Elliptic-difference-type Painlevé equations

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Supported by the Australian Research Council

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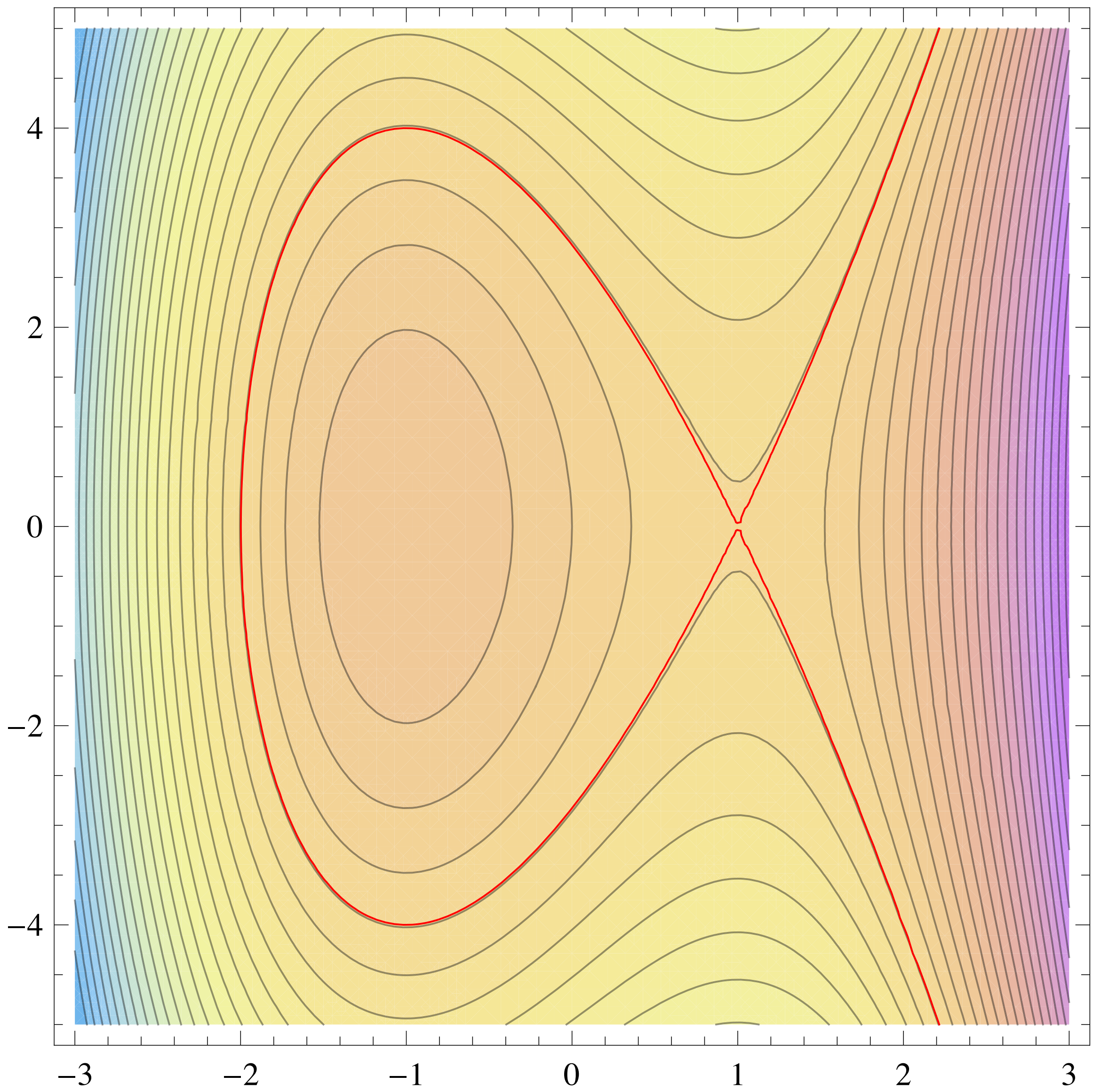
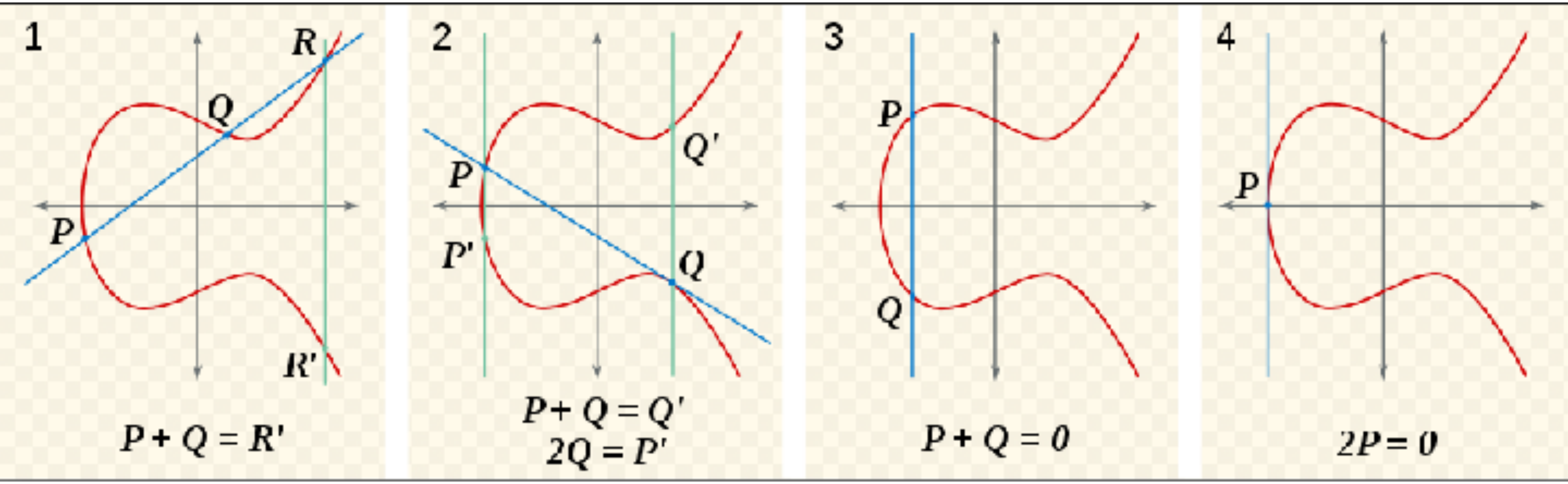


A pencil of Weierstrass cubic curves: $y^2 = 4x^3 - g_2 x - g_3$

Iterations on elliptic curves

Addition theorems give iterations on elliptic curves.



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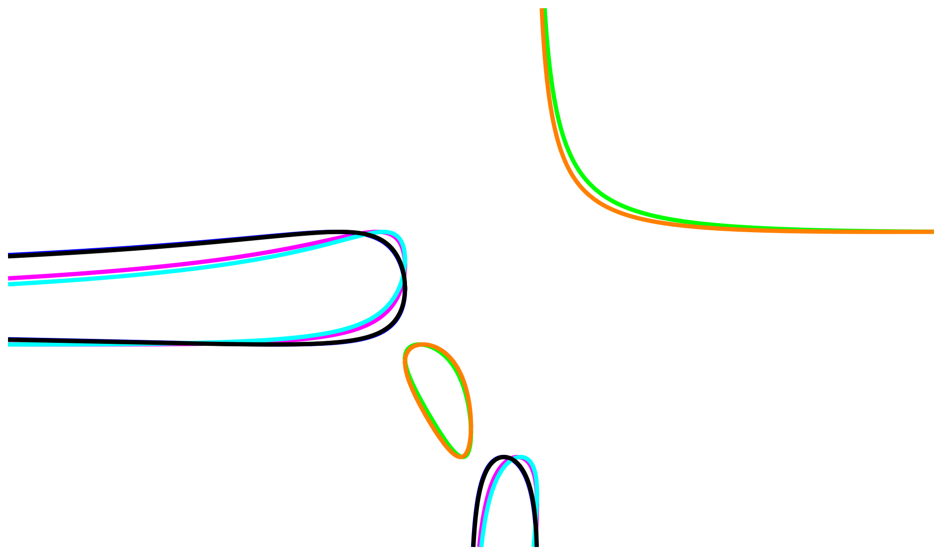
But such iterations occur on a single, fixed curve.

Discrete Painlevé equations iterate *between* elliptic curves.

Consider iterations from one curve to another.

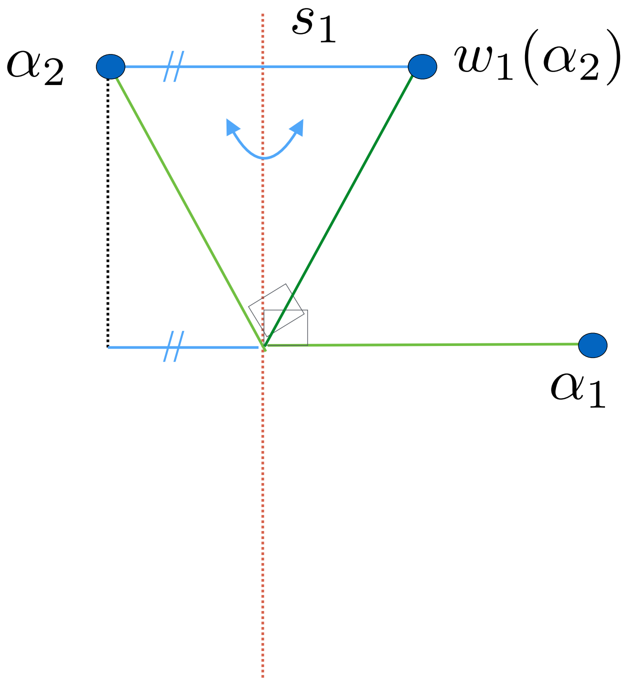
$$(x+y+\wp(2t))(4\wp(2t)\,xy-g_3)=\left(xy+\wp(2t)(x+y)+\frac{g_2}{4}\right)^2$$



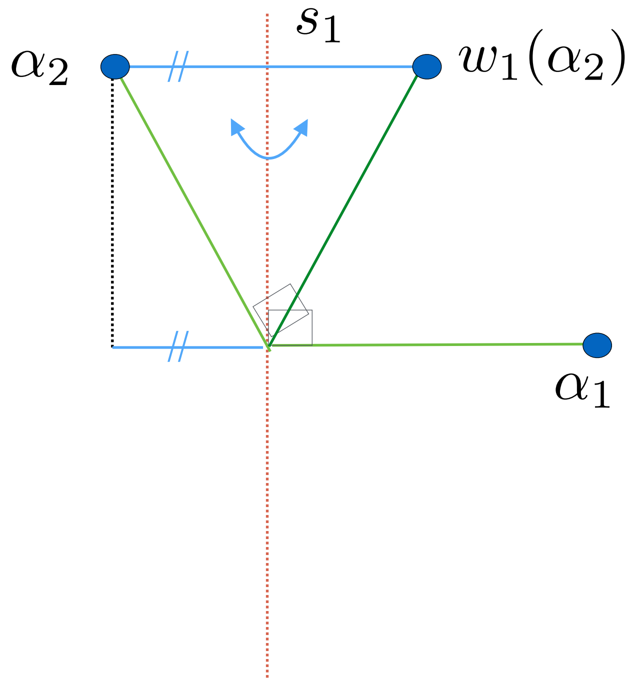
For lemniscatic case.

To characterise all of these, we need reflection groups.

