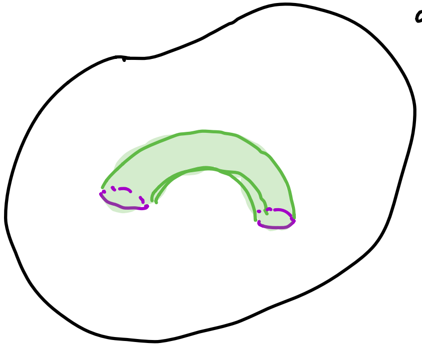
Handle attachments in dimension 3:

$k=0$

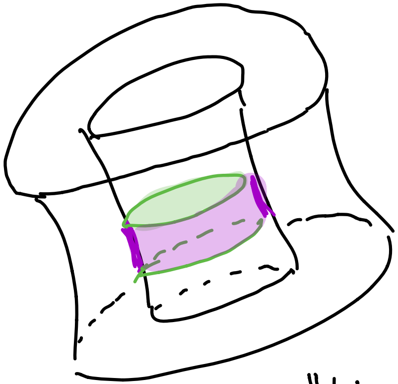
$\emptyset \mapsto$ 

creating connected component

"attaching a handle"



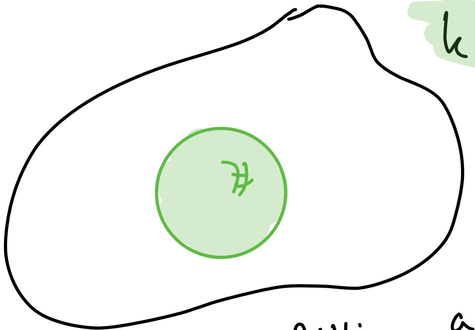
$k=1$



$k=2$

"blocking a pipe"

$k=3$



filling a void

HANDLE DECOMPOSITIONS

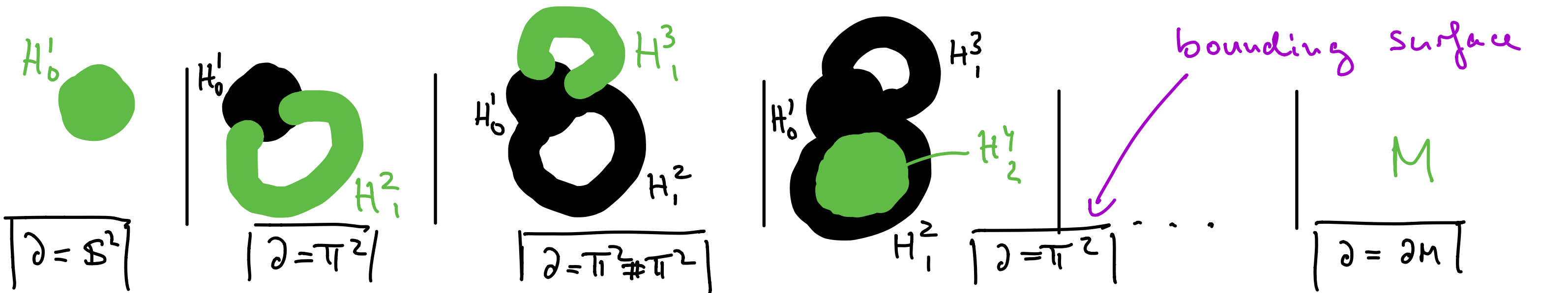
$\boxed{\text{Thm: Every 3-manifold can be decomposed into handles}}$

