Symmetry through Geometry

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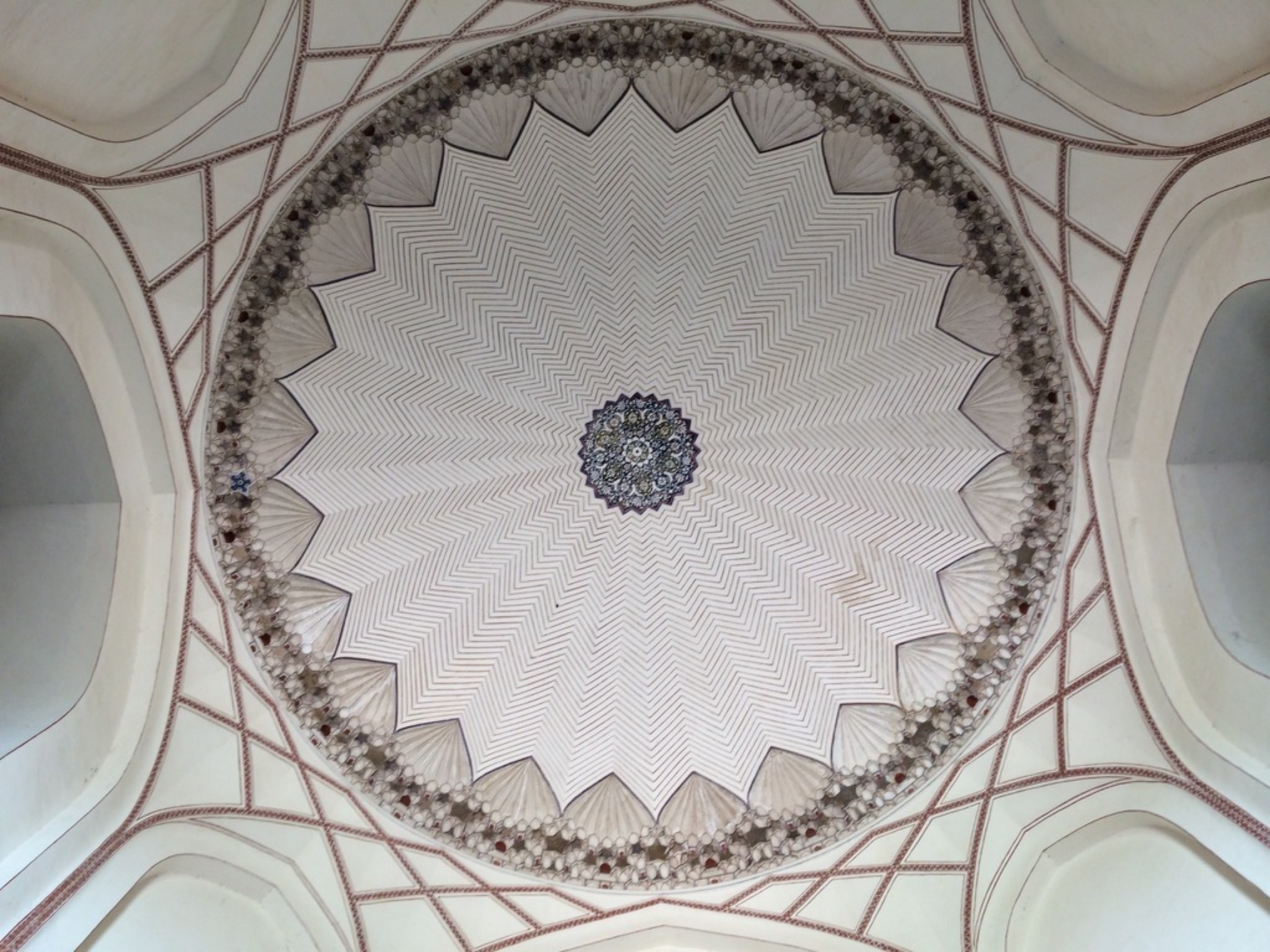
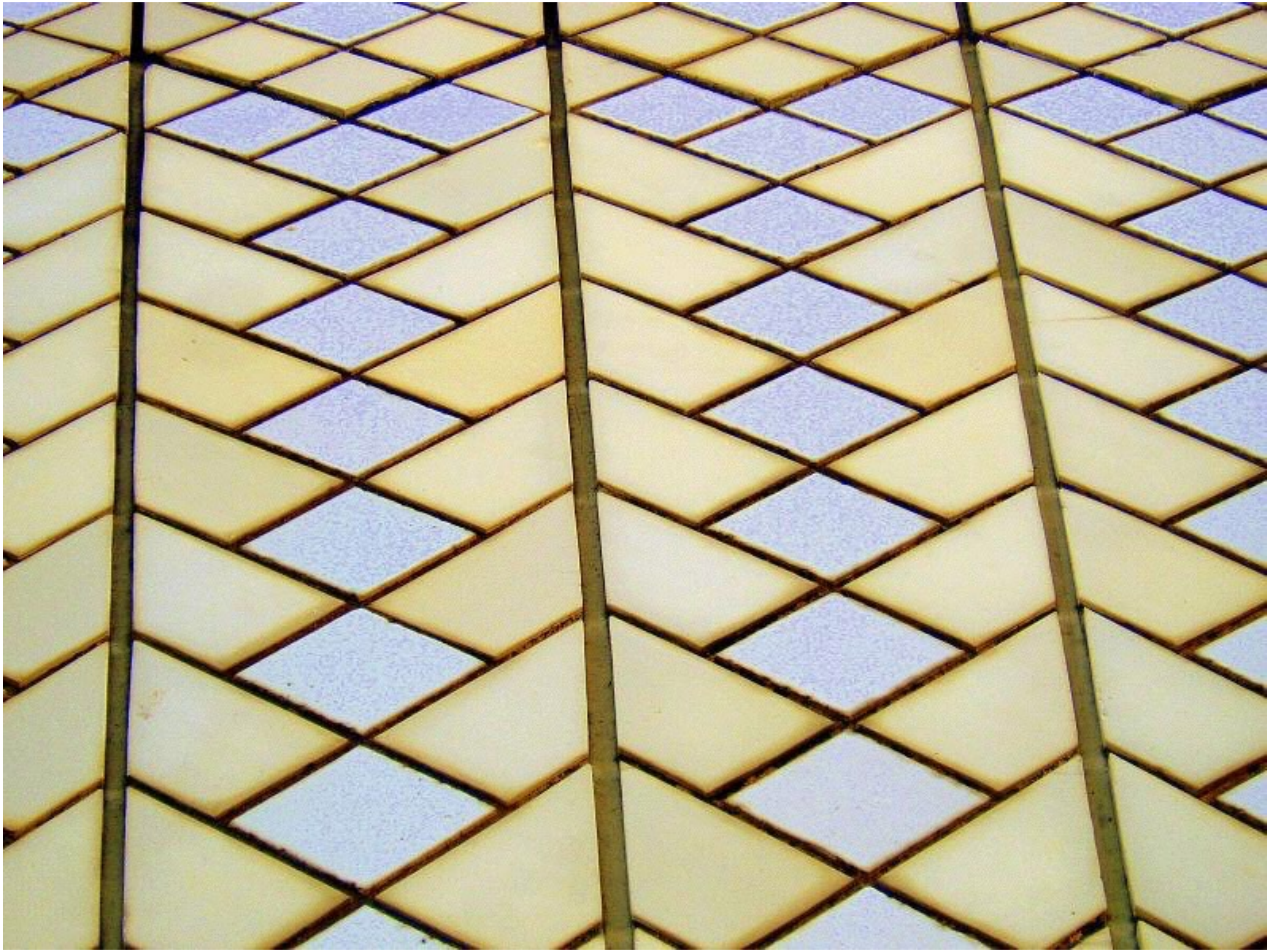
Belur, Karnataka, India



Humayun's Tomb, Delhi, India



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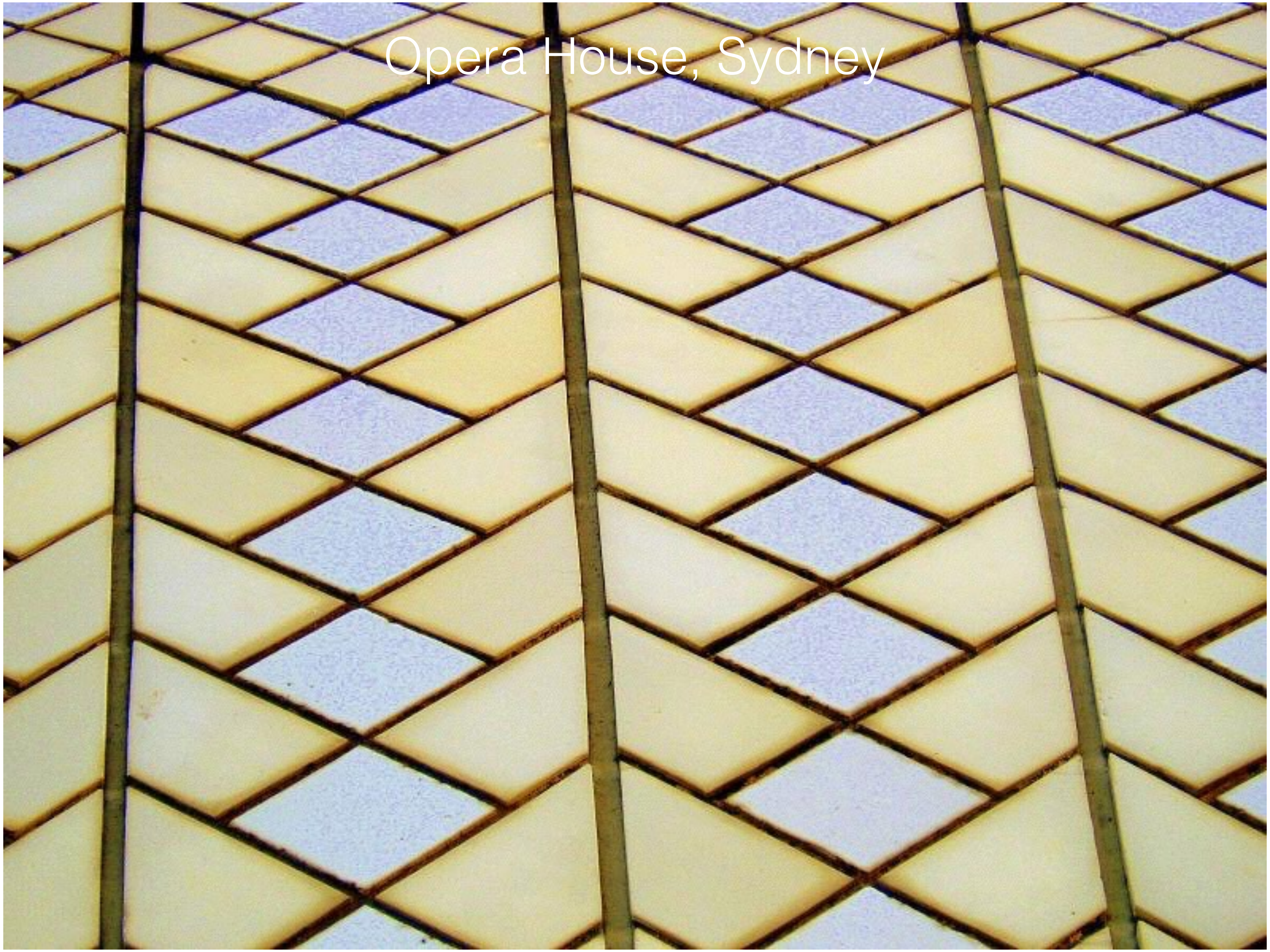

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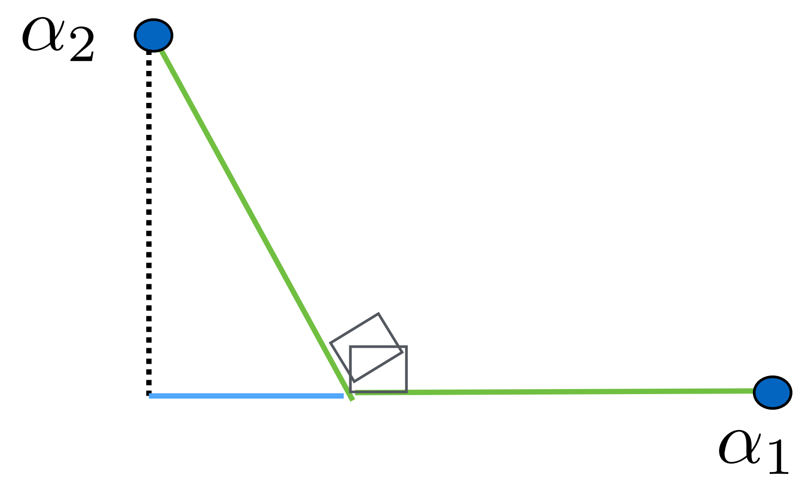
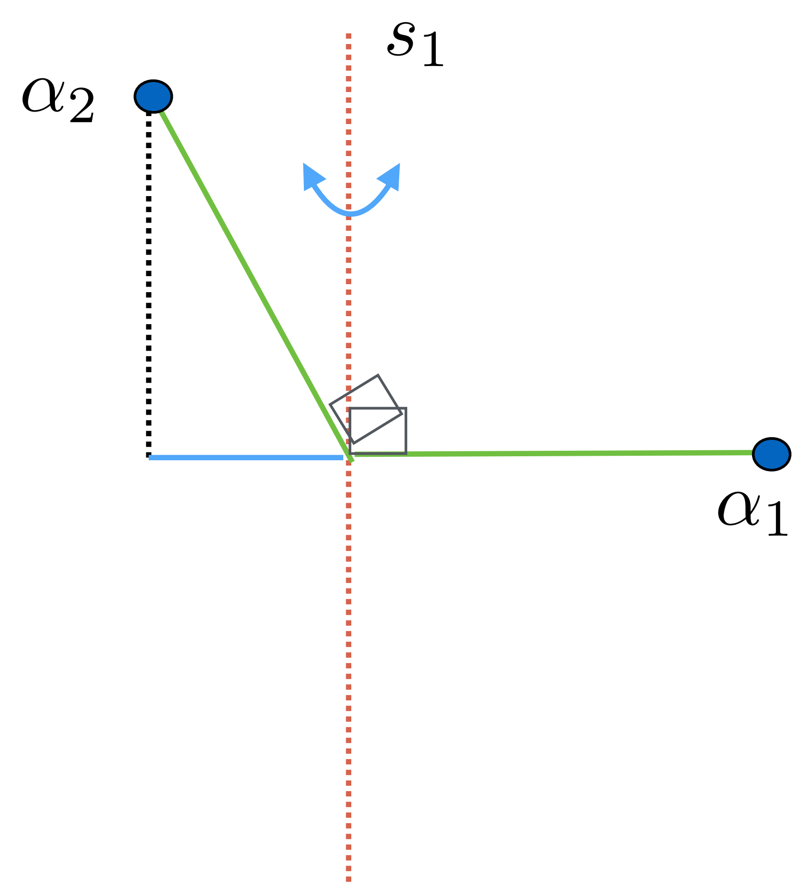
A Reflection



A Reflection



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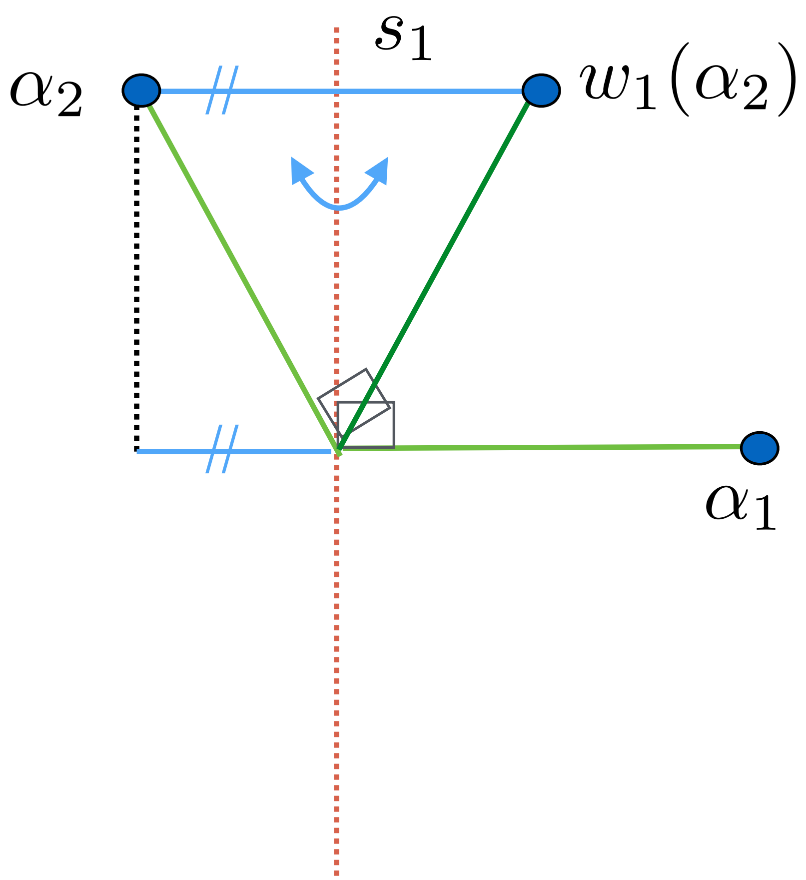
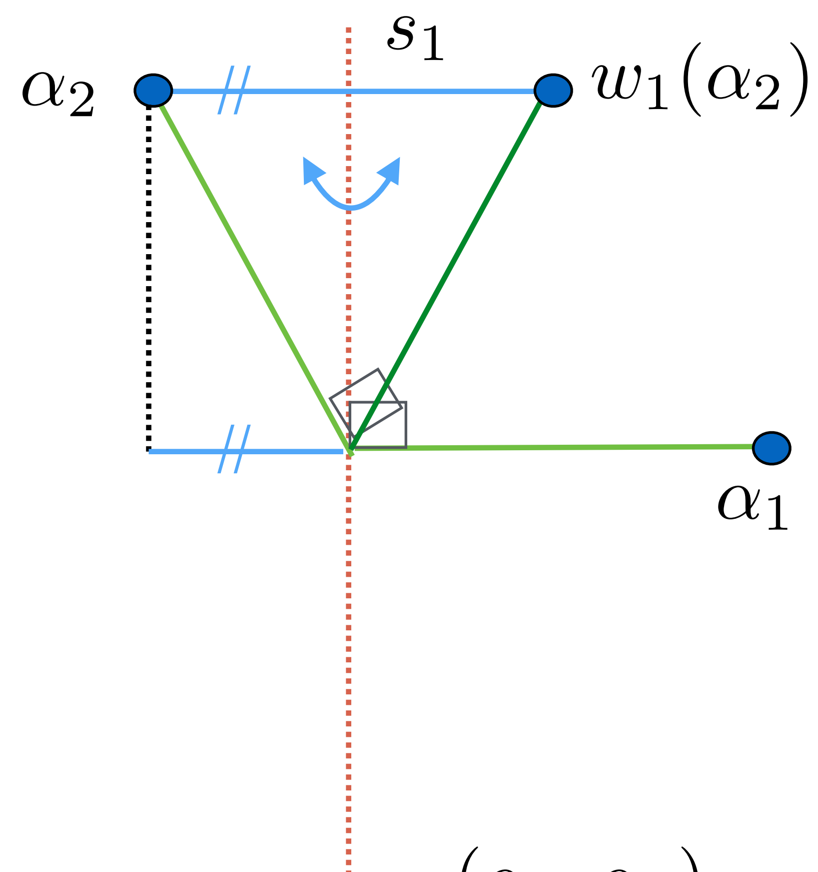
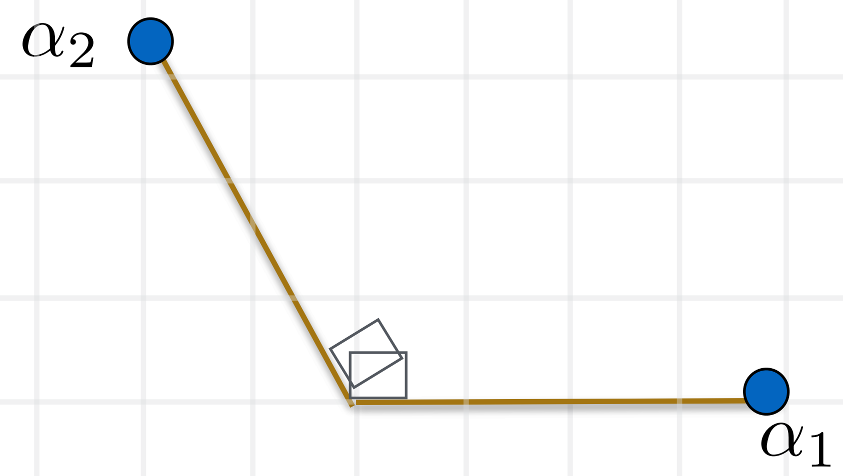


A Reflection



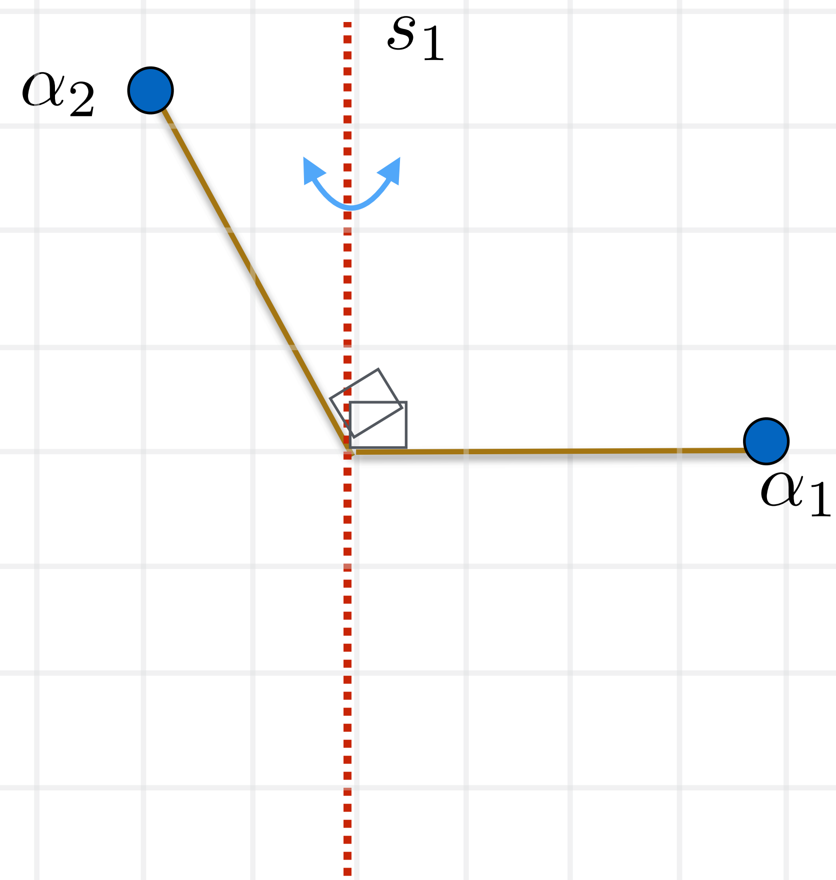
$$
\begin{aligned}
w_1(\alpha_2) &= \alpha_2 - 2\,\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\,\alpha_1 \\
&= (-1, \sqrt{3}) + (2, 0) \\
&= (1, \sqrt{3})
\end{aligned}
$$

Root System



α_1 and α_2 are “simple” roots

Root System



α_1 and α_2 are “simple” roots

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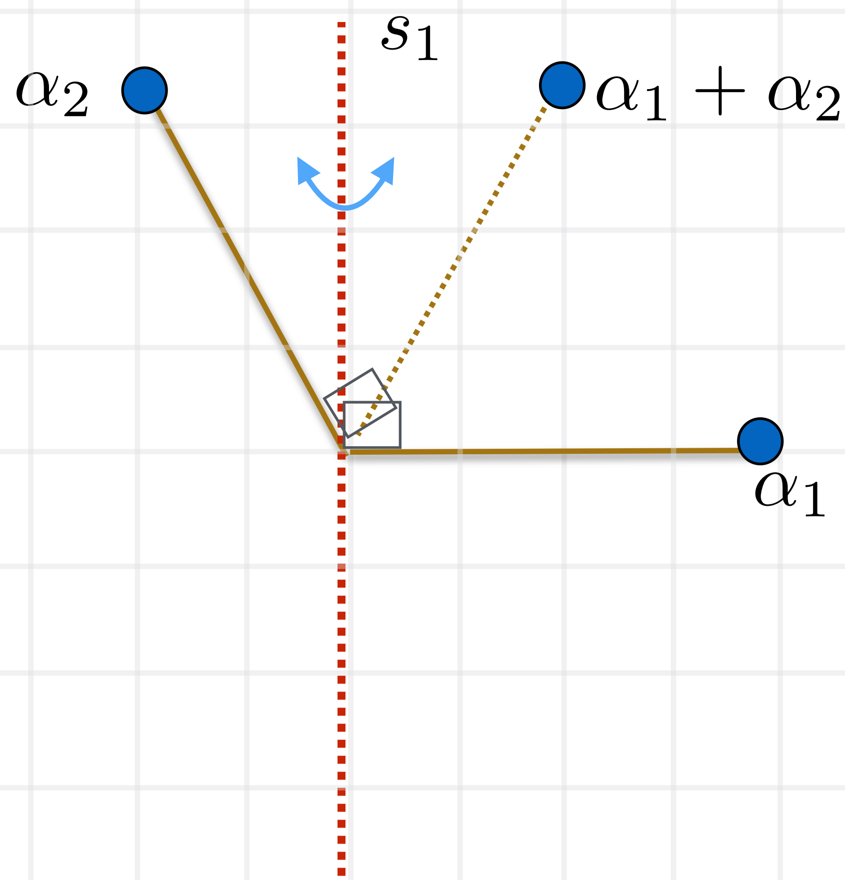

