Analytic Results for Cellular Automata

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The Game of Life

Conway's game of life is a grid of black (live) and white (dead) squares where the state of each square evolves according to rules applied to its eight neighbouring squares:

1. Any live cell with fewer than two live neighbours dies (under-population).
2. Any live cell with two or three live neighbours lives on to the next generation.
3. Any live cell with more than three live neighbours dies (overcrowding).
4. Any dead cell with exactly three live neighbours becomes a live cell (reproduction).
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Conway’s Game of Life is a grid of black (live) and white (dead) squares that evolve according to the following rules:

- Any live cell with less than 2 live neighbours dies.
- Any live cell with more than 3 live neighbours dies.

The Game of Life

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One-dimensional Automata
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• Each cell has 2 neighbours
One-dimensional Automata

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• There are 8 configurations
One-dimensional Automata

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Wolfram’s Classification
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- Rule 0
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- Rule 1
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Consider a line of cells, each either black or white, whose states evolve according to rules that depend on the initial state of a cell and its two neighbours.

Wolfram showed there are 256 possible choices, numbered from 0 to 255.
Wolfram’s Classification

- Wolfram showed that there are 256 CA
- Rule 0
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Consider a line of cells, each either black or white, whose states evolve according to rules that depend on the initial state of a cell and its two neighbours. Wolfram showed there are 256 possible choices, numbered from 0 to 255.
Rule 1
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One initial black cell evolves like
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One initial black cell evolves like

\[
\begin{align*}
\text{Rule 1:} \quad u_j(t+1) &= 1 - \max \left( u_j(t), u_{j-1}(t), u_{j+1}(t), 0 \right) \\
\text{So the block } \left( \ldots, 0, 0, 1, 0, 0, \ldots \right) \text{ gives } \left( \ldots, 1, 0, 0, 0, 1, \ldots \right).
\end{align*}
\]
• CA can also be written as equations on a state \( u_i(t) \) where \( u_i = 0 \) means white and \( u_i = 1 \) means black.

• Rule 1 is then

\[
u_j(t + 1) = 1 - \max(u_{j-1}(t), u_j(t), u_{j+1}(t), 0)
\]
Rule 30

One black cell evolves under Rule 30 as

Rule 30 --

From MathWorld


1 of 2

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After 250 iterations

Used as a random number generator in Mathematica
• HIV/AIDS clinical data is modelled by Zorzenon dos Santos et al PRL (2001) 168102 as a CA with four rules.

• It reproduced the three-phase cycle in clinical data for the first time.

• Result relies on two crucial parameters in the model.

• What happens when these change? No one knows.

FIG. 3 (color). Four snapshots of parts of the lattice configuration for different time steps: (a)–(d) correspond to 5, 18, 25, and 200 weeks, respectively. We have adopted the same parameters used in Fig. 2. The color codes for the different states of the cell are the following: healthy=blue, infected-A1=yellow, infected-A2=green, and dead=red.
Empirical Automata

- Wooton *Nature* **413** (2001) 841 studied the population of Californian mussels in the intertidal zone of the north-west Pacific coast of the USA.
Agreement with data

- Agreement with data is almost unbelievably close.

- How would this agreement change when the parameters change?
The Emperor’s New Theory

• “The equations of a good theory are taken to represent physical reality because they can be used to make predictions...”

from *The emperor’s new theory*

*The Economist* 06 Jan 2002 363 #8275 p. 79.

• Do predictable CA exist?
The box and ball system appears to be integrable.

An example of solitons in filter parity rules first suggested by Park, Steiglitz and Thurston, Physica D x1986y.

How can we decide when a given CA is integrable?