A Reflection

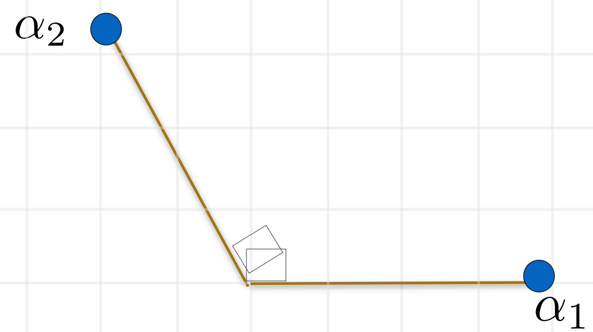


A Reflection



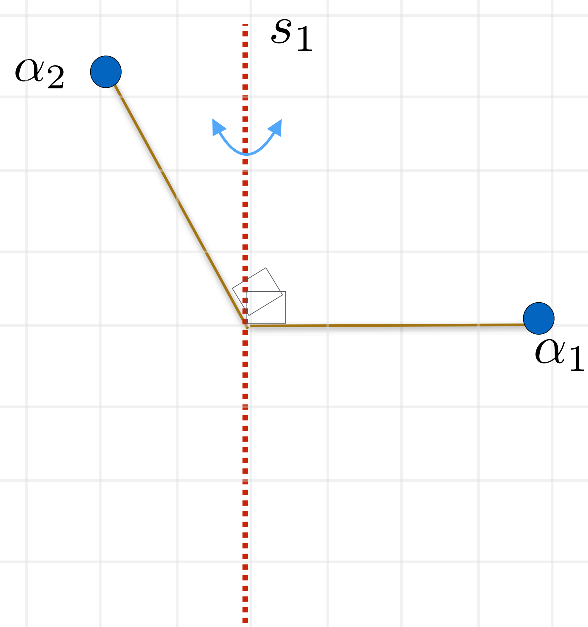
$$w_1(\alpha_2) = \alpha_2 - 2\,\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\,\alpha_1$$
$$= (-1, \sqrt{3}) + (2, 0)$$
$$= (1, \sqrt{3})$$

Root System



α_1 and α_2 are "simple" roots

Root System



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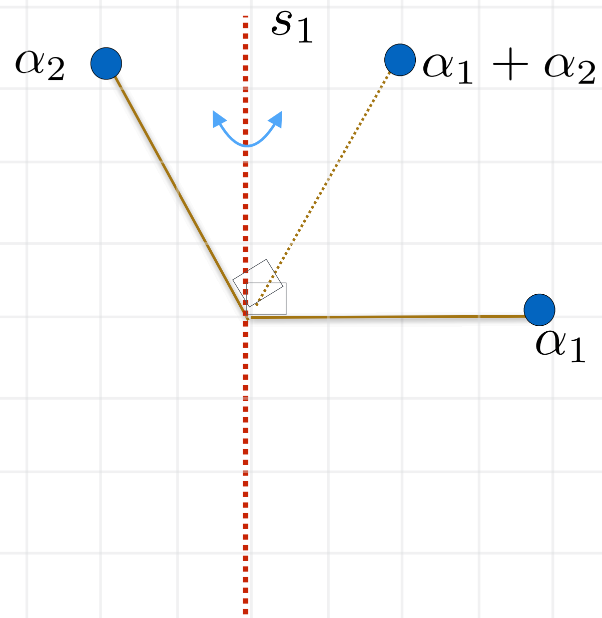

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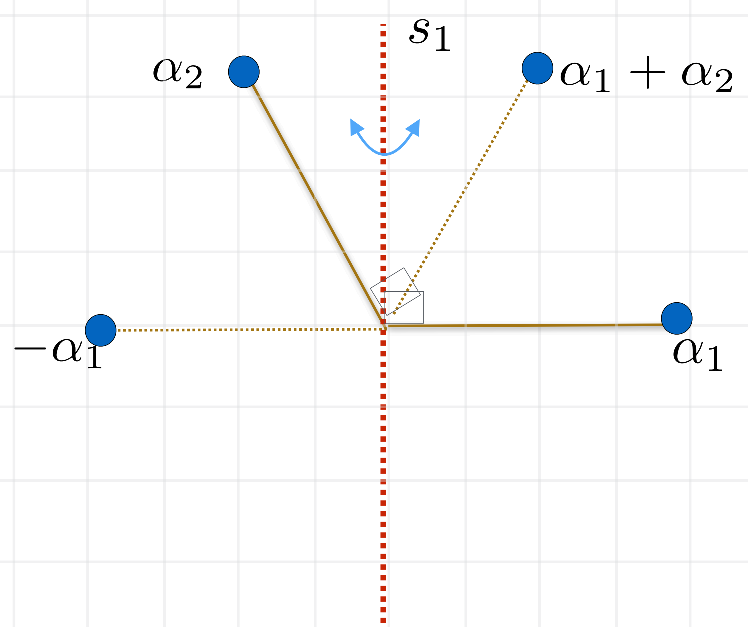

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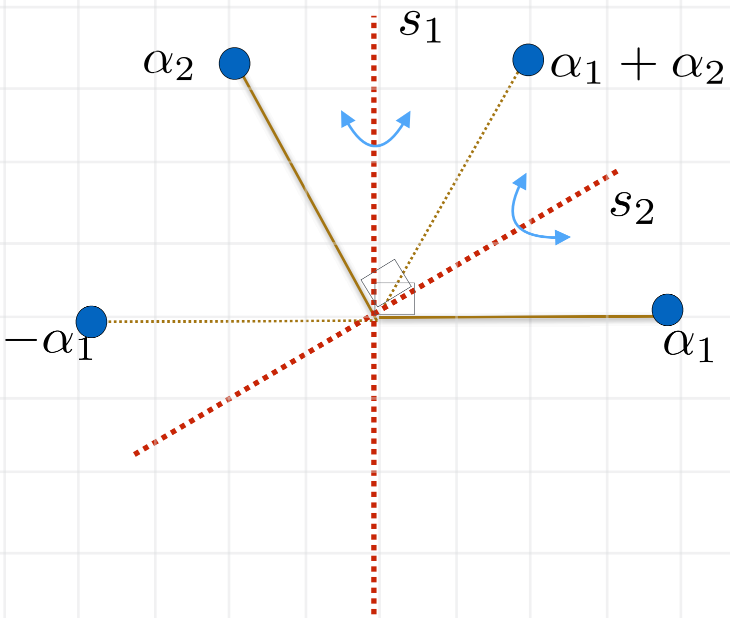

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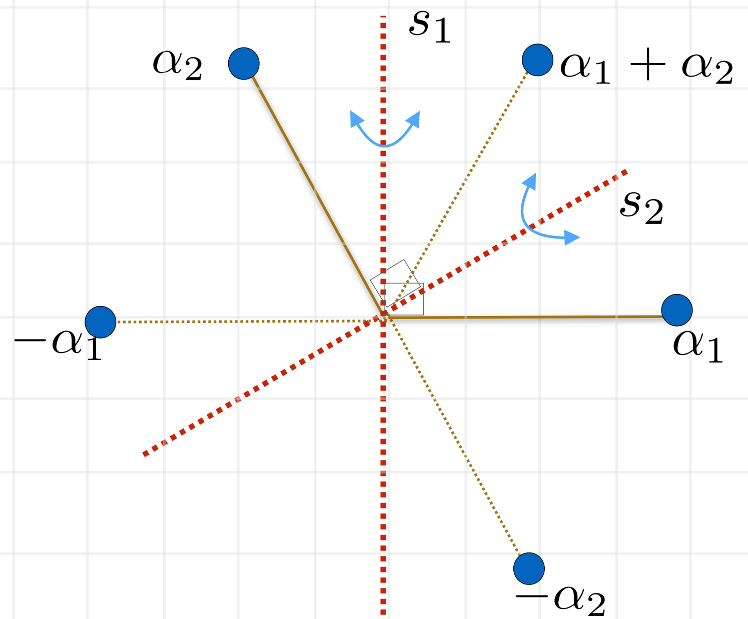

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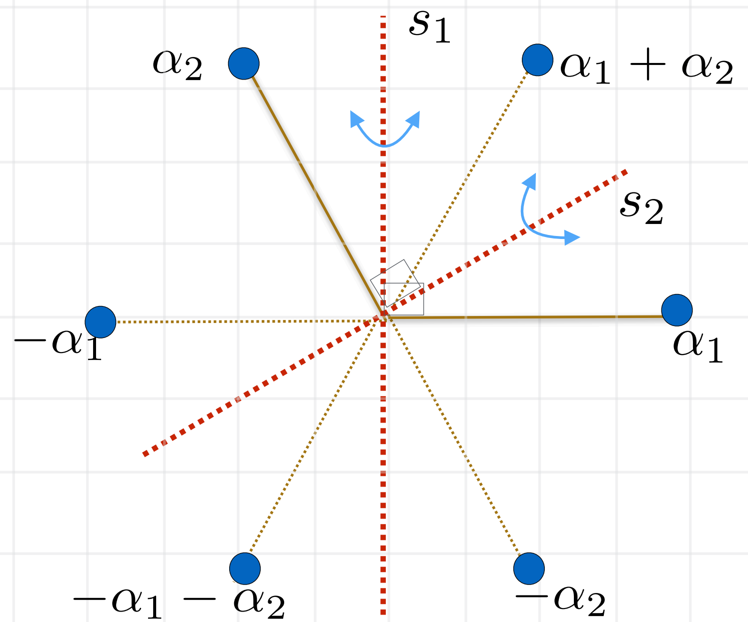

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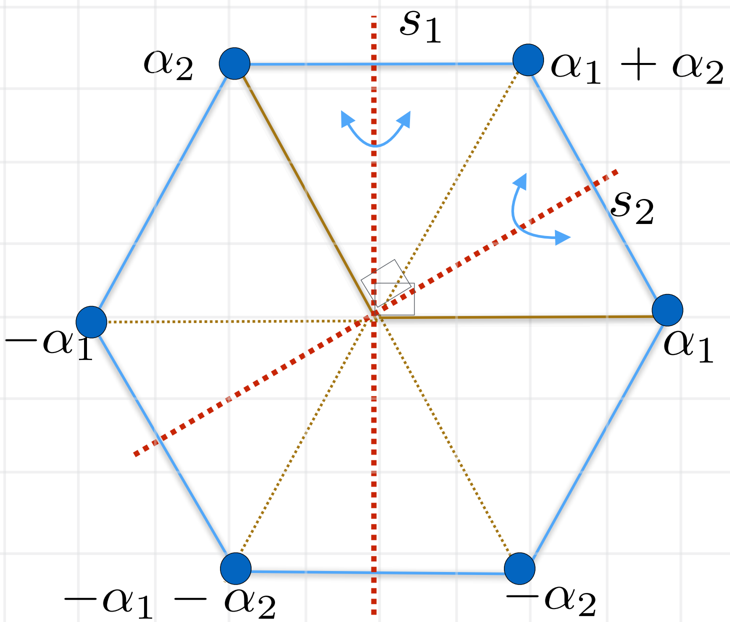


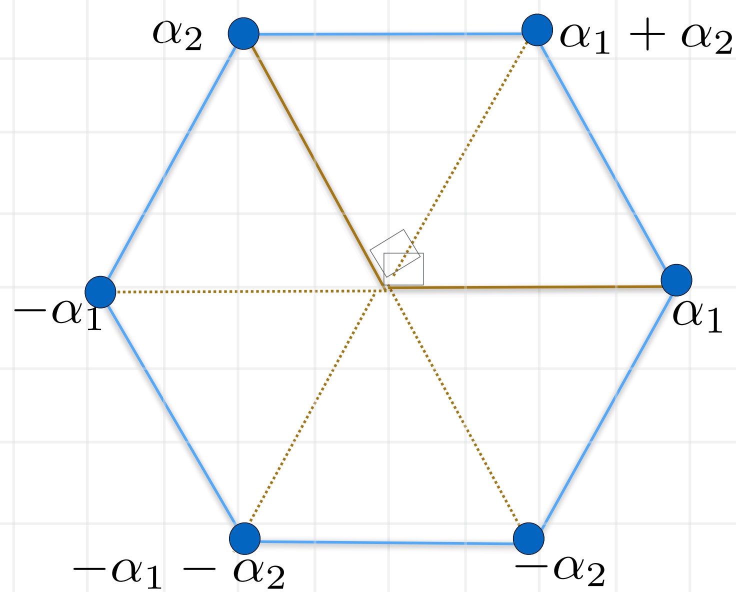
α_1 and α_2 are “simple” roots

Reflection Groups

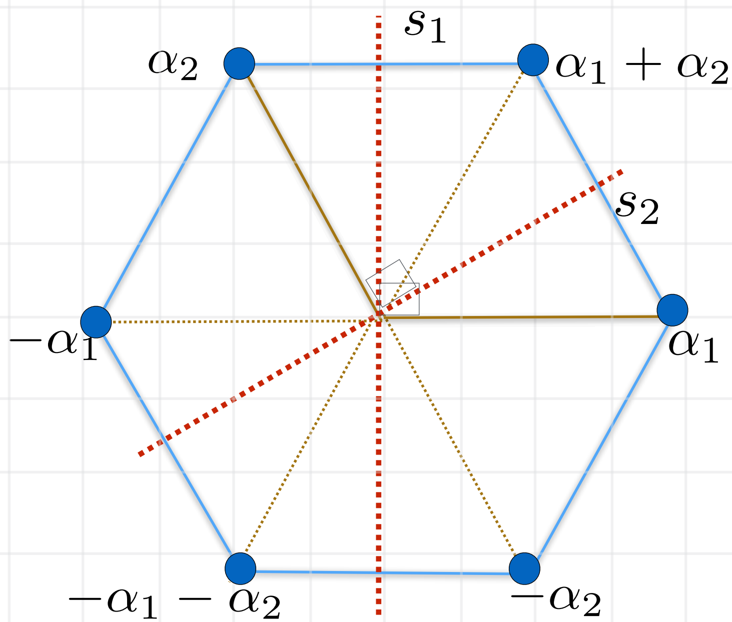
- Roots: $\alpha_1, \alpha_2, \ldots, \alpha_n$
- Reflections: $w_i(\alpha_j) = \alpha_j - 2\,\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\,\alpha_i$
- Co-roots: $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$
- Weights: $h_1, h_2, \ldots, h_n$