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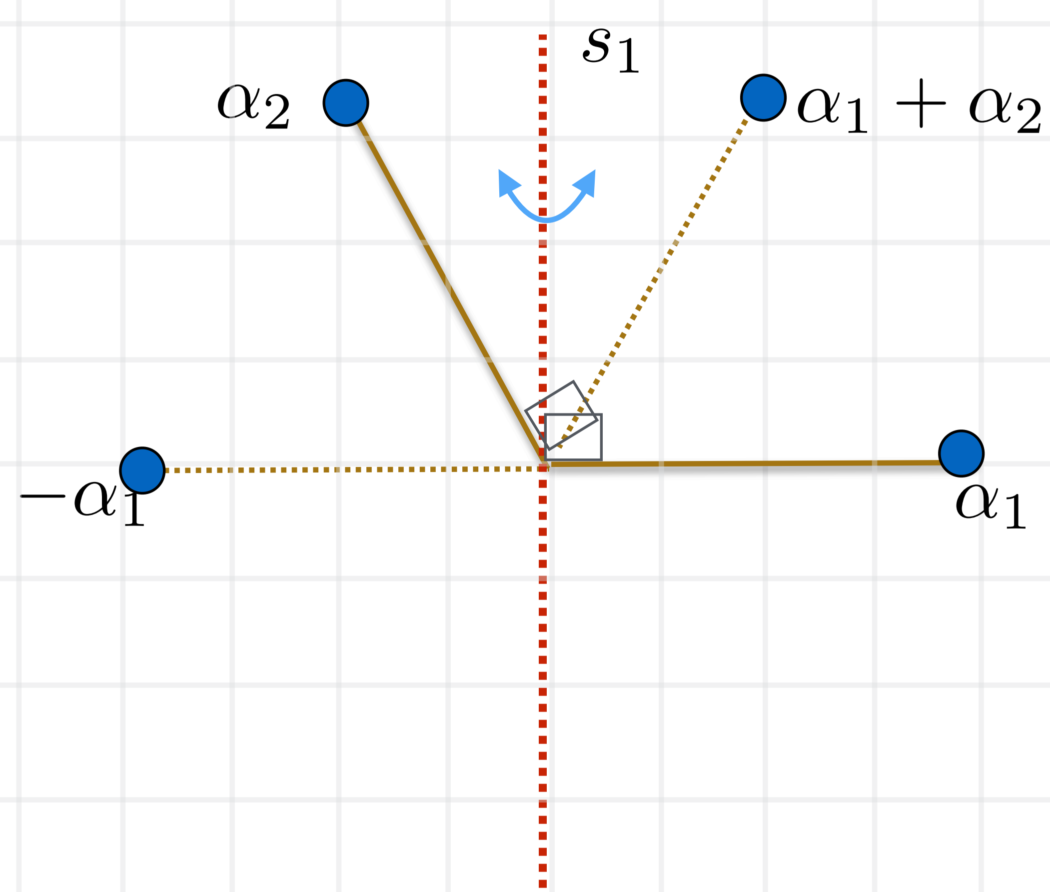
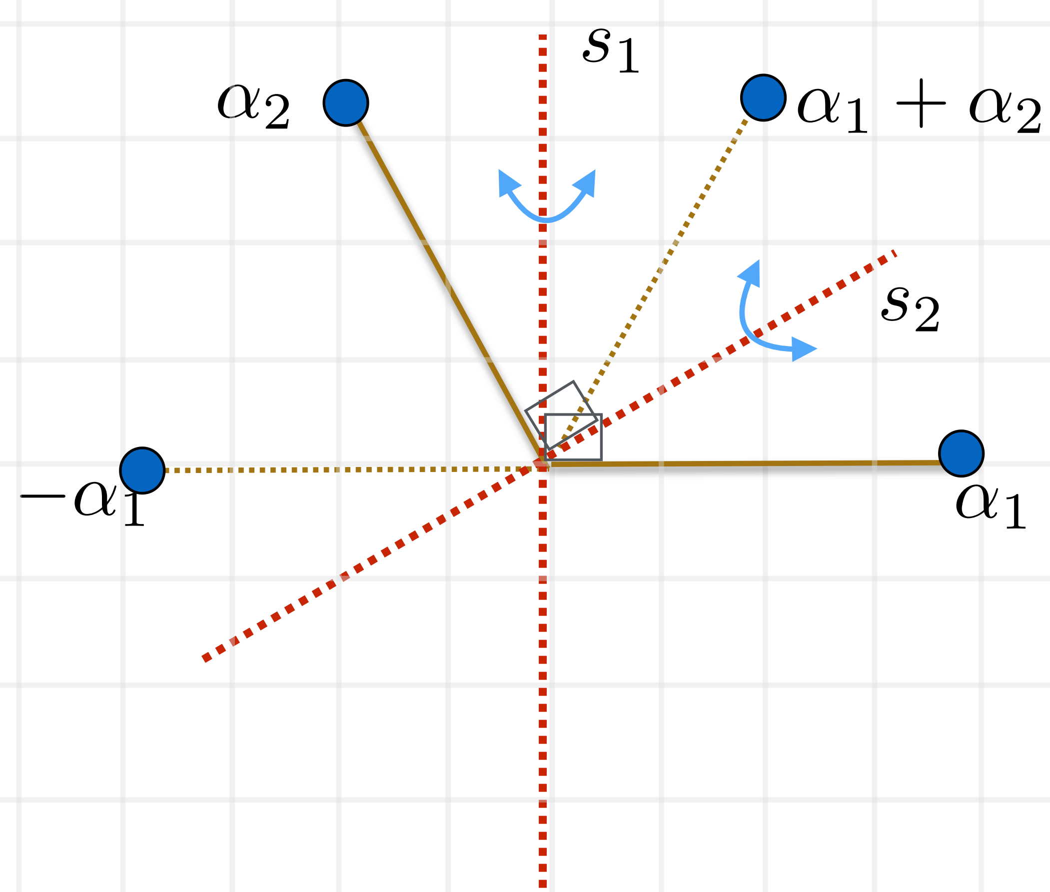
Root System



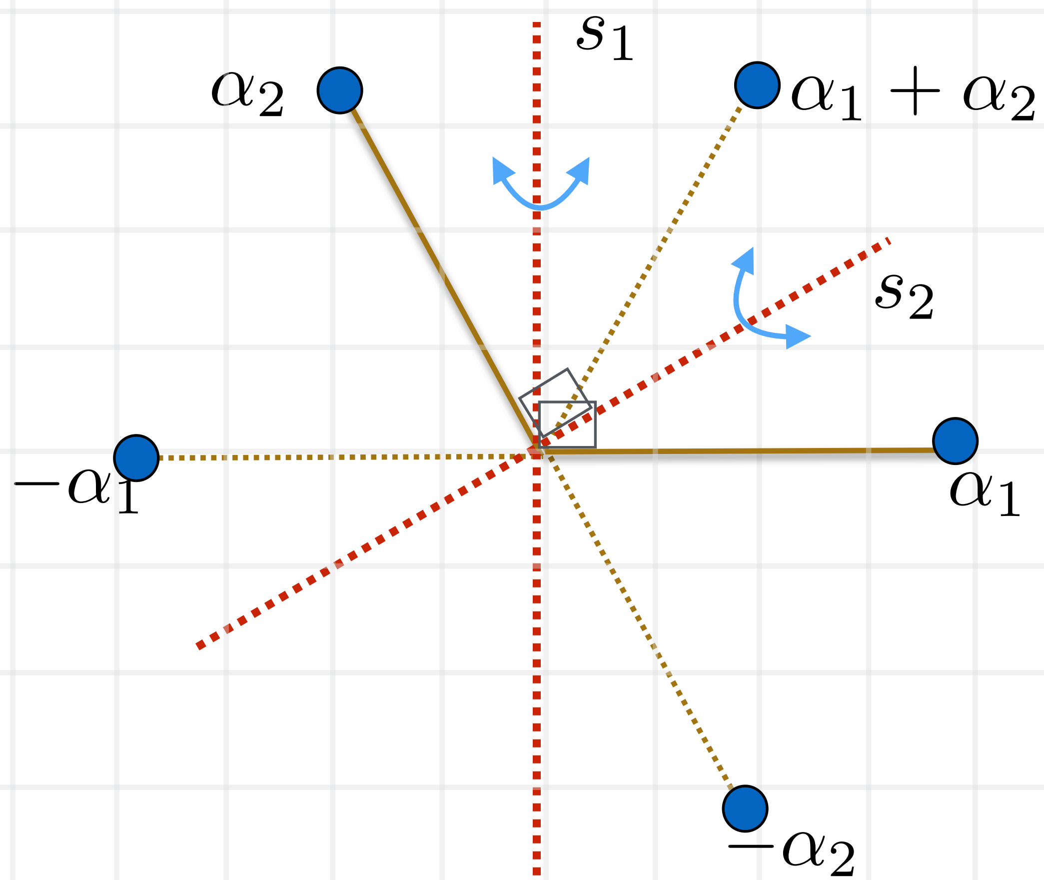
α_1 and α_2 are "simple" roots

Root System



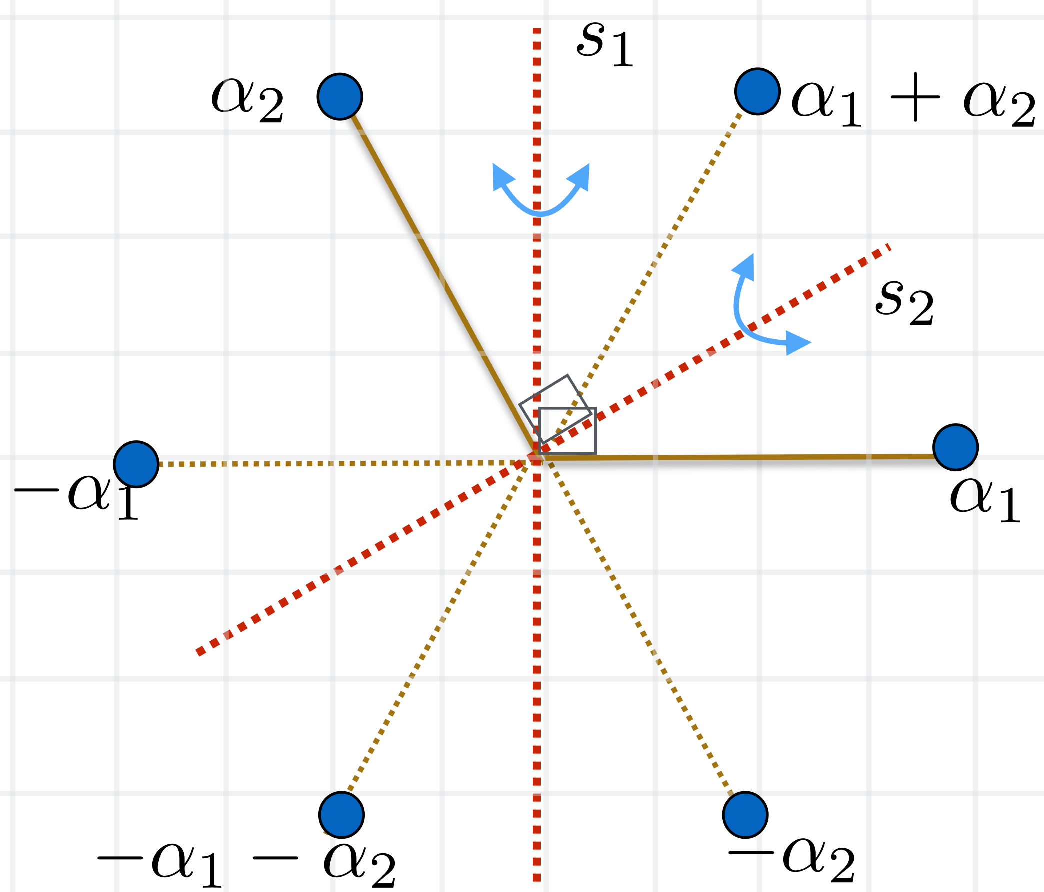
α_1 and α_2 are “simple” roots

Root System



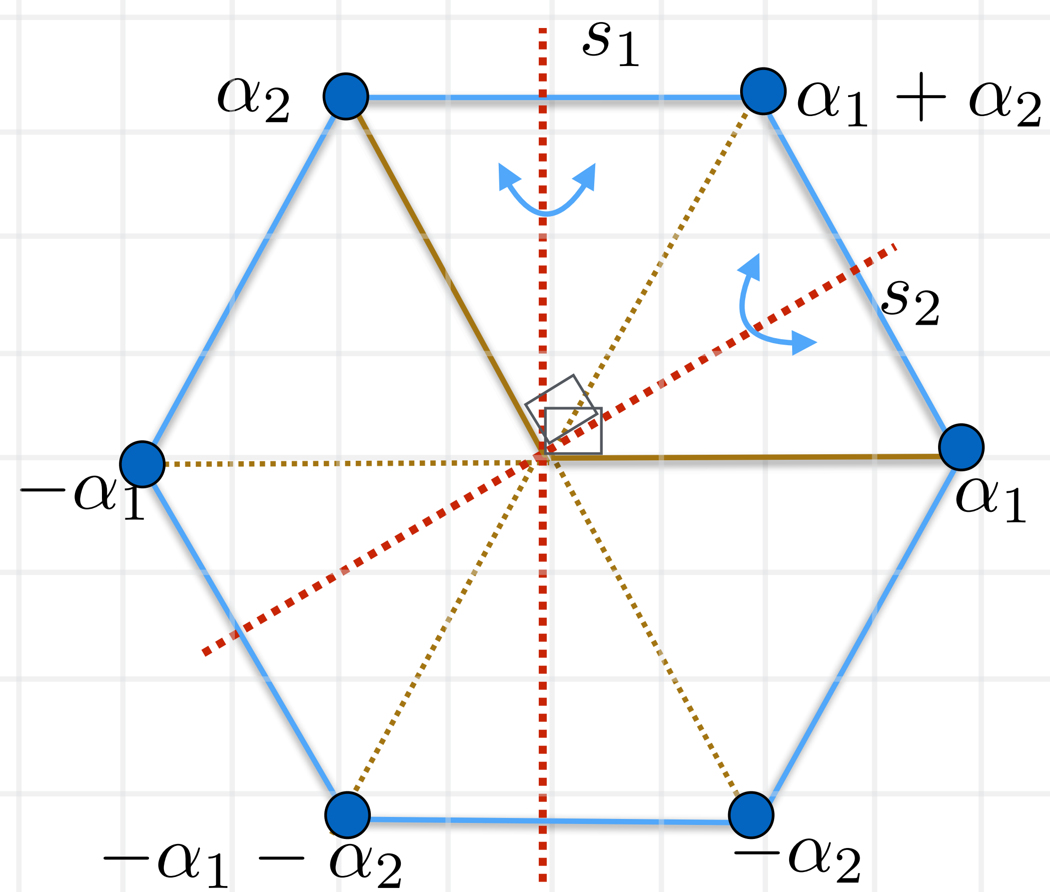
α_1 and α_2 are “simple” roots

Root System



α_1 and α_2 are “simple” roots

Root System



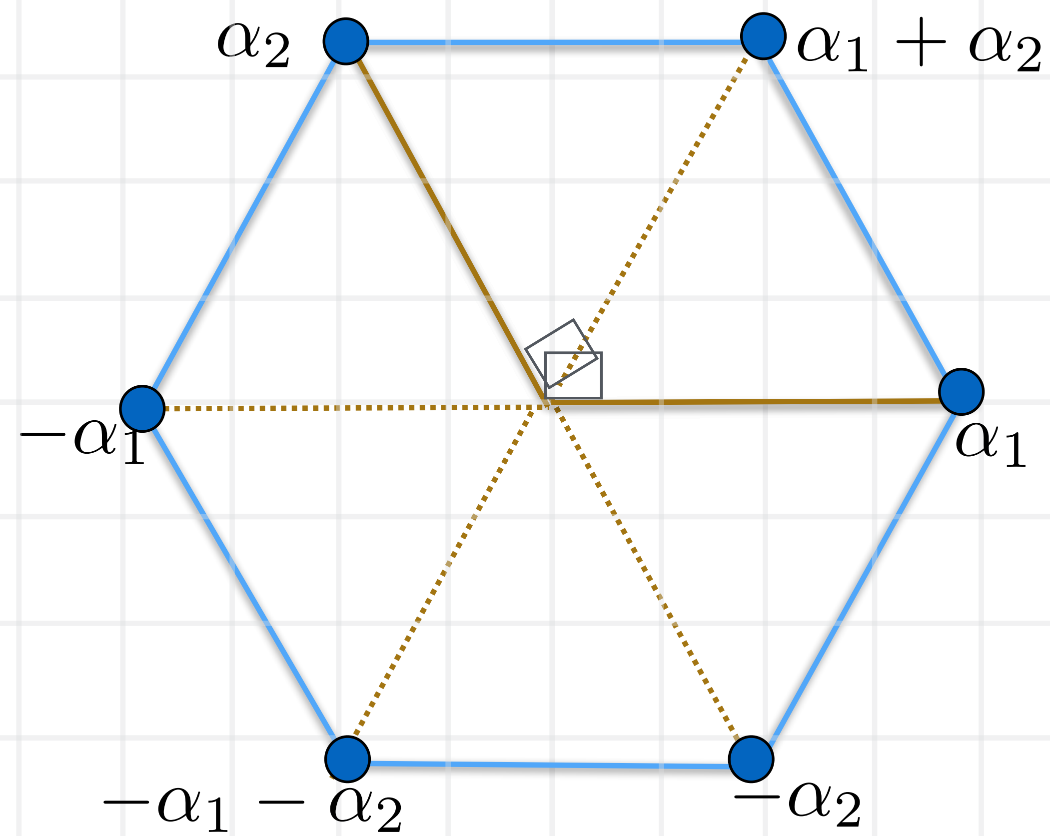
α_1 and α_2 are "simple" roots

Reflection Groups

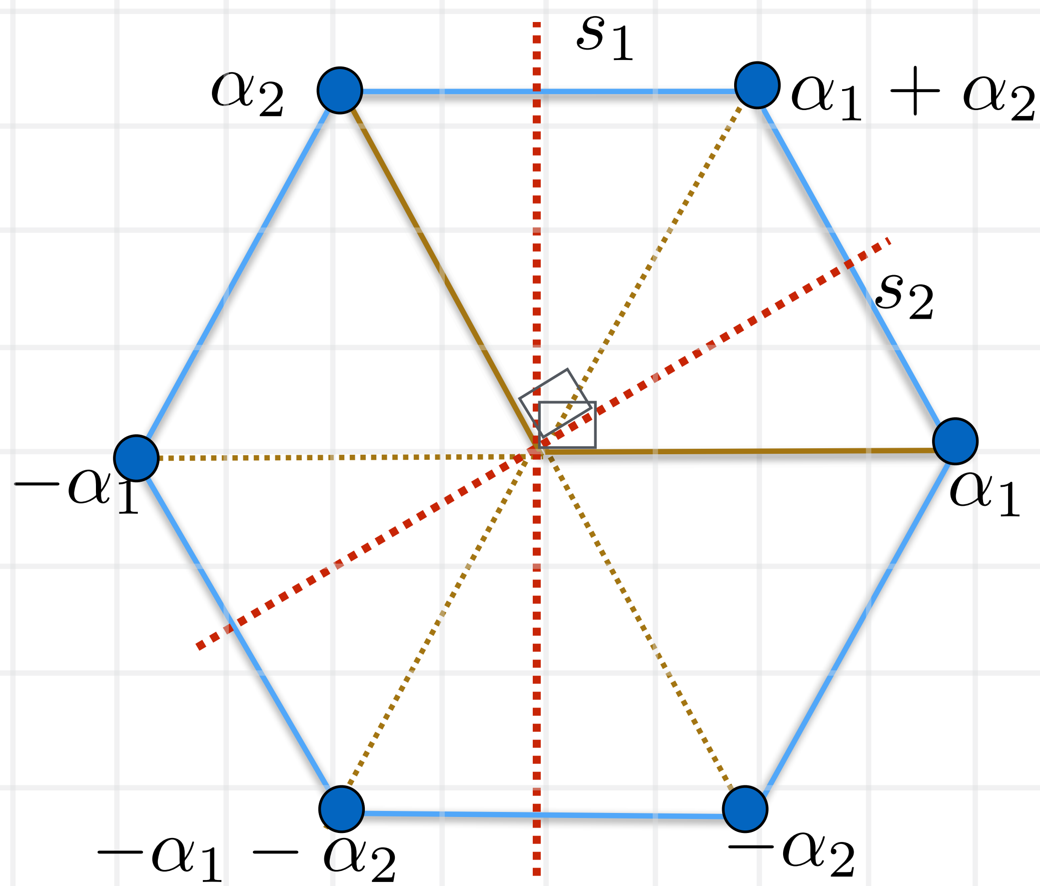
- Roots: $\alpha_1, \alpha_2, \ldots, \alpha_n$
- Reflections: $w_i(\alpha_j) = \alpha_j - 2\,\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\,\alpha_i$
- Co-roots: $\check{\alpha}_i = 2\,\frac{\alpha_i}{(\alpha_i, \alpha_i)}$
- Weights: $h_1, h_2, \ldots, h_n$

$$(h_i, \check{\alpha}_i) = \delta_{ij}$$

A$_2$

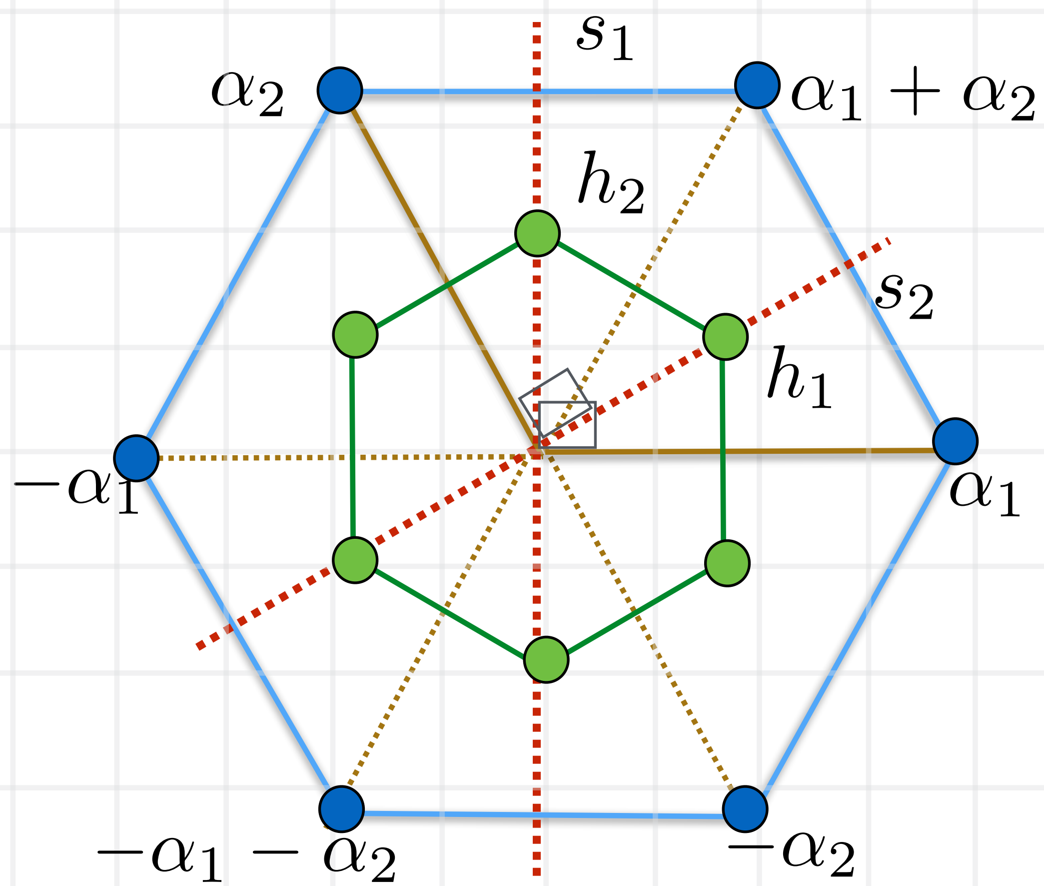


A_2



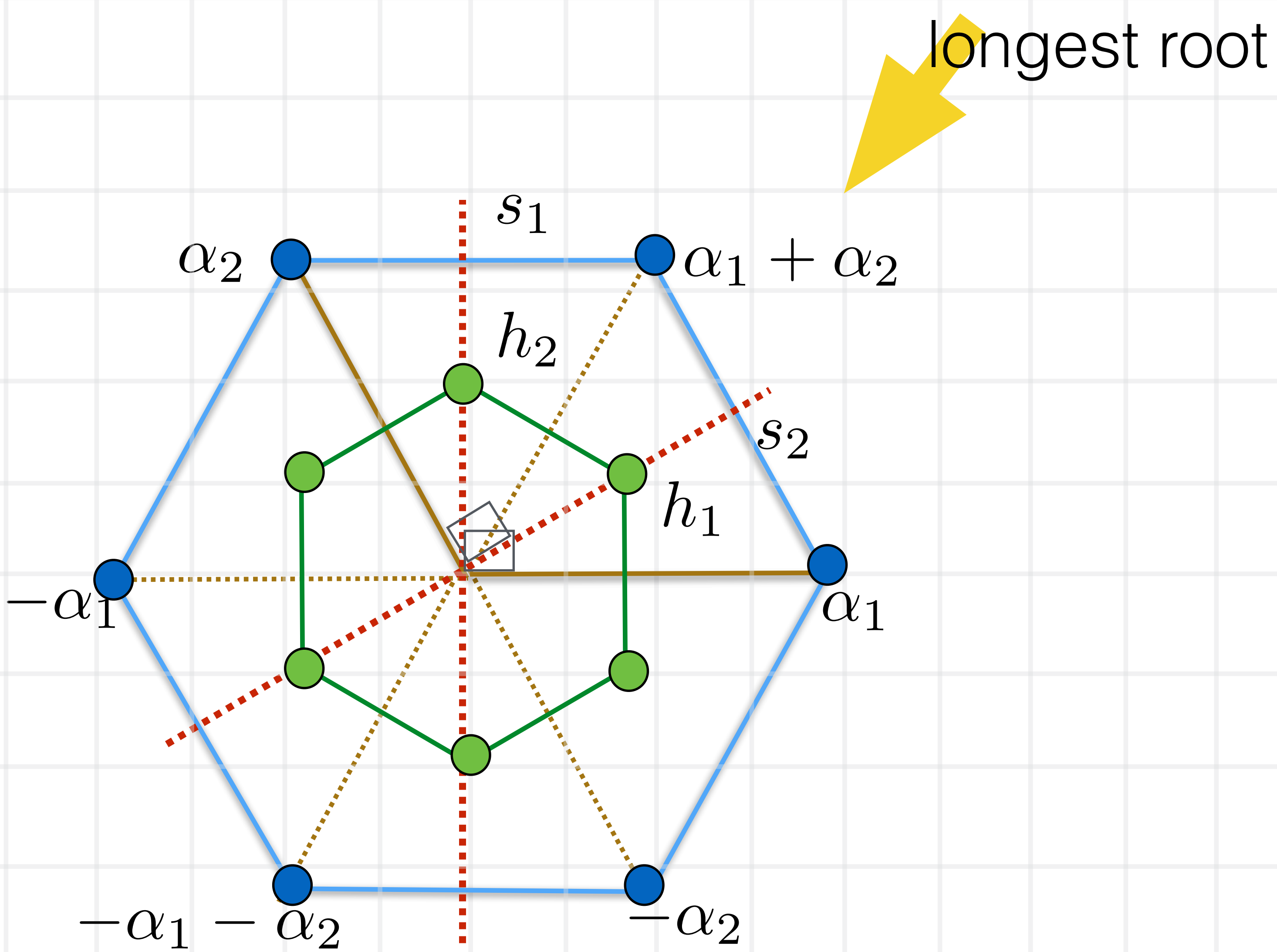
s_1
α_2
$\alpha_1 + \alpha_2$
s_2
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2



