Localized standing waves in inhomogeneous Schrödinger equations

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Motivation - Bose-Einstein Condensates (BECs)

- Cool a dilute gas of weakly interacting bosons to near absolute zero (10^{-7} K).
- Quantum entanglement occurs:
  - All atoms drop into the lowest quantum state.
  - The length of the wave functions is longer than the distance between the atoms.
  - Have around 1000 atoms all with the same wave function.
- Macroscopic quantum effects are observable.
- Predicted in 1920's (Bose and Einstein).
- Found experimentally in 1990's (Cornell, Wieman, and Ketterle).

- Trap the gas in a $1 - D$ magnetic trap and tune the frequency and amplitude of its wave function.
- Release it and take as many pictures as you can in the 10^{-9}s that follow.
- Two phenomena can result from instabilities
  - "Burns" - damage equipment
  - "Bose-novas" - supernova like implosion followed by an explosion - not good for equipment but perhaps could give a quantum model simulation of a supernova.

Mathematical Model

- We begin with an inhomogeneous nonlinear Schrödinger equation

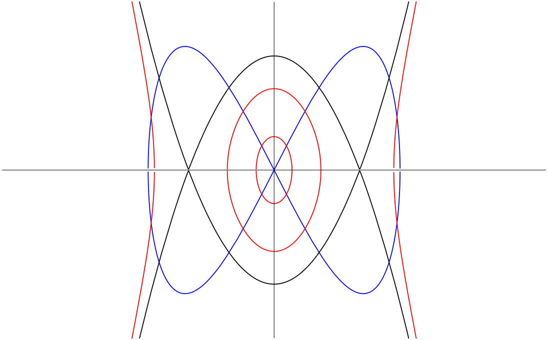
$$\begin{aligned} &i\Psi_t + \Psi_{xx} + |\Psi|^2\Psi = V\Psi \qquad &|x| > L, \\ &i\Psi_t + \Psi_{xx} - |\Psi|^2\Psi = 0 \qquad &|x| < L, \end{aligned} \tag{1}$$

- Pass to a rotating frame and consider solutions of the form $\Psi(x,t) = e^{-i\omega t}\psi(x,t)$
- Standing wave solutions are real, t independent solutions $u(x)$ to the ODE

$$\begin{aligned} u_{xx} &= (V-\omega)u - u^3 \qquad &|x| > L, \\ u_{xx} &= -\omega u + u^3 \qquad &|x| < L. \end{aligned} \tag{2}$$

A composite phase portrait

- An 'outer' system with a homoclinic orbit (requires that $(V - \omega) > 0$)
- An 'inner' system with periodic orbits and a heteroclinic orbit.

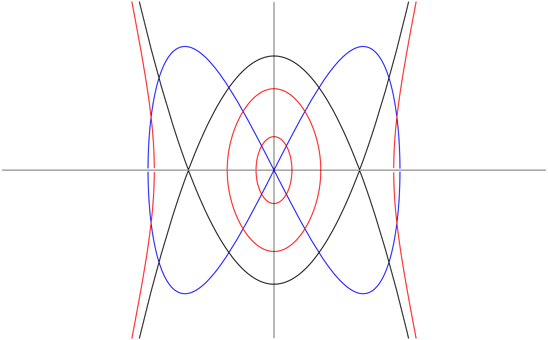


- Solutions begin on the blue curve and then 'flip' to one of the red curves, and then 'flip' back.

Theorem

Positive, unstable orbits appear when $\dfrac{\omega}{V} < \dfrac{3}{4}$.

- Focus on 'symmetric' orbits - symmetric about one of the axes in the phase plane.
- Positive solutions are ground states in the context of BECs.
- Unstable excited states appear for all $\omega < V$.