  $(h_i, \alpha_j^\vee) = \delta_{ij}$

A_2 again

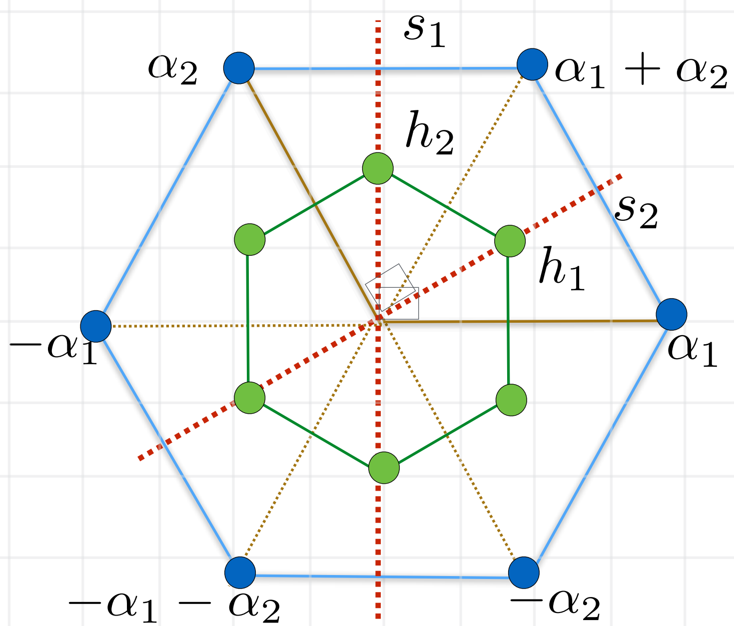


A_2 again



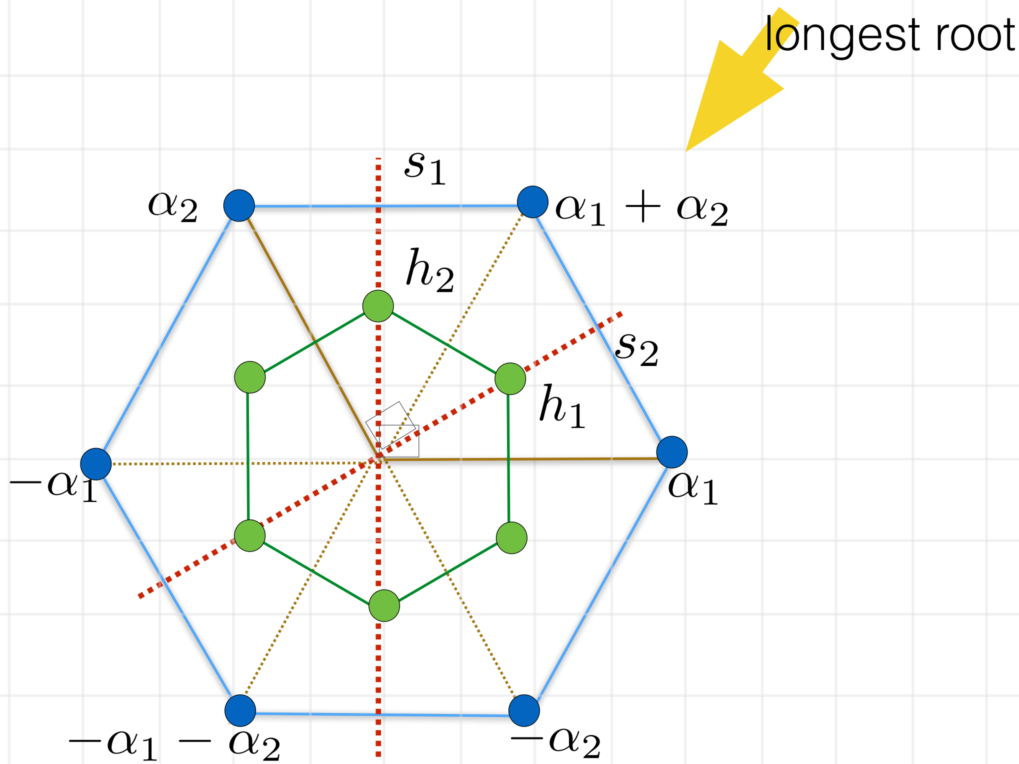
s_1
α_2
$\alpha_1 + \alpha_2$
s_2
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2 again



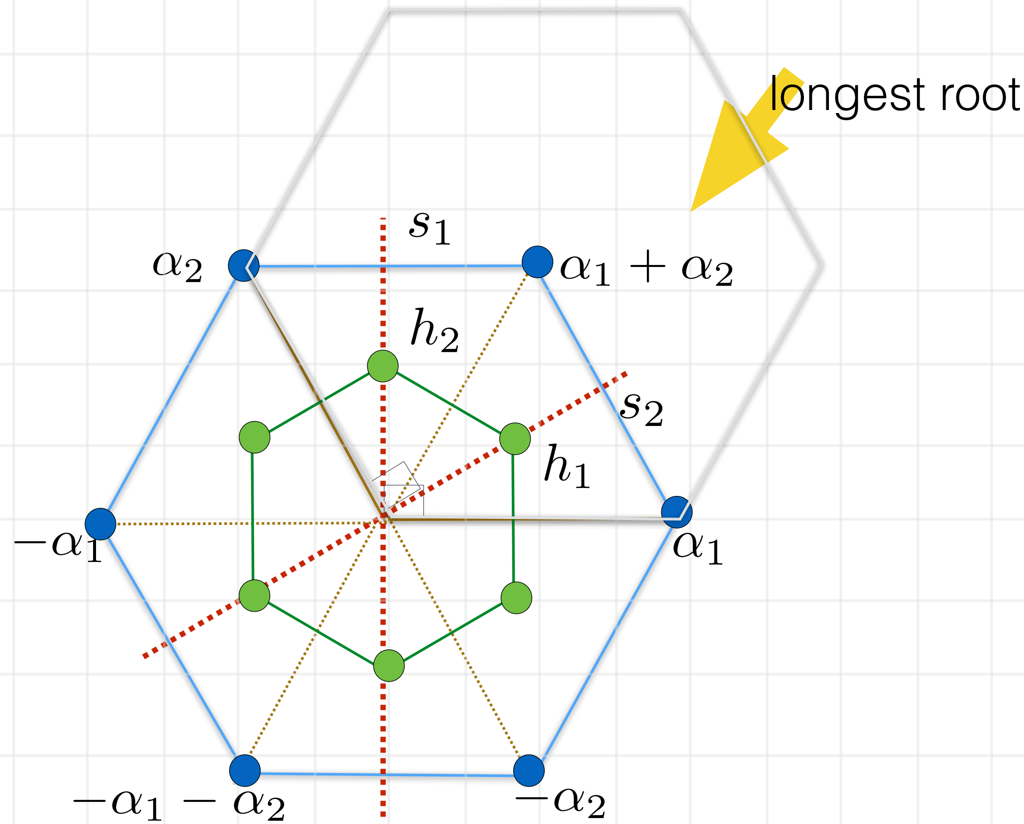
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2 again



longest root
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2 again


longest root
s_1
α_2
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h_2
s_2
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$-\alpha_1$
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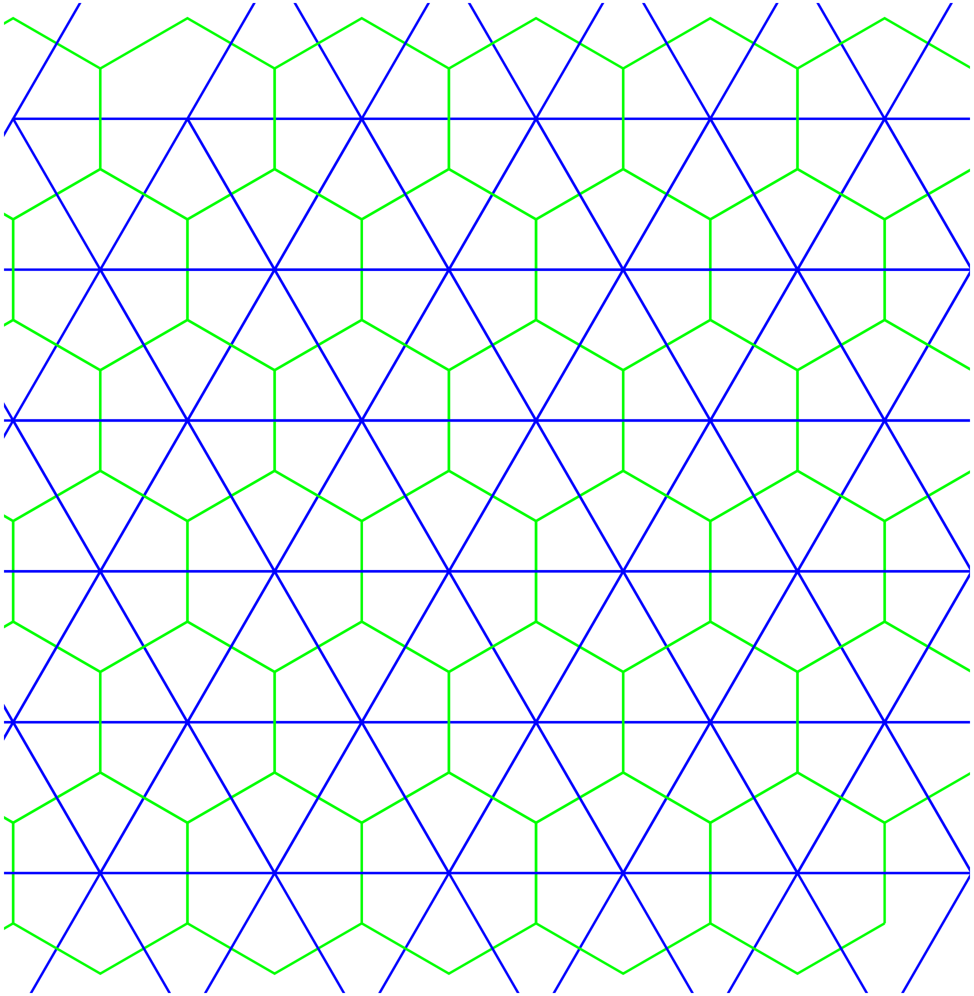
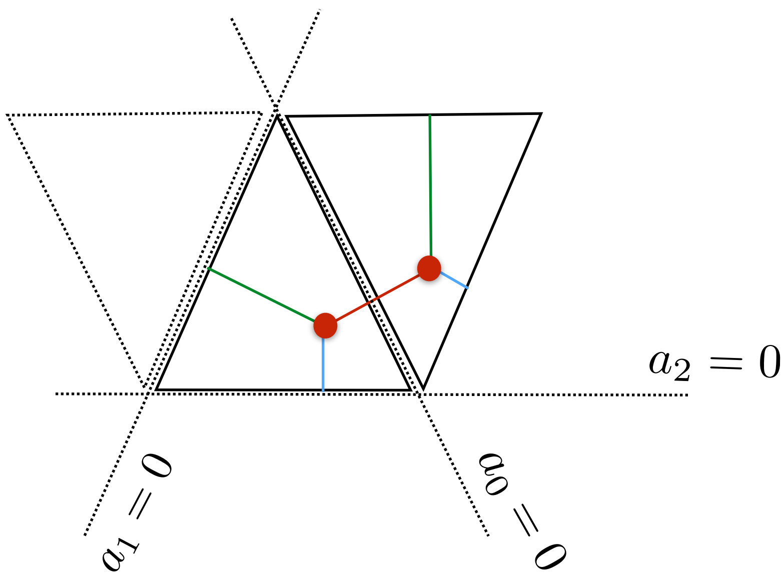
Weight and Root Lattices

$A_2^{(1)}$



On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



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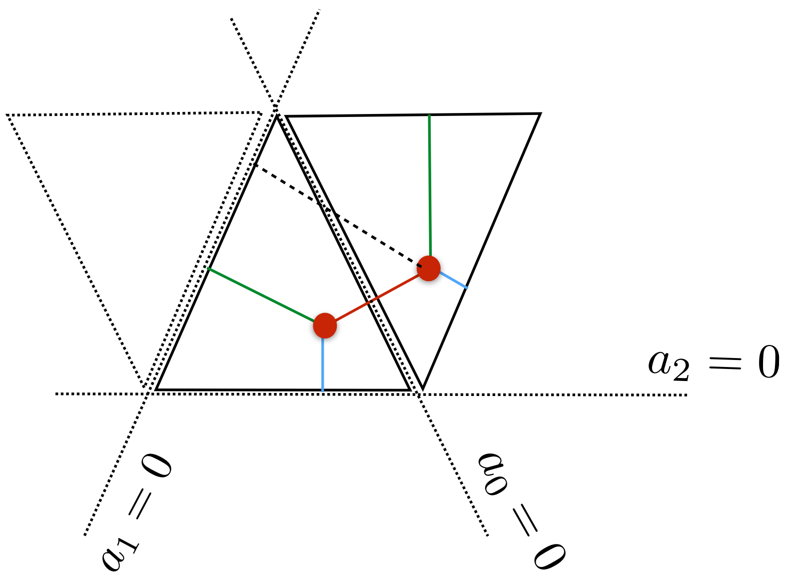