Given initial data, how can we predict what happens in $n$ steps, where $n$ is an arbitrary positive integer?
From PDE to CA
From PDE to CA

Korteweg-de Vries Eqn

\[ u_t + 6u u_x + u_{xxx} = 0 \]
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Lotka-Volterra Eqn
\[ \partial_t b_j(t) = b_j(t) (b_{j+1}(t) - b_{j-1}(t)) \]
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\[ \partial_t b_j(t) = b_j(t) (b_{j+1}(t) - b_{j-1}(t)) \]

Discrete Lotka-Volterra Eqn
\[ \frac{c_j(t + 1)}{c_j(t)} = \frac{1 + \delta c_{j-1}(t)}{1 + \delta c_{j+1}(t + 1)} \]
From PDE to CA

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CA
\[ u_j(t+1) = \min \left( 1 - u_j(t), \sum_{t=-\infty}^{j-1} (u_j(t) - u_j(t+1)) \right) \]
Parity Filter Rule
Parity Filter Rule

• When the next state of a cell depends on the state of all cells to the left, the CA is called a “parity filter rule”.
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• When the next state of a cell depends on the state of all cells to the left, the CA is called a “parity filter rule”.
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• The “ultra-discrete” version of the KdV has solitons...
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• The “ultra-discrete” version of the KdV has solitons...

...011110000011001000000000000000...
...000001111000110100000000000000...
...000000000111001011100000000000...
...000000000111001011100000000000...
...000000000110100011110000000000...
...000000000101100000111110000000...
...00000000000000101100000111110000...
...000000000000000101100000111111111...

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What is “ultra-discretization”? 
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• Given any variable $x$, any parameter $a$, replace them by exponentials
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$$x = \exp(X/\epsilon)$$
What is “ultra-discretization”? 

• Given any variable $x$, any parameter $a$, replace them by exponentials

\[ x = \exp\left(\frac{X}{\epsilon}\right) \]
\[ a = \exp\left(\frac{A}{\epsilon}\right) \]
What is “ultra-discretization”?

• Given any variable \( x \), any parameter \( a \), replace them by exponentials

\[
x = \exp\left(\frac{X}{\epsilon}\right)
\]
\[
a = \exp\left(\frac{A}{\epsilon}\right)
\]

• Then use
What is “ultra-discretization”? 

• Given any variable $x$, any parameter $a$, replace them by exponentials

\[
\begin{align*}
  x &= \exp\left(\frac{X}{\epsilon}\right) \\
  a &= \exp\left(\frac{A}{\epsilon}\right)
\end{align*}
\]

• Then use

\[
\lim_{\epsilon \to 0^+} \epsilon \log \left(1 + e^{X/\epsilon}\right)
\]

\[
= \begin{cases} 
  X & \text{if } X > 0 \\
  0 & \text{otherwise}
\end{cases}
\]

\[
= \max(0, X).
\]
In this limit

\[
\begin{align*}
  a + b & \mapsto \max(A, B) \\
  ab & \mapsto A + B \\
  a/b & \mapsto A - B
\end{align*}
\]

Associativity and distributivity hold.
• But some new objects have to be defined.

\[
\begin{align*}
  a + 0 & \mapsto \max(A, -\infty) \\
         & = A \\
(a + b) - b & \mapsto \max(\max(A, B), B + \eta) \\
           & = A.
\end{align*}
\]

They lead to a new invertible max-plus algebra developed by Ochiai and Nacher (2005).
Riccati Equation and its Avatars

- The ODE is linearizable

\[
x'(t) = -x(t)^2 + t
\]

\[
x = \frac{y'}{y}
\]

\[
\Rightarrow y'' = t y
\]
Discrete Riccati Equation

- The discrete equation is also linearizable

\[
\overline{x} := x(n + 1) = \frac{\alpha x(n) + \beta}{\gamma + x(n)}
\]

\[
x(n) = \frac{F(n)}{G(n)}
\]

\[
\Rightarrow \begin{cases} 
F(n + 1) = \alpha F(n) + \beta G(n) \\
G(n + 1) = \gamma G(n) + F(n)
\end{cases}
\]
Ultra-discrete Riccati Equation
Ultra-discrete Riccati Equation

• Appears to be linear already
Ultra-discrete Riccati Equation

- Appears to be linear already

\[
\overline{X} = A + X - \max(X, 0)
\]

\[(\beta = 0, \gamma = 1)\]

