Unstable Standing Waves in Inhomogeneous Schrödinger Equations

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Motivation - Bose-Einstein Condensates (BECs)

- Cool a dilute gas of weakly interacting bosons to near absolute zero (10^{-7} K).
- Quantum entanglement occurs:
  - All atoms drop into the lowest quantum state.
  - The length of the wave functions is longer than the distance between the atoms.
  - Have between $10^3 - 10^5$ atoms all with the same wave function.
- Macroscopic quantum effects are observable.
- Predicted in 1920's (Bose and Einstein).

- Found experimentally in 1990's by Cornell, and Wieman, (CU Boulder NIST-JILA) and Ketterle (MIT).
- Trap the gas in a $1 - D$ magnetic trap and tune the frequency and amplitude of its wave function.
- Release it and take as many pictures as you can in the 10^{-9}s that follow.
- Two phenomena can result from instabilities
    - "Burns" - damage equipment
    - "Bose-novas" - supernova like - implosion followed by an explosion - not good for equipment but perhaps could give a quantum model simulation of a supernova.

Setup

- We begin with an inhomogeneous nonlinear Schrödinger equation

$$\begin{array}{ll} i\Psi_t + \Psi_{xx} + |\Psi|^2\Psi = V\Psi & x \in \mathbb{R} \setminus U_I, \\ i\Psi_t + \Psi_{xx} - |\Psi|^2\Psi = 0 & x \in U_I \text{ a collection of intervals} \end{array} \tag{1}$$

- Pass to a rotating frame and consider solutions of the form $\Psi(x,t) = e^{-i\omega t}\psi(x,t), \omega \in \mathbb{R}_+$
- Standing wave solutions are real, t independent solutions $u(x)$ to the ODE

$$\begin{array}{rclc} u_{xx} & = & (V-\omega)u - u^3 & x \in \mathbb{R} \setminus U_I, \\ u_{xx} & = & -\omega u + u^3 & x \in U_I. \end{array} \tag{2}$$

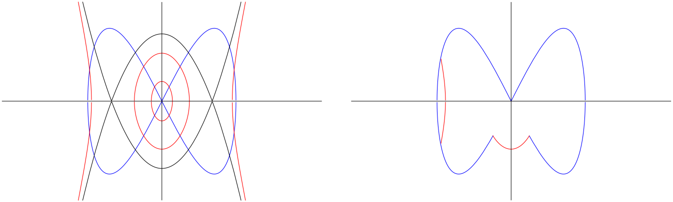
A Composite Phase Portrait

- Use a dynamical systems approach and build a ‘composite phase portrait’

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 \qquad x \in \mathbb{R} \setminus U_I, \\ u_{xx} &= -\omega u + u^3 \qquad x \in U_I. \end{aligned} \tag{3}$$

- An ‘outer’ system with a homoclinic orbit.
- An ‘inner’ system with periodic orbits and a heteroclinic orbit.
- Require that $(V-\omega) > 0$, so that we can find u such that $u, u_x \to 0$ as $x \to \pm\infty$.
- Solutions begin in the outer system and then ‘flip’ to the inner system, and then flip back to the outer system, repeating for each component of U_I.

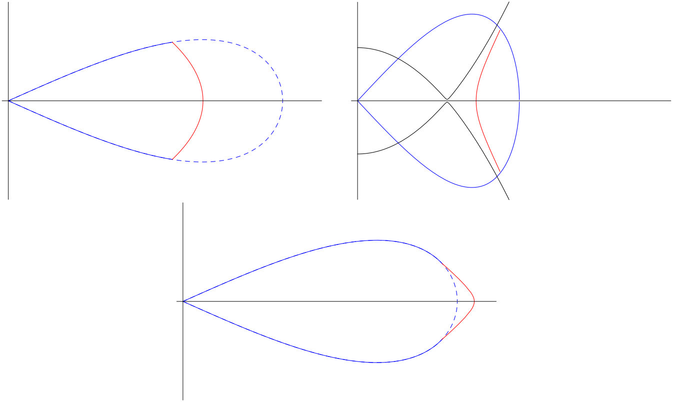
$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 & x &\in \mathbb{R}\setminus U_I, \\ u_{xx} &= -\omega u + u^3 & x &\in U_I. \end{aligned} \tag{4}$$



- Solutions begin on the blue curve and then ‘flip’ to one of the red curves, and then ‘flip’ back to the blue curve. Then back to the red. Etc. But always decay to the origin.