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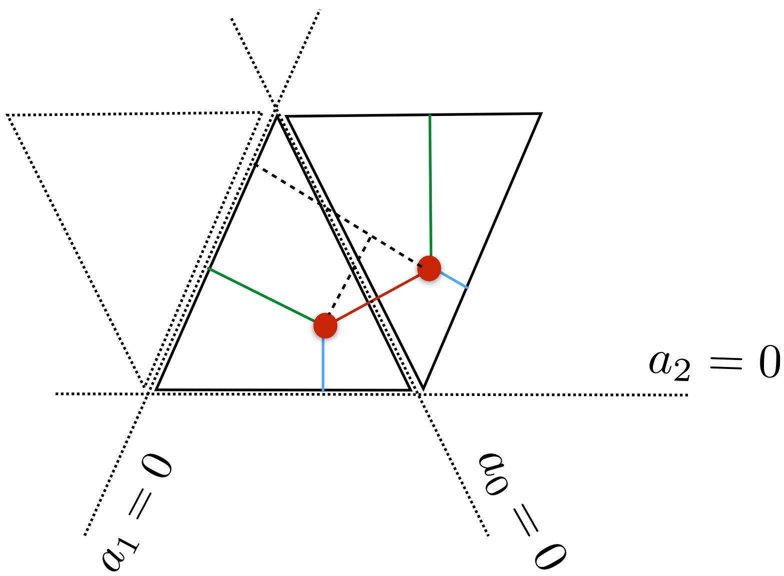

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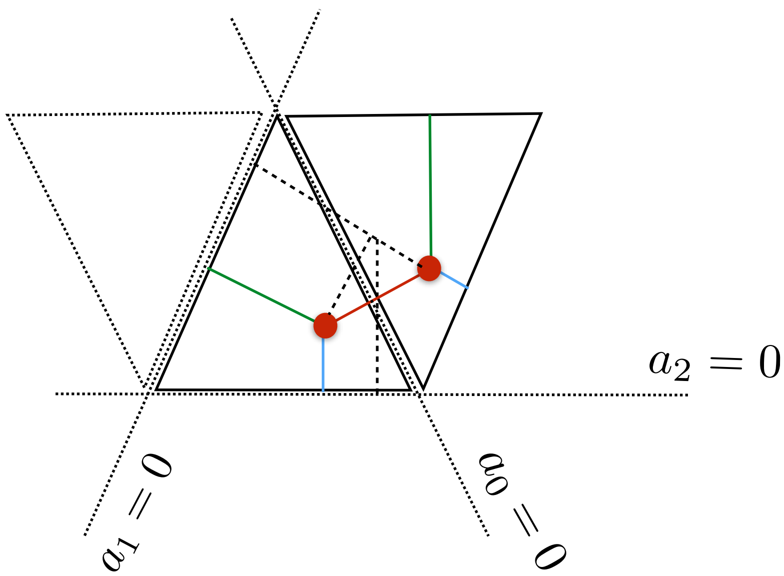

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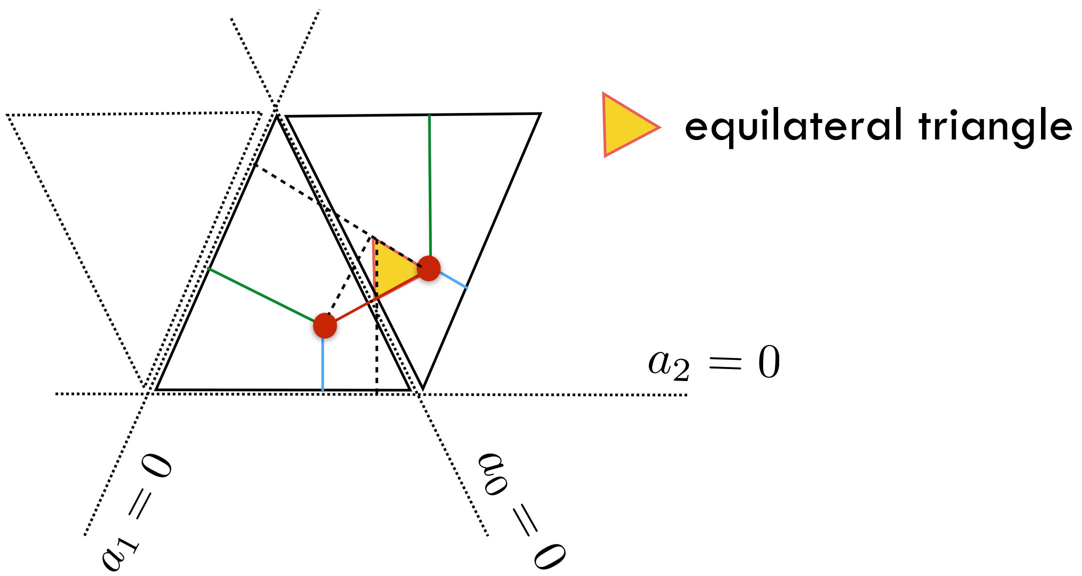

On the Lattice

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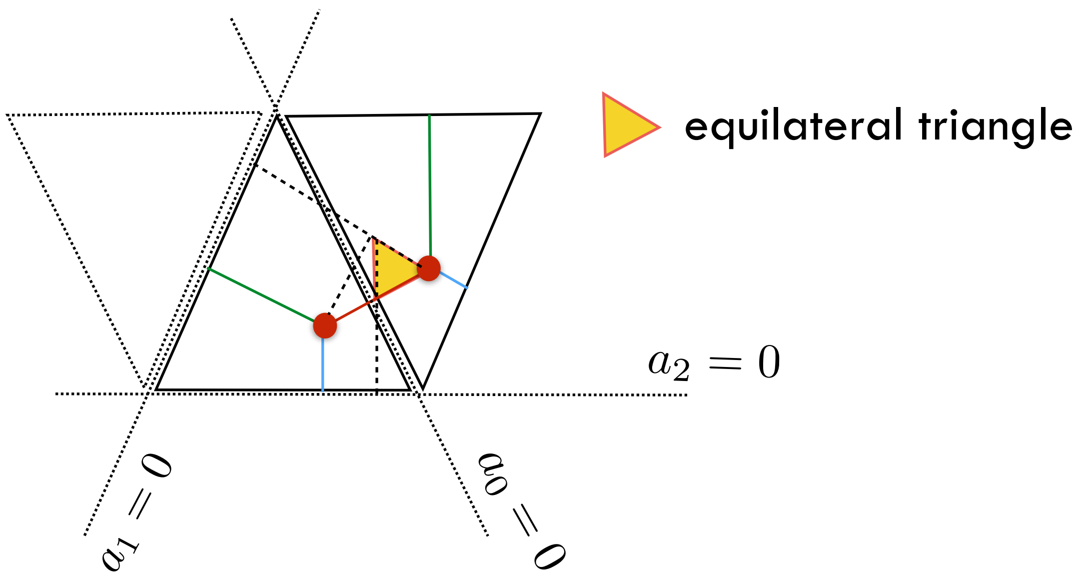
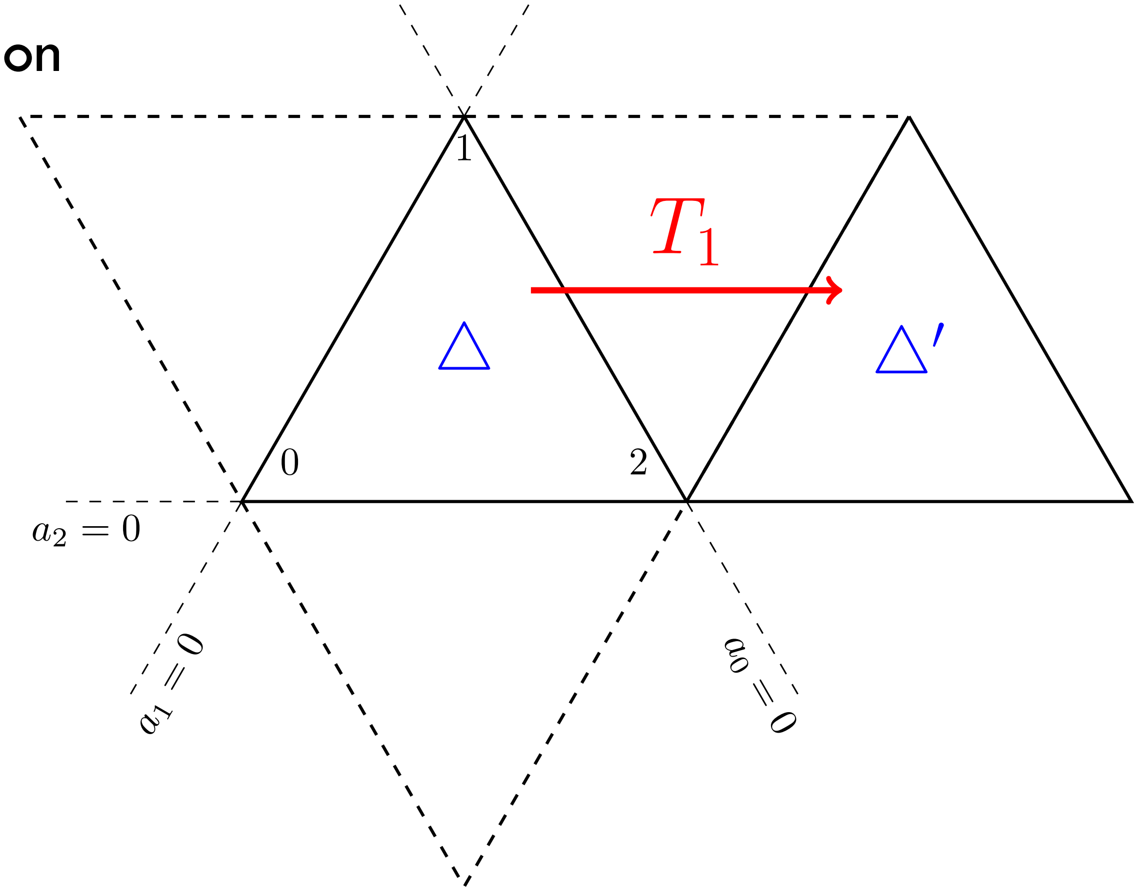


$$s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$$

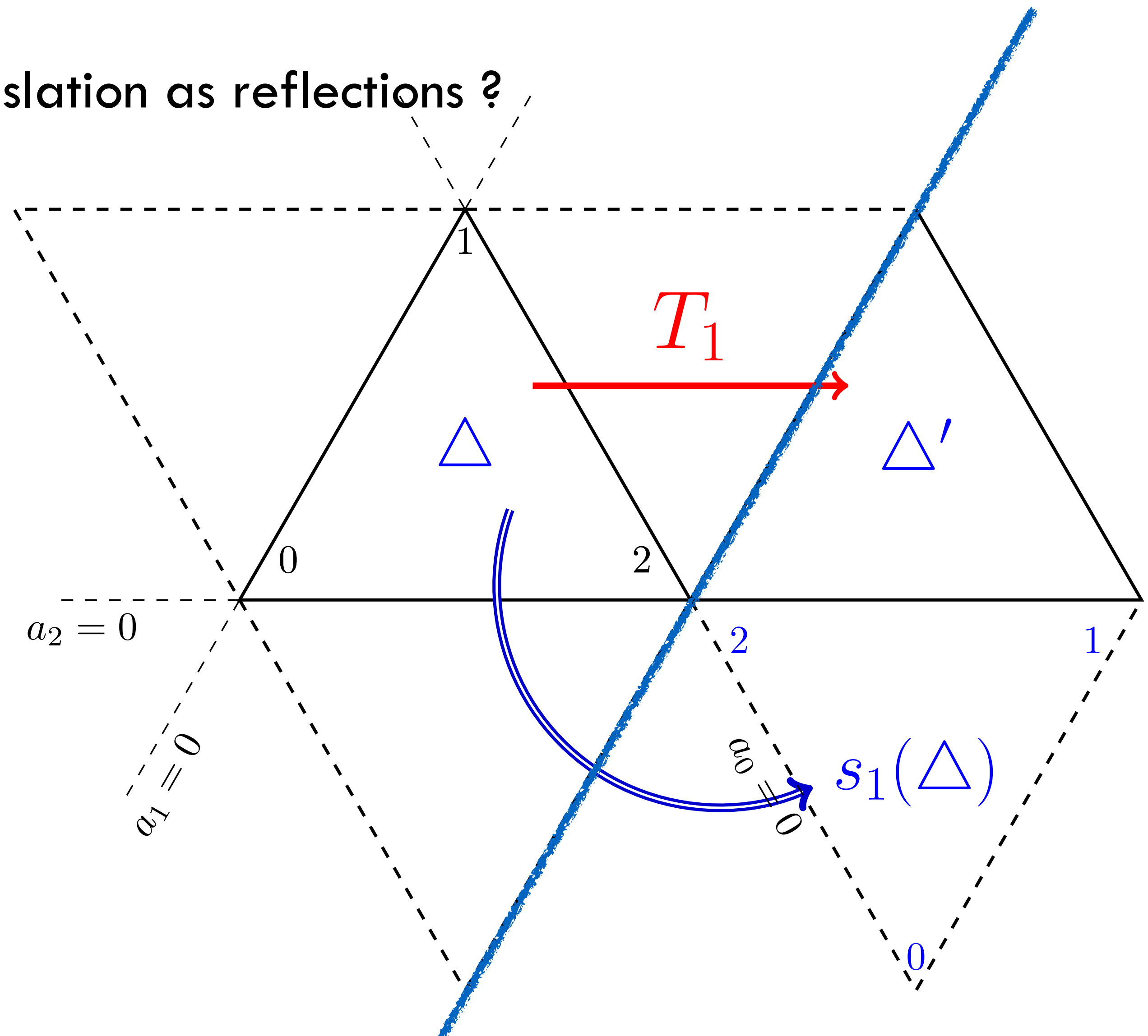
Discrete Dynamics I

- Translation



Discrete Dynamics II

- Translation as reflections ?