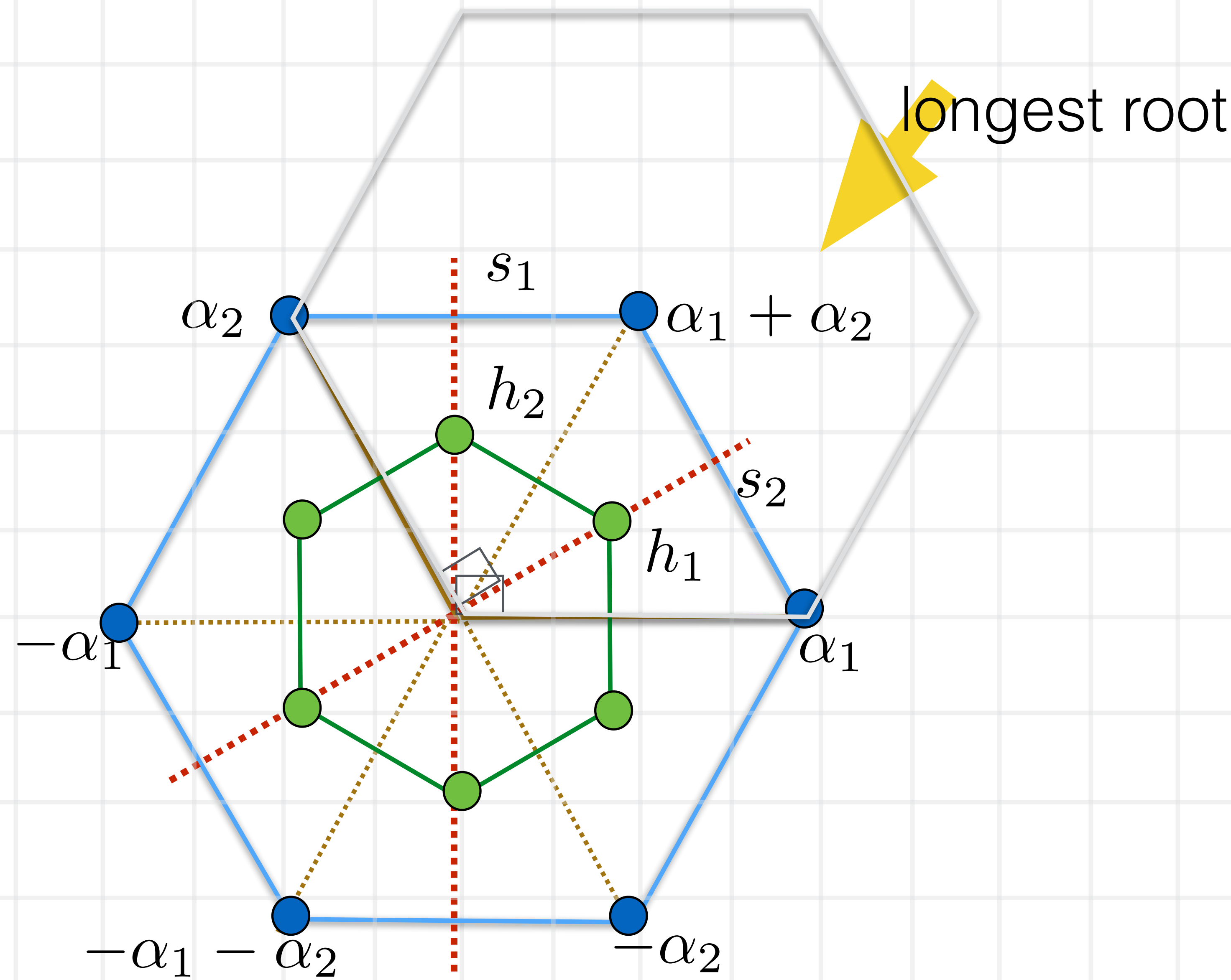
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A$_2$



longest root
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

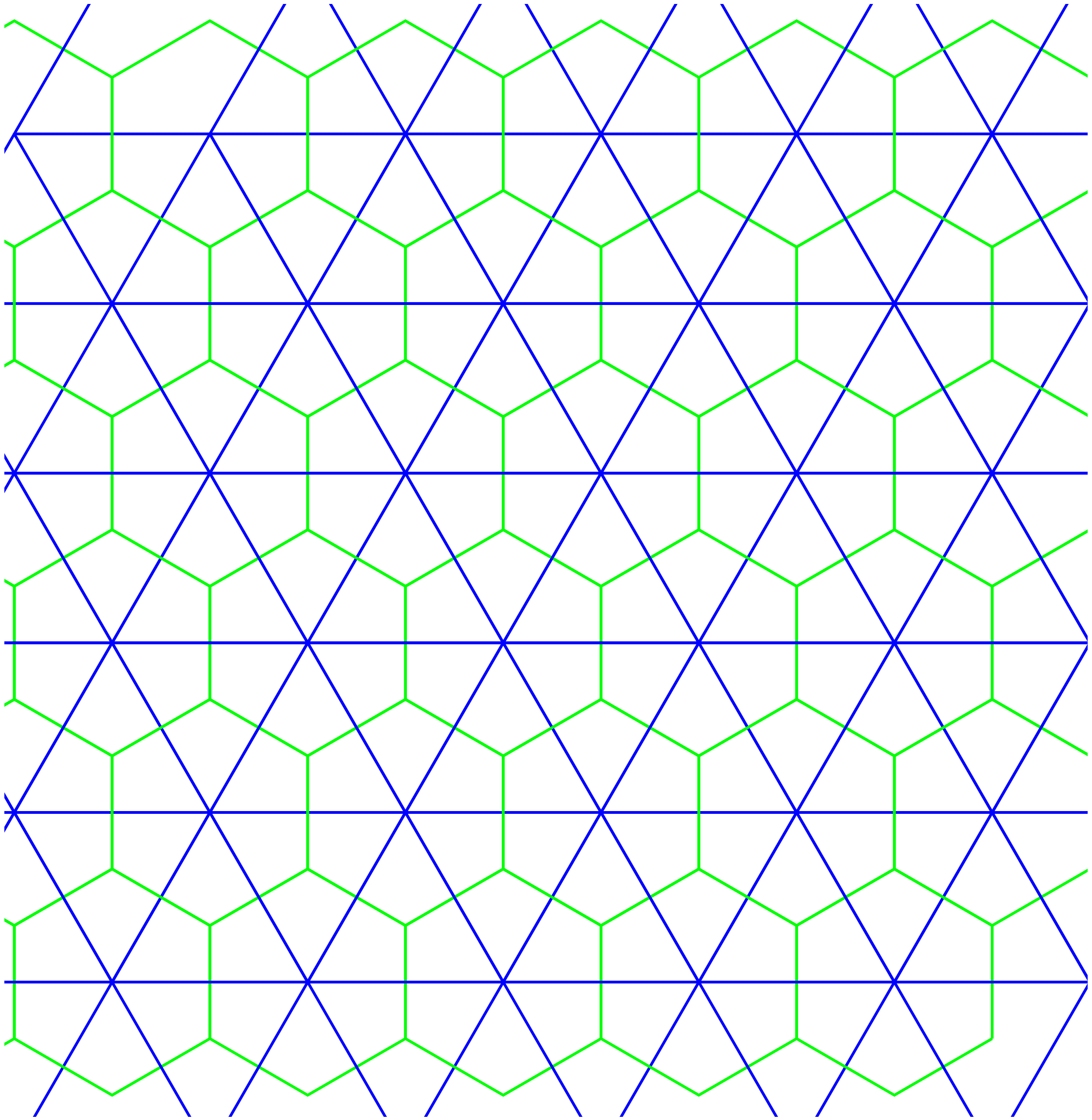
A2



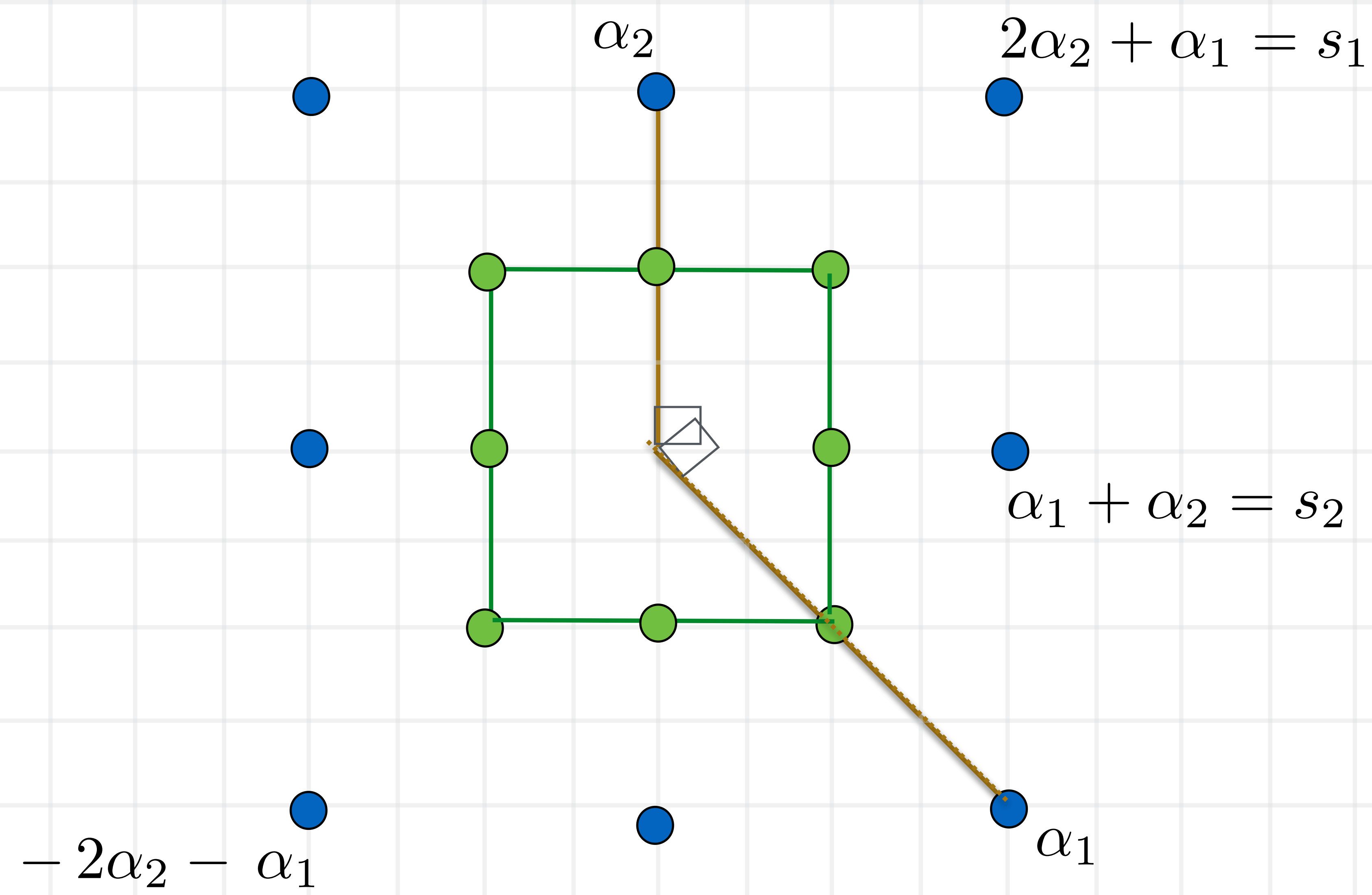
longest root
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