The case when $U_I = (-L, L)$

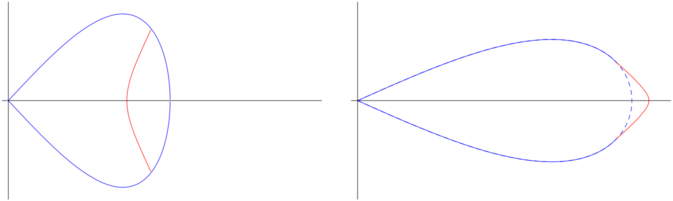
- The real value L corresponds to the length of the trap.
- Here there are are only two 'flips'. One at $-L$ and again at L.
- Relative position of the heteroclinic orbit of the inner system and the homoclinic orbit of the outer system determines when unstable solitons can occur.



Theorem

If $U_I = (-L, L)$, then positive, unstable orbits appear when $\dfrac{\omega}{V} < \dfrac{3}{4}$.

- Focus on ‘symmetric’ orbits - symmetric about one (or both) of the axes in the phase plane.
- Positive solutions are ground states in the context of BECs.
- Unstable excited states appear for all $\omega < V$.



- The proof relies on a theorem of Jones to reduce everything to geometric conditions on the phase portrait.
- Let

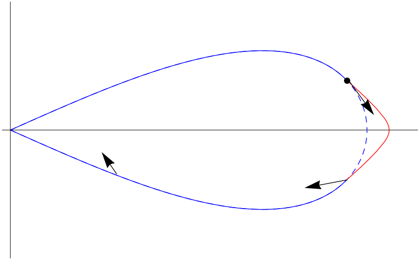
$$\begin{aligned} Q &= \text{the number of zeros of the standing wave } u(x). \\ P &= \text{the number of zeros of a solution to the} \\ &\quad\ \text{variational equation along } u(x). \end{aligned}$$

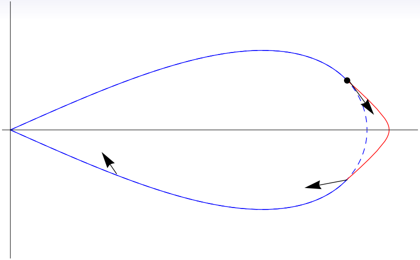
Theorem (Jones)

If $|P - Q| \neq 0, 1$ then the standing wave is unstable.

An Example

- This is a Maslov index calculation. $|P - Q| - 1$ is a lower bound for the Maslov index of the solution.
- Q is easy to determine (below $Q = 0$).
- P is the number of times a vector initially tangent to the solution in the phase plane is pushed through the vertical as the base point moves along the orbit (below $P = 2$).

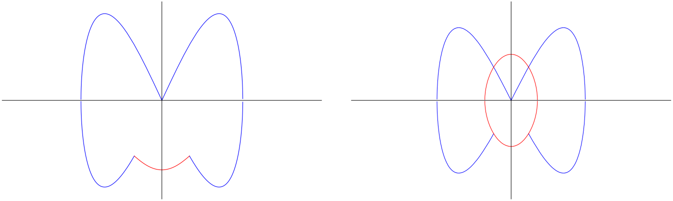




- To calculate P, use a dynamical systems approach to analyze the variational equation along the orbit (u, u_x).
- Use the fact that variational flows are orientation preserving.
- Use the cross product of the tangent vectors to the inner and outer systems to determine when a vector has passed through the vertical i.e. P has increased.
- Analyze the long time dynamics of the variational equation near the origin in the phase plane.