Discrete Dynamics II

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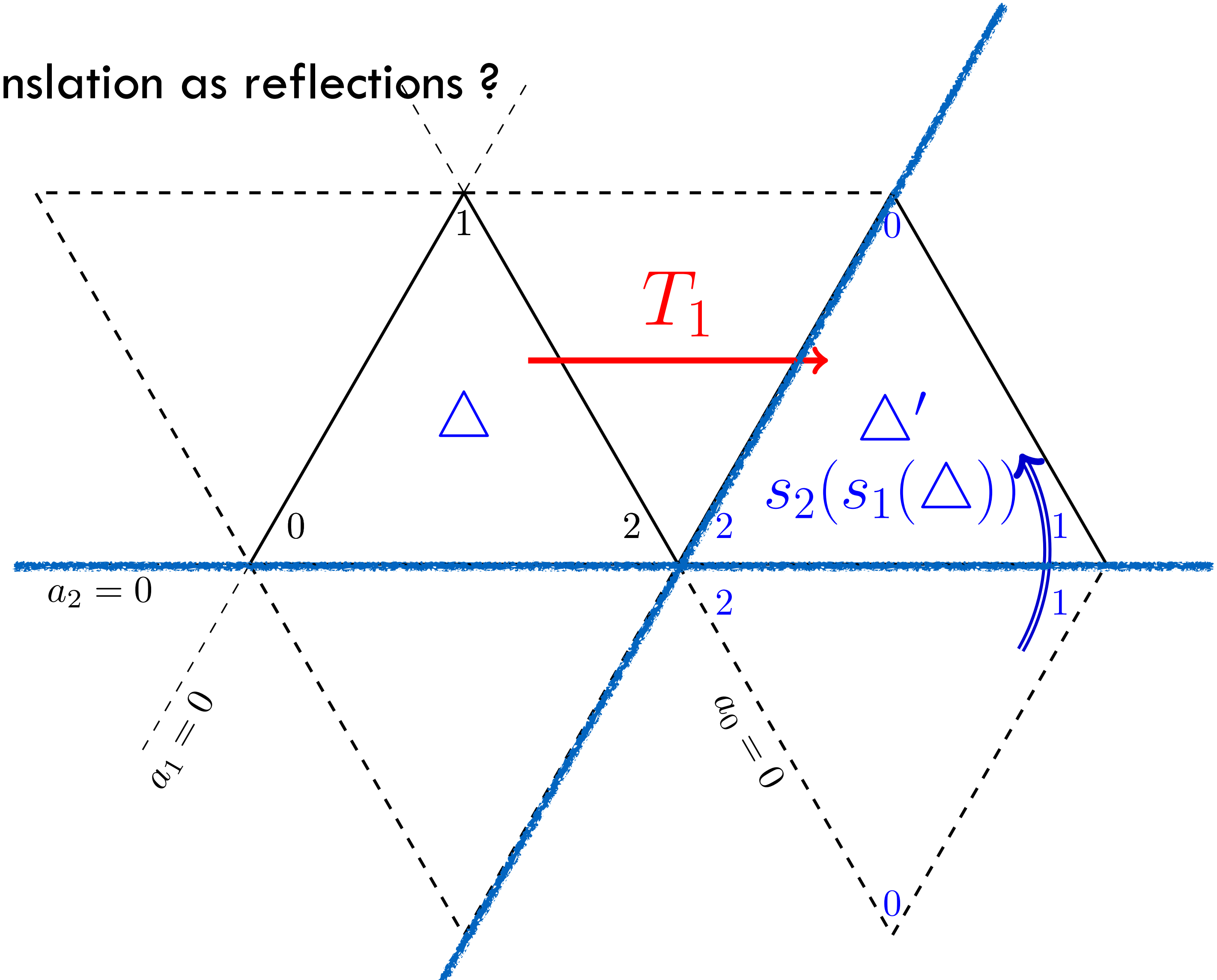

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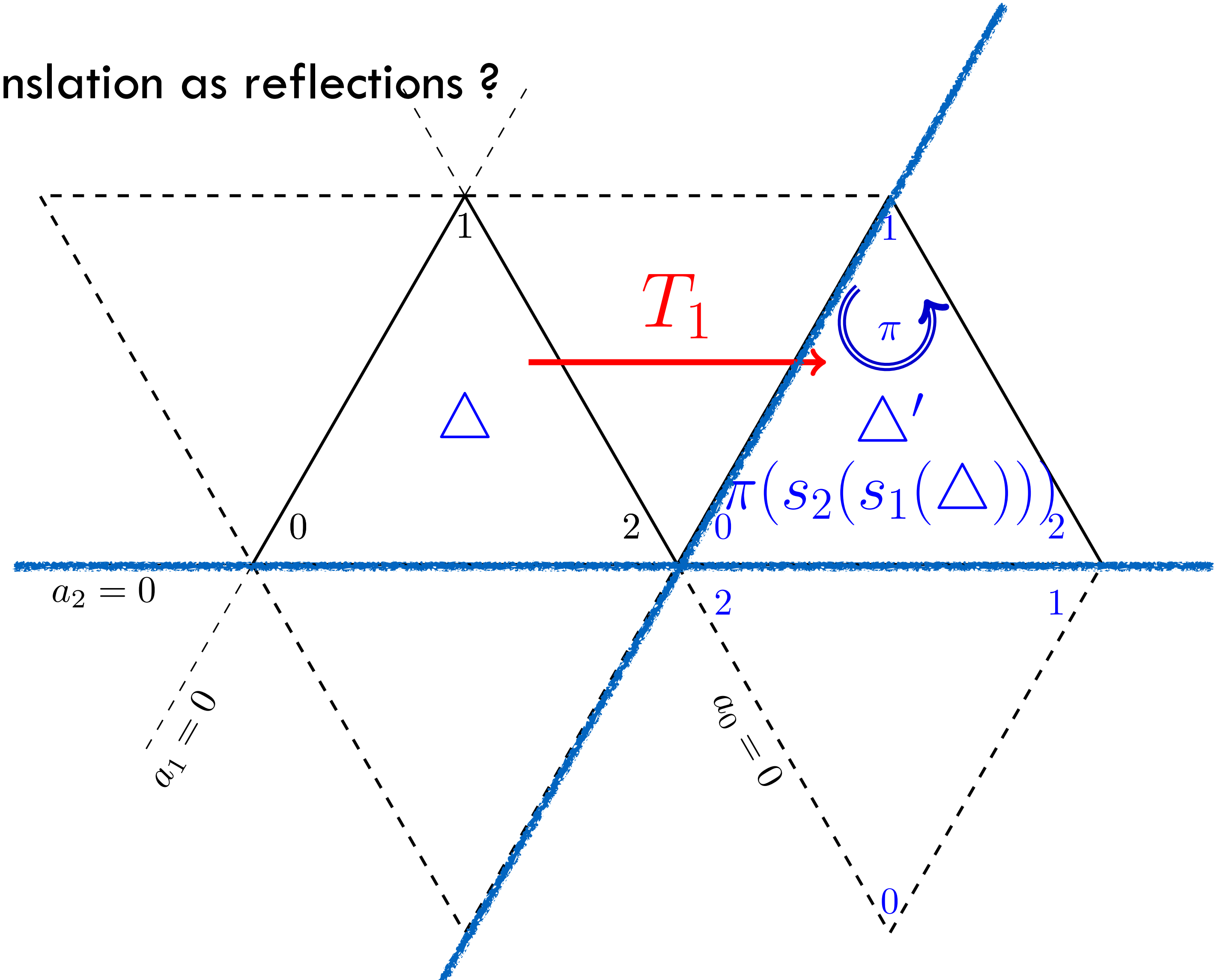

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Noumi 2004

Cremona Isometries

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s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
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s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Noumi 2004

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \qquad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

Translations again

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This is a *discrete first Painlevé* equation,

Translations again

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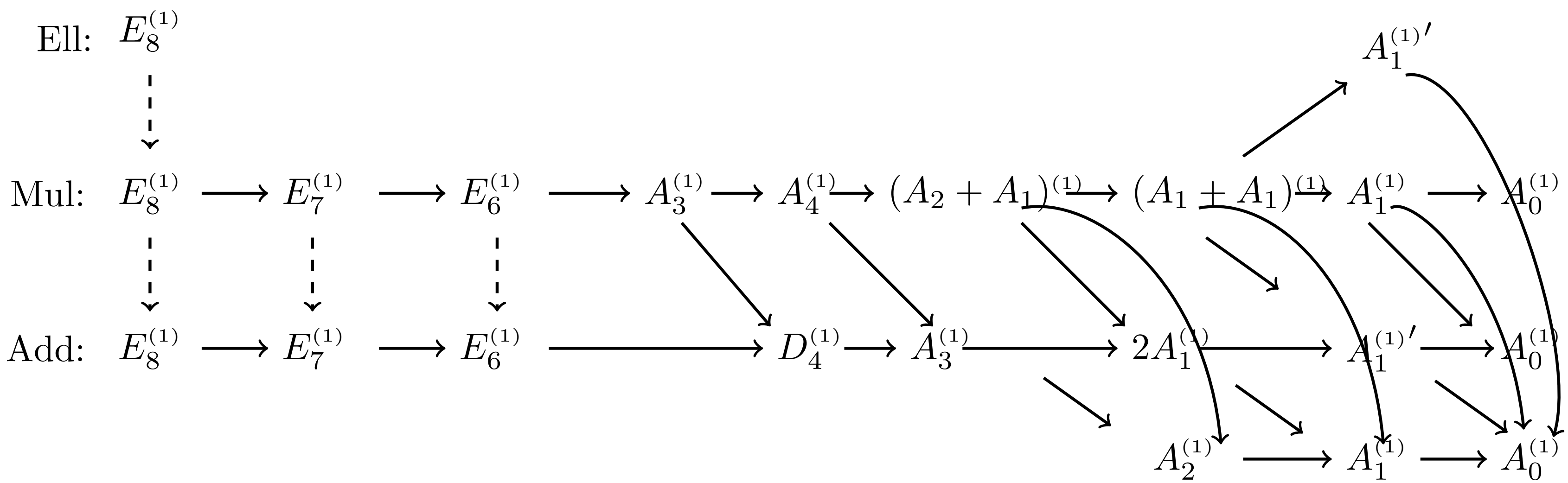
This is a *discrete first Painlevé* equation,
also called the *string* equation.

Sakai (2001) described equations obtained by such translations on all lattices.