cut up manifold into handles respecting attachment maps

$$M = \{ H^1_{i_1}, H^2_{i_2}, \dots, H^n_{i_n} \}$$

types of handle

Idea: Build up M handle by handle by handle attachments



HANDLE DECOMPOSITIONS

Special case: Attach all 0- and 1- handles first

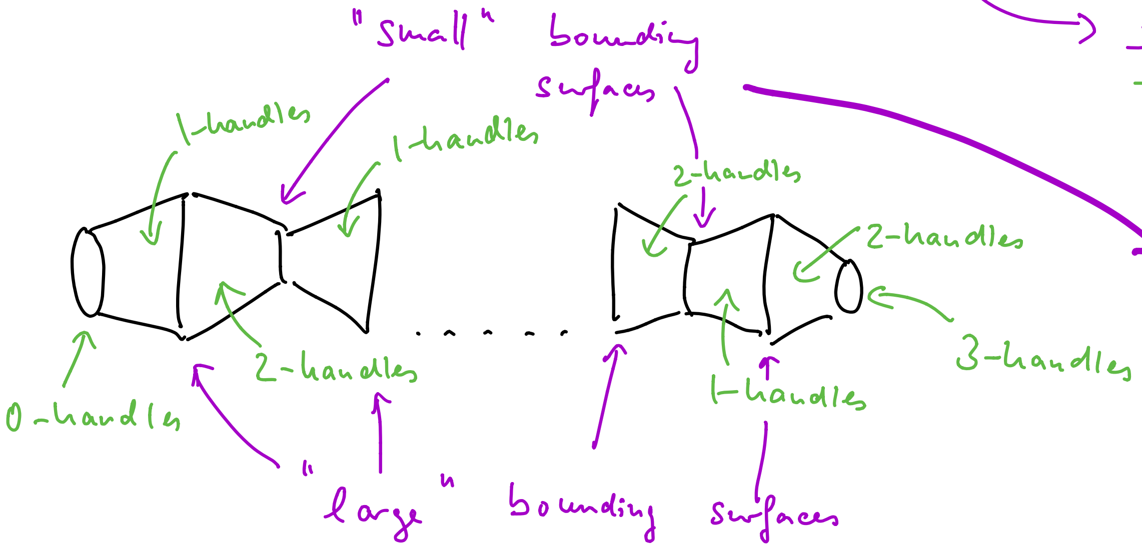
M = handle body $\cup_\partial$ handle body

Heegaard splitting

Schar lemann & Thompson: (1994)

Try to attach 1-handles as late as possible.