Excited States

- Determination of instability is a topological argument. It does not require solutions to be positive.
- Left is an example of an unstable excited state ($Q = 1, P = 3$).
- Can produce an unstable solution with any desired number of zeroes (right is an example with $Q = 2n + 1, P = 2n + 3$).

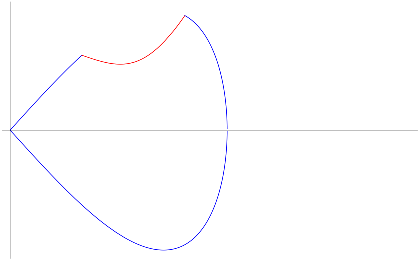


Stability

- When $P - Q = 0, 1$, solutions are not necessarily stable.
- Further criteria are needed. (V-K, and G-S-S for ground states.)
    - Conservation of energy.
    - Conservation of mass.
- Above criteria only apply to ground states.
- Stability of excited states with $P - Q = 0, 1$ is unclear.
- Maslov index characterization of stability of excited states unknown.

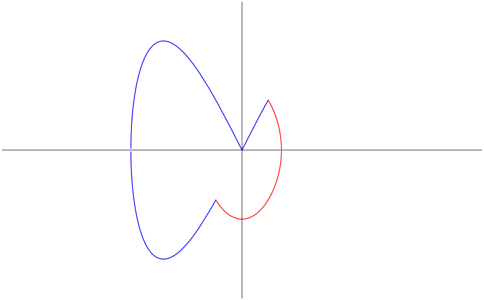
Asymmetric States

- Asymmetric states have been difficult to analyze. Often $P - Q = 0, 1$.



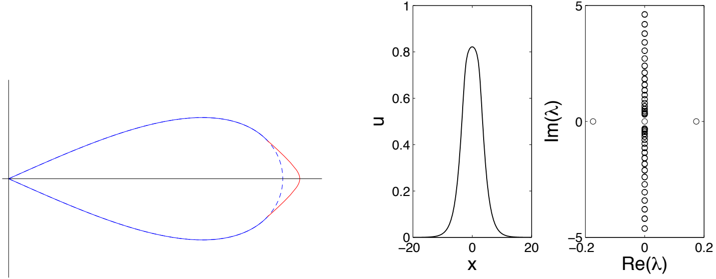
- Ground states can use (G-S-S and V-K) criteria.

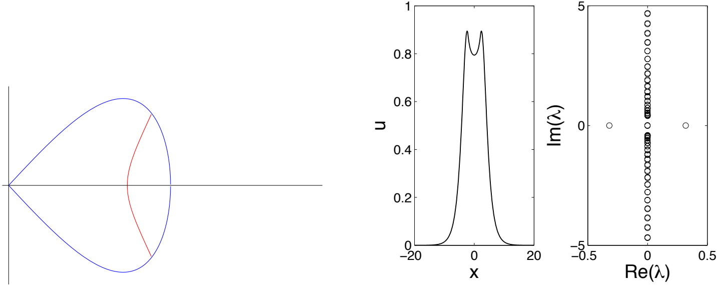
- Below is an example of an asymmetric excited state with $Q = 1, P = 2$.



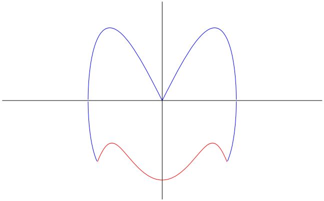
- Stability is unknown. Numerically all eigenvalues are purely imaginary.

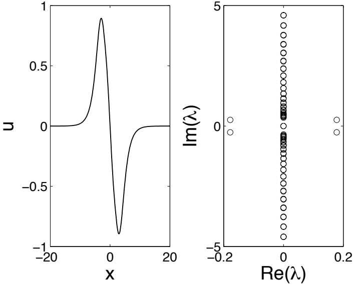
Solution Profiles and Phase Portraits with $P - Q = 2$.