Translation by longest root

$A_2^{(1)}$

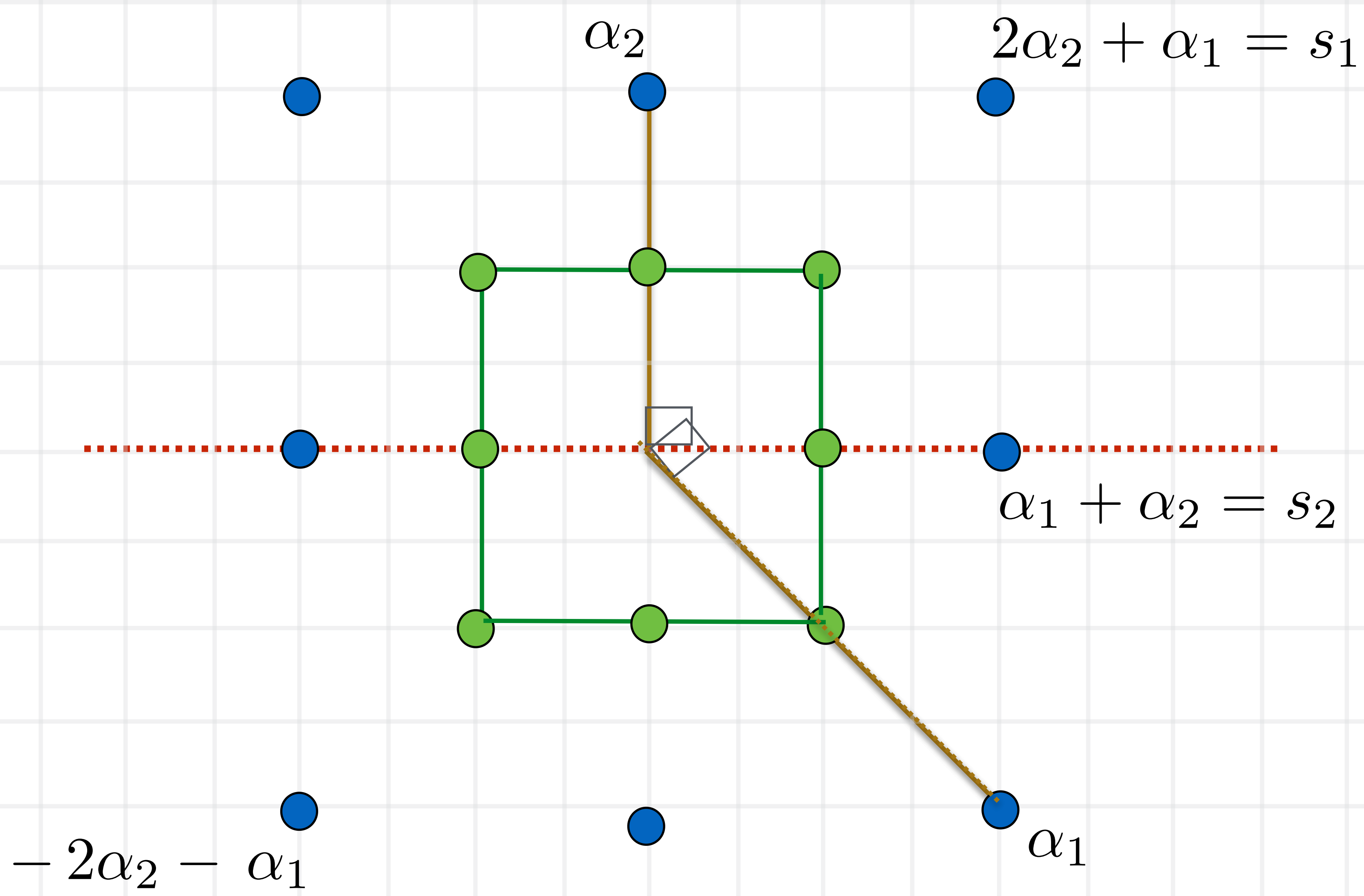


B_2

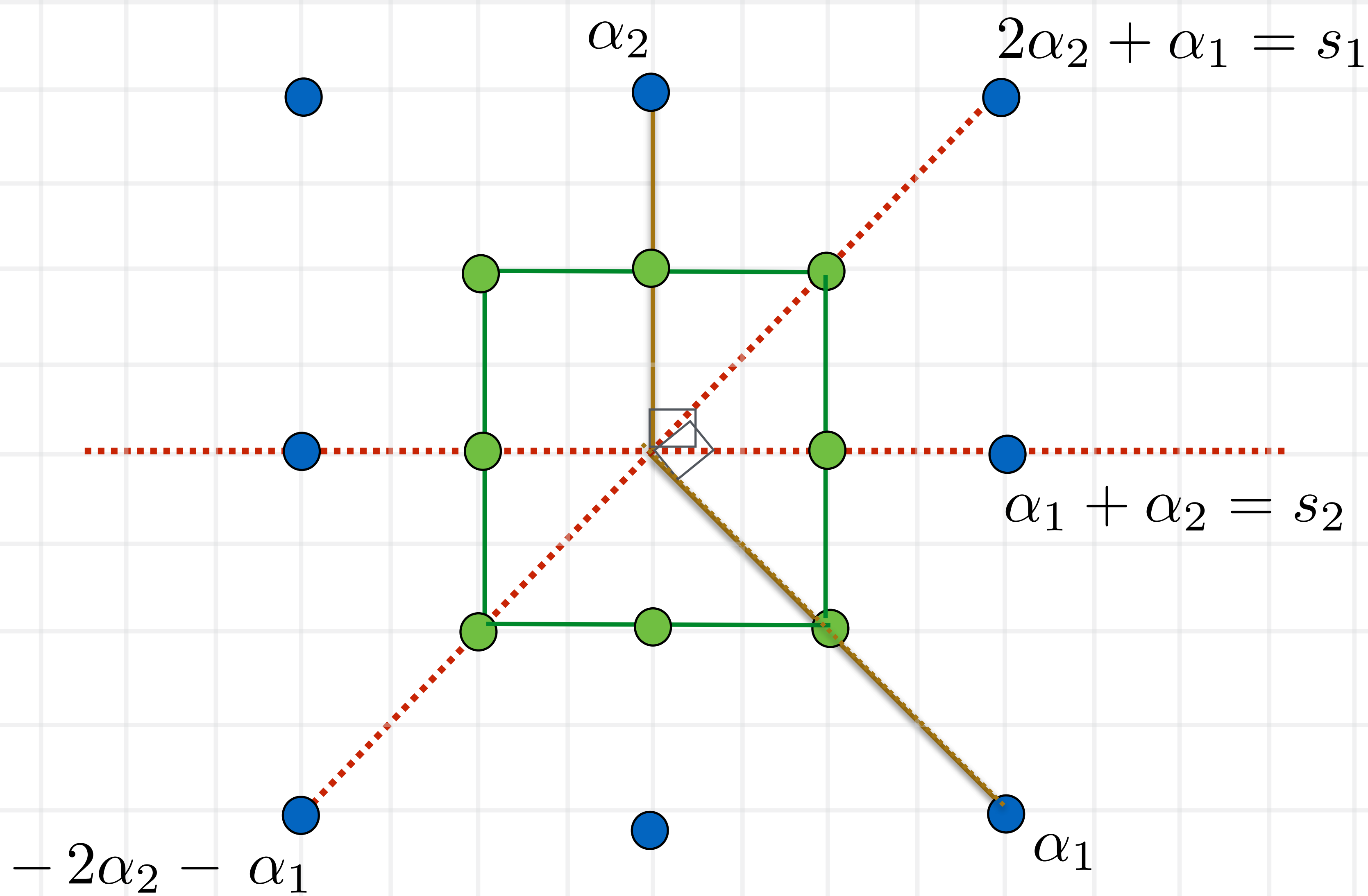


α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

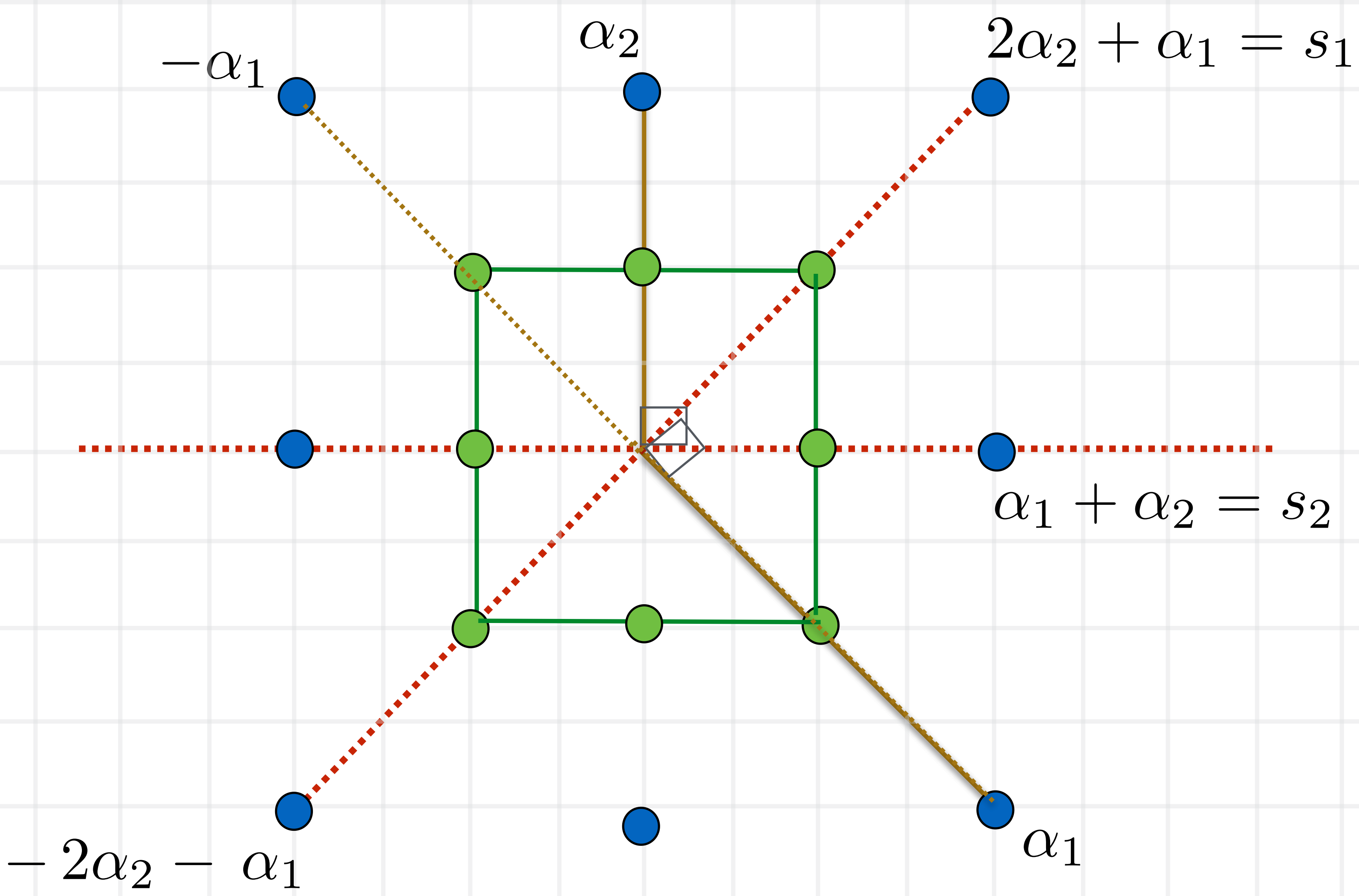
B_2


α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

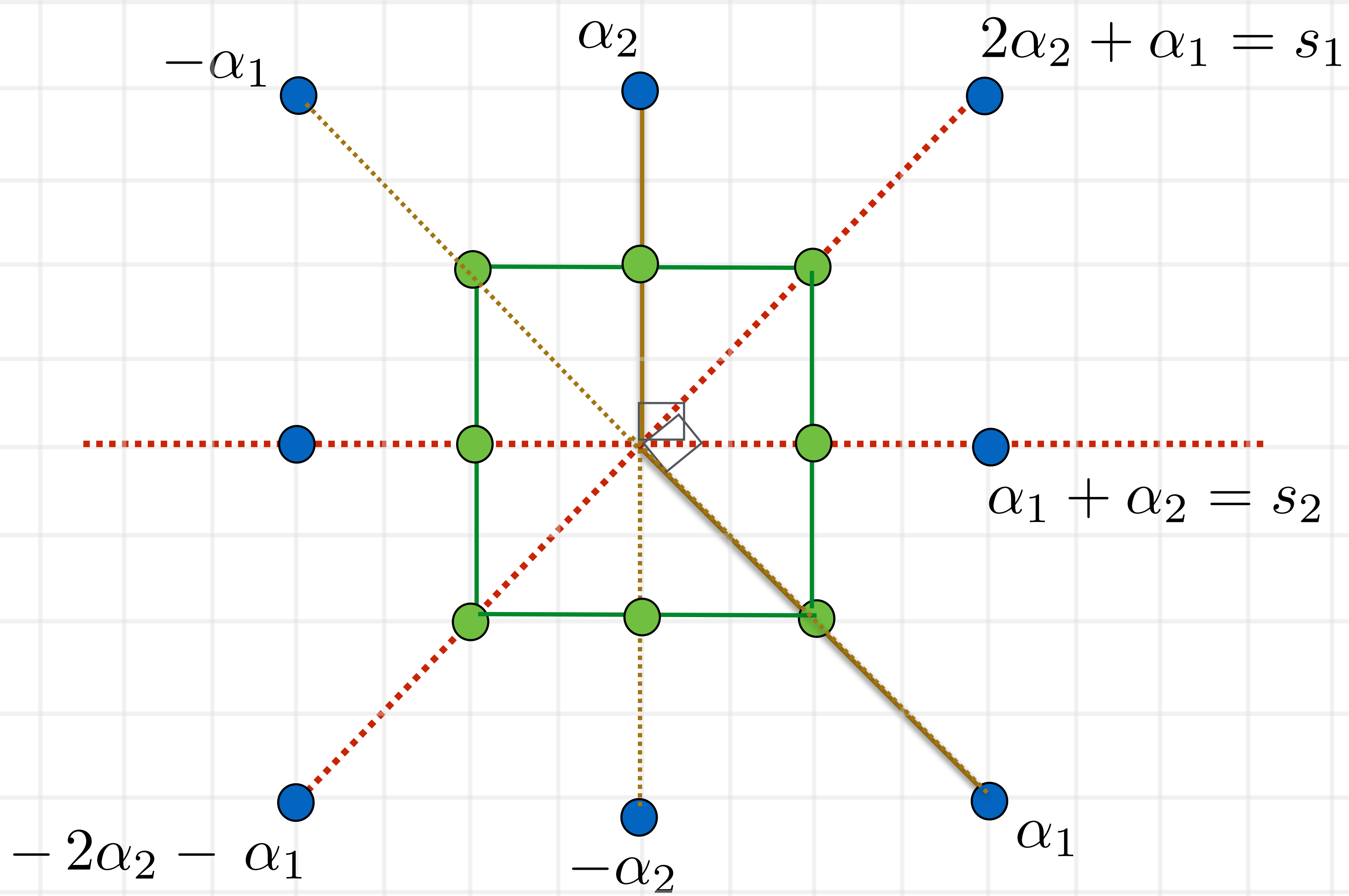
B_2


α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B$_2$

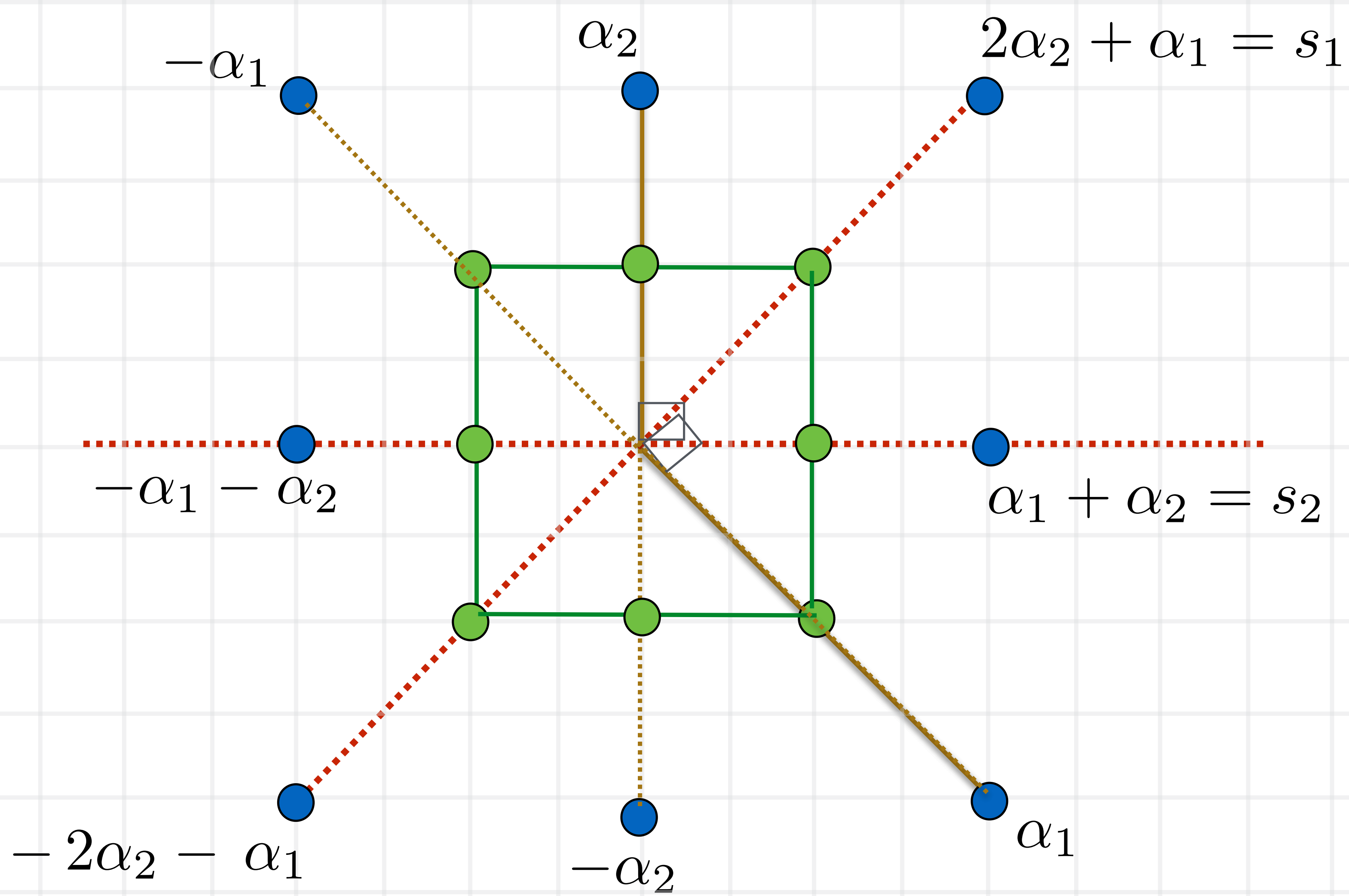

$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B_2



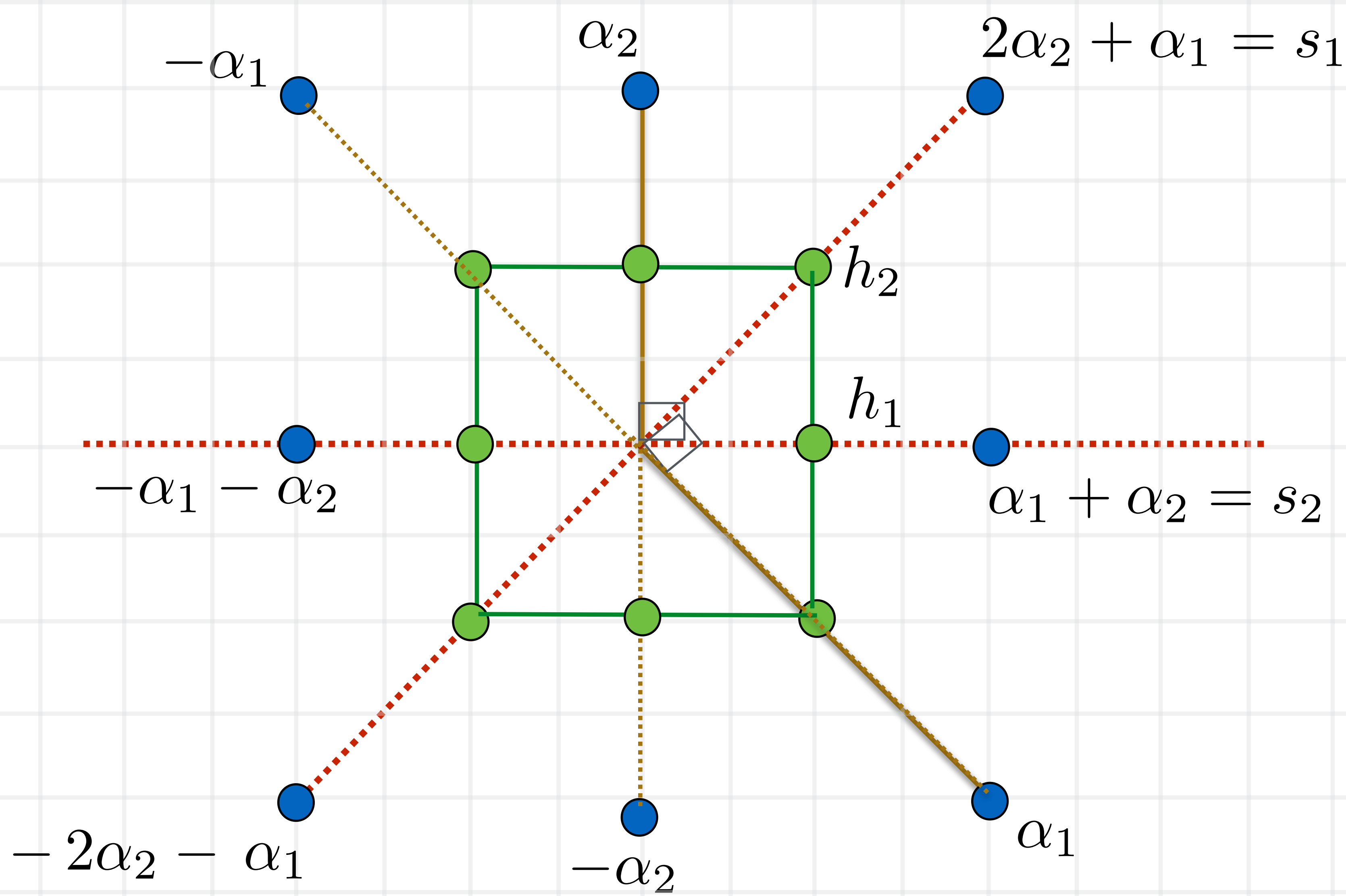
$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

B$_2$



$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
$-\alpha_1 - \alpha_2$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

B_2



α_2
$-\alpha_1$
$2\alpha_2 + \alpha_1 = s_1$
h_2
h_1
$-\alpha_1 - \alpha_2$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

Crystallographic Property

$$(\alpha_i, \check{\alpha}_j) \in \mathbb{Z}$$

$$\Rightarrow \quad (\alpha_i, \check{\alpha}_j)(\check{\alpha}_i, \alpha_j) = 4\cos^2(\theta_{\alpha_i \alpha_j}) \in \mathbb{N}$$

$$\Rightarrow \quad \cos(\theta_{\alpha_i \alpha_j}) = 0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}, \pm 1$$

$$\Rightarrow \quad \theta_{\alpha_i \alpha_j} = \pi - \theta_{s_i s_j} = \pi - \frac{\pi}{m_{ij}}$$

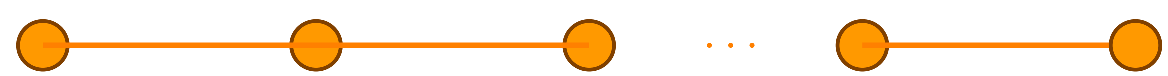
$$\Rightarrow \quad m_{ij} = 2, 3, 4, 6$$

Dynkin Diagrams

A_2

B_2

A_n



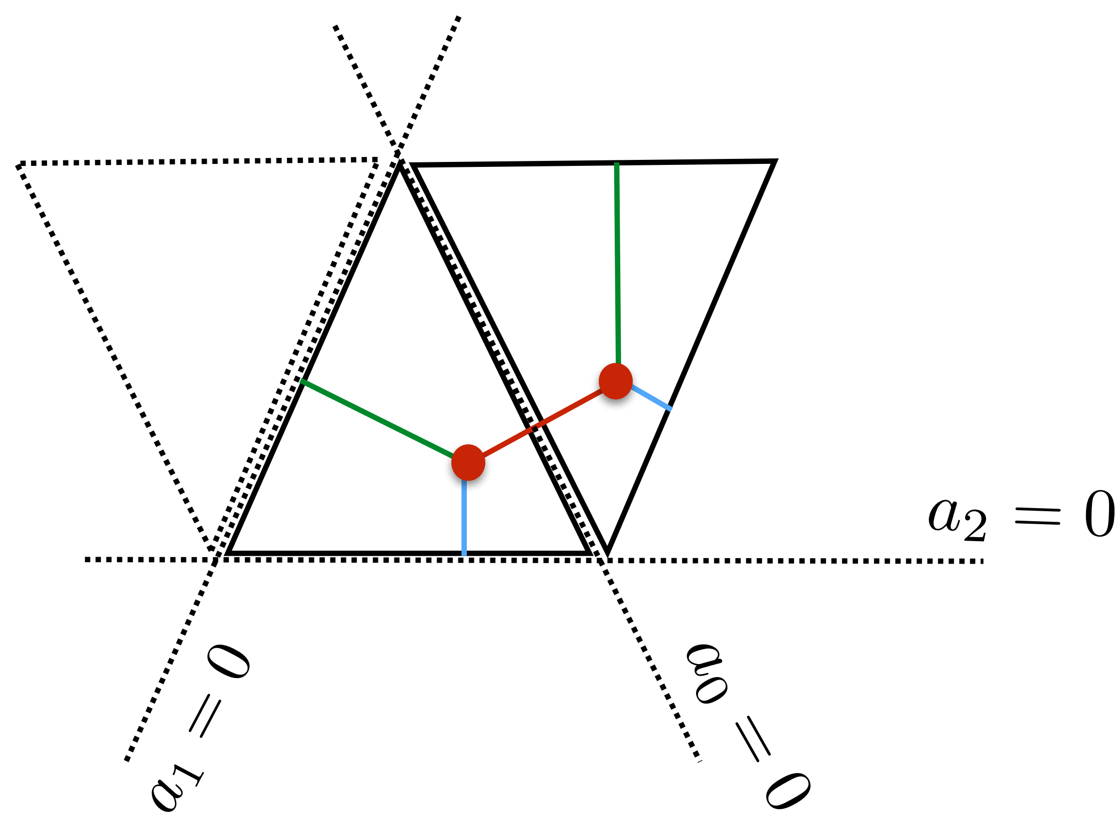
B_{n+1}

Part I

- Lattices
- Dynamics on N-cubes
- Symmetry reductions

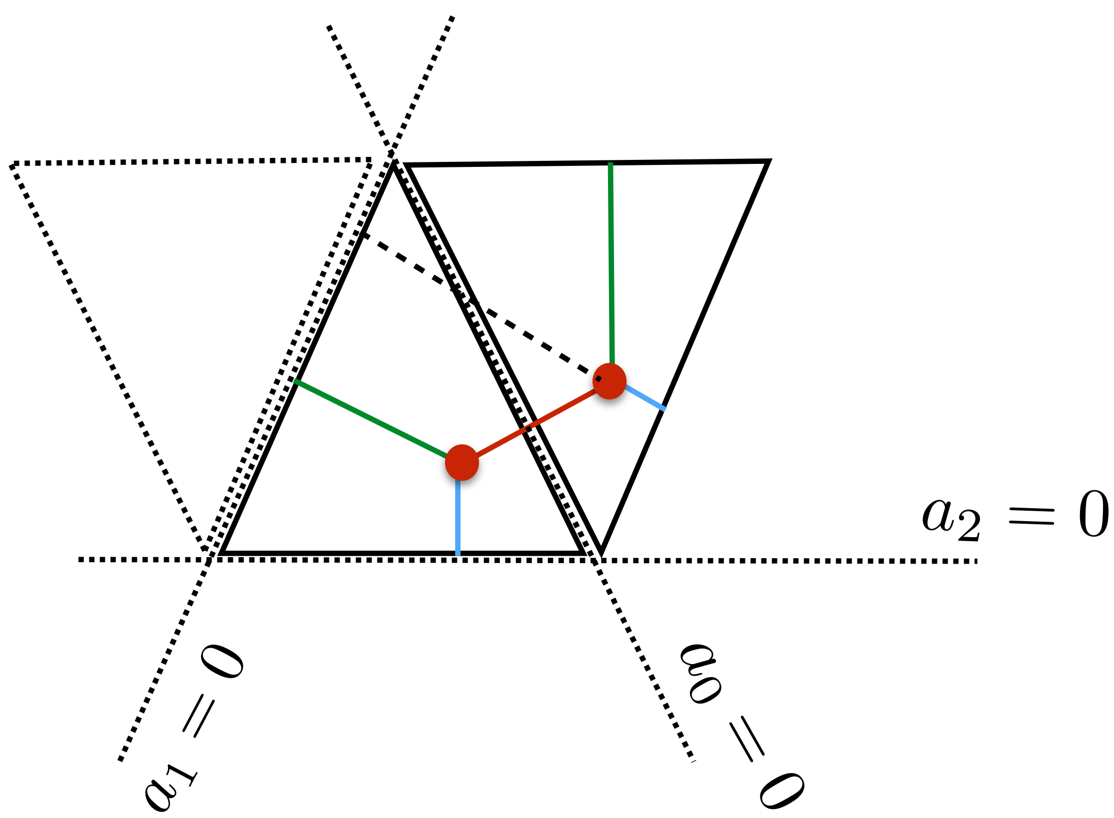
On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



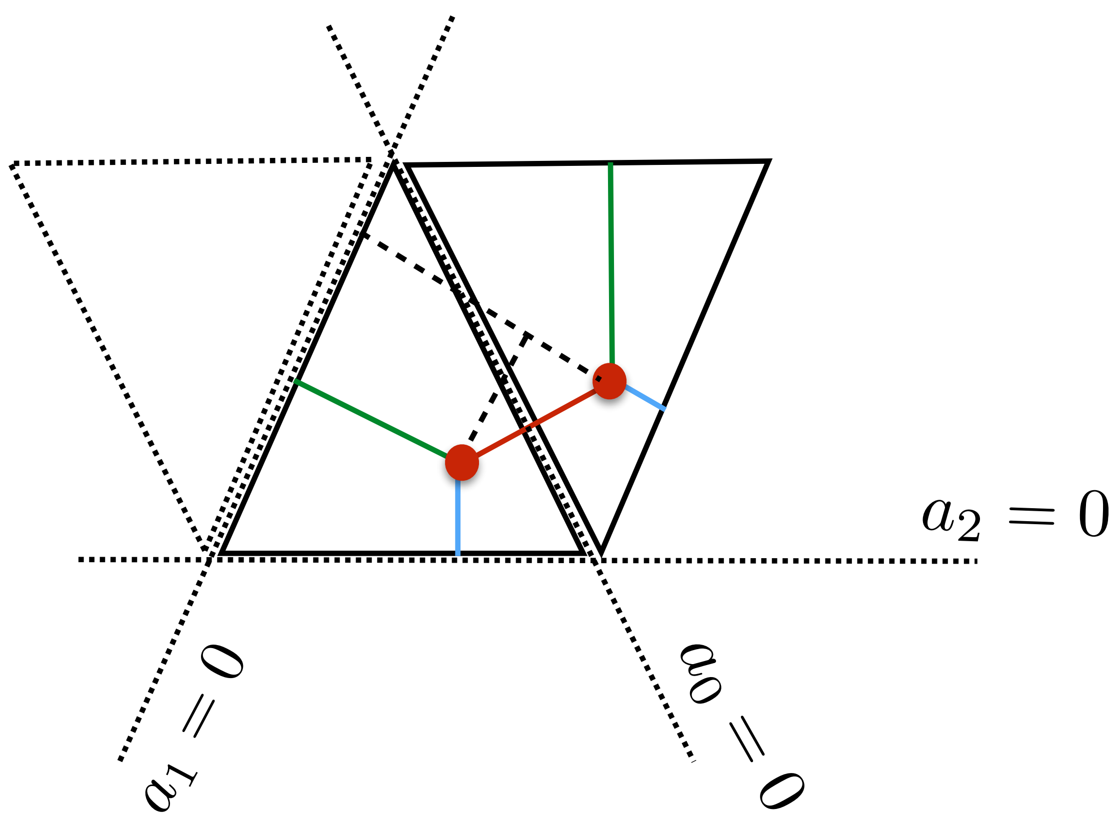
On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



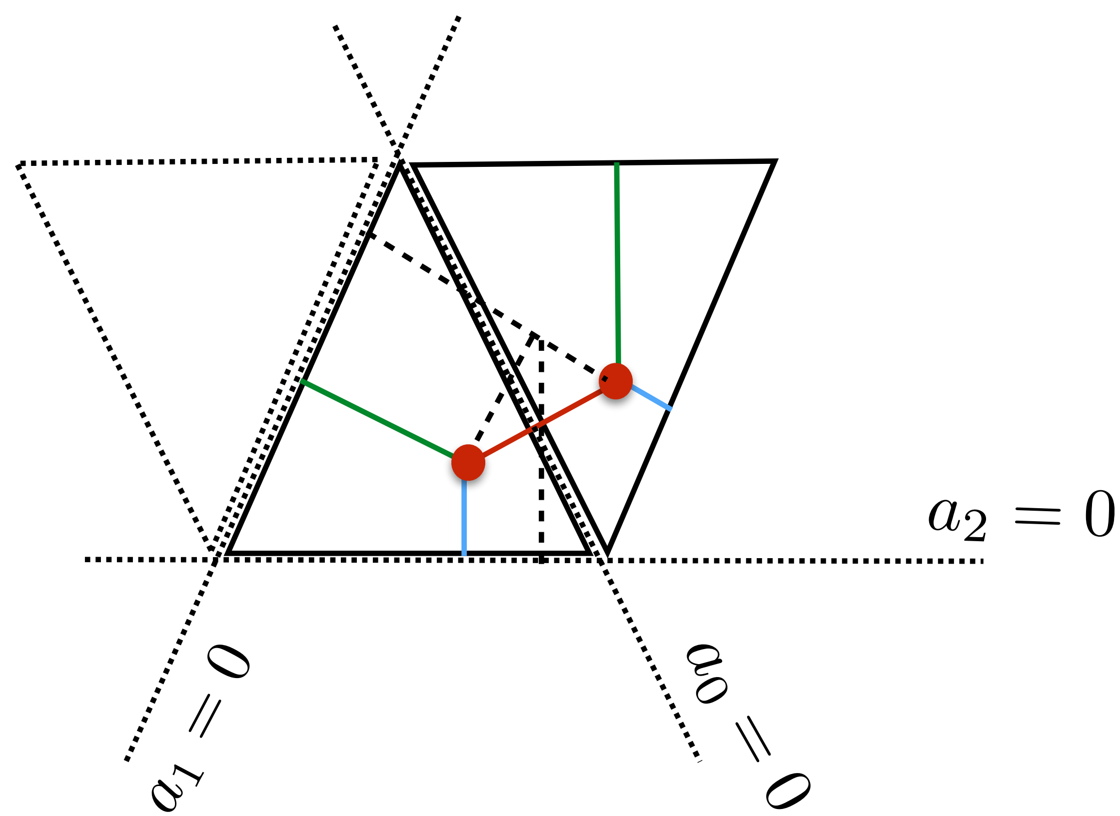
On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



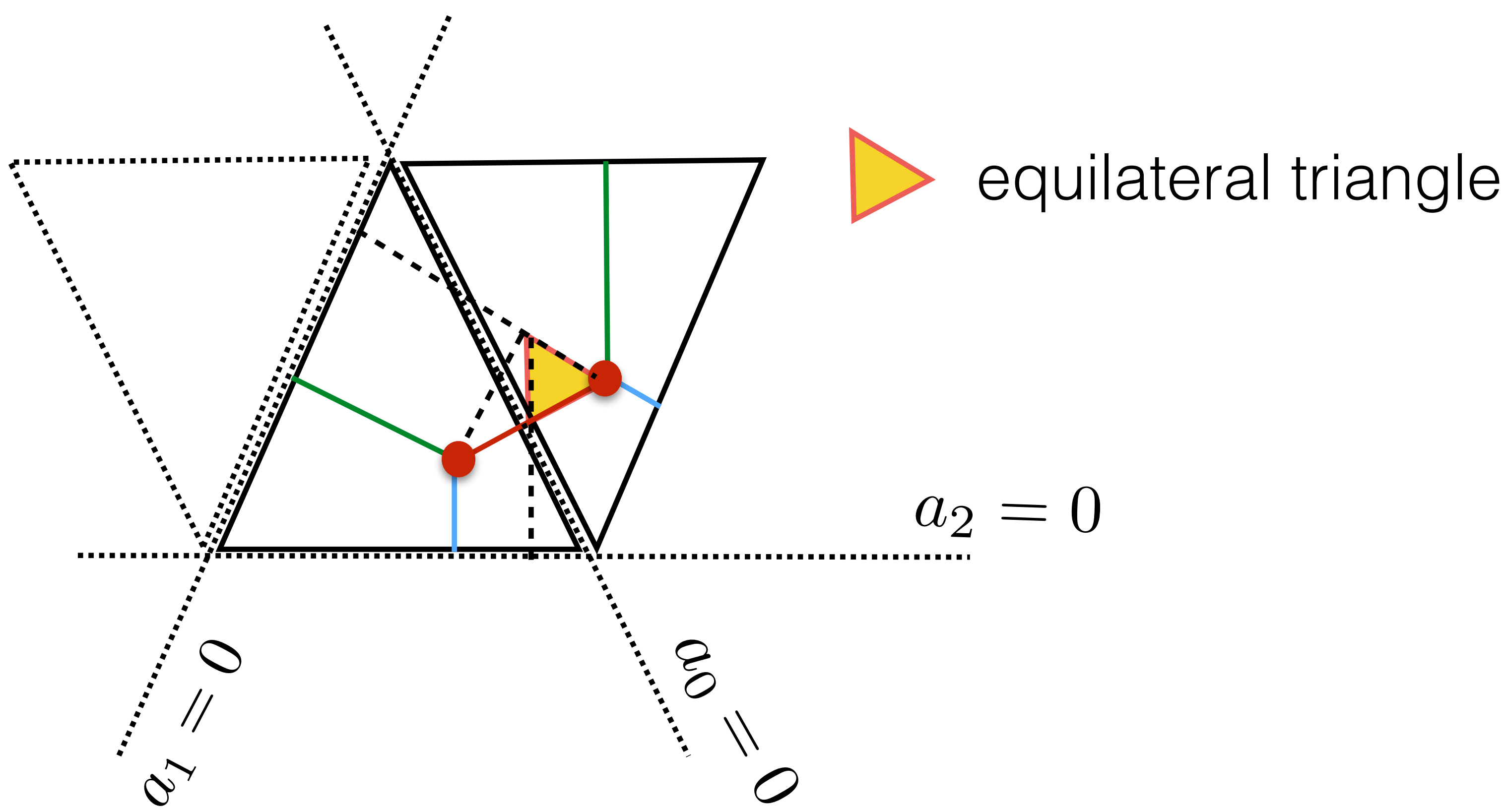
On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



