

Stability and Instability in Kink Wave Solutions to the Sine-Gordon Equation

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Introduction

- The Sine-Gordon equation:

$$u_{tt} = u_{xx} + \sin u$$

- Used to model magnetic flux propagation, DNA modeling, crystal dislocation, 'splay wave' propagation along a lipid membrane.
- A nice physical model is the continuum limit of the 'ribbon -pendulum' - a line of pendula weakly coupled to their nearest neighbor via Hooke's law.
- References:
 - (i) Georgiev DD, Papaioanou SN, Glazebrook JF. *Neuronic system inside neurons: molecular biology and biophysics of neuronal microtubules*. Biomedical Reviews 2004; 15: 67-75.
 - (ii) Georgiev DD. *Solitonic effects of the local electromagnetic field on neuronal microtubules ? tubulin tail sine-Gordon solitons could control MAP attachment sites and microtubule motor protein function*. CogPrints ID 3894
 - (iii) wikipedia.org/wiki/Sine_Gordon_equation

Setup

- The sine-Gordon equation:

$$u_{tt} = u_{xx} + \sin u$$

- Looking for travelling waves, set $z = x + ct$, and consider solutions of the form $v(x, t) = v(x + ct, t) = v(z, t)$.
- So v solves

$$(c^2 - 1)v_{zz} + 2cv_{zt} + v_{tt} = \sin v$$

- Solutions to (3) that depend only on z will solve the pendulum equation:

$$(c^2 - 1)v_{zz} = \sin v$$

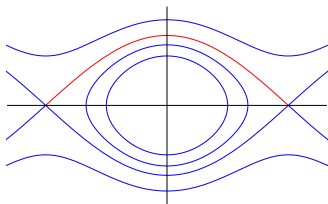
- Investigate the spectrum of the linearized operator corresponding to

$$(c^2 - 1)v_{zz} + 2cv_{zt} + v_{tt} = \sin v$$

around (kink wave) solutions to

$$(c^2 - 1)v_{zz} = \sin v$$

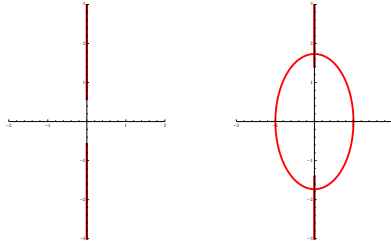
- *Kink waves* correspond to heteroclinic orbits of the pendulum equation.



- *Subluminal kink waves* are when $c^2 - 1 < 0$.
- *Superluminal kink waves* are when $c^2 - 1 > 0$.

Theorem

Subluminal kink waves are stable. Superluminal kink waves are unstable.



Real Eigenvalues

- Formally the linearization of the travelling wave equation about a kink-wave solution $v(z)$ is:

$$(c^2 - 1)p_{zz} + 2cp_{zt} + p_{tt} = (\cos v)p$$

- Writing $p_t = q$ leads to the eigenvalue problem:

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = (\cos v)p - (c^2 - 1)p_{zz} - 2cq_z = \lambda \begin{pmatrix} p \\ q \end{pmatrix}$$

- Which reduces to the ODE:

$$(c^2 - 1)p_{zz} + 2c\lambda p_z + (\lambda^2 - \cos v)p = 0.$$

- Consider L^2 perturbations of kink waves.
- **Definition:** λ is an eigenvalue with eigenfunction p if there is a solution to the ODE

$$p_{zz} + \frac{2c\lambda}{c^2 - 1} p_z + \frac{\lambda^2 - \cos v}{c^2 - 1} p = 0.$$

with

$$\lim_{z \rightarrow \pm\infty} p = 0.$$

- The next step is to reformulate this eigenvalue condition in an equivalent, geometric way.