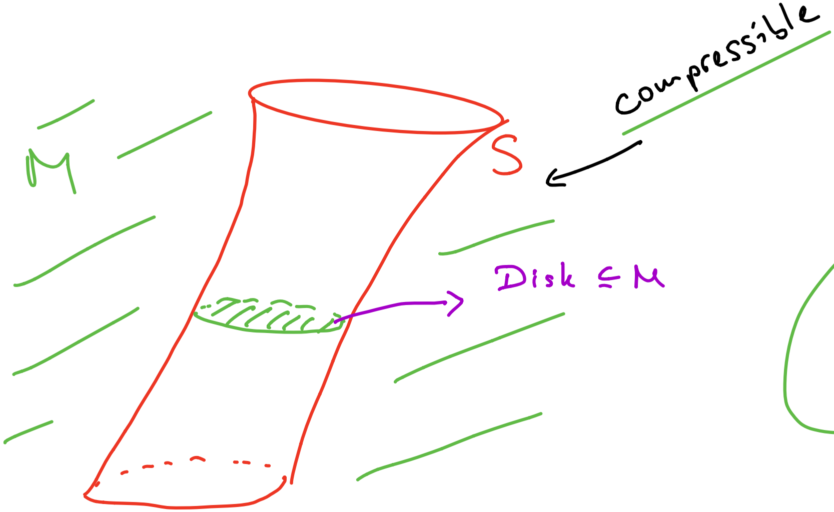
Put M into thin position

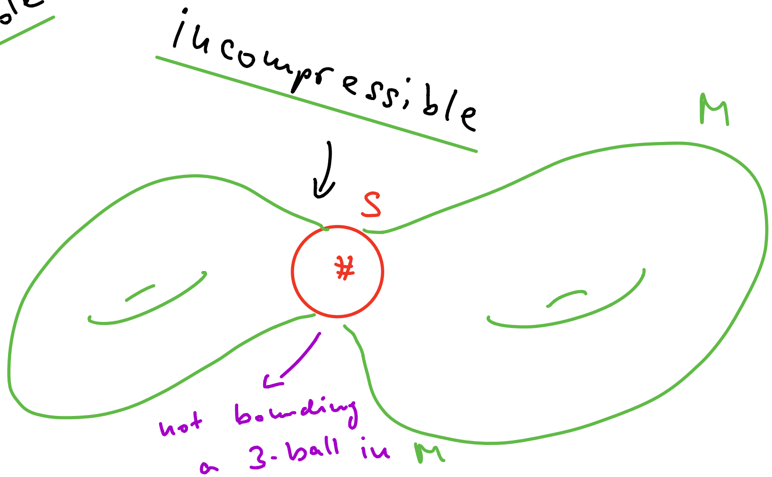


Small bounding surfaces are incompressible

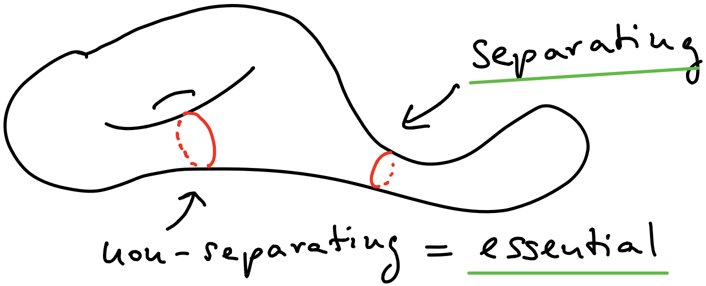
HANDLE DECOMPOSITIONS

• <u>Dim = 3</u> : Surfaces S in 3-manifolds M





• <u>Dim = 2</u> : Curves on surfaces:



HANDLE DECOMPOSITIONS

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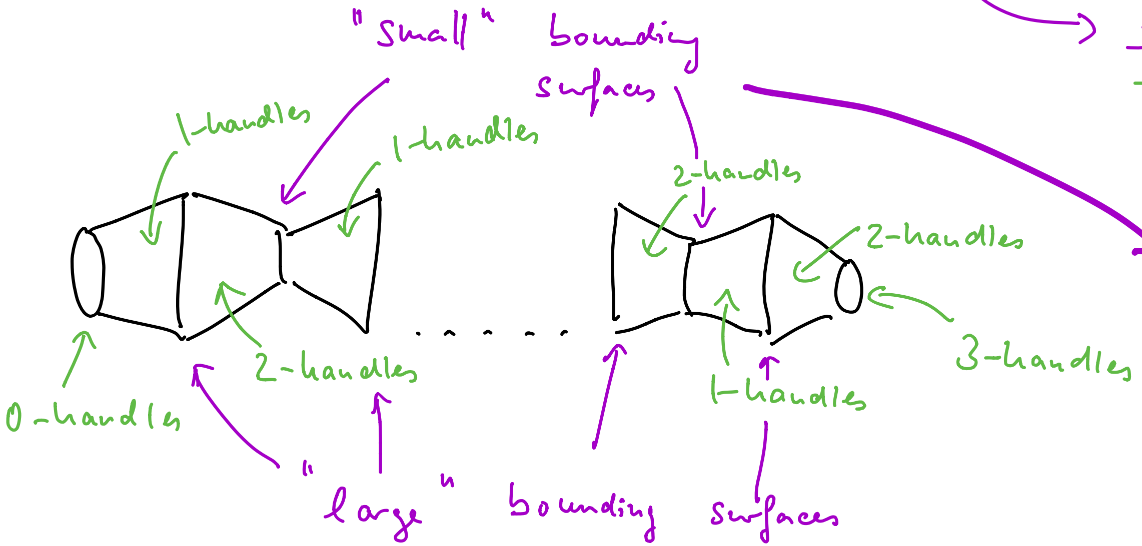
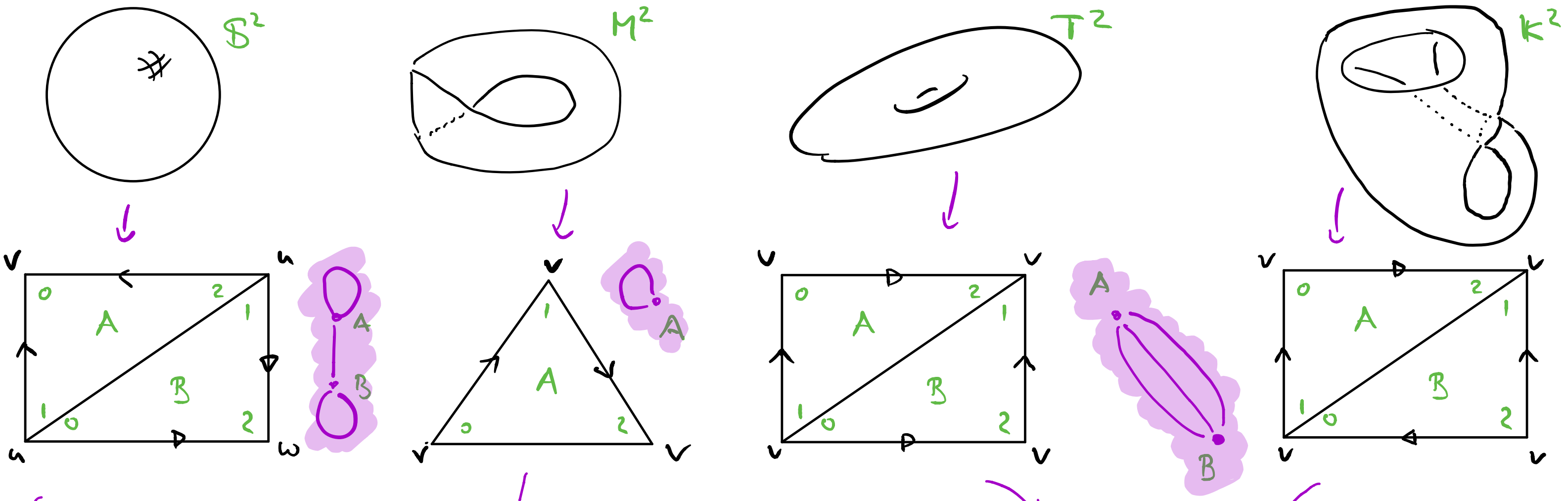