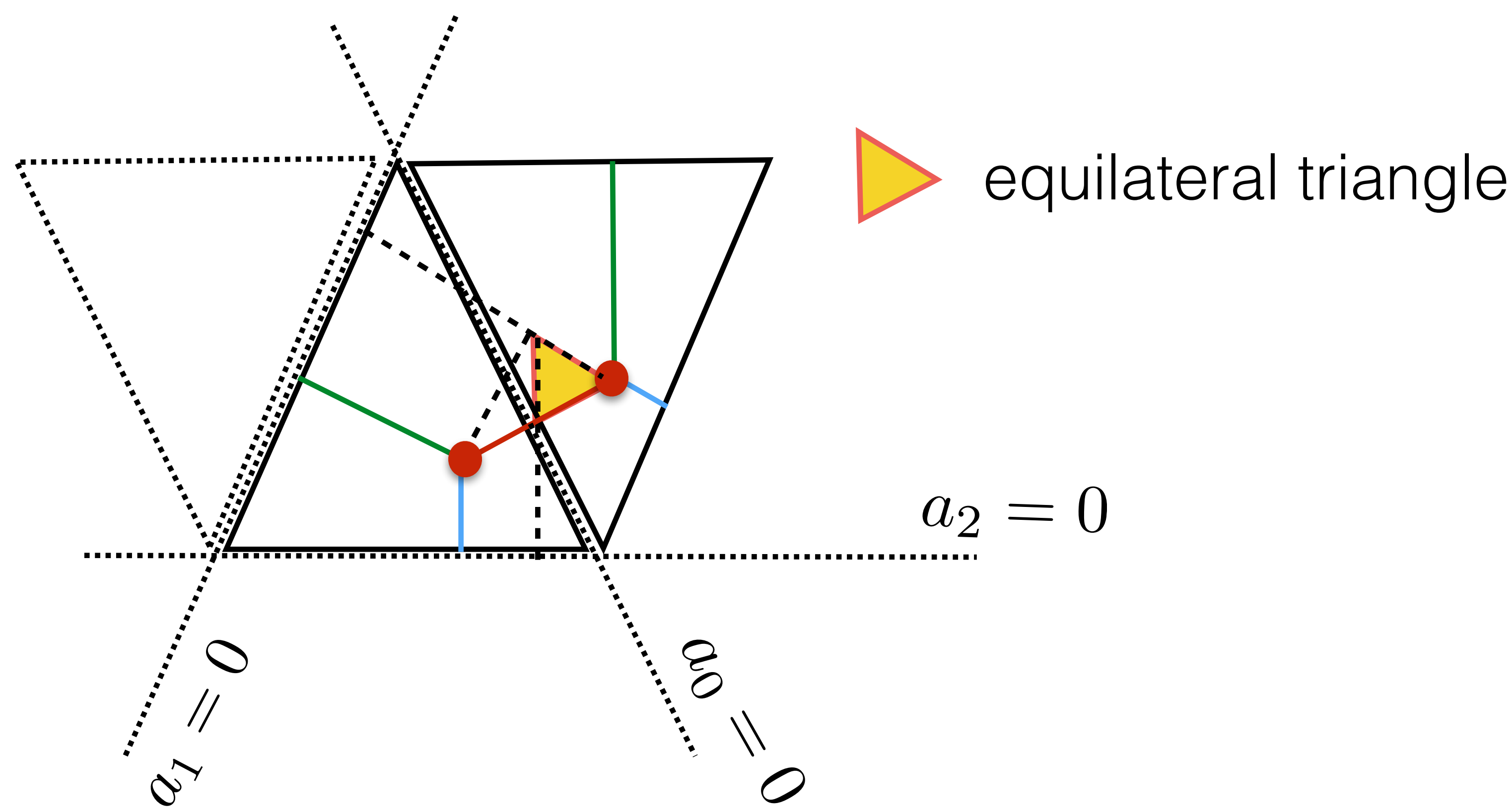
On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



On the Lattice

- Define s_0, s_1, s_2 to be reflections across each edge



$$s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$$

Coxeter Relations

$$\widetilde{\mathcal{W}}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$$

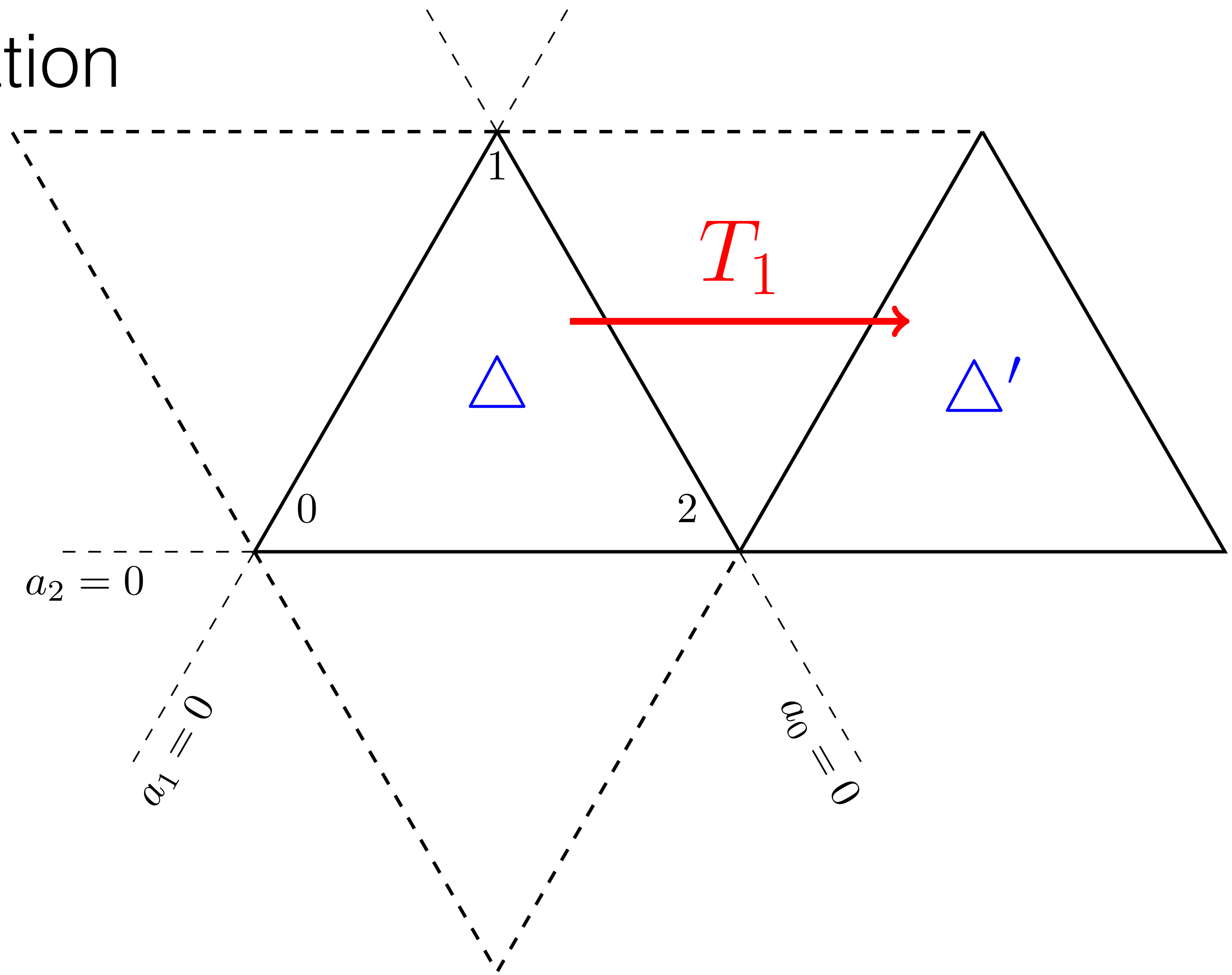
$$\left.\begin{aligned} & s_j^2 = 1 \\ & (s_j\, s_{j+1})^3 = 1 \\ & \pi\, s_j = s_{j+1}\, \pi \end{aligned}\right\} \quad j \in \mathbb{N} \bmod 3$$

π : diagram automorphism

$$\pi^3 = 1$$

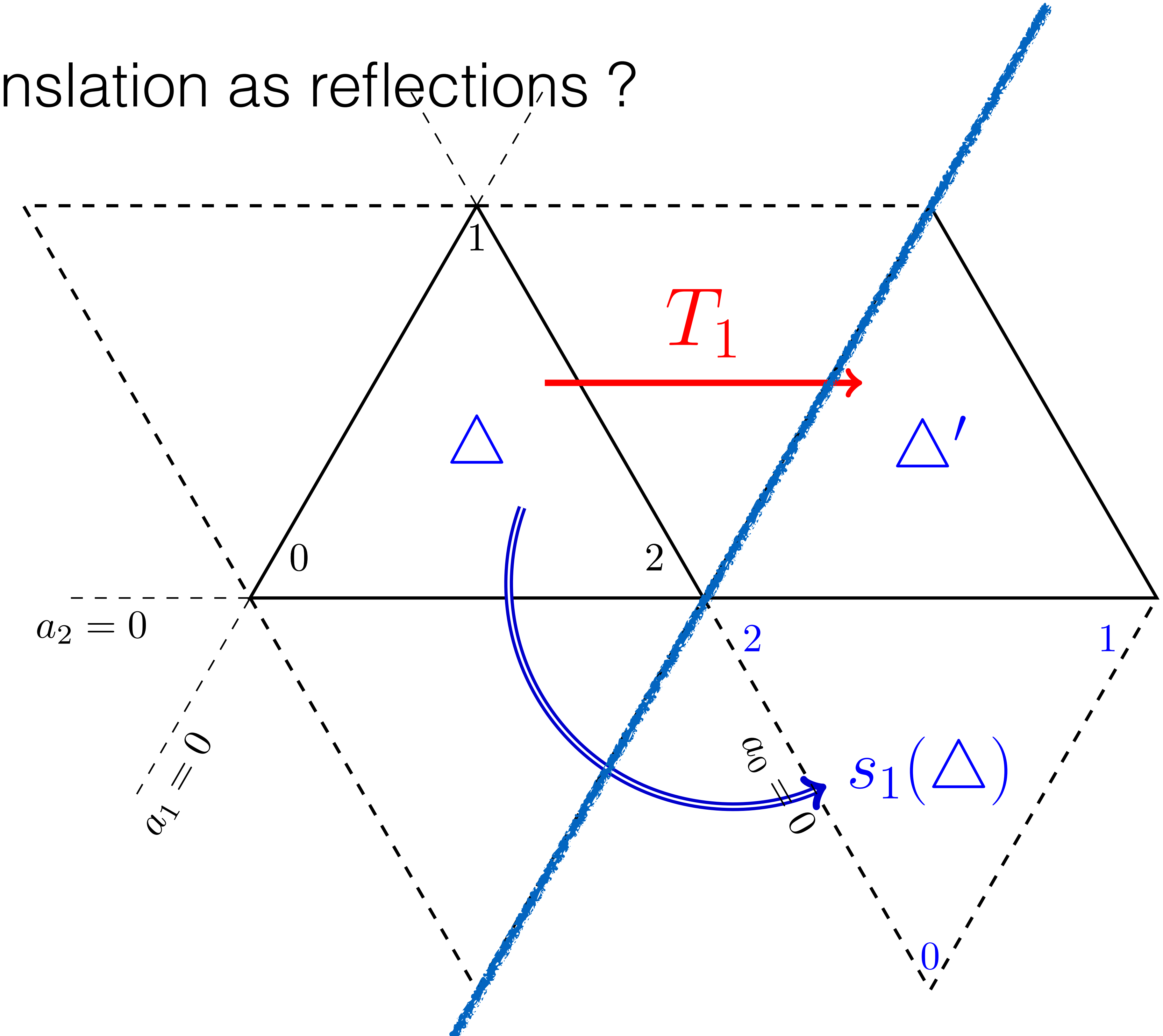
Discrete Dynamics I

- Translation



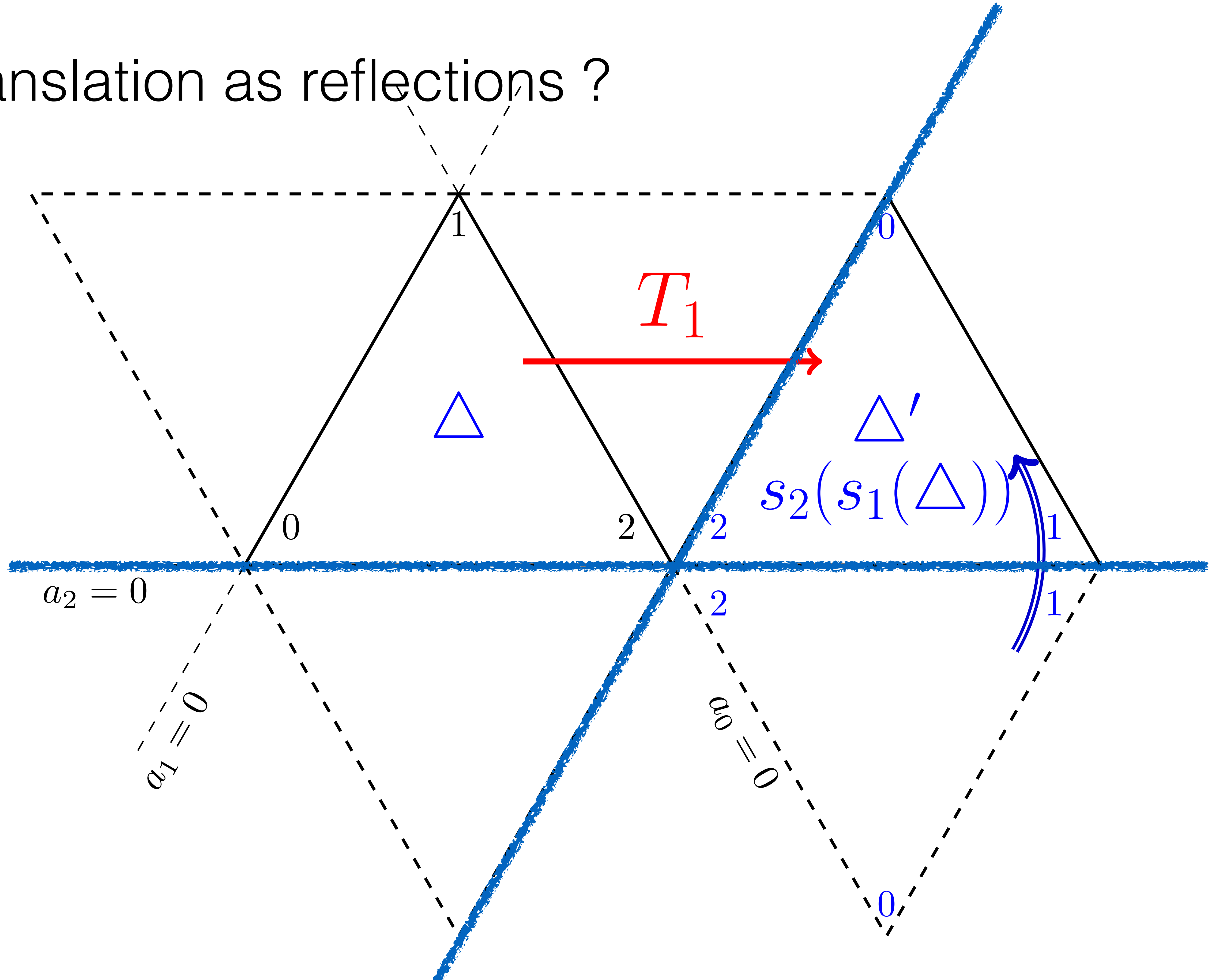
Discrete Dynamics II

- Translation as reflections ?



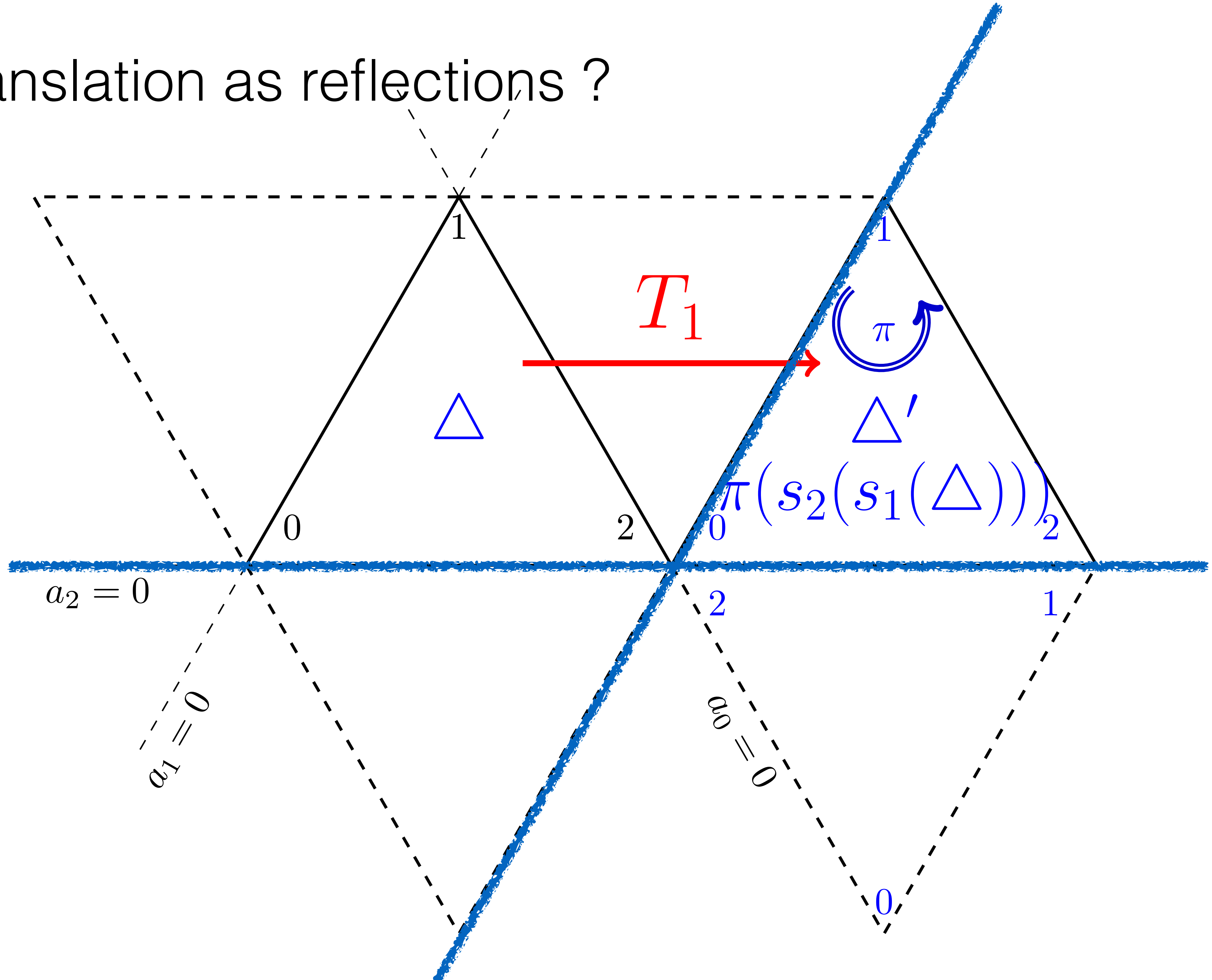
Discrete Dynamics II

- Translation as reflections ?



Discrete Dynamics II

- Translation as reflections ?



Discrete Dynamics III

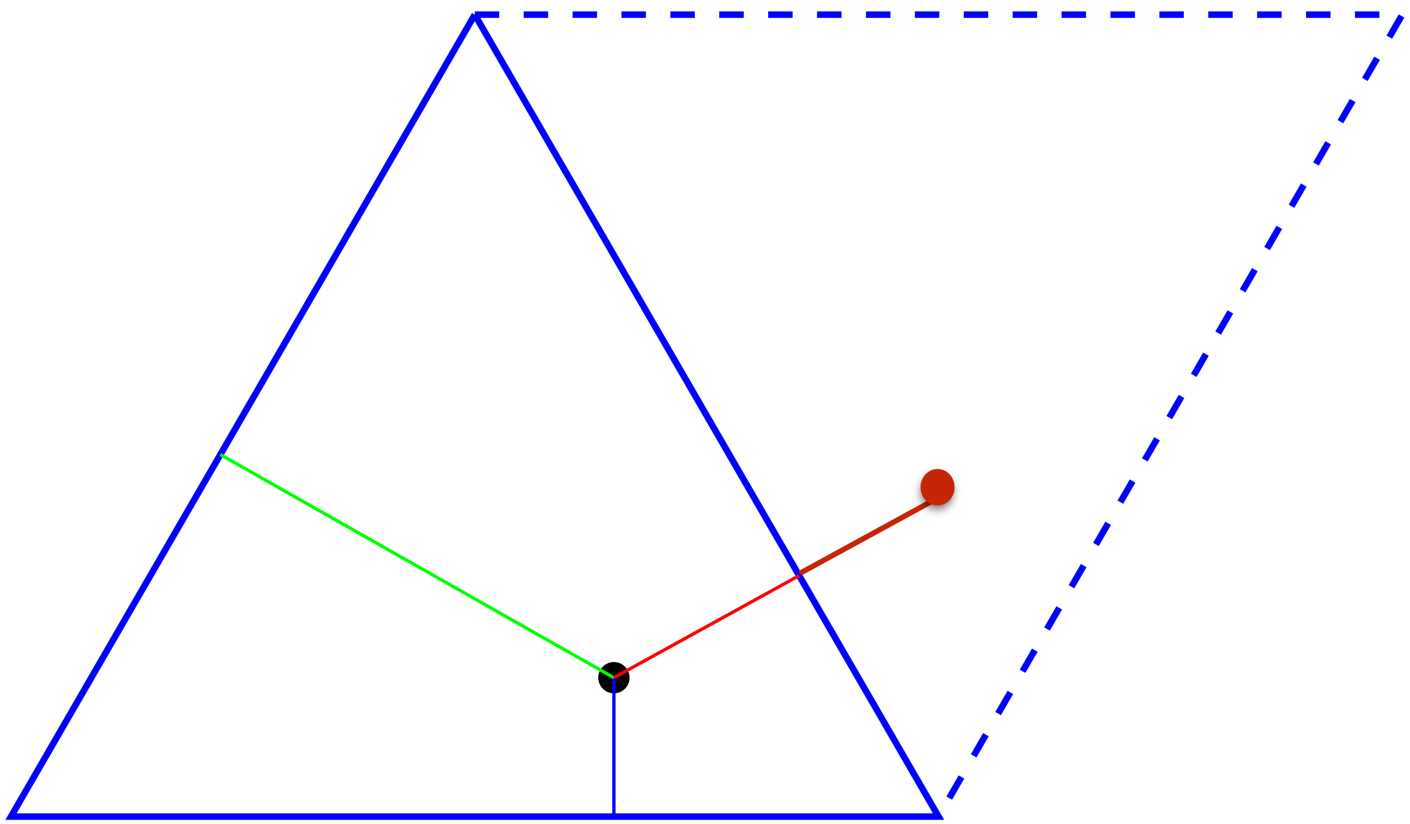
- Translations as reflections
  + diagram automorphism