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Ultra-discrete Riccati Equation

• Appears to be linear already

\[
\bar{X} = A + X - \max(X, 0)
\]

(\(\beta = 0, \gamma = 1\))

with solution (for positive \(A\))
Ultra-discrete Riccati Equation

• Appears to be linear already

\[ \overline{X} = A + X - \max(X, 0) \]

\((\beta = 0, \gamma = 1)\)

with solution (for positive \(A\))

\[ X = \begin{cases} 
A & \forall n \text{ if } X_0 > 0 \\
NA + X_0 & \forall n \leq N \text{ if } X_0 \in \mathcal{I}_N \\
A & \forall n > N \text{ if } X_0 \in \mathcal{I}_N
\end{cases} \]

\(\mathcal{I}_N = \{ -NA \leq X_0 \leq -(N-1)A < 0 \}\)

\(\beta = 0, \gamma = 1\)
Is it a CA?
Is it a CA?

• If $A$ and initial value are integer, then all iterates are integer. So it is like a CA, but has an infinite number of discrete states.
Is it a CA?

- If $A$ and initial value are integer, then all iterates are integer. So it is like a CA, but has an infinite number of discrete states.

- For some equations, such as the KdV equation, we can project to a finite number of states, so it is a CA.
Painlevé Equations

- Reductions of soliton equations lead to the Painlevé equations. The first Painlevé equation is

\[ y'' = 6y^2 + x \]

They appear as universal classes of distributions in random matrix theory describing
+ scattering of neutrons off a heavy nucleus,
+ zeroes of the Riemann zeta function,
+ bus arrival times in Cuernavaca, Mexico,
+ aircraft boarding times ...

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In the Complex Plane
Discrete PI

One discrete version:

\[ x_{n+1} + x_n + x_{n-1} = \frac{a_n + b}{x_n} + c \]
Discrete Painlevé Equations
Discrete Painlevé Equations

- PI

- PII
Discrete Painlevé Equations

- P\textsubscript{I} 
  \[ y'' = 6y^2 + t \]

- P\textsubscript{II}
Discrete Painlevé Equations

- Pl
  \[ y'' = 6y^2 + t \]

- PII
  \[ y''' = 2y^3 + ty + \alpha \]
Discrete Painlevé Equations

- P\textsc{I}
  \[ y'' = 6y^2 + t \]
  \[ \bar{x}x = \frac{\alpha q^n}{x} + \frac{1}{x^2} \]

- P\textsc{II}
  \[ y'' = 2y^3 + ty + \alpha \]
Discrete Painlevé Equations

- **PI**
  \[
  y'' = 6y^2 + t \\
  xx = \frac{\alpha q^n}{x} + \frac{1}{x^2}
  \]

- **PII**
  \[
  y''' = 2y^3 + ty + \alpha \\
  xx = \frac{(\alpha q^n + x)}{(1 + \beta q^n x)}
  \]
Discrete Painlevé Equations

• PI

\[ y'' = 6y^2 + t \]
\[ \overline{xx} = \frac{\alpha q^n}{x} + \frac{1}{x^2} \]
\[ \overline{X} + \underline{X} + 2X = \max (X + Qn + a, 0) \]

• PII

\[ y'' = 2y^3 + ty + \alpha \]
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- **PI**
  \[ y'' = 6y^2 + t \]
  \[ \bar{xx} = \frac{\alpha q^n}{x} + \frac{1}{x^2} \]
  \[ \bar{X} + X + 2X = \max (X + Qn + a, 0) \]

- **PII**
  \[ y'' = 2y^3 + ty + \alpha \]
  \[ \bar{xx} = \frac{(\alpha q^n + x)}{(1 + \beta q^n x)} \]
  \[ \bar{X} + X - X = \max (Qn + a - X, 0) \]
  \[ - \max (X + Qn + b, 0) \]
Are there others?
Are there others?

- How do we know these are integrable?
Are there others?