Discrete Painlevé Equations

Symmetry groups



Sakai 2001
Rains 2016

Discrete Painlevé Equations

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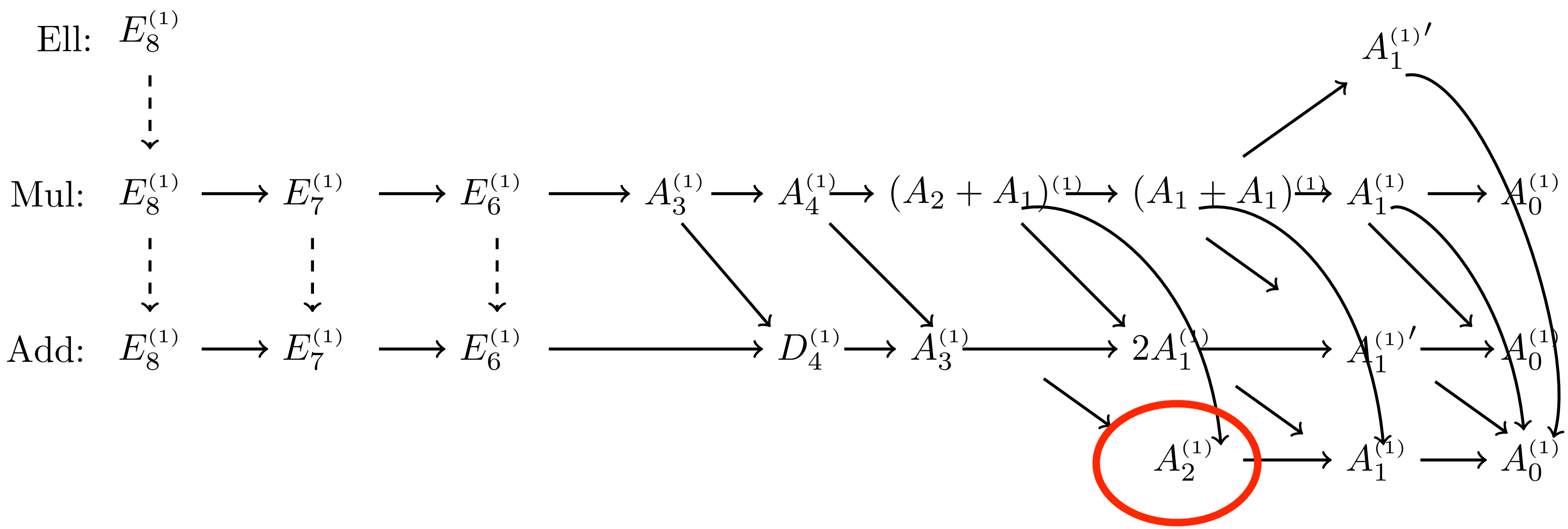

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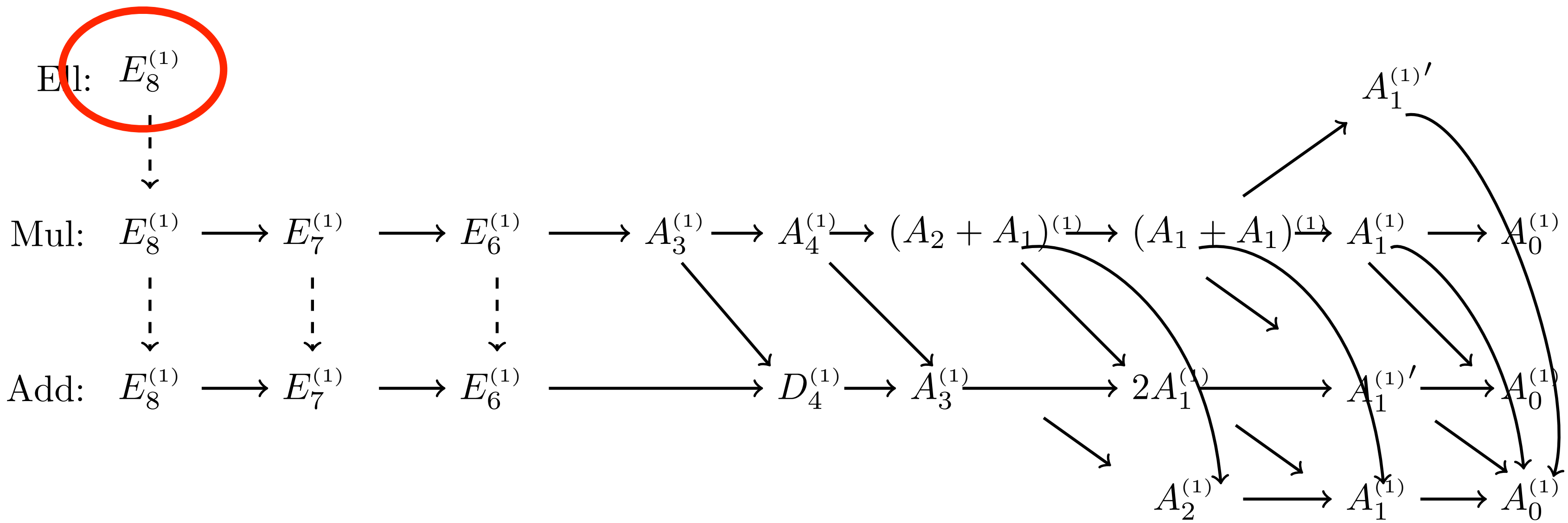
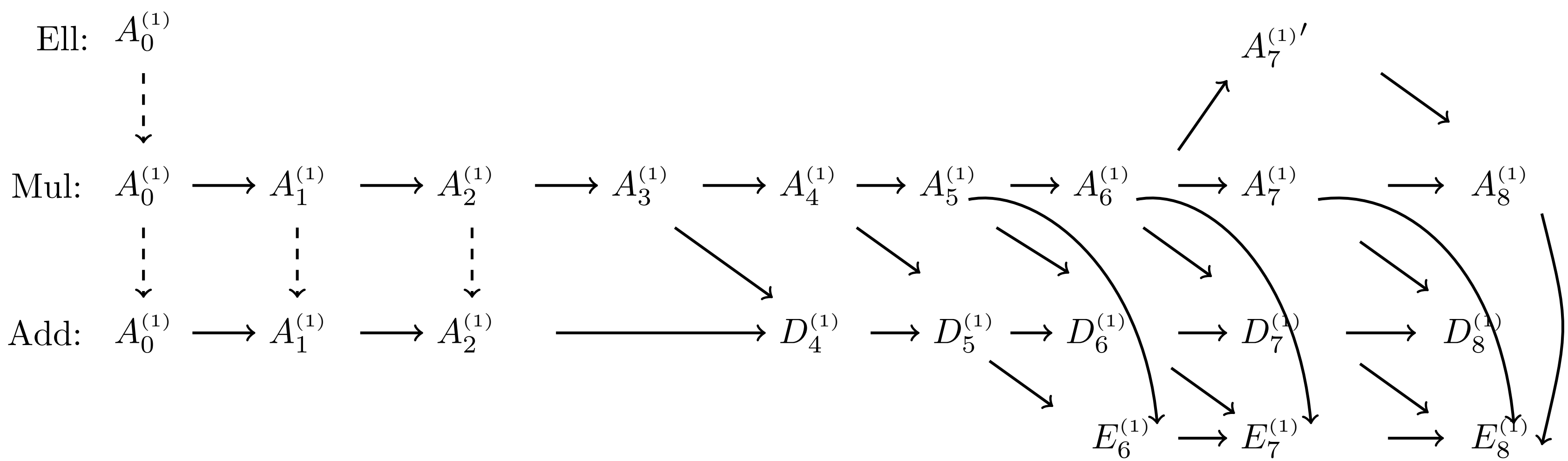


Sakai 2001
Rains 2016

Discrete Painlevé Equations

Initial Value Spaces



Sakai 2001
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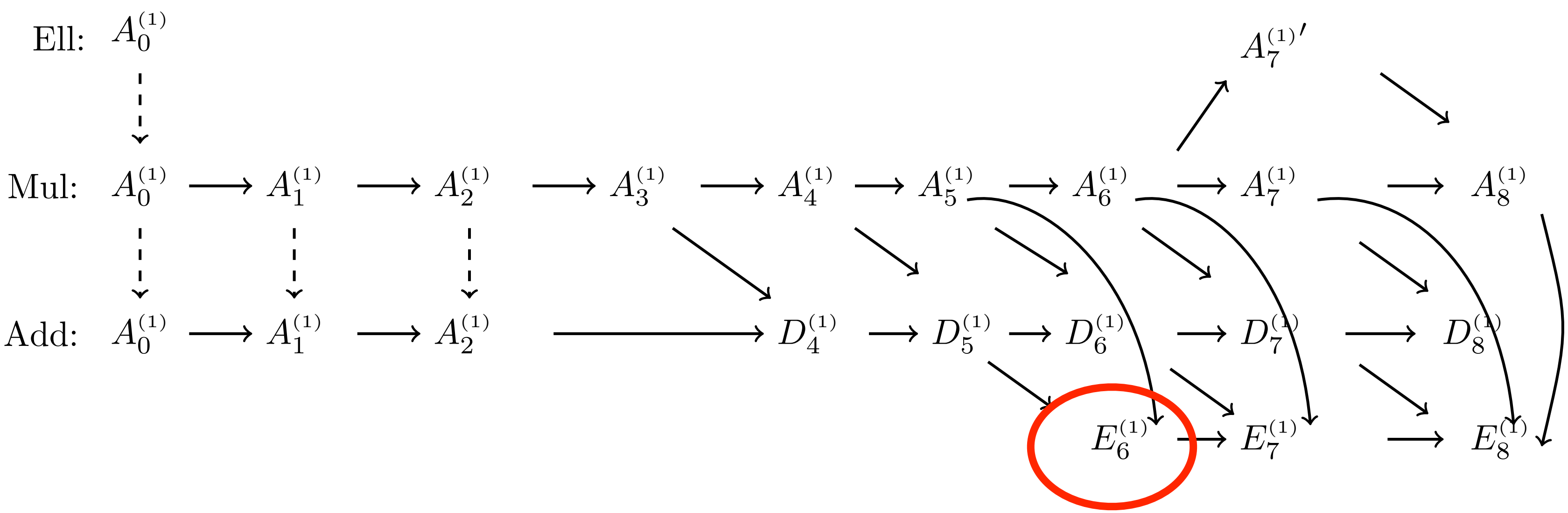

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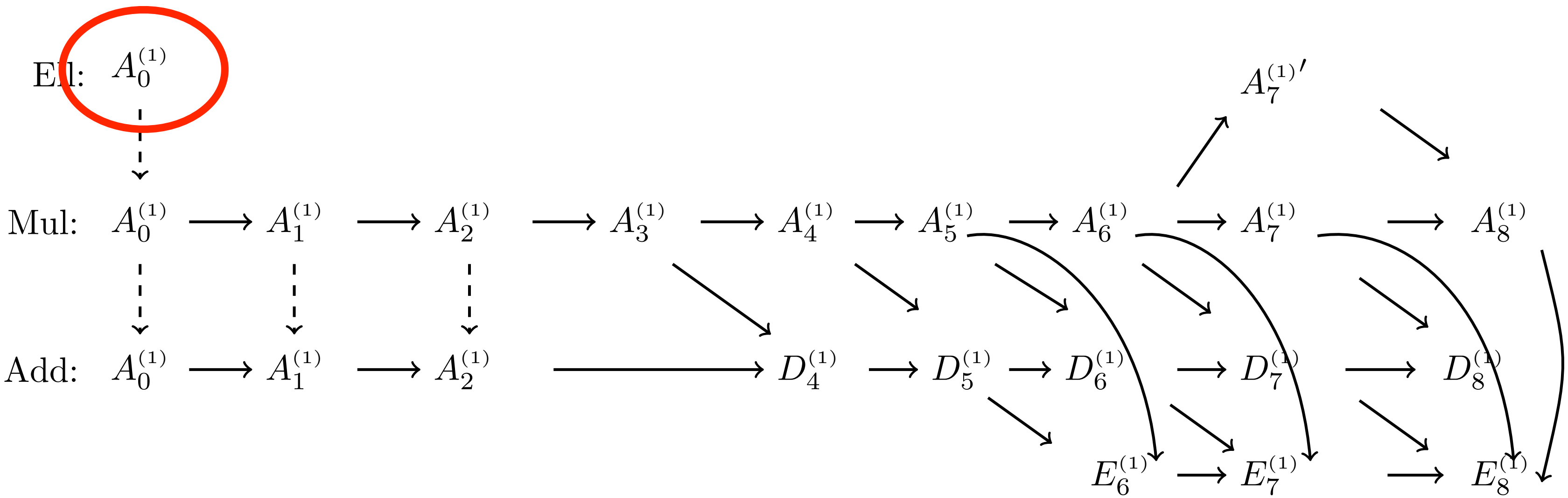

Sakai 2001
Rains 2016

The equation at the top of these diagrams is an elliptic-difference equation.