$$T_1 = \pi \, s_2 \, s_1$$

$$T_2 = s_1 \, \pi \, s_2$$

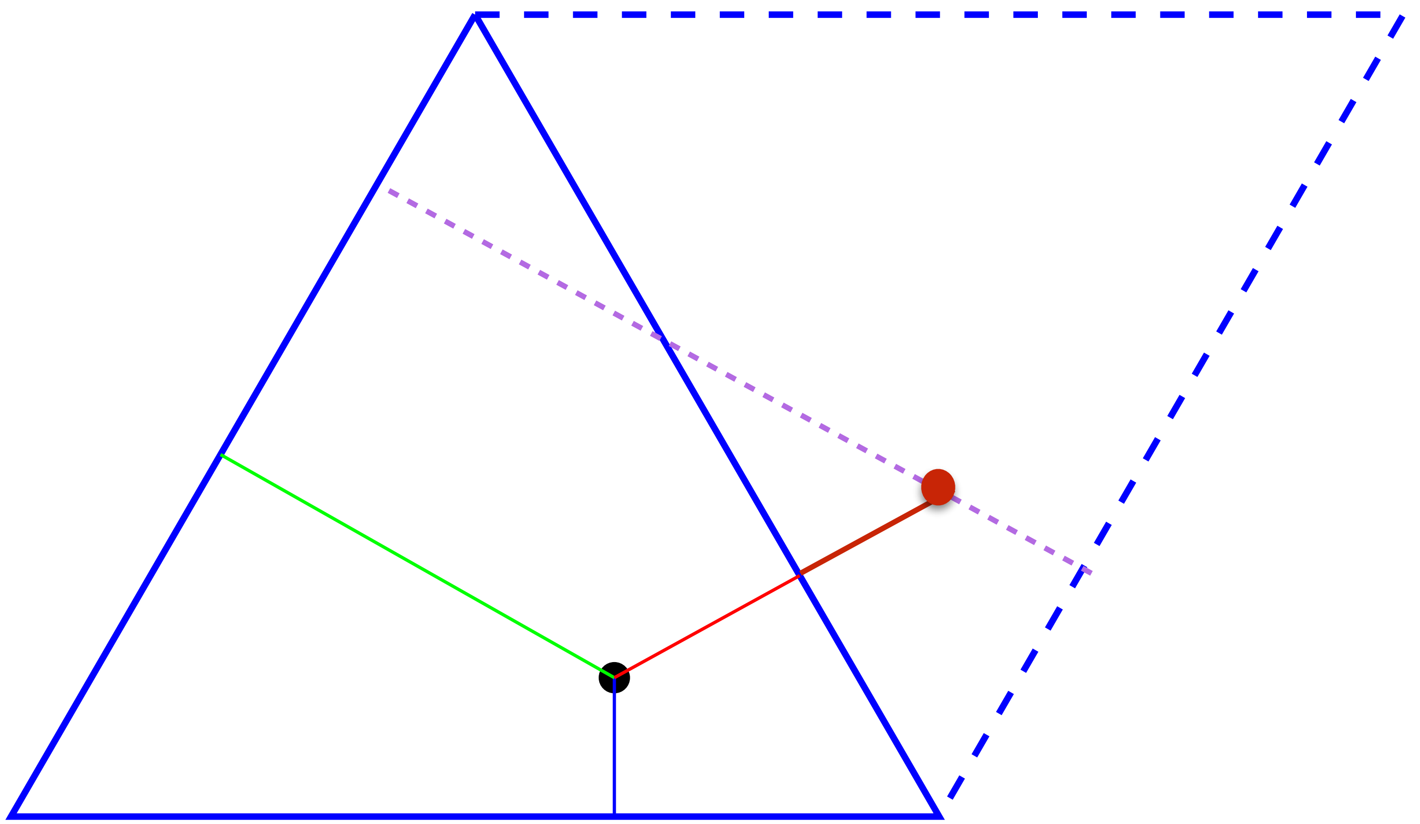
$$T_0 = s_2 \, s_1 \, \pi$$

Constancy of coordinates



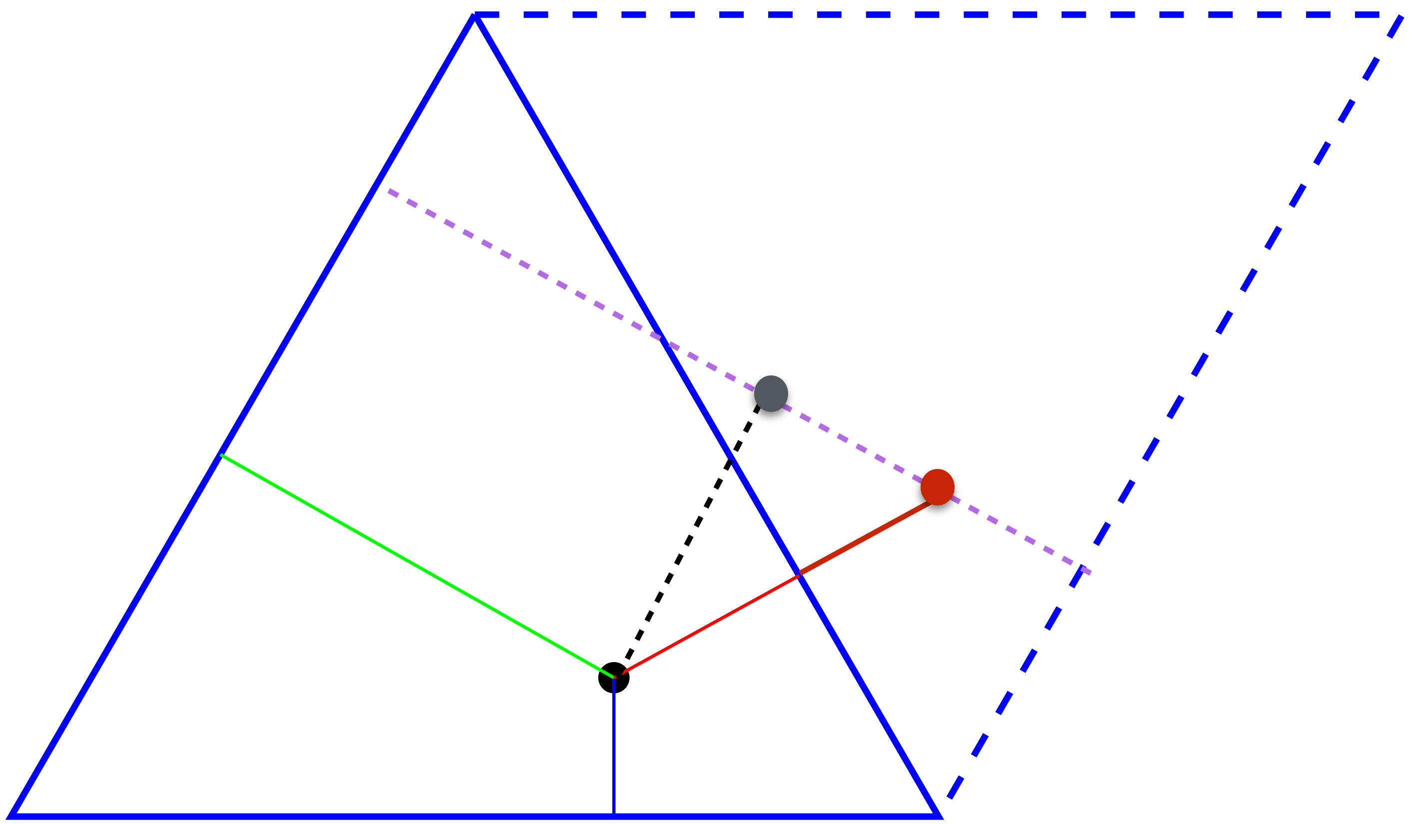
$$a_0 + a_1 + a_2 = k$$

Constancy of coordinates



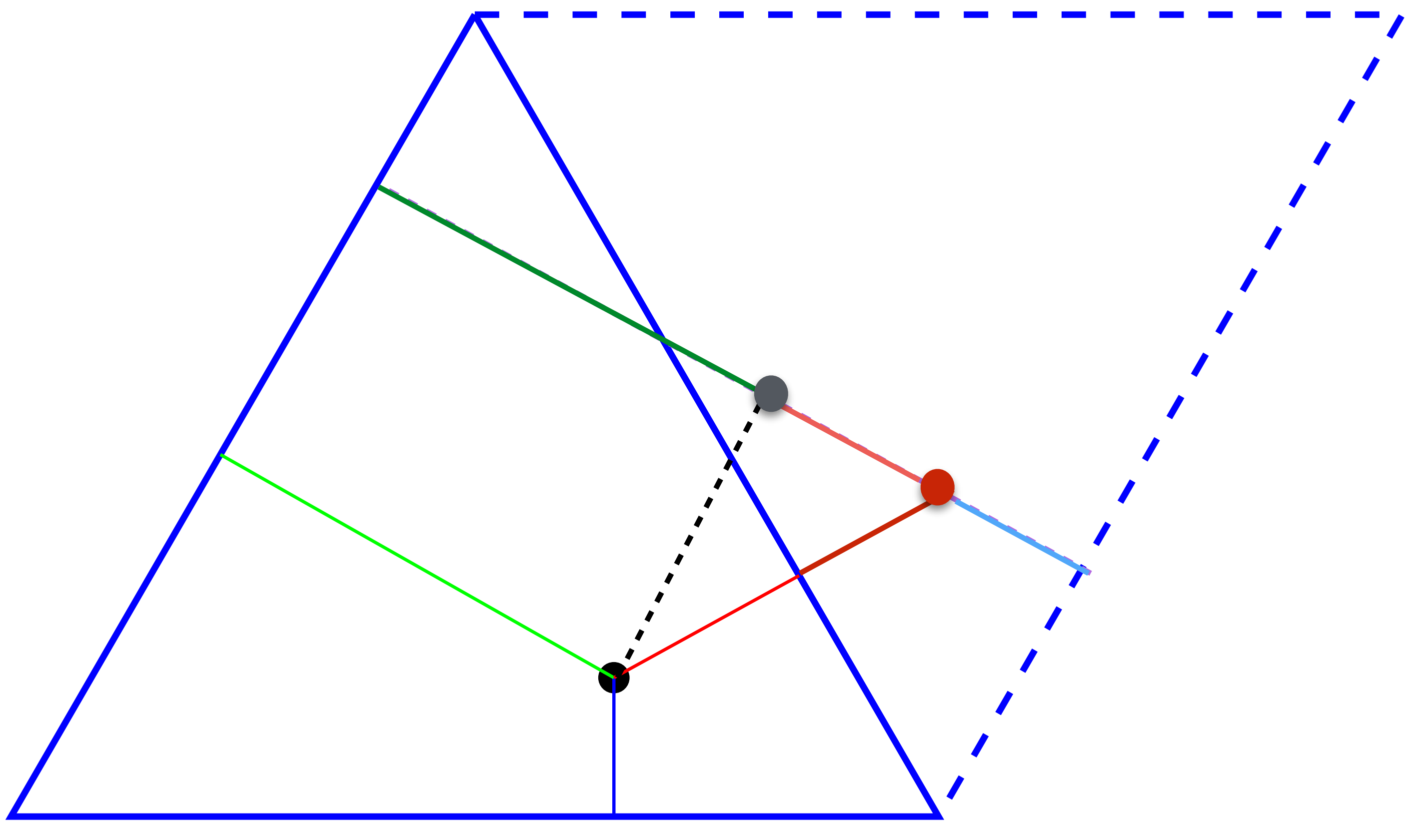
$$a_0 + a_1 + a_2 = k$$

Constancy of coordinates



$$a_0 + a_1 + a_2 = k$$

Constancy of coordinates



$$a_0 + a_1 + a_2 = k$$

Translations

So we have

$$\begin{aligned} T_1(a_0) &= \pi\, s_2\, s_1(a_0) \\ &= \pi\, s_2\,(a_0 + a_1) \\ &= \pi\,(a_0 + a_1 + 2a_2) \\ &= a_1 + a_2 + 2\,a_0 = a_0 + k \end{aligned}$$

$\Rightarrow$

$$T_1(a_0) = a_0 + k, \quad T_1(a_1) = a_1 - k, \quad T_1(a_2) = a_2$$

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Noumi 2004

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Noumi 2004

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \qquad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

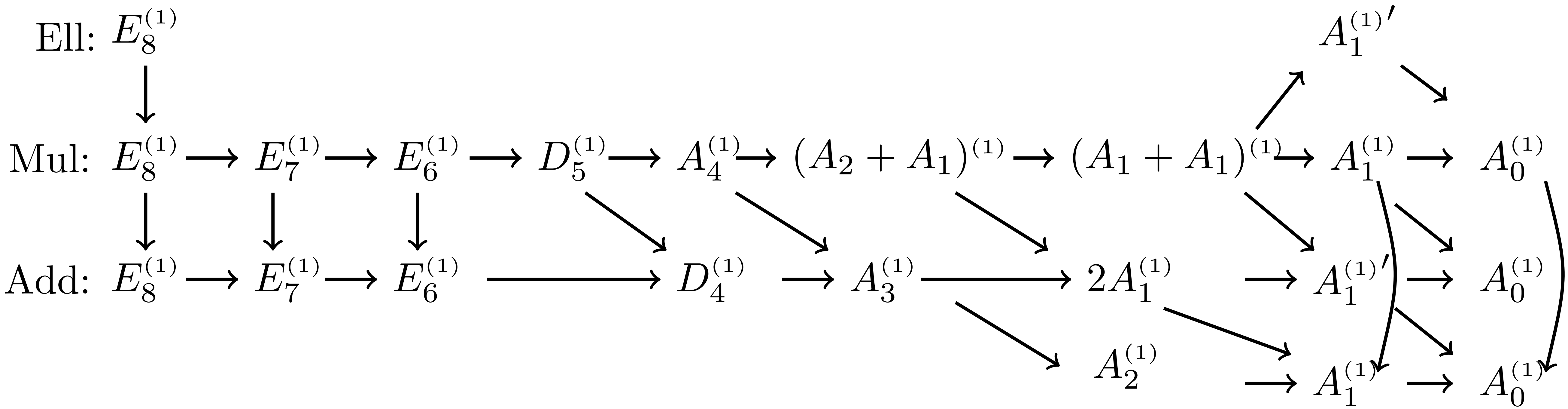
$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \quad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

This is a discrete Painlevé equation.

Sakai described all such equations.

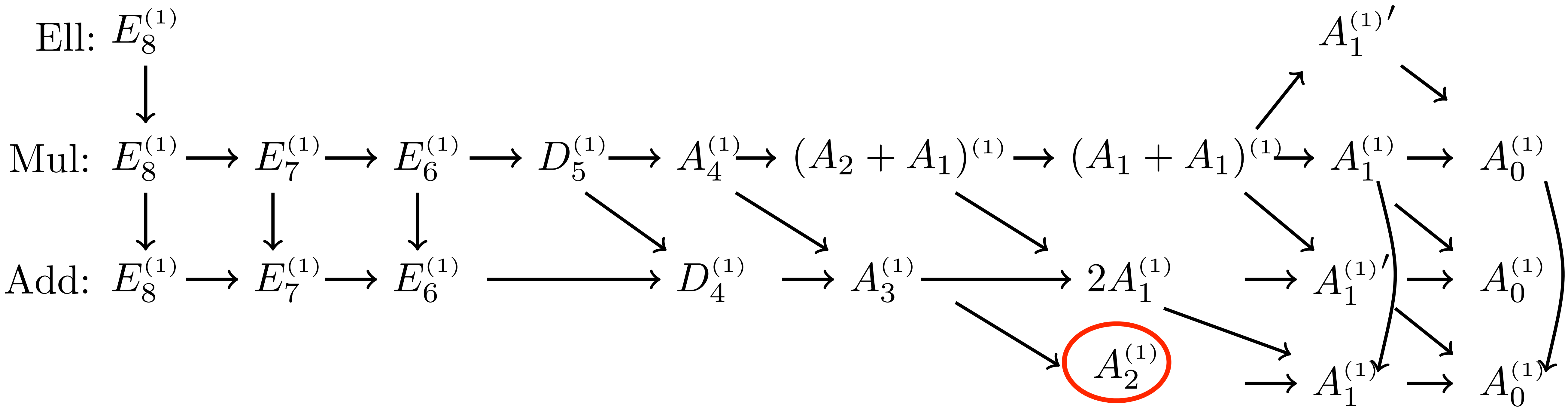
Sakai's Description I



Symmetry groups of Painlevé equations

Sakai 2001

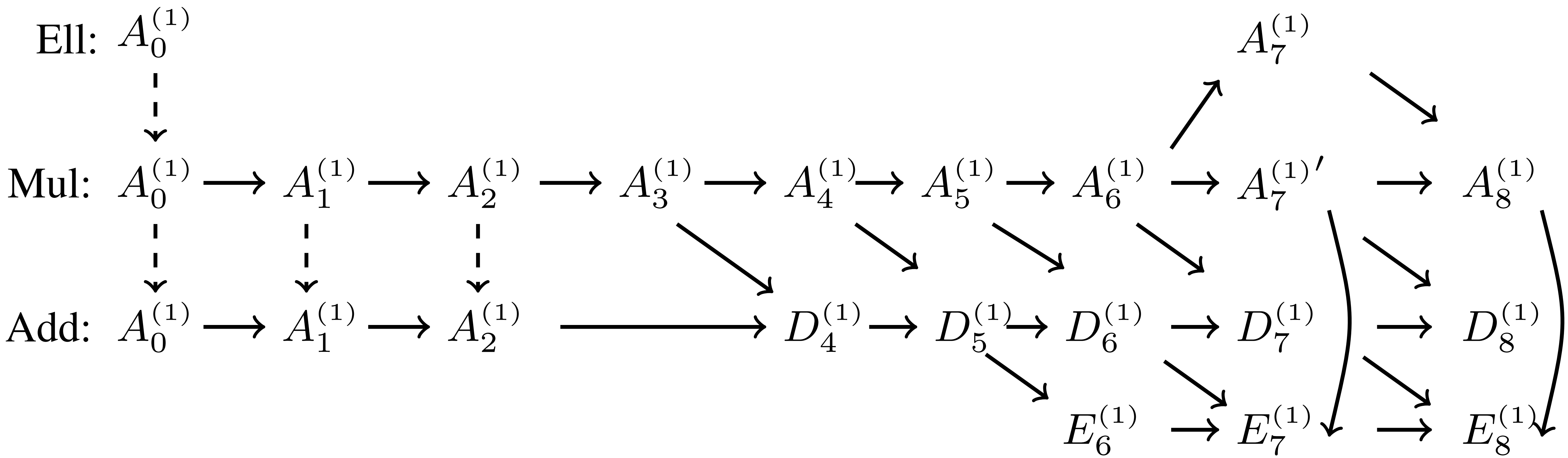
Sakai's Description I



Symmetry groups of Painlevé equations

Sakai 2001

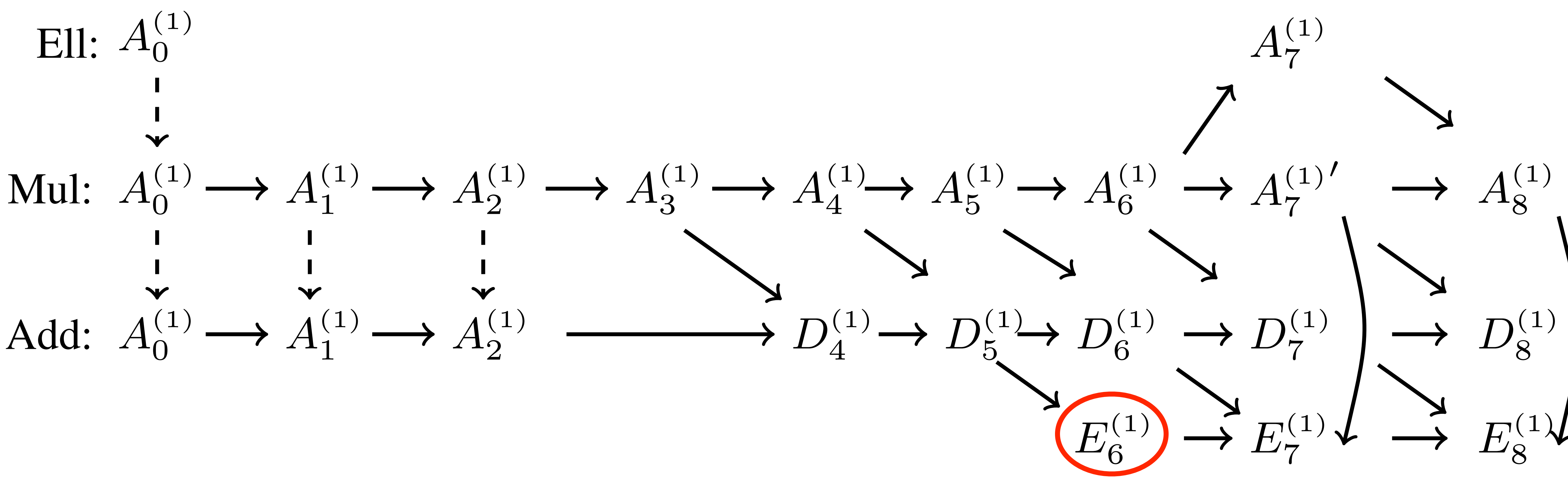
Sakai's Description II



Initial-value spaces of Painlevé equations

Sakai 2001

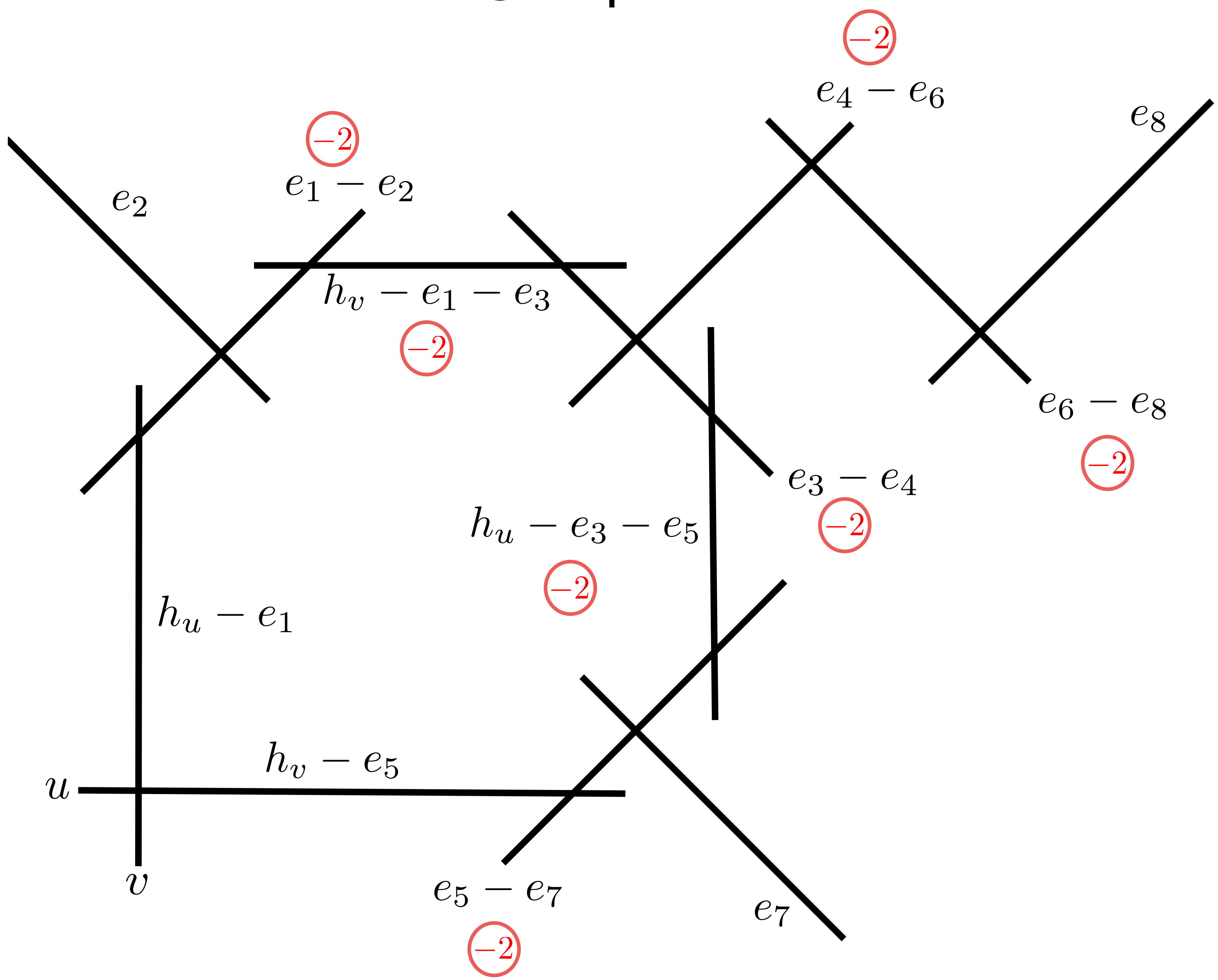
Sakai’s Description II



Initial-value spaces of Painlevé equations

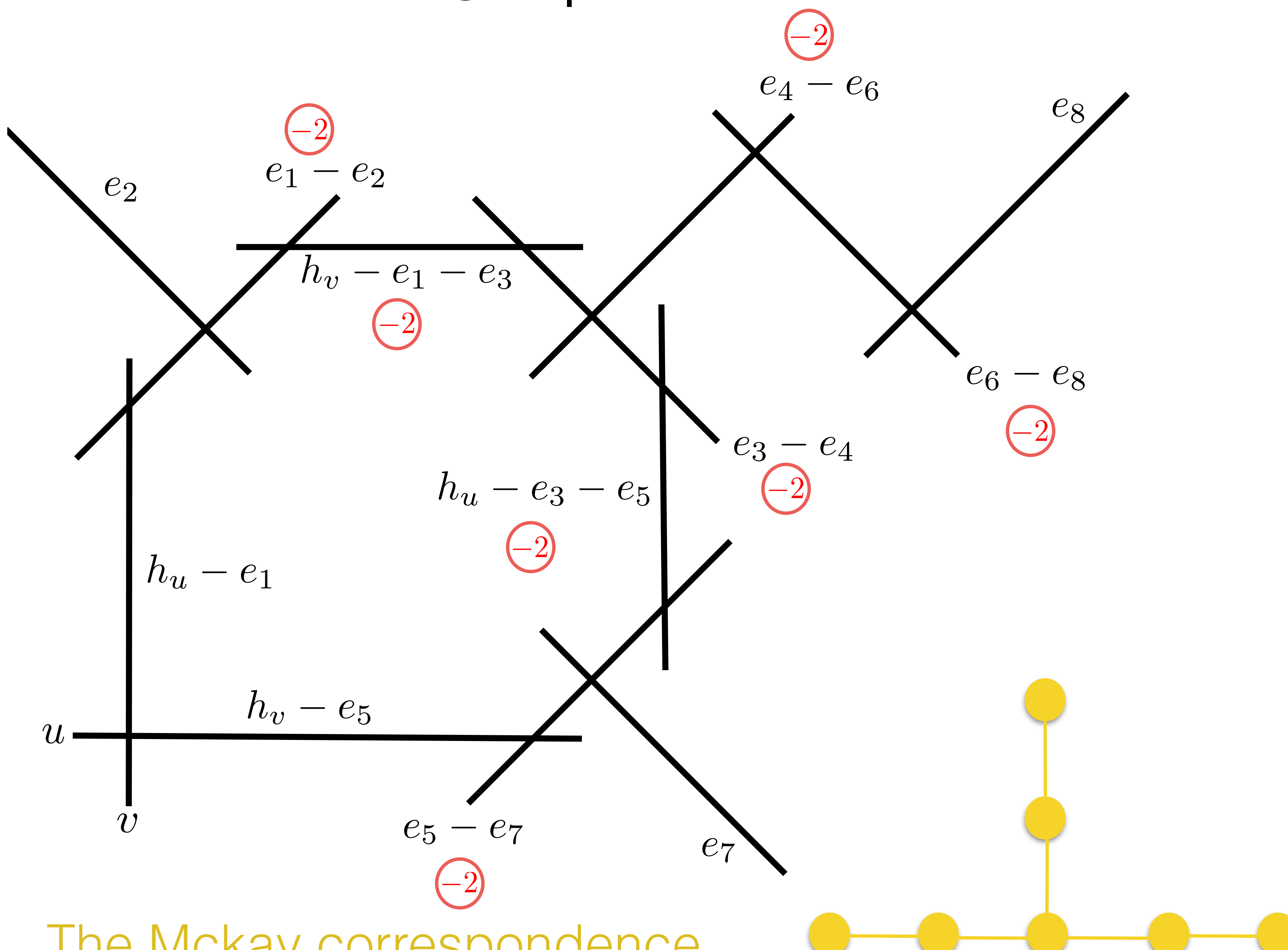
Sakai 2001

dP$_I$



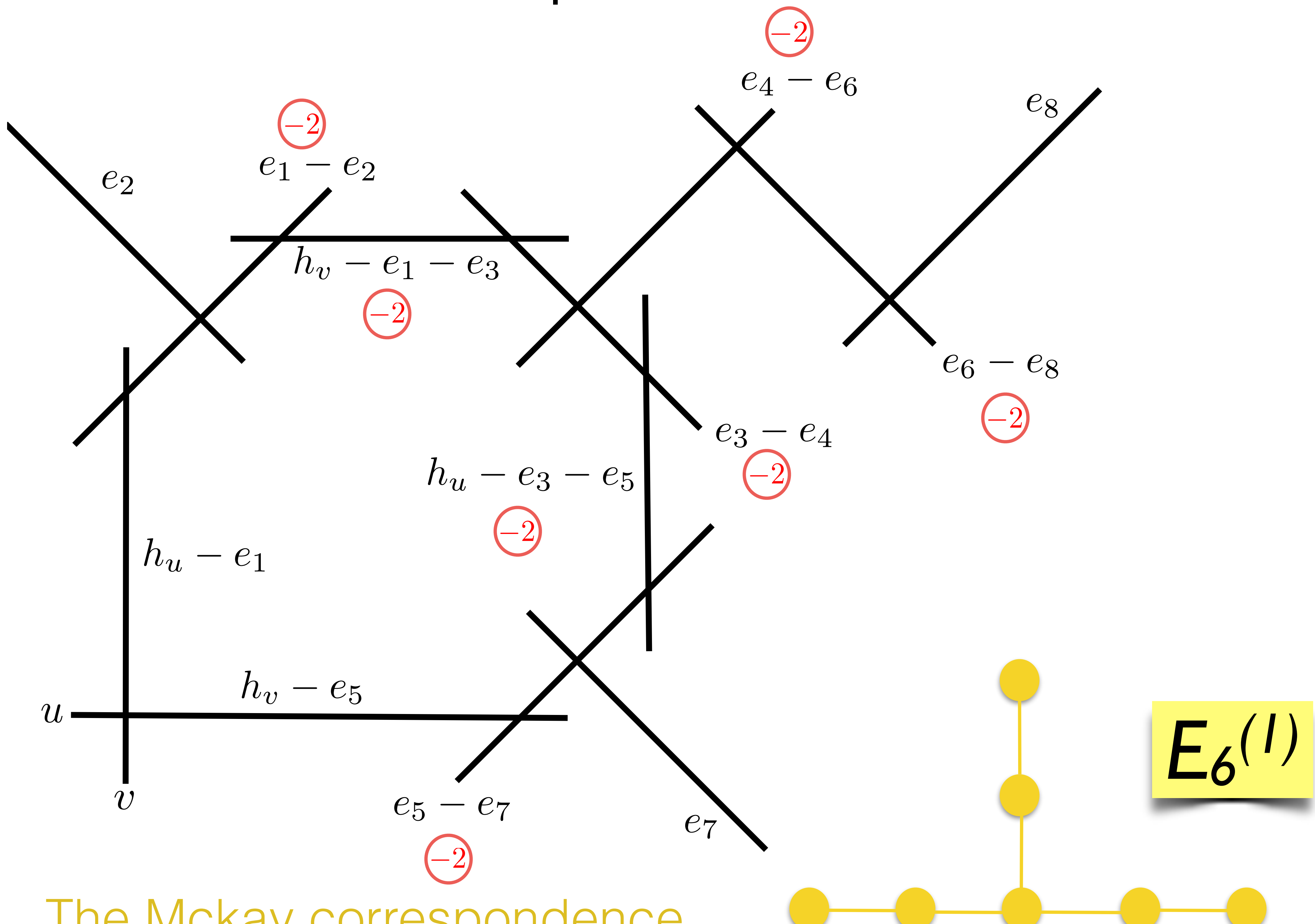
-2
$e_4 - e_6$
e_8
-2
$e_1 - e_2$
e_2
$h_v - e_1 - e_3$
-2
$e_6 - e_8$
-2
$e_3 - e_4$
-2
$h_u - e_3 - e_5$
-2
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
-2
e_7