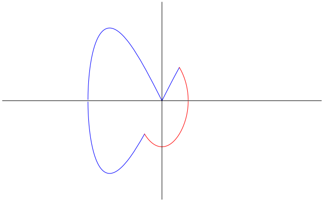


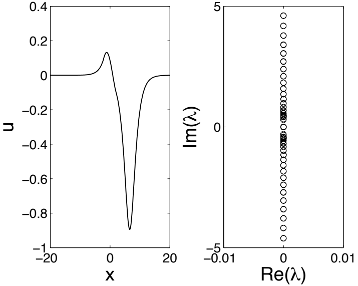


Solution Profiles and Phase Portraits with $P - Q = 1$.

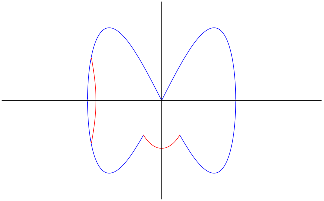


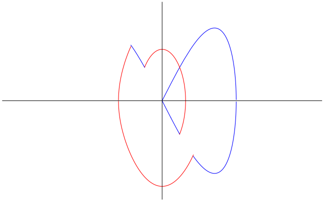






- Can also have any number of of jumps between outer and inner systems.
- Below are when U_I is two intervals. Left is unstable. Right is stable (numerically).





Gap Solitons

- Gap solitons are when U_I has an infinite number of components.

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 \qquad x \in \mathbb{R} \setminus U_I, \\ u_{xx} &= -\omega u + u^3 \qquad x \in U_I. \end{aligned} \tag{5}$$

- We'll focus on the case when U_I is symmetric about 0.

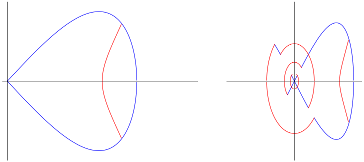
$$U_I = \dots(-x_4,-x_3)\cup(-x_2,-x_1)\cup(-L,L)\cup(x_1,x_2)\cup(x_3,x_4)\dots$$

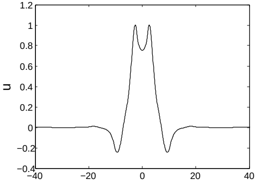
- The initial interval $(-x_0, x_0) = (-L, L)$ can be arbitrary (or not).
- The others $(\pm x_i, \pm x_{i+1})$ occur with (some) regularity.
- Designate the x_is later on.

- View a gap soliton as a ‘central’ solution, i.e. a solution to

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 \qquad x \in \mathbb{R} \setminus (-L, L), \\ u_{xx} &= -\omega u + u^3 \qquad x \in (-L, L). \end{aligned} \tag{6}$$

- With 'ripples' in the tails.





- Use phase portrait techniques and dynamical systems approach from before.
- Create a family of (unstable) solitons which
  - Decay to zero,
  - Have a constant, finite Maslov index. (Actually $P - Q$ is the same).
  - Converge (in H^1) to a gap soliton.
- To do this (and confirm instability) requires choosing the x_i's and initial condition correctly.

A Linear Inner System

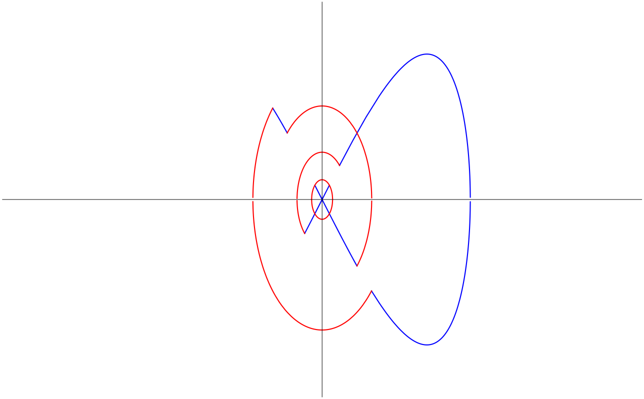
- When the inner system is linear

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 & x \in \mathbb{R} \setminus U_I, \\ u_{xx} &= -\omega u & x \in U_I \end{aligned} \tag{7}$$

$$U_I = \ldots \cup (-x_2, -x_1) \cup (-L, L) \cup (x_1, x_2) \cup \ldots$$

- Still have a homoclinic orbit in the outer system.
- Choose x_i's so that length of the intervals is $\frac{\pi}{\sqrt{\omega}}$.
- Choose initial conditions so that $(u(x_1), u_x(x_1))$ is on the homoclinic orbit.