Small bounding surfaces are incompressible

TRIANGULATIONS (2-D)



2 triangles
3 edges
3 vertices

$\chi(\mathbb{S}^2) = 2$

	01	02	12
A	A(02)	A(01)	B(01)
B	A(12)	B(12)	B(02)

1 triangle
2 edges
1 vertex

$\chi(M^2) = 0$

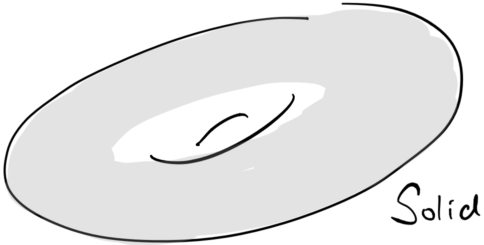
	01	02	12
A	A(12)		A(01)

2 triangles
3 edges
1 vertex

$\chi(\mathbb{T}^2) = \chi(\mathbb{K}^2) = 0$

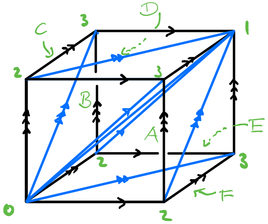
	01	02	12
A	B(12)	B(02)	B(01)
B	A(12)	A(20)	A(01)

TRIANGULATIONS (3-D)

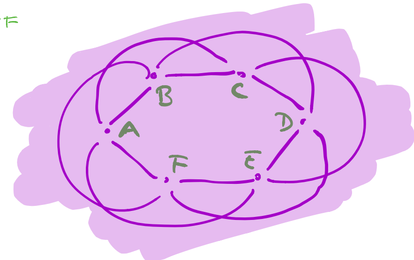


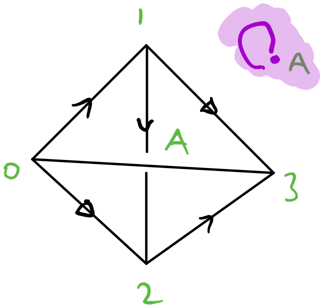
$\mathbb{B}^2 \times \mathbb{S}^1$

Solid torus



$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^3$





	012	013	023	123
A	F(012)	B(013)	E(231)	C(302)
B	C(012)	A(013)	D(231)	F(302)
C	B(012)	D(013)	A(231)	E(302)
D	E(012)	C(013)	F(231)	B(302)
E	D(012)	F(013)	C(231)	A(302)
F	A(012)	E(013)	B(231)	D(302)

	012	013	023	123
A	A(123)			A(012)

$\longrightarrow$ There are ~ 13.400 closed 3-manifolds admitting a triangulation of up to 11 tetrahedra.