- The proof relies on a theorem of Jones to reduce everything to geometric conditions on the phase portrait.
- Let

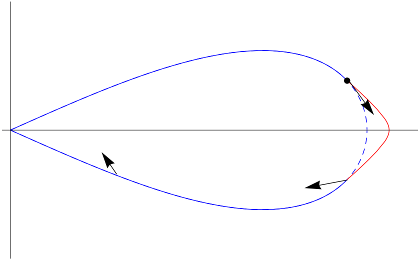
$$\begin{aligned} Q &= \text{the number of zeros of the standing wave } u(x). \\ P &= \text{the number of zeros of a solution to the} \\ &\quad\ \text{variational equation along } u(x). \end{aligned}$$

Theorem (Jones)

If $|P - Q| \neq 0, 1$ *then the standing wave is unstable.*

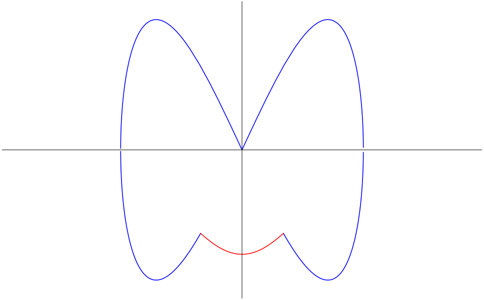
An Example

- This is a Maslov index calculation.
- Q is easy to determine (below $Q = 0$).
- P is the number of times a vector initially tangent to the solution in the phase plane is pushed through the vertical as the base point moves along the orbit (below $P = 2$).

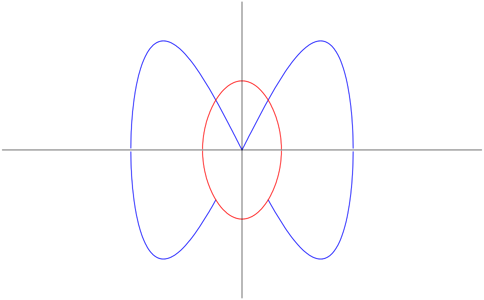


Excited States

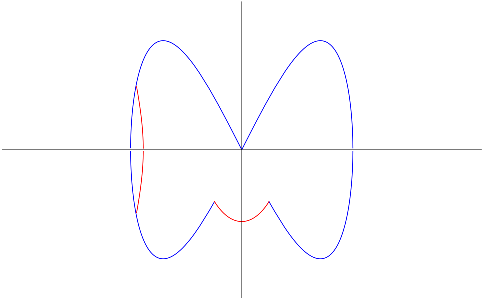
- Determination of instability is a topological argument, and does not require solutions to be positive.
- Below is an example of an unstable excited state ($Q = 1, P = 3$).



- Can produce an unstable solution with any desired number of zeroes (below is an example with $Q = 2n + 1, P = 2n + 3$).



- Can also have any number of of jumps between outer and inner systems.

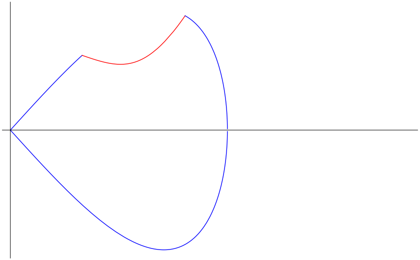


Stability

- When $|P - Q| = 0, 1$, solutions are not necessarily stable.
- Further criteria are needed. (V-K, and G-S-S for ground states.)
  - Conservation of energy.
  - Conservation of mass.
- Above criteria only apply to ground states.
- Stability of excited states is unclear.
- Maslov index characterization of stability unknown.
- Stabilize an excited state via the introduction of defects.

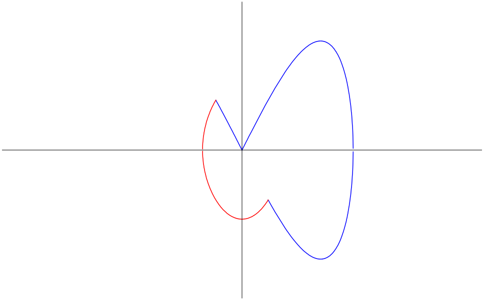
Asymmetric States

- Asymmetric states have been difficult to analyze. Often $|P - Q| = 0, 1$.



- Ground states can use (G-S-S and V-K) criteria.

- Below is an example of an asymmetric excited state with $Q = 1, P = 2$.



- Stability is unknown. Numerically all eigenvalues are purely imaginary.

Thank You