dP$_I$



The Mckay correspondence

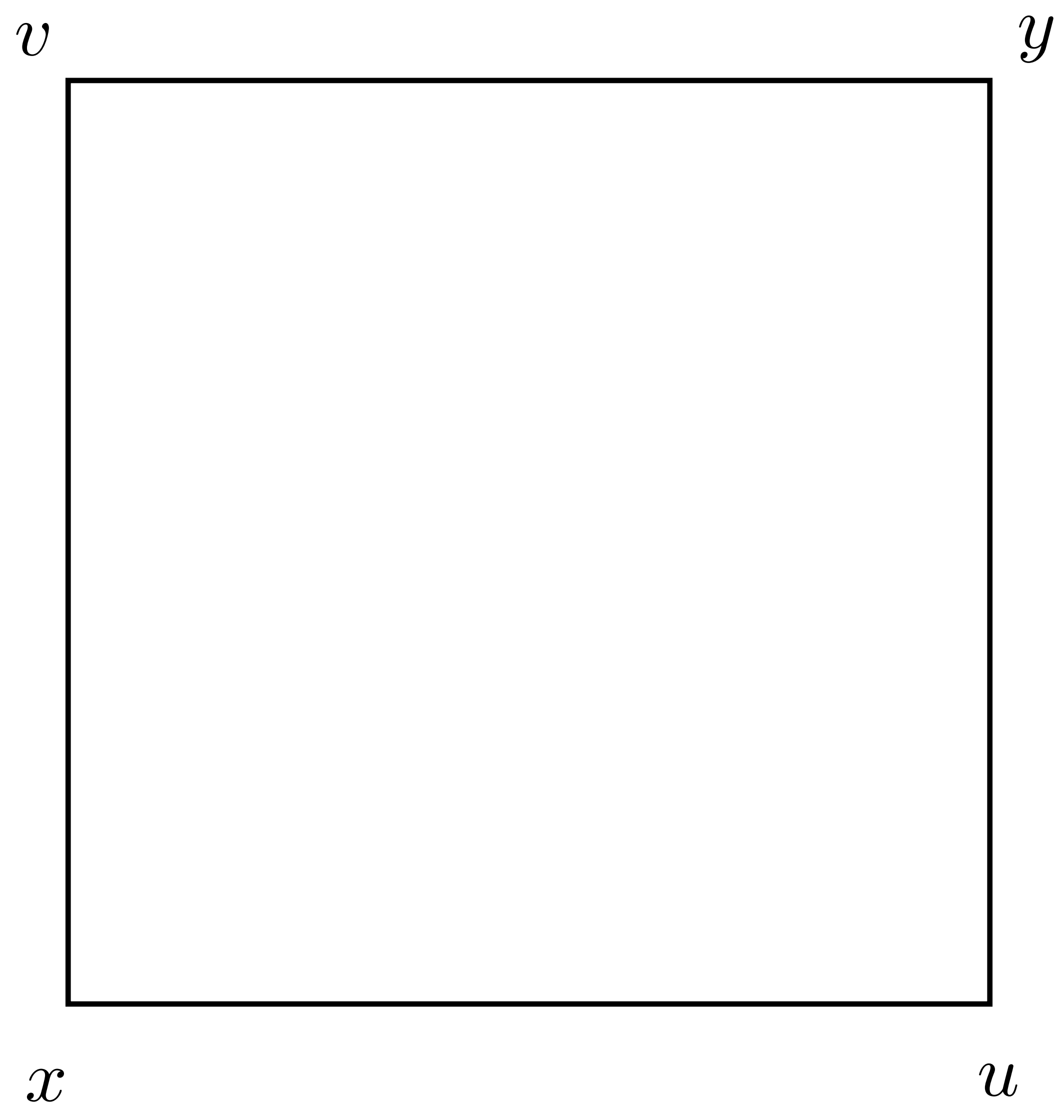
dP_I



The Mckay correspondence

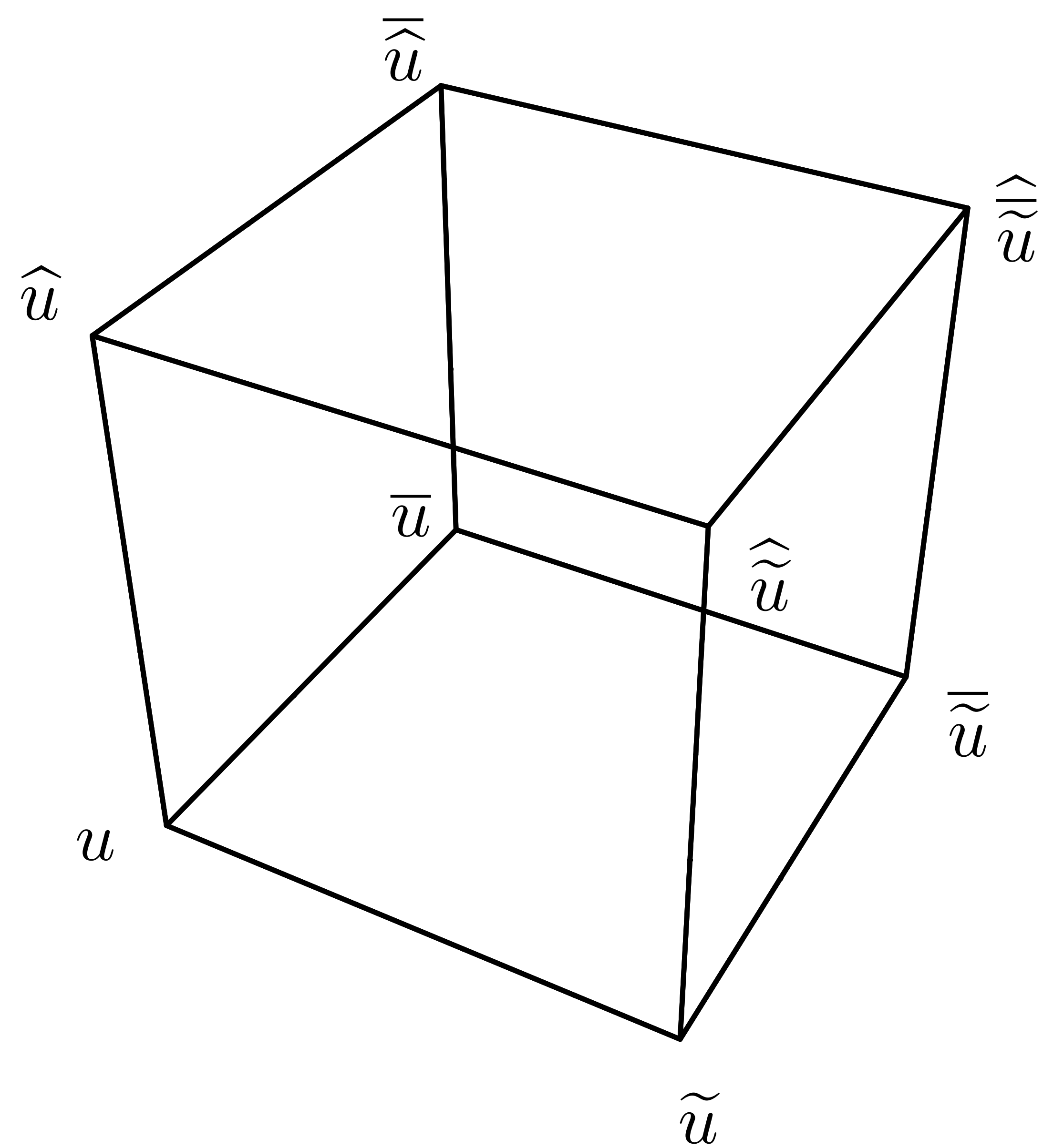
Part 2

- Lattices
- Dynamics on *N*-cubes
- Symmetry reductions

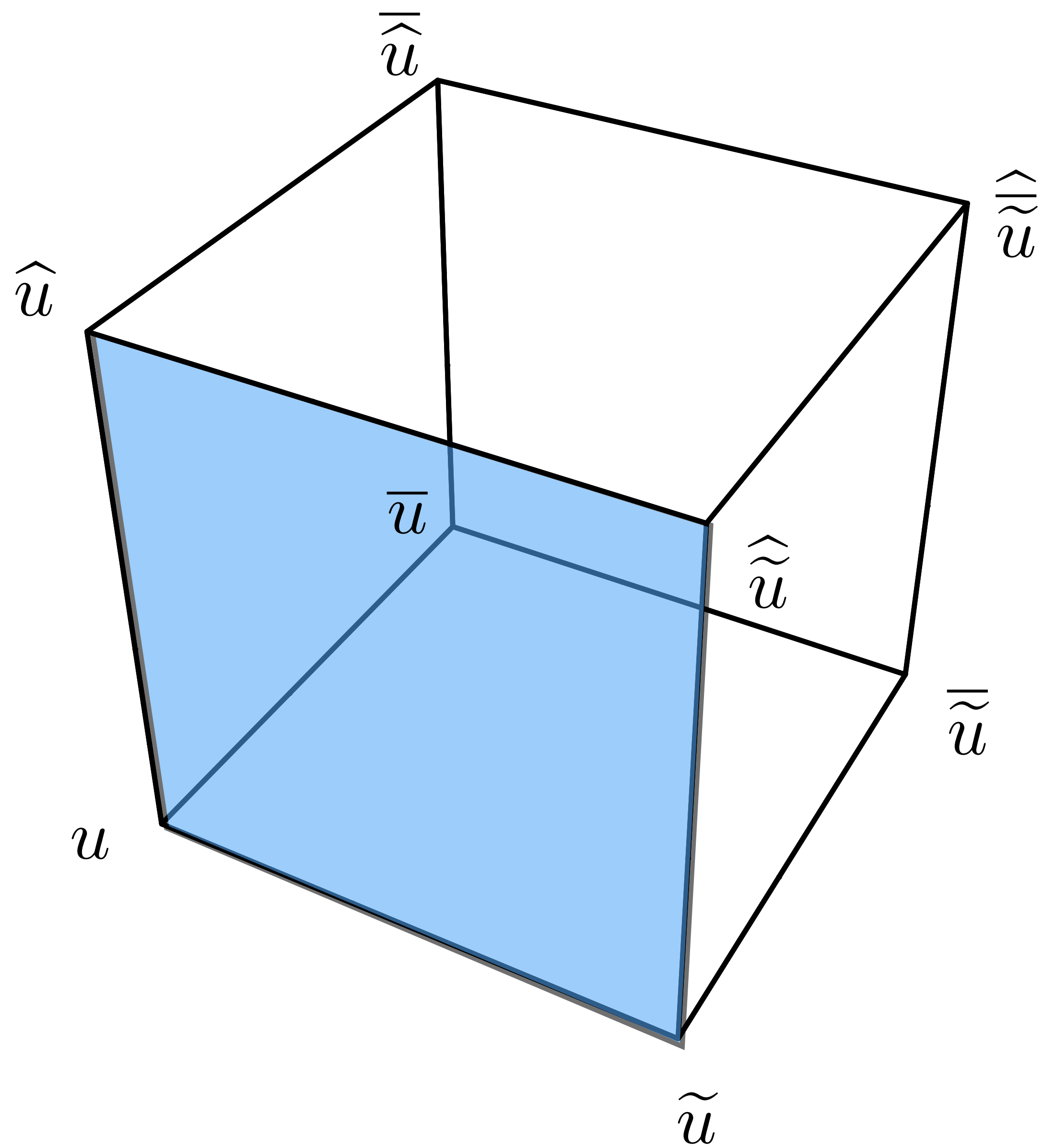


$$Q(x, u, v, y) = 0$$

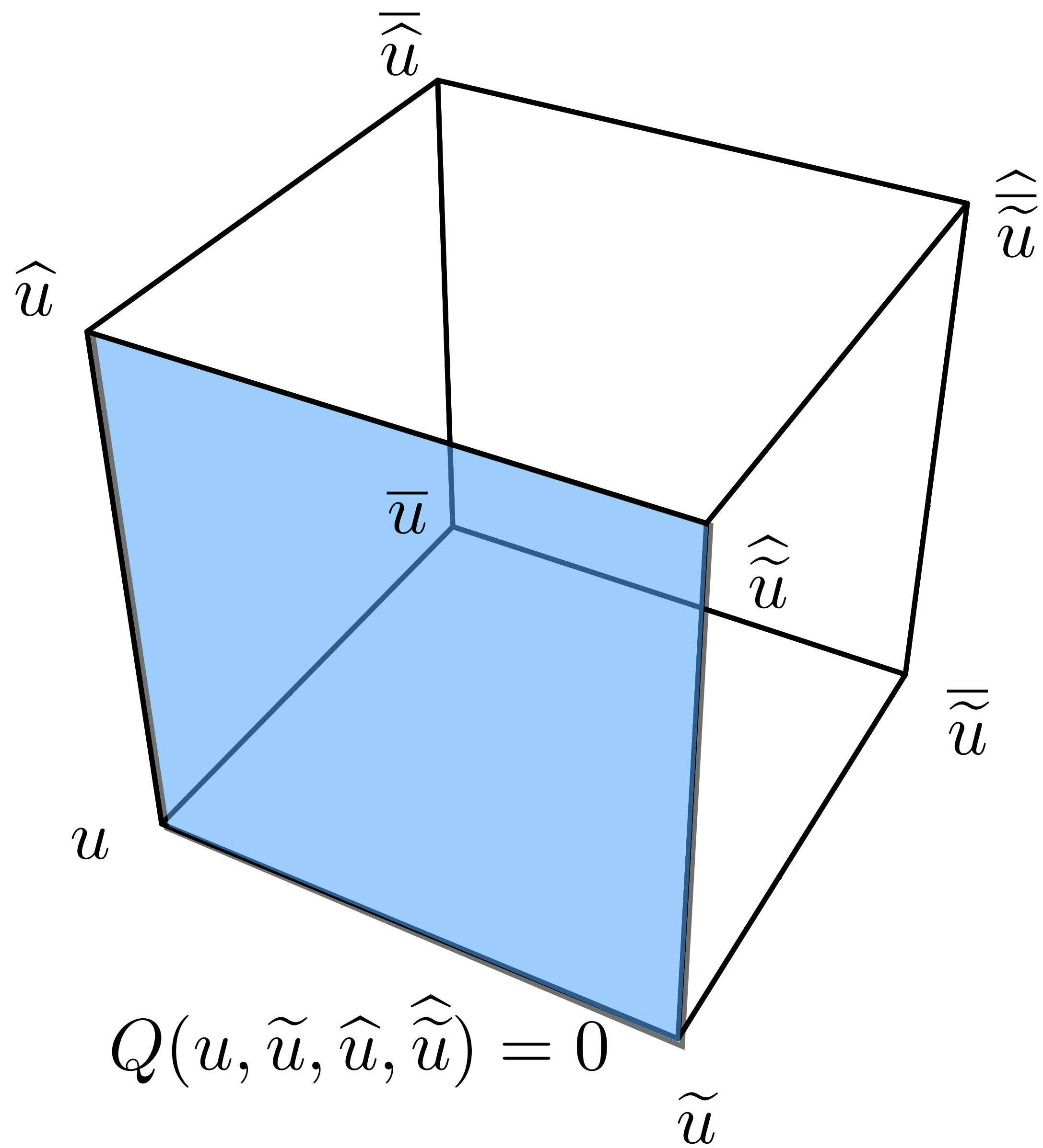
Consistency



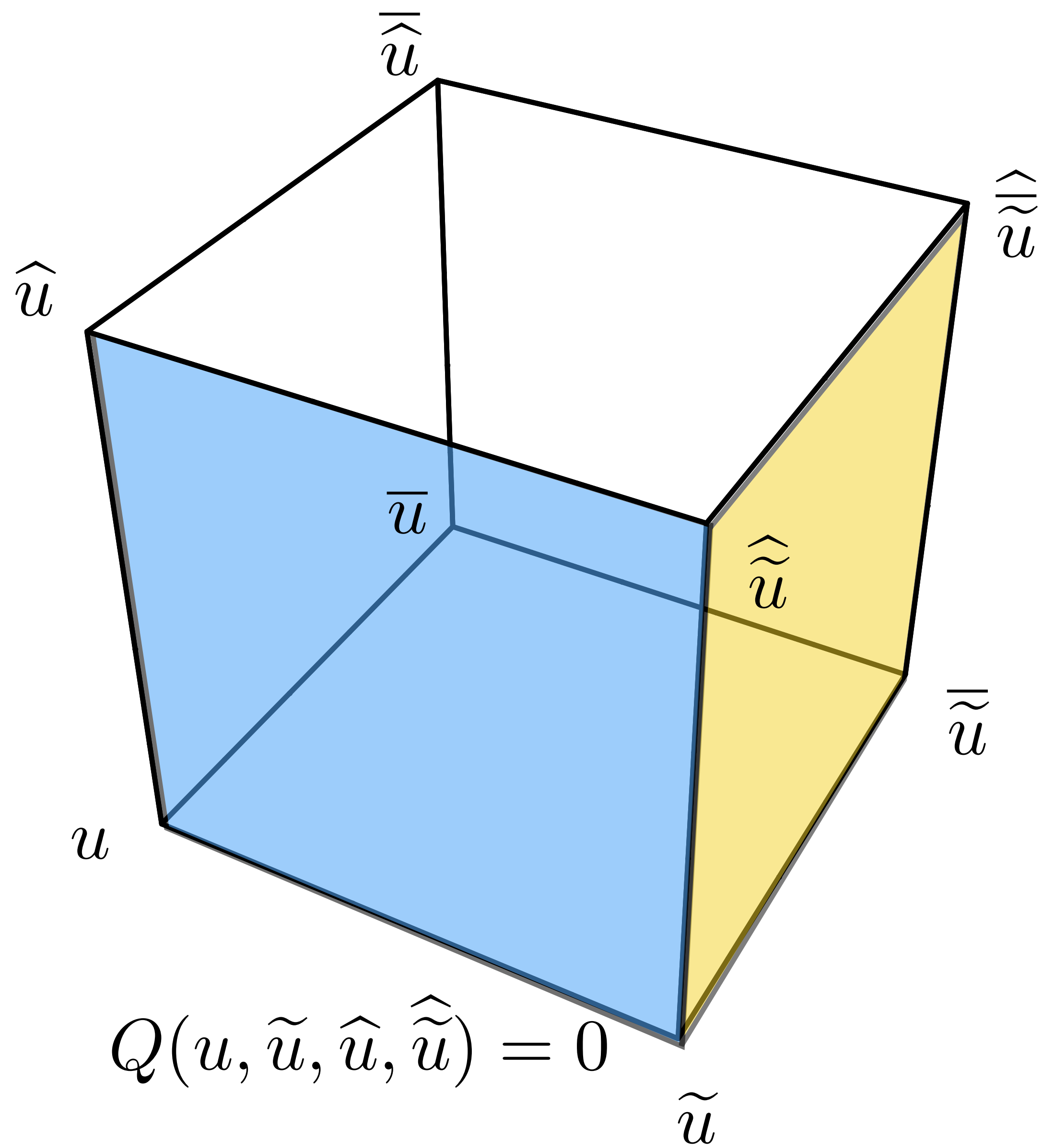
Consistency



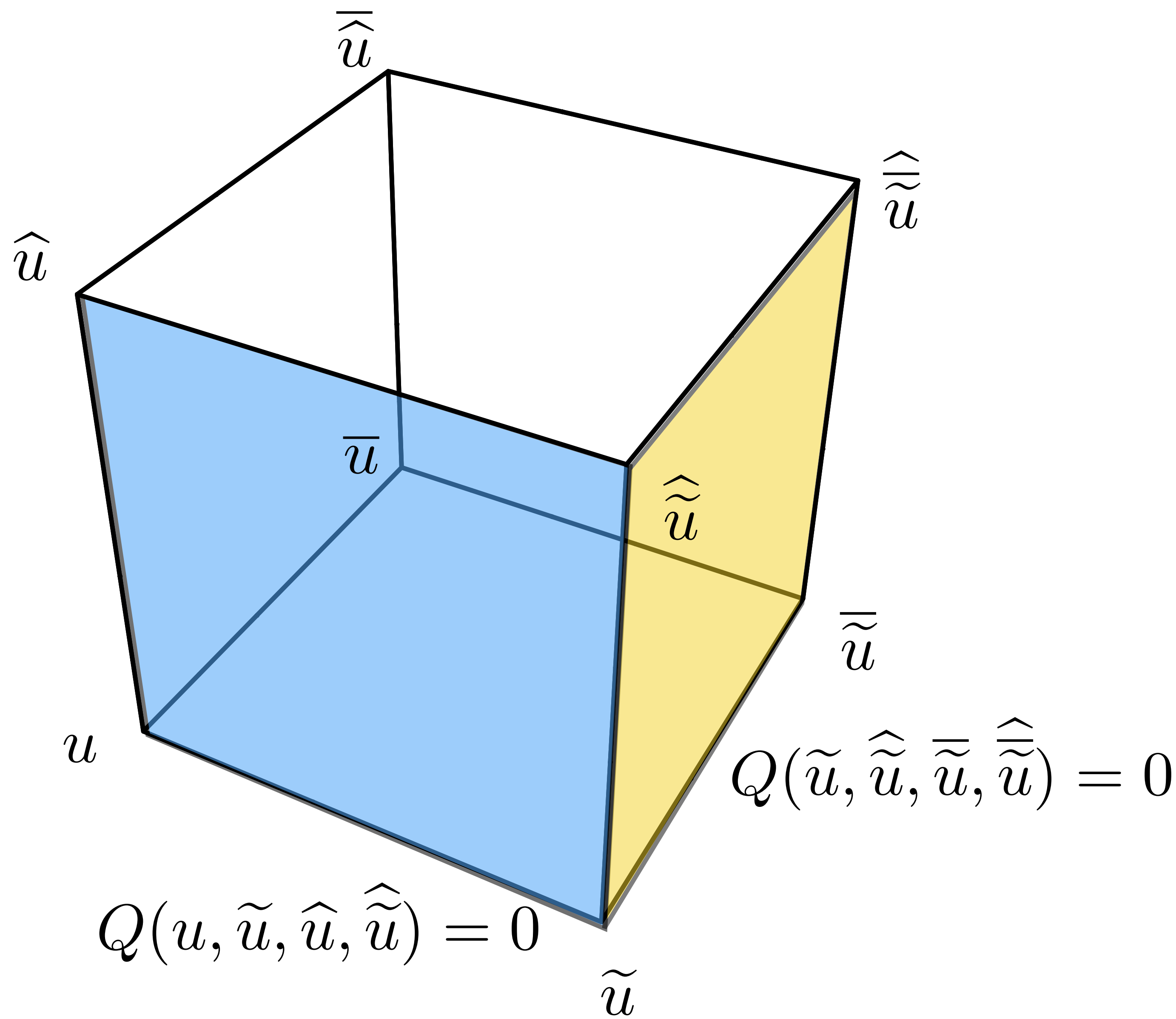
Consistency



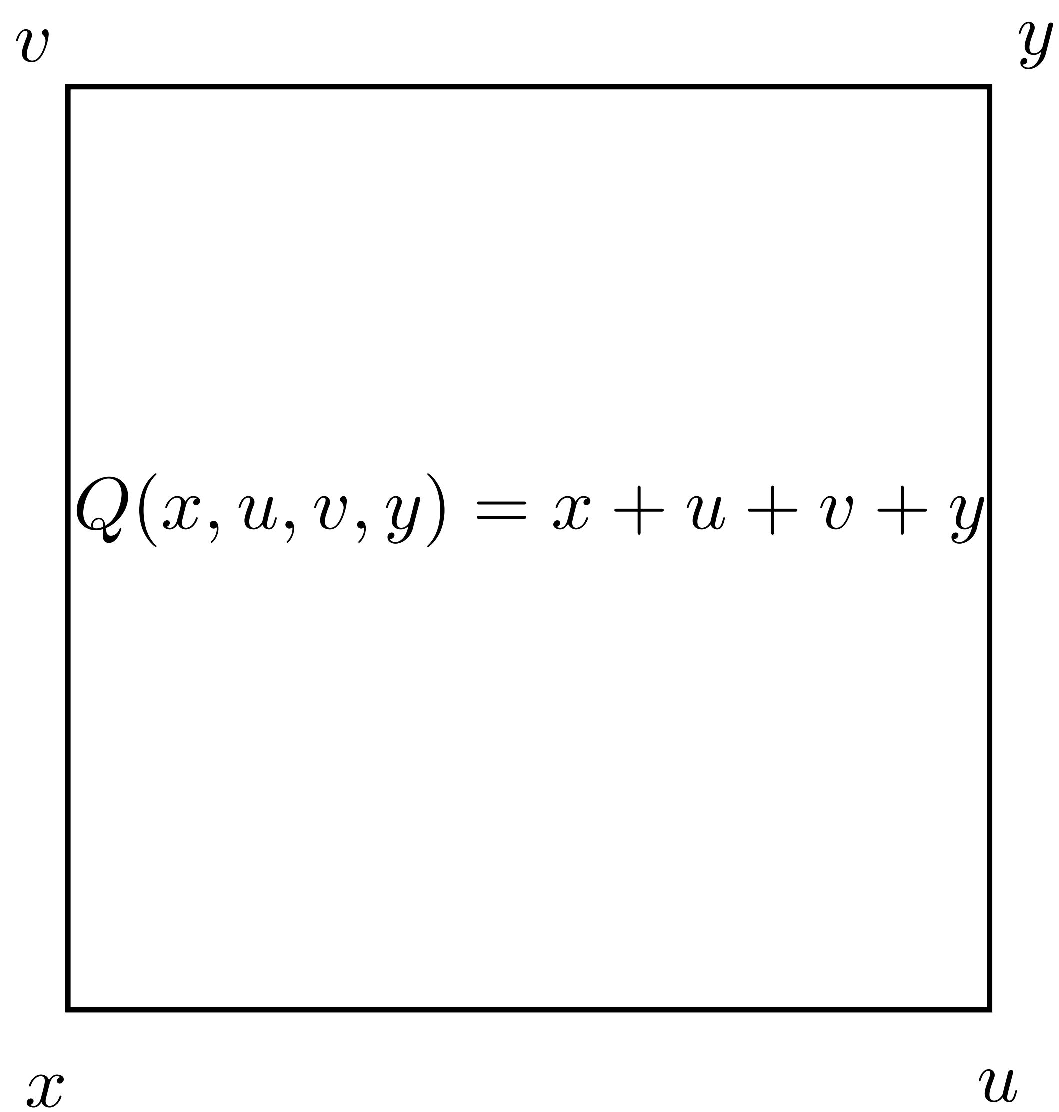
Consistency



Consistency



Linear Case



$$x + u + v + y = 0$$

Linear Consistency

Consider $Q(x,u,v,y) = x + u + v + y$

$$u + \widetilde{u} + \widehat{u} + \widehat{\widetilde{u}} = 0$$

$$u + \widetilde{u} + \overline{u} + \overline{\widetilde{u}} = 0$$

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All 3 paths to the last vertex lead to the same value:

$$\widehat{\overline{\widetilde{u}}} = 2u + \widetilde{u} + \overline{u} + \widehat{u}$$