38 MINS.

CANONICAL HANDLE DECOMPOSITION

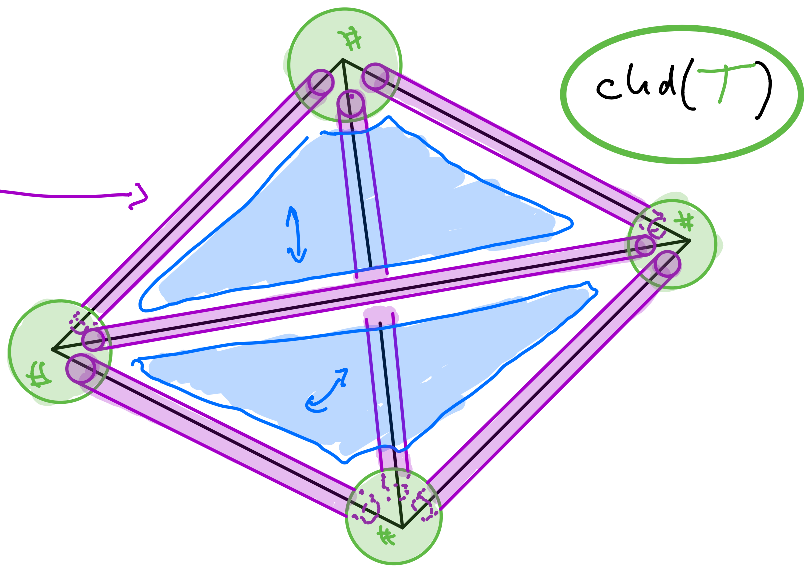
Connecting triangulations and handle decompositions

$\mathcal{T}$ triangulation:

0-handles	$\longleftrightarrow$	tetrahedra
1-handles	$\longleftrightarrow$	triangles
2-handles	$\longleftrightarrow$	edges
3-handles	$\longleftrightarrow$	vertices

0- & 1-handles: "thickened dual graph"

2- & 3-handles: "thickened graph"



Thm (Agol): There exist 3-mfds. M

(a) requiring an unbounded number of handles

(b) without incompressible surfaces

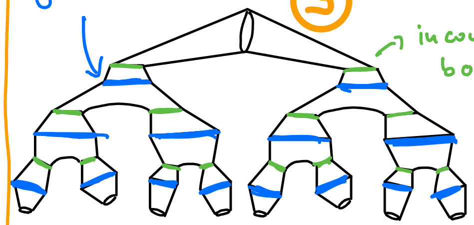
Thm (Saito, Scharlemann, Thompson): Thin position works for arbitrary graph layouts

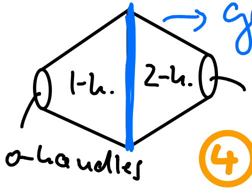
(A)

(B)

M does not have incompr. surfaces

genus $\leq 24(k+1)$



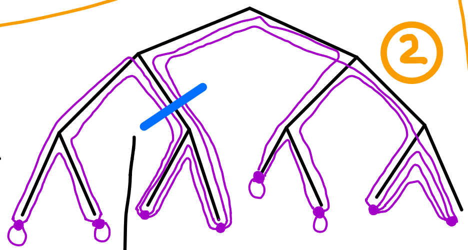


END

k unbounded

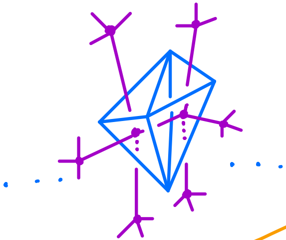
START

Triangulation T of M with dualgraph $(T) \leq k$



$\mathcal{H} = \mathrm{chd}(T)$

$tw \leq k$



$tw \leq k \Rightarrow \max |\text{cutset}| \leq 4(k+1)$

THANK YOU

Burton, Maria, S. Algorithms and complexity for Turaev-Viro invariants. Journal of Applied and Computational Topology, 2018.

Huszár, S., Wagner On the treewidth of triangulated 3-manifold. Journal of Computational Geometry, 2019.

Huszár, S. 3-manifold triangulations of small treewidth. Symposium of Computational Geometry, 2019.