Geometric Reformulation of the Eigenvalue Problem

- Rewrite (7) as a system. Set $' = \frac{d}{dz}$. Also let $p = p_1$, and $p_2 = p_1'$:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{\cos v - \lambda^2}{c^2 - 1} & \frac{-2c\lambda}{c^2 - 1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} =: A(z, \lambda) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

- Define

$$A(\lambda) := \lim_{z \rightarrow \pm\infty} A(z, \lambda) = \begin{pmatrix} 0 & 1 \\ \frac{(\mp 1) - \lambda^2}{c^2 - 1} & \frac{-2c\lambda}{c^2 - 1} \end{pmatrix}$$

- $A(\lambda)$ has a stable and an unstable subspace denoted by ξ^s and ξ^u respectively.
- **Note:** The \mp in $A(\lambda)$ depends on sub or superluminal case, not $\pm\infty$.

- **Geometric Eigenvalue Condition:** Given that $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ solves the ODE

$$\lim_{z \rightarrow \pm\infty} p_1 = 0$$

is equivalent to

$$\lim_{z \rightarrow -\infty} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow \xi^u, \text{ and } \lim_{z \rightarrow \infty} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow \xi^s$$

- Compactify by reparameterizing the extended phase plane with a new variable $z(\tau)$, $\tau \in [-1, 1]$, so that flow can be continuously extended on $\mathbb{R}^2 \times [-1, 1]$
- Then consider induced flow on $S^1 \times [-1, 1]$.
- The stable and unstable subspaces ξ^u and ξ^s are fixed points of this flow, with (unique) 1-d unstable and stable manifolds (respectively).

The Winding Number of a Solution

- Consider the line $\ell(z)$ defined by

$$\ell(z) = \left\{ \left\langle \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\rangle \mid \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \text{ solves the ODE and} \right. \\ \left. \lim_{z \rightarrow -\infty} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \rightarrow \xi^u \right\}$$

- $\ell(z)$ is a curve on S^1 .
- The idea is to look at how $\ell(z)$ can cross the point ξ^s on S^1 .

Lemma

For all values of λ , $\ell(z)$ can cross ξ^s in only one direction.

Sketch of proof.

- Define $\theta(z)$ as the angle that $\ell(z)$ makes with the p_1 axis.
- Show that

$$\left. \frac{d\theta}{dz} \right|_{\ell(z)=\xi^s} = K^2 \frac{\cos v + 1}{c^2 - 1}$$



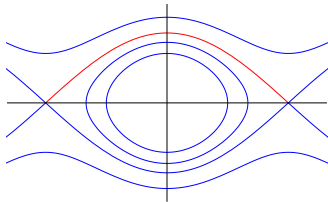
- Moreover, an implicit function theorem argument can be used to locally write the location z where $\ell(z)$ crosses ξ^s as a function $z = z(\lambda)$, for all values of λ .
- Because of this, we have that the number of crossings of $\ell(z)$ with respect to the stable subspace is monotone in λ .

Corollary

If X_1 is the number of times $\ell(z)$ crosses ξ^s when $\lambda = \lambda_1$, as z ranges from $-\infty$ to ∞ and X_2 is the number of times $\ell(z)$ crosses ξ^s when $\lambda = \lambda_2$, then $|X_1 - X_2| =$ the number of eigenvalues of the linearized operator in (λ_1, λ_2) .

- When $\lambda \gg 1$, $\ell(z)$ does not cross ξ^s .
- When $\lambda = 0$, $\ell(z)$ is the solution to the equation of variations associated with the heteroclinic orbit of the pendulum equation, with the initial condition that $\ell(z)$ be tangent to the orbit.

Conclusion: No Real Eigenvalues



- There are no real positive eigenvalues to the linearized operator in either the sub or superluminal case.
- Same arguments hold for $\lambda \ll -1$. So have no real eigenvalues except $\lambda = 0$.

Complex Eigenvalues

- For complex eigenvalues off the imaginary axis consider again the ODE:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{\cos v - \lambda^2}{c^2 - 1} & \frac{-2c\lambda}{c^2 - 1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

- Make a convenient (Liouville) coordinate transformation.
- Look at associated Riccati equation on (charts of) $\mathbb{C}P^1$.
- Keep track of the relevant subspaces $\xi^u \rightarrow \eta^u$ and $\xi^s \rightarrow \eta^s$.