- Below is an example of a phase portrait of a gap soliton with linear inner system.



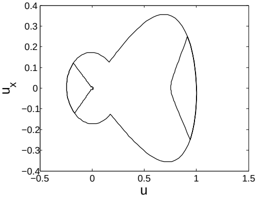
- Chose $L = 0$ in this example.

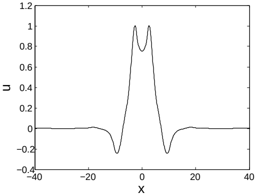
- Set $U_0 = (-L, L)$.
- Let $U_n = U_0 \cup$ the first $2n$ components on either side of U_0.
- E.g. $U_1 = \ldots (-x_4, -x_3) \cup (-x_2, -x_1) \cup (-L, L) \cup (x_1, x_2) \cup (x_3, x_4) \ldots$
- In order to readily find more unstable solutions, we use a nonlinear inner system for U_0.
- A (linear) gap soliton is then the limit of solutions to

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 & x &\in \mathbb{R} \setminus U_n, \\ u_{xx} &= u^3 - \omega u & x &\in U_0 \\ u_{xx} &= -\omega u & x &\in U_n \setminus U_0 \end{aligned} \tag{8}$$

- Below is a phase portrait, as well as a solution profile of a gap soliton constructed in this manner.

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 & x &\in \mathbb{R} \setminus U_n, \\ u_{xx} &= u^3 - \omega u & x &\in U_0 \\ u_{xx} &= -\omega u & x &\in U_n \setminus U_0 \end{aligned} \tag{9}$$





- Let f_n be the solution to

$$\begin{array}{rcl} u_{xx} &=& (V-\omega)u-u^3 \qquad x\in\mathbb{R}\setminus U_n, \\ u_{xx} &=& u^3-\omega u \qquad x\in U_0 \\ u_{xx} &=& -\omega u \qquad x\in U_n\setminus U_0 \end{array} \qquad (10)$$

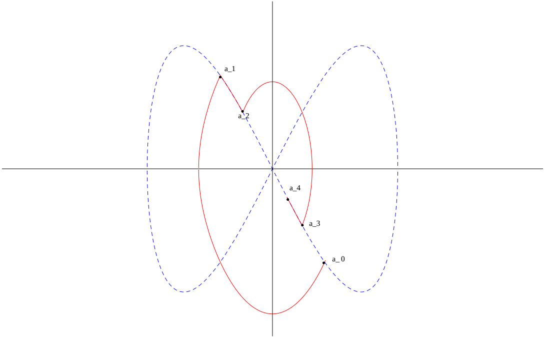
  as constructed.
- Let f be the limit of the $f_n's$ (in H^1).

Theorem

The Maslov indices of all the f_n's, and their limit f (a gap soliton) are equal. In particular if f_0 is unstable, then they all are.

Sketch of Proof

- Show that ripples contribute nothing to the phase portrait calculation of the Maslov index.
- Contribution to both P and Q from each piece is 2.



- In fact, modulo 2π, there is no effect whatsoever on the variational equation by the ‘detour’.

An Alternate Sketch Using Topology

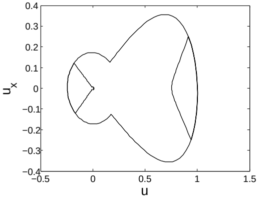
- A solution in H^1 can be thought of as a curve in $\Lambda(2)$, the space of Lagrangian planes.
- The Maslov index of the solution is the (fixed end point) homotopy class of this curve in $\pi_1(\Lambda(2)) \approx \mathbb{Z}$.
- $|P - Q| - 1$ is a lower bound for the Maslov index.
- The Maslov index being nontrivial means that there is a real, positive eigenvalue to the linearized operator about the soliton. (See theorem 2.)
- The curves of the solutions with the ripples are all homotopic to the curve of the central solution (without the ripples).