Notation

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \quad \Leftrightarrow \quad w = \frac{az+b}{cz+d}$$

$$M(h,\kappa_1,\kappa_2,s) = M_0(h,\kappa_1,\kappa_2,s)\, M_1(h,\kappa_1,\kappa_2,s)\, M_2(h,\kappa_1,\kappa_2,s),$$

$$M_0(h,\kappa_1,\kappa_2,s) = \begin{pmatrix} -\wp\big(2\,s - \frac{(-\kappa_1+\kappa_2)}{2}\big) & \wp\big(2\,s - \frac{(\kappa_1-\kappa_2)}{2}\big) \\ 1 & 1 \end{pmatrix}$$

$$M_1(h,\kappa_1,\kappa_2,s) =$$

$$\begin{aligned} \mathrm{diag}\Big(&\big(h - \wp(\kappa_2)\big)\big(\wp(2\,s) - \wp(2\,s-\kappa_2)\big) \\ &\times \big(\wp\big(2\,s - (\kappa_1+\kappa_2)/2\big) - \wp\big(2\,s - (\kappa_1-\kappa_2)/2\big)\big), \\ &\big(h - \wp(\kappa_1)\big)\big(\wp(2\,s) - \wp(2\,s-\kappa_1)\big) \\ &\times \big(\wp\big(2\,s - (\kappa_1+\kappa_2)/2\big) - \wp\big(2\,s - (-\kappa_1+\kappa_2)/2\big)\big)\Big), \end{aligned}$$

$$M_2(h,\kappa_1,\kappa_2,s) = \begin{pmatrix} 1 & -\wp\big(2\,s-\kappa_1\big) \\ 1 & -\wp\big(2\,s-\kappa_2\big) \end{pmatrix}$$

Sakai's elliptic difference equation

$$\overline{g} = M\left(f, c_7, c_8, t - \sum_{i=1}^{6} c_i/4\right) M\left(f, c_5, c_6, t - \sum_{i=1}^{4} c_i/4\right)$$
$$\times M\left(f, c_3, c_4, t - \sum_{i=1}^{2} c_i/4\right) M\left(f, c_1, c_2, t\right)\, g$$
$$\underline{f} = M\left(g, d_7, d_8, t - \sum_{i=1}^{6} d_i/4\right) M\left(g, d_5, d_6, t - \sum_{i=1}^{4} d_i/4\right)$$
$$\times M\left(g, d_3, d_4, t - \sum_{i=1}^{2} d_i/4\right) M\left(g, d_1, d_2, t\right)\, f$$

where

$$\overline{g} = g(t+\lambda), \underline{f} = f(t-\lambda)$$

$$\lambda = \frac{1}{2}\,\sum_{i=1}^{8} b_i, c_i = b_i + t, d_i = t - b_i.$$

Murata, Sakai, Yoneda, 2003

Geometry of Sakai's equation

It has base points:

$$p_i : \quad \big(\wp(t+b_i), \wp(t-b_i)\big), \; i = 1, \ldots, 8$$

which lie on the curve:

$$(x+y+\wp(2t))(4\wp(2t)\, xy - g_3) = \left(xy + \wp(2t)(x+y) + \frac{g_2}{4}\right)^2$$

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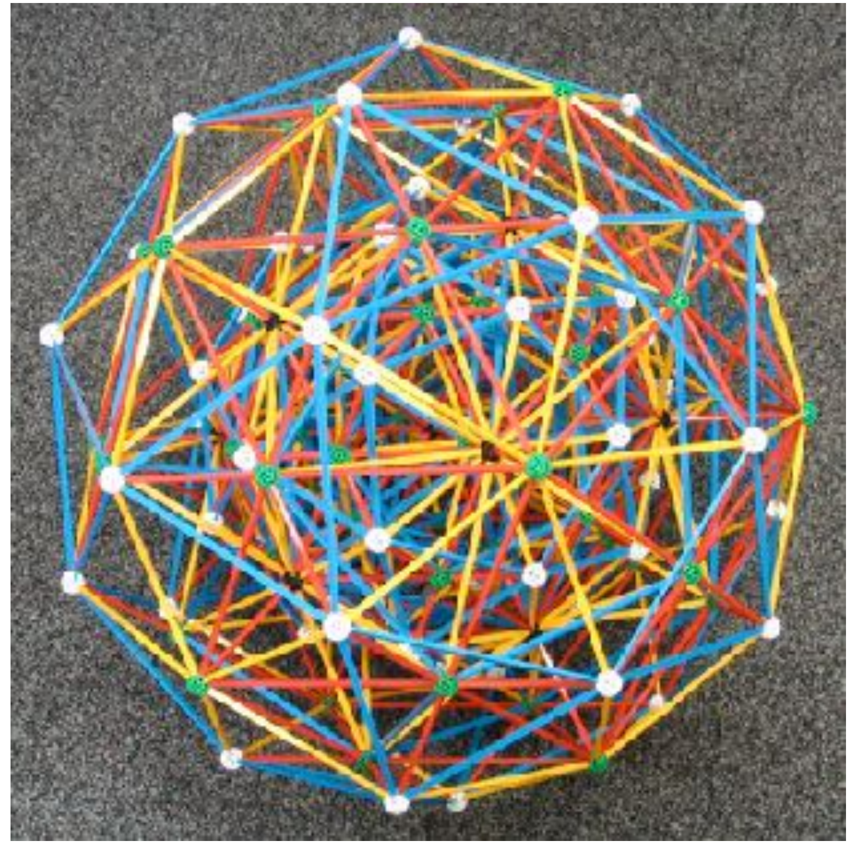
- Are there other elliptic-difference equations?
- To answer this, we have to analyse translations in the $E_8^{(1)}$ lattice.

E_8

- A root system containing 240 root vectors spanning 8 dimensions.
- All root vectors have the same length $\sqrt{2}$.
- The roots span a polytope, known as the 4_{21} polytope.



E_8 Weight Lattice

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- For each vertex, 240 nearest-neighbours, reached by vectors of squared length 2.

⇒

Sakai's elliptic difference equation

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→ Sakai's elliptic difference equation

- For each vertex, 2160 next-nearest-neighbours, reached by vectors of squared length 4.

→ A new elliptic difference equation

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- Atkinson, Howes, J. and Nakazono (2016)

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RCG equation

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$$
\begin{aligned}
&\mathrm{cn}(\gamma_n)\mathrm{dn}(\gamma_n)\big(1-k^2\mathrm{sn}^4(z_n)\big)u_n\big(u_{n+1}+u_{n-1}\big)\\
&-\mathrm{cn}(z_n)\mathrm{dn}(z_n)\big(1-k^2\mathrm{sn}^2(z_n)\mathrm{sn}^2(\gamma_n)\big)\big(u_{n+1}u_{n-1}+{u_n}^2\big)\\
&+\big(\mathrm{cn}^2(z_n)-\mathrm{cn}^2(\gamma_n)\big)\mathrm{cn}(z_n)\mathrm{dn}(z_n)\big(1+k^2{u_n}^2u_{n+1}u_{n-1}\big)=0
\end{aligned}
$$

where

$$
z_n=(\gamma_e+\gamma_o)n+z_0,\quad \gamma_n=\begin{cases}\gamma_e, & \text{for } n=2\,j\\ \gamma_o, & \text{for } n=2j+1.\end{cases}
$$

Unusual features of the RCG equation

Unusual features of the RCG equation

- The singularity structure of the RCG equation differs from that of its autonomous version.
- The "deautonomization" of the autonomous equation (keeping the same singularities) leads to a *q*-difference equation, not an *ell*-difference one.
- The initial value space of the autonomous equation is $A_1^{(1)}$, while that of the RCG equation is $A_0^{(1)}$.

Geometry of RCG equation I

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- Atkinson, Howes, J. and Nakazono (2016) showed that
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  - but, its symmetry group is $F_4^{(1)}$, a *subgroup* of $E_8^{(1)}$
  - and, time iteration is *not* given by translation on this lattice.

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$$
\begin{array}{llll}
\text{Ell:} & E_8^{(1)} & & \\
 & \downarrow & & \\
\text{Mul:} & E_8^{(1)} \longrightarrow & E_7^{(1)} \longrightarrow & E_6^{(1)} \\
 & \downarrow & \downarrow & \downarrow \\
\text{Add:} & E_8^{(1)} \longrightarrow & E_7^{(1)} \longrightarrow & E_6^{(1)}
\end{array}
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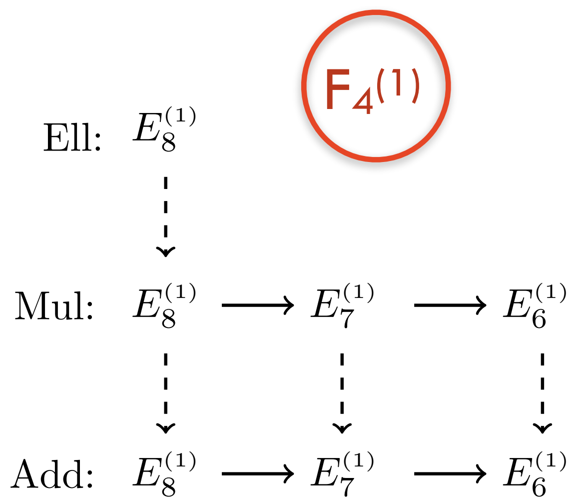

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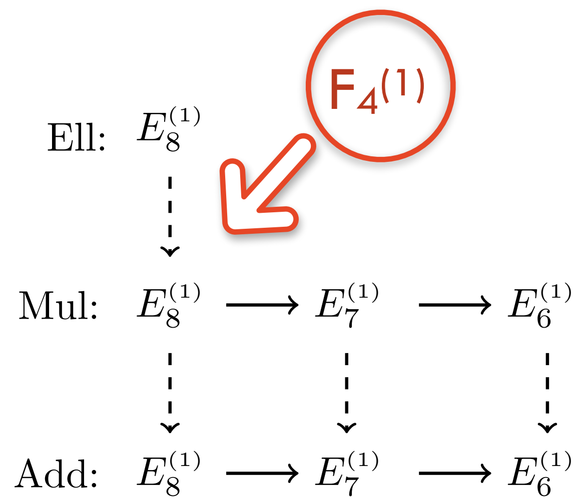


Geometry of RCG equation II

The RCG equation has 8 base points

$$p_1 : (x, y) = \big(\mathrm{cd}(\gamma_o + \kappa), \mathrm{cd}(z_0 - \gamma_e - \gamma_o + \kappa)\big),$$

$$p_2 : (x, y) = \big(\mathrm{cd}(\gamma_o + \mathrm{i}K'), \mathrm{cd}(z_0 - \gamma_e - \gamma_o + \mathrm{i}K')\big),$$

$$p_3 : (x, y) = \big(\mathrm{cd}(\gamma_o + 2K), \mathrm{cd}(z_0 - \gamma_e - \gamma_o + 2K)\big),$$

$$p_4 : (x, y) = \big(\mathrm{cd}(\gamma_o), \mathrm{cd}(z_0 - \gamma_e - \gamma_o)\big),$$

$$p_5 : (x, y) = \big(\mathrm{cd}(z_0 + \kappa), \mathrm{cd}(\gamma_e + \kappa)\big),$$

$$p_6 : (x, y) = \big(\mathrm{cd}(z_0 + \mathrm{i}K'), \mathrm{cd}(\gamma_e + \mathrm{i}K')\big),$$

$$p_7 : (x, y) = \big(\mathrm{cd}(z_0 + 2K), \mathrm{cd}(\gamma_e + 2K)\big),$$

$$p_8 : (x, y) = \big(\mathrm{cd}(z_0), \mathrm{cd}(\gamma_e)\big),$$

where $K(k)$, $K'(k)$ are complete elliptic integrals.

Atkinson, Howes, J. and Nakazono, 2016

Geometry of RCG equation III

The base points lie on elliptic curves:

$$\mathrm{sn}(z_0-\gamma_e)^2\left(1+k^2x^2y^2\right)+2\,\mathrm{cn}(z_0-\gamma_e)\mathrm{dn}(z_0-\gamma_e)xy-(x^2+y^2)=0$$

which is iterated by

$$\tilde{}: z_0 \mapsto \ z_0+2(\gamma_e+\gamma_o)$$

Atkinson, Howes, J. and Nakazono, 2016

Generalization

Consider 8 base points

$$p_i \ : \ (x,y) = \big(\mathrm{cd}(c_i+\eta), \mathrm{cd}(\eta - c_i)\big), \ \ i = 1, \ldots, 8$$

which lie on the curve

$$\mathrm{sn}(2\eta)^2\,(1+k^2\,x^2\,y^2) + 2\mathrm{cn}(2\eta)\mathrm{dn}(2\eta)xy - (x^2+y^2) = 0$$

iterated by

$$T_{J,1} : (c_i, c_{i+4}, \eta) \mapsto (c_i - \lambda, c_{i+4} + \lambda + 4\,\kappa, \eta + \lambda - 2\kappa), \ \ i = 1, \ldots, 4$$

with $\lambda = \sum_{i=1}^{8} c_i$ remaining invariant.