- How do we know these are integrable?
- How do we find new integrable ultra-discrete equations?
Are there others?

• How do we know these are integrable?
• How do we find new integrable ultra-discrete equations?
• We look at their singularities!
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]

- Start with arbitrary and
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]

• Start with arbitrary \( X_{n-1} \) and
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]

- Start with arbitrary \( X_{n-1} \) and \( X_n = -K + \epsilon \)
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]

- Start with arbitrary \( X_{n-1} \) and \( X_n = -K + \epsilon \)
- Then the iterates for are
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]

- Start with arbitrary \( X_{n-1} \) and \( X_n = -K + \epsilon \)
- Then the iterates for \( X_{n-1} > 2|K| \) are
\[ X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0) \]

- Start with arbitrary \( X_{n-1} \) and \( X_n = -K + \epsilon \)

- Then the iterates for \( X_{n-1} > 2|K| \) are

<table>
<thead>
<tr>
<th>( X_n )</th>
<th>( X_{n+1} )</th>
<th>( X_{n+2} )</th>
<th>( X_{n+3} )</th>
<th>( X_{n+4} )</th>
<th>( X_{n+5} )</th>
<th>( X_{n+6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon &gt; 0 )</td>
<td>(-K + \epsilon)</td>
<td>( K - X_{n-1} )</td>
<td>( X_{n-1} - \epsilon )</td>
<td>( K - X_{n-1} + \epsilon )</td>
<td>(-K - \epsilon)</td>
<td>( X_{n-1} )</td>
</tr>
<tr>
<td>( \epsilon &lt; 0 )</td>
<td>(-K + \epsilon)</td>
<td>( K - X_{n-1} - \epsilon )</td>
<td>( X_{n-1} )</td>
<td>( X_{n-1} + \epsilon )</td>
<td>(-K - \epsilon)</td>
<td>( X_{n-1} )</td>
</tr>
</tbody>
</table>
Smooth dependence

- Iterates of
  \[ X(n + 1) + \sigma X(n) + X(n - 1) = \max(X(n) + K, 0) \]
  are smooth only if
  \[ \sigma = 0, 1, 2 \]
  These are also the cases in which there exist conserved quantities.
Non-constant Coeffts
Non-constant Coeffts

• Demanding smooth dependence on initial data implies integrability.
Non-constant Coeffts

• Demanding smooth dependence on initial data implies integrability.

• Can we extend this one more step to find ultra-discrete Painlevé equations?
Yes!

• Consider

\[ X_{n+1} + X_{n-1} = a + \max(b - X_n, 0) - \max(b + X_n, 0) \]

This has conserved quantity

\[ I = \max(b + X(n-1) + X(n), X(n-1), X, a - X(n-1), a - X(n), a + b - X(n) - X(n-1)) \]
Extension

- If we extend to

\[ X_{n+1} + X_{n-1} = a + \max(\phi(n) - X_n, 0) \]
\[ - \max(\phi(n) + X_n, 0) \]

asking for smooth dependence near \( \phi(n_0) \)

shows that

\[ \phi(n_0 + 3) - \phi(n_0 + 2) - \phi(n_0 + 1) + \phi(n_0) = 0 \]

- Exactly one of the ultra-discrete Painlevé equations
Two-dimensions

The same correspondence between singularities and integrability occurs on a two-dimensional lattice.

Check it out for the lattice KdV:

\[
\begin{align*}
    \frac{u^{i+1}_{j+1}}{u_j} &= u_j + \max \left( u^{i+1}_j - 1, 0 \right) \\
    &= \max \left( u^i_{j+1} - 1, 0 \right)
\end{align*}
\]
Summary
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• Very few analytic results are known for CA.
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• We have provided analytic tools for studying singularities and isomonodromic systems.
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• There is a path from continuous to discrete equations which preserves certain structures.
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• *Tropical* mathematical theories need be developed.
• We have provided analytic tools for studying singularities and isomonodromic systems.
• [www.maths.usyd.edu.au/u/nalini](http://www.maths.usyd.edu.au/u/nalini)