The Fully Nonlinear Case

- Proof techniques in linear case do not rely on explicit knowledge of solutions to inner system.
- Can extend this to when inner system is fully nonlinear and same result holds.
- The choosing of the $x_i's$ from a numerical viewpoint becomes more difficult.
- Choose x_i's so as to travel between invariant manifolds of the outer system.
- Initial condition is still on the homoclinic orbit.
- The lengths of the components will still be fairly regular. The limit of the lengths of the components of $U_I \to \frac{\pi}{\sqrt{\omega}}$.

- Gap solitons are called as such because the introduction of the (almost) periodic varying nonlinear term introduces band gaps to the spectrum of the Schrödinger operator.
- If ω is chosen so as to be in the band gap, a soliton can be formed.
- For a fixed V and a fixed length of intervals U_I, there is a range of ω that will produce gap solitons.
- The gap soliton as constructed has ω in the exact center of a band gap.

- There are actually many solutions that decay to zero (for any fixed V and ω).
- Each one corresponds to a different length of the components of the U_I which contain the value ω in a band gap.



- This means that instability is robust. The initial condition and the lengths of components of U_I do not have to be exact, and theorem 3 will still hold.

Stability of Gap Solitons

- Stability criteria for gap solitons is unknown.
- Conjecture that for *certain* values of the lengths of components of U_I (for a fixed ω), stability of 'central' soliton will not be disrupted by the ripples in the tails.
- For a fixed ω, the lengths of the components that don't disrupt stability will be close to $\frac{\pi}{\sqrt{\omega}}$.
- How close is unclear. Lengths of intervals for which ω is near the edge of a band gap, introduce eigenvalues with positive real part.

Surface Gap Solitons

- These techniques also apply to so-called 'surface gap solitons'
- These are when $U_I = (-L, L) \cup (x_1, x_2) \cup (x_3, x_4), \ldots.$
- Theorem 3 still holds. If central soliton is unstable then so is the surface gap soliton.
- Numerically, initial condition is easier to choose.
- Still have difficulty with the (precise) interval length in the fully nonlinear case.

Summary I

For the first part of the talk, $U_I = (-L, L)$.

- Can determine instability of standing waves to inhomogeneous NLS equations using the Maslov index and phase portrait techniques.
- Shape of the orbit's curve in the phase plane determines instability.
- Beyond symmetric ground state orbits, there is an array of excited and asymmetric states.
- Complete stability criteria of these excited states and excited asymmetric states has not been fully developed.

Summary II

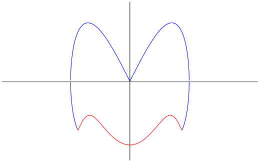
In the second part of the talk U_I was a possibly infinite collection of intervals.

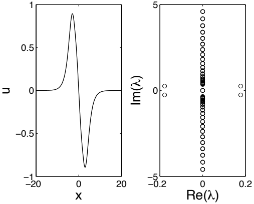
- View a gap soliton as a central soliton plus tails with ripples.
- Created a family of (potentially) unstable standing waves whose limit is a gap soliton.
- Ripples don't contribute to the Maslov index.
- Instability is robust. Stability is more delicate.
- Result and techniques also work for surface gap solitons.

Further

- Is there a way to explicitly determine the appropriate lengths of the components of U_l so as to guarantee existence of a gap soliton?
- Can geometric stability criteria be established for excited states and gap solitons?
- Is there a way to explicitly determine the appropriate lengths of the components of U_l so as to not disrupt stability?
- Can construct asymmetric gap solitons as well. Stability properties are unclear here also.

- These equations when U_I is a single interval are also used to model light propagation down a fiber optic cable.
- For a fixed L and the correct tuning of ω and V, multiple solitons can be present.
- So far only one of them is (potentially) stable - ground state
- Is there a way to stabilize an excited state via the introduction of a defect?





Thank You