No Complex Eigenvalues Either

- Geometric interpretation of eigenvalue condition still holds.
- The existence of an eigenvalue is equivalent to a heteroclinic orbit between η^u and η^s .
- η^u and η^s lie on opposite sides of the real axis, and the Riccati flow is going in the wrong direction on the real axis.
- So no heteroclinic orbits possible (regardless of sign of $c^2 - 1$).
- Hence no complex eigenvalues off the imaginary axis in either case.

Essential Spectrum

- The linearized operator about a kink wave solution is a compact perturbation of

$$\mathcal{L}_- \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \psi \\ -\varphi - (c^2 - 1)\varphi_{zz} - 2c\psi_z \end{pmatrix}$$

in the subluminal case and

$$\mathcal{L}_+ \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \psi \\ \varphi - (c^2 - 1)\varphi_{zz} - 2c\psi_z \end{pmatrix}.$$

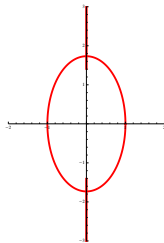
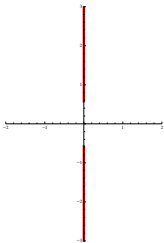
in the superluminal case

- We can then compute the essential spectrum of \mathcal{L}_\pm directly:

$\sigma_{\text{ess}}(\mathcal{L}_-) = \{\mu \equiv i\beta \mid \beta \in (-\infty, -c] \cup [c, \infty)\}$ and,

$\sigma_{\text{ess}}(\mathcal{L}_+) = \{\mu \equiv \alpha + i\beta \mid \alpha^2 + \frac{\beta^2}{c^2} = 1\} \cup$

$\{\mu \equiv i\beta \mid \beta \in (-\infty, -\sqrt{c^2 - 1}] \cup [\sqrt{c^2 - 1}, \infty)\}$



Summary

Theorem

Subluminal kink waves are stable. Superluminal kink waves are unstable.

- Show no real eigenvalues via winding number argument (no difference between sub and superluminal case)
- Show no complex eigenvalues via Riccati equation argument (difference between sub and superluminal case - though no eigenvalues in either)
- Calculate essential spectrum.
- It is the presence of essential spectrum with positive real part which makes the superluminal waves unstable.

Further

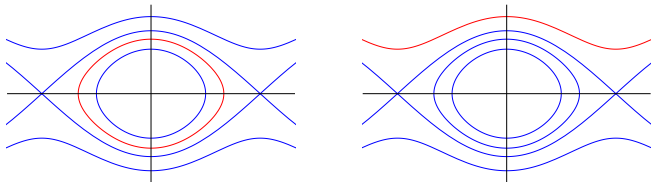
- The geometric methods to show a lack of eigenvalues will work on standing or travelling waves in a wide class of 1-spatial dimension equations.

$$u_t = u_{xx} + f(u) + g(u_x)$$

- Can extend the winding number argument to systems - Maslov index.
- Riccati equations for systems mean more charts. Becomes harder to analyze.
- Can we extend these arguments to more than one space dimension?

Further Still

- Consider 'periodic' wave trains - orbits other than the heteroclinic orbit.
- Geometric techniques used in this result fail due to lack of constant asymptotic behavior.



- In addition to sub and superluminal waves have *librational* (inside) and *rotational* (outside) waves. Both have periodic eigenvalue problem.
- Spectral picture is more complicated.

Thank You

The Lorentz Transform

- Classically the Lorentz boost can only be done in the subluminal case: $\tau = \frac{t - cx}{\sqrt{1 - c^2}}$
- Even if we set $\tau = t - cx$, the Lorentz boost fails to detect the presence of the unstable essential spectrum in the superluminal case. Travelling wave sine-Gordon equation becomes:

$$(c^2 - 1)u_{zz} - (c^2 - 1)u_{\tau\tau} - \sin u = 0$$

- The Lorentz boost does not, in general preserve the spectrum of our linear operator.
- The Lorentz boost fails horribly in the periodic case.