J. and Nakazono, 2017

A new elliptic difference equation

$$\left(\frac{k\,\mathrm{cd}\,(\eta - c_8 + \kappa)\,\overline{y} + 1}{k\,\mathrm{cd}\,(\eta - c_7 + \kappa)\,\overline{y} + 1}\right)\left(\frac{\tilde{x} - \mathrm{cd}\left(\eta - c_7 + \frac{c_{5678}}{2} + \lambda + \kappa\right)}{\tilde{x} - \mathrm{cd}\left(\eta - c_8 + \frac{c_{5678}}{2} + \lambda + \kappa\right)}\right)$$

$$= G_{\frac{c_{5678}-2c_5+\lambda}{2}, \frac{c_{5678}-2c_6+\lambda}{2}, \frac{c_{5678}-2c_7+\lambda}{2}, \frac{c_{5678}-2c_8+\lambda}{2}, \eta + \frac{\lambda}{2} + \kappa}$$

$$\times \frac{P_{\frac{c_{5678}-2c_5+\lambda}{2}, \frac{c_{5678}-2c_6+\lambda}{2}, \frac{c_{5678}-2c_7+\lambda}{2}, \eta + \frac{\lambda}{2} + \kappa}(\tilde{x}, \tilde{y})}{P_{\frac{c_{5678}-2c_5+\lambda}{2}, \frac{c_{5678}-2c_6+\lambda}{2}, \frac{c_{5678}-2c_8+\lambda}{2}, \eta + \frac{\lambda}{2} + \kappa}(\tilde{x}, \tilde{y})},$$

$$\left(\frac{k\,\mathrm{cd}\,(\eta + c_4 + \kappa)\,\overline{x} + 1}{k\,\mathrm{cd}\,(\eta + c_3 + \kappa)\,\overline{x} + 1}\right)\left(\frac{k\,\mathrm{cd}\,(\eta - c_3 + 2\lambda + \kappa)\,\overline{y} + 1}{k\,\mathrm{cd}\,(\eta - c_4 + 2\lambda + \kappa)\,\overline{y} + 1}\right)$$

$$= G_{\eta - c_1 + \frac{c_{1234}}{4} + \lambda, \eta - c_2 + \frac{c_{1234}}{4} + \lambda, \eta - c_3 + \frac{c_{1234}}{4} + \lambda, \eta - c_4 + \frac{c_{1234}}{4} + \lambda, \frac{c_{5678}+2\lambda}{4} + \kappa}$$

$$\times \frac{P_{\eta - c_1 + \frac{c_{1234}}{4} + \lambda, \eta - c_2 + \frac{c_{1234}}{4} + \lambda, \eta - c_3 + \frac{c_{1234}}{4} + \lambda, \frac{c_{5678}+2\lambda}{4} + \kappa}\left(\frac{-1}{k\overline{y}}, \tilde{x}\right)}{P_{\eta - c_1 + \frac{c_{1234}}{4} + \lambda, \eta - c_2 + \frac{c_{1234}}{4} + \lambda, \eta - c_4 + \frac{c_{1234}}{4} + \lambda, \frac{c_{5678}+2\lambda}{4} + \kappa}\left(\frac{-1}{k\overline{y}}, \tilde{x}\right)}$$

Where

$\tilde{x} = R_{J,1}(x), \tilde{y} = R_{J,1}(y)$ are defined by

$$\left(\frac{k\,\mathrm{cd}\left(\eta + c_8 - \frac{c_{5678}}{2}\right)\tilde{y} + 1}{k\,\mathrm{cd}\left(\eta + c_7 - \frac{c_{5678}}{2}\right)\tilde{y} + 1}\right)\left(\frac{x - \mathrm{cd}\left(\eta + c_7\right)}{x - \mathrm{cd}\left(\eta + c_8\right)}\right) = G_{c_5,c_6,c_7,c_8,\eta}\,\frac{P_{c_5,c_6,c_7,\eta}\left(x,y\right)}{P_{c_5,c_6,c_8,\eta}\left(x,y\right)},$$

$$\left(\frac{k\,\mathrm{cd}\left(\eta - c_4 + \frac{c_{1234}}{2}\right)\tilde{x} + 1}{k\,\mathrm{cd}\left(\eta - c_3 + \frac{c_{1234}}{2}\right)\tilde{x} + 1}\right)\left(\frac{k\,\mathrm{cd}\left(\eta + c_3 + \frac{c_{5678}}{2}\right)\tilde{y} + 1}{k\,\mathrm{cd}\left(\eta + c_4 + \frac{c_{5678}}{2}\right)\tilde{y} + 1}\right)$$

$$= G_{\eta+c_1+\frac{c_{5678}}{4},\eta+c_2+\frac{c_{5678}}{4},\eta+c_3+\frac{c_{5678}}{4},\eta+c_4+\frac{c_{5678}}{4},\frac{c_{5678}}{4}}$$

$$\times\frac{P_{\eta+c_1+\frac{c_{5678}}{4},\eta+c_2+\frac{c_{5678}}{4},\eta+c_3+\frac{c_{5678}}{4},\frac{c_{5678}}{4}}\left(\frac{-1}{k\tilde{y}},x\right)}{P_{\eta+c_1+\frac{c_{5678}}{4},\eta+c_2+\frac{c_{5678}}{4},\eta+c_4+\frac{c_{5678}}{4},\frac{c_{5678}}{4}}\left(\frac{-1}{k\tilde{y}},x\right)}.$$

$$G_{a_1,a_2,a_3,a_4,b} = \left(\frac{1-\frac{\mathrm{cd}\left(a_4+\frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(a_2+\frac{a_1+a_2}{2}\right)}}{1-\frac{\mathrm{cd}\left(a_3+\frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(a_2+\frac{a_1+a_2}{2}\right)}}\right)\left(\frac{1-\frac{\mathrm{cd}(b-a_4)}{\mathrm{cd}(b-a_1)}}{1-\frac{\mathrm{cd}(b-a_3)}{\mathrm{cd}(b-a_1)}}\right)\left(\frac{1-\frac{\mathrm{cd}\left(b+a_4-\frac{a_1+a_2+a_3+a_4}{2}\right)}{\mathrm{cd}\left(b+a_2+\frac{a_1+a_2+a_3+a_4}{2}\right)}}{1-\frac{\mathrm{cd}\left(b+a_3-\frac{a_1+a_2+a_3+a_4}{2}\right)}{\mathrm{cd}\left(b+a_2+\frac{a_1+a_2+a_3+a_4}{2}\right)}}\right)\left(\frac{1-\frac{\mathrm{cd}\left(a_3+\frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(2b+a_2-\frac{a_1+a_2}{2}\right)}}{1-\frac{\mathrm{cd}\left(a_4+\frac{a_1+a_2}{2}\right)}{\mathrm{cd}\left(2b+a_2-\frac{a_1+a_2}{2}\right)}}\right),$$

$$\begin{aligned}
&Q_{a_1,a_2,a_3,a_4,a_5,b}\left(X\right)\\
&= \left(\mathrm{cd}\left(b+a_3-\frac{a_5}{2}\right)-\mathrm{cd}\left(b+a_2+\frac{a_5}{2}\right)\right)\left(\mathrm{cd}\left(b+a_1+\frac{a_5}{2}\right)-\mathrm{cd}\left(b+a_4+\frac{a_5}{2}\right)\right)\\
&\quad\times\left(\mathrm{cd}\left(b+a_4\right)\mathrm{cd}\left(b+a_1\right)+\mathrm{cd}\left(b+a_2\right)X\right)+\left(\mathrm{cd}\left(b+a_3-\frac{a_5}{2}\right)-\mathrm{cd}\left(b+a_1+\frac{a_5}{2}\right)\right)\\
&\quad\times\left(\mathrm{cd}\left(b+a_4+\frac{a_5}{2}\right)-\mathrm{cd}\left(b+a_2+\frac{a_5}{2}\right)\right)\left(\mathrm{cd}\left(b+a_4\right)\mathrm{cd}\left(b+a_2\right)+\mathrm{cd}\left(b+a_1\right)X\right)\\
&-\left(\mathrm{cd}\left(b+a_3-\frac{a_5}{2}\right)-\mathrm{cd}\left(b+a_4+\frac{a_5}{2}\right)\right)\left(\mathrm{cd}\left(b+a_1+\frac{a_5}{2}\right)-\mathrm{cd}\left(b+a_2+\frac{a_5}{2}\right)\right)\\
&\quad\times\left(\mathrm{cd}\left(b+a_1\right)\mathrm{cd}\left(b+a_2\right)+\mathrm{cd}\left(b+a_4\right)X\right),
\end{aligned}$$

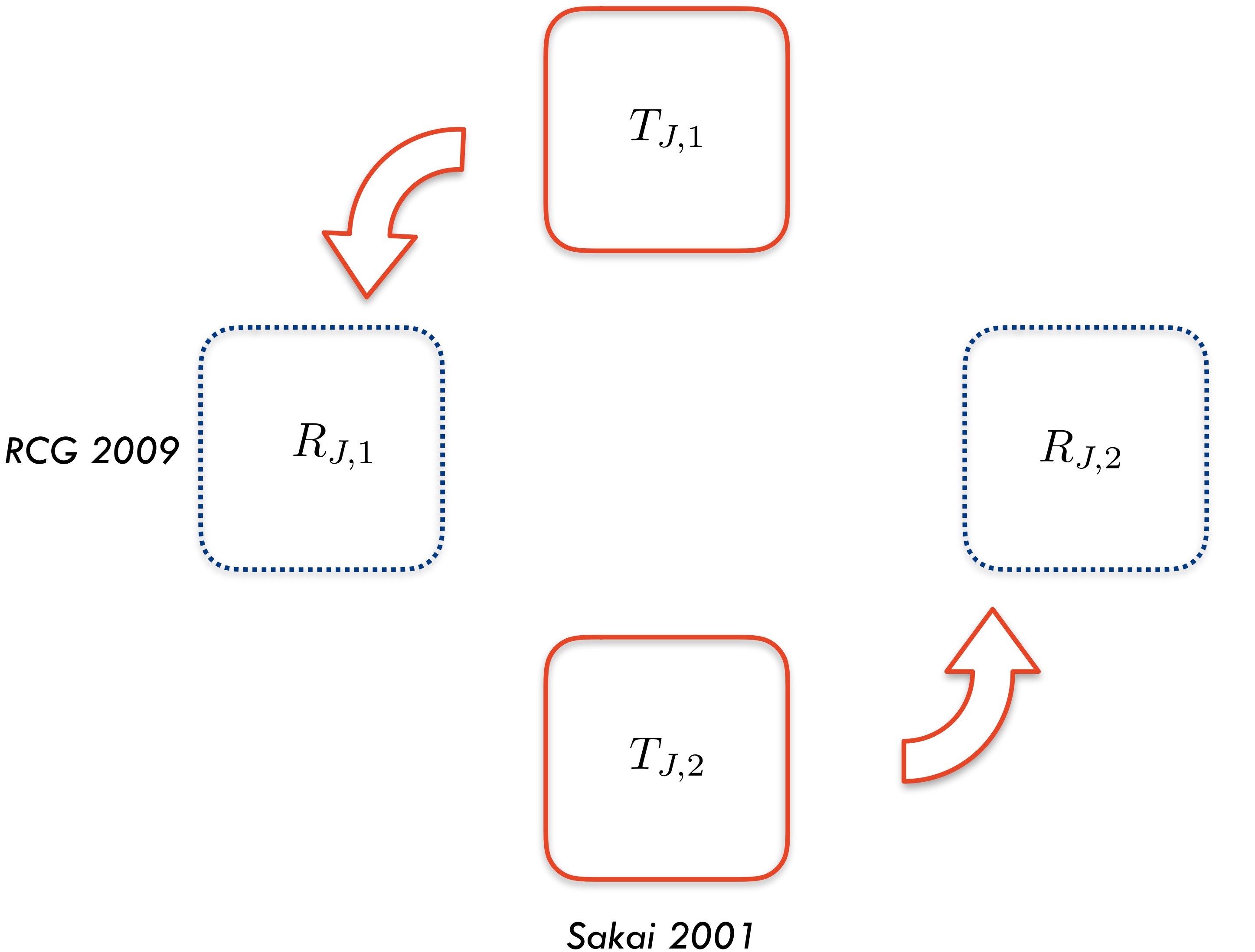
$$P_{a_1,a_2,a_3,b}\left(X,Y\right) = C_1XY + C_2X + C_3Y + C_4,$$

The RCG equation is a *projective reduction* of the new elliptic difference equation.

$$T_{J,1} = R^2_{J,1}$$

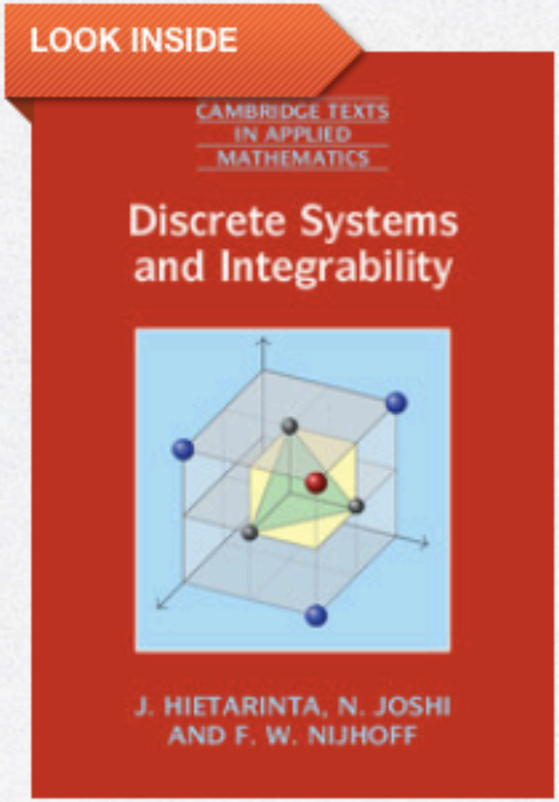
J. and Nakazono, 2017

At least four elliptic-difference equations



Summary

- Four elliptic-difference Painlevé equations are now known.
- Two of these are "projective reductions" of the remaining equations.
- We showed that at least 1 has a symmetry group given by $F_4^{(1)}$ - the first non-simply laced group appearing in the classification.
- Many questions remain open: (i) properties of these new equations, (ii) the question of completeness, (iii) other reductions of Q4…



Discrete Systems and Integrability

Part of Cambridge Texts in Applied Mathematics

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DATE PUBLISHED: August 2016
AVAILABILITY: In stock
FORMAT: Paperback
ISBN: 9781107669482

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