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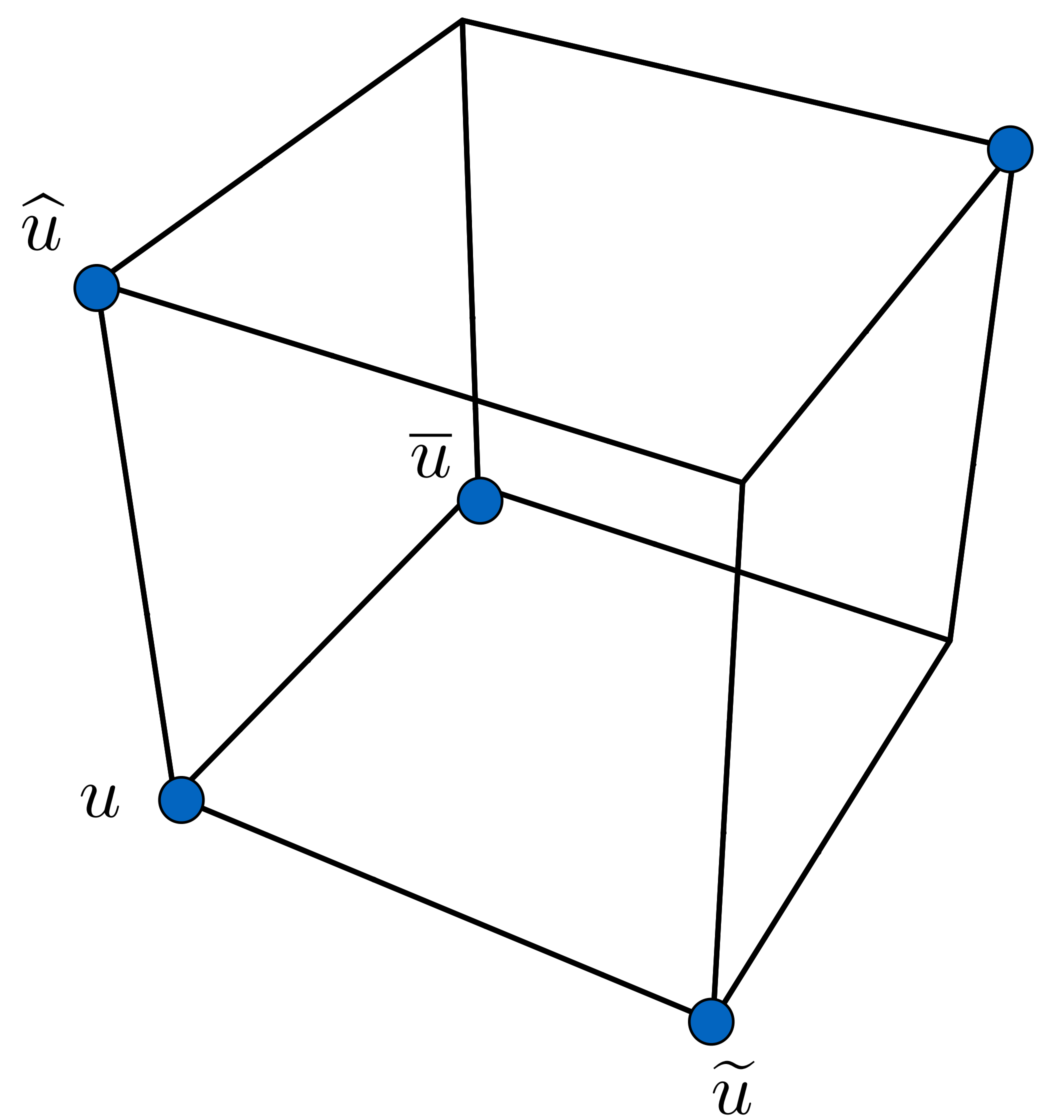
All 3 paths to the last vertex lead to the same value:

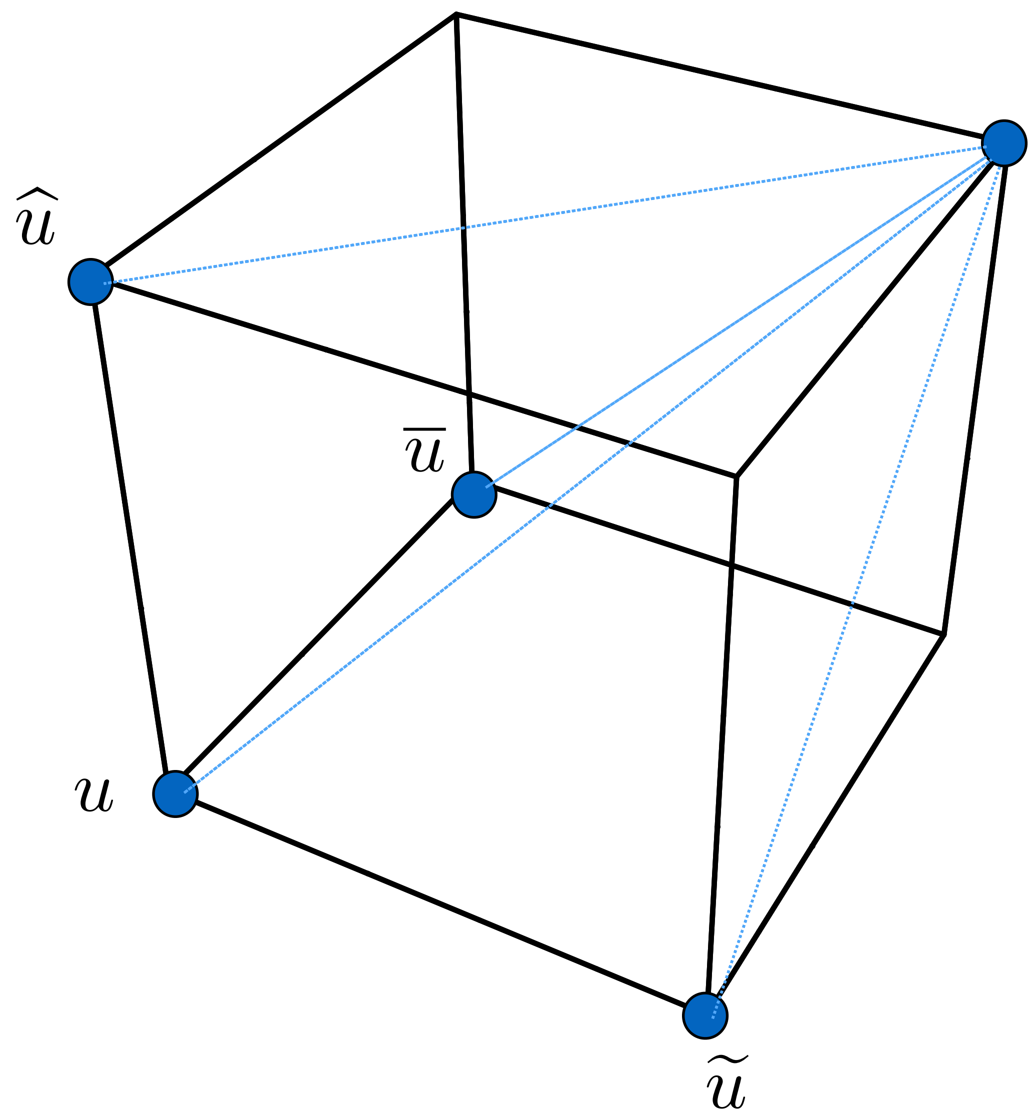
$$\widehat{\overline{\widetilde{u}}} = 2u + \widetilde{u} + \overline{u} + \widehat{u}$$

Tetrahedral Condition



The result depends only on 4 earlier vertices to which it is not connected by an edge.

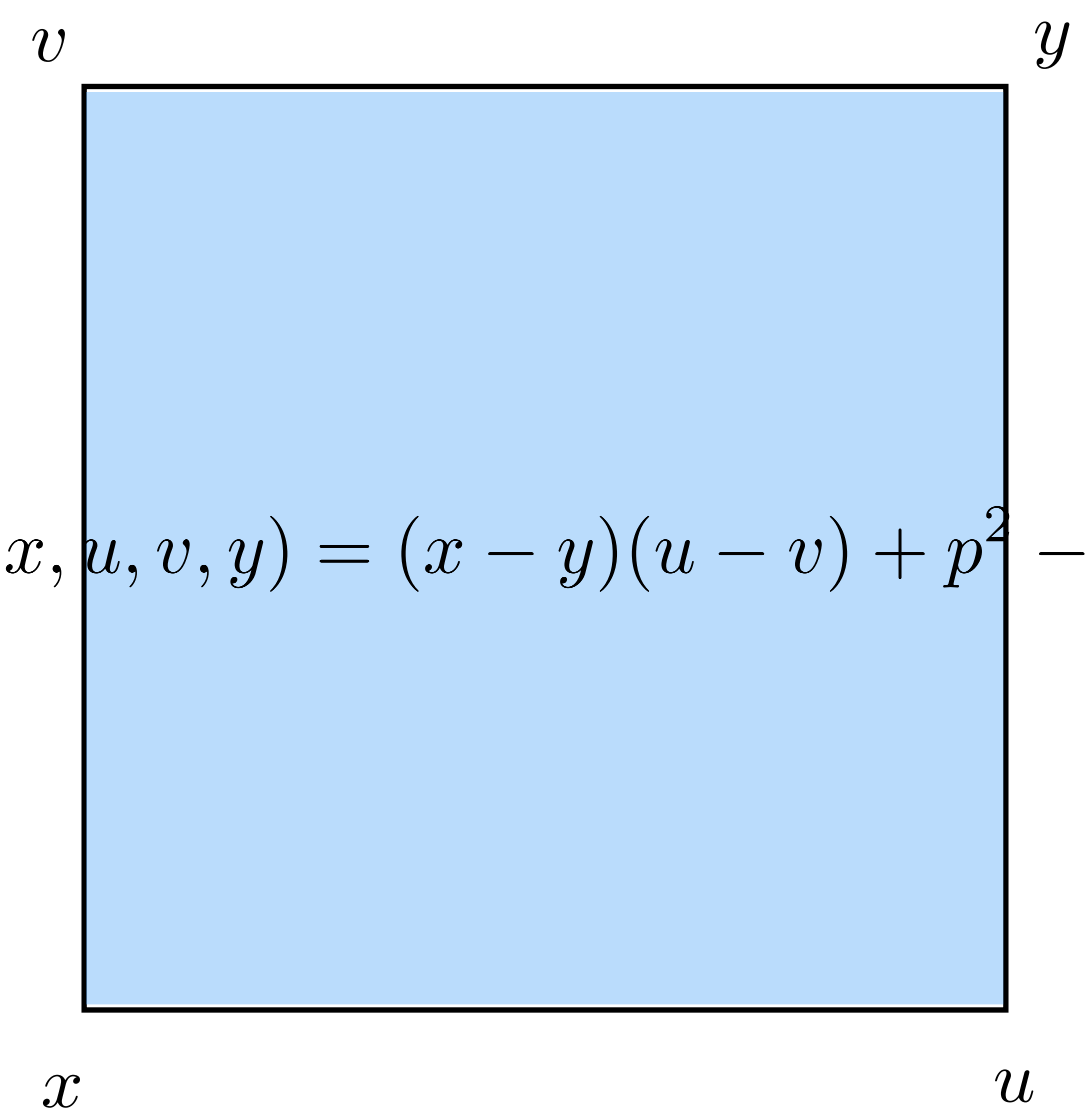
Tetrahedral Condition



The result depends only on 4 earlier vertices
to which it is not connected by an edge.

Are there more examples?

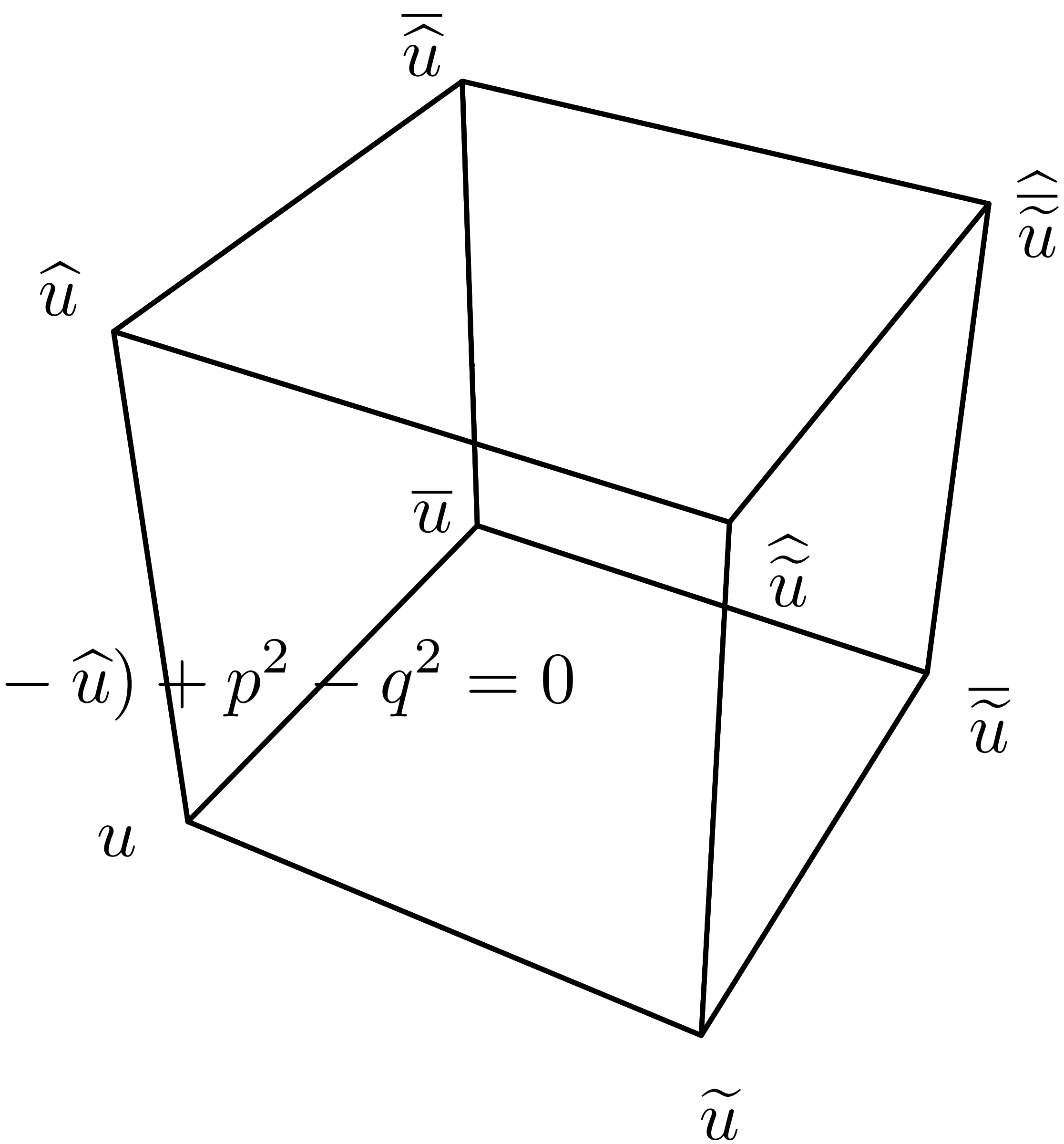
Non-Linear Case



$$Q(x,u,v,y) = (x-y)(u-v) + p^2 - q^2$$

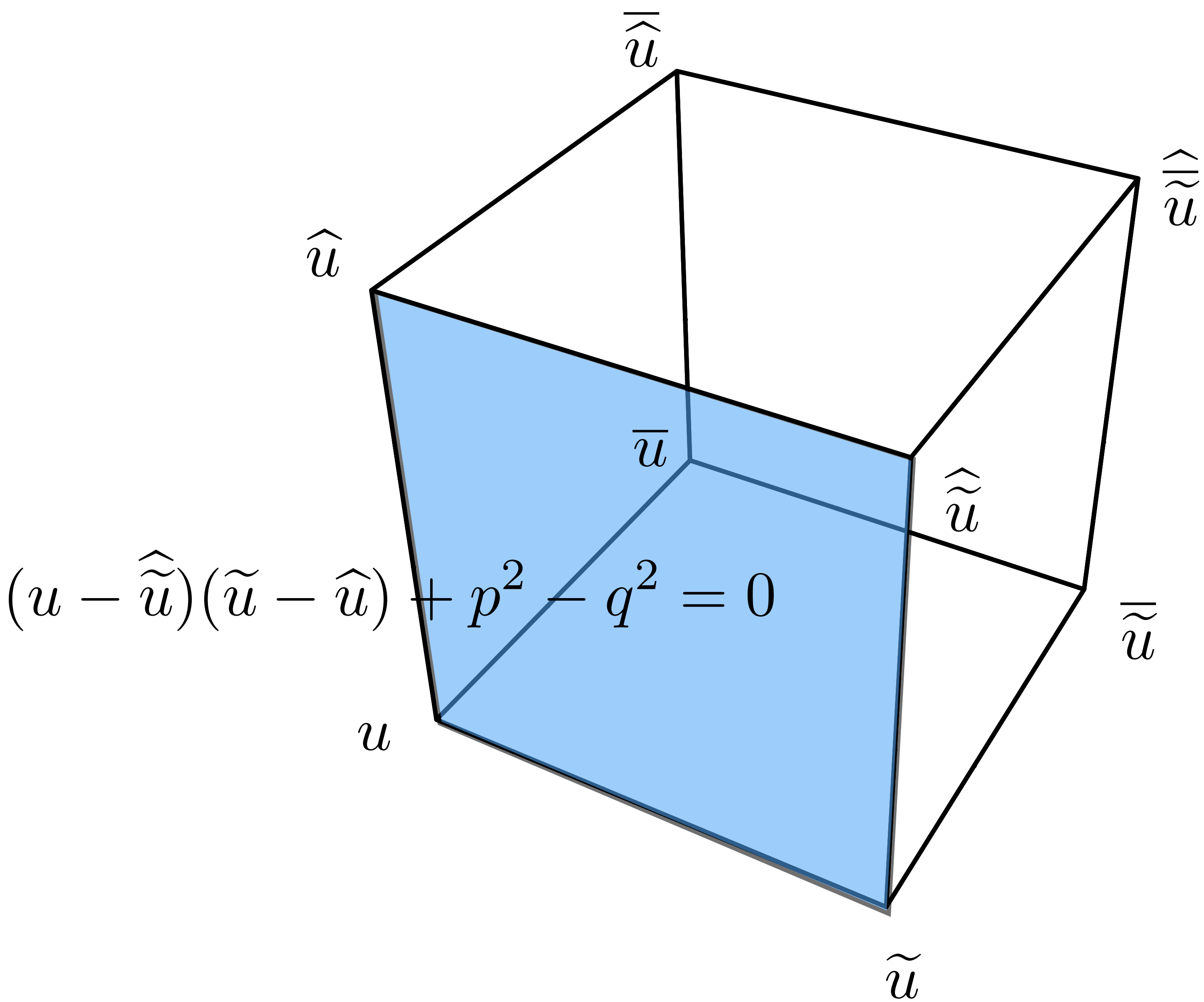
Nijhoff, Quispel, Capel,1983
Nijhoff, Quispel, van der Linden, Capel,1983

Non-linear Consistency

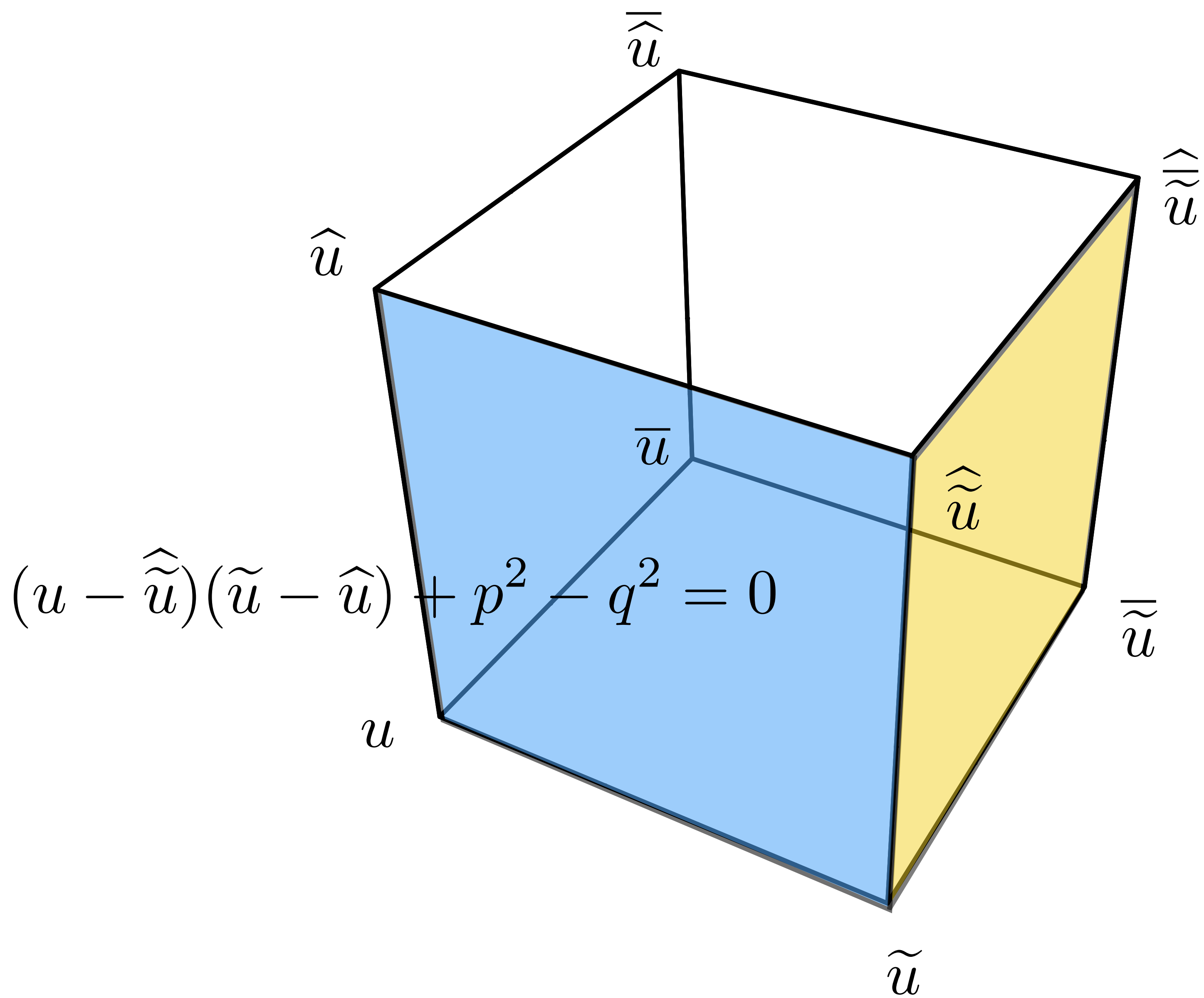


$$(u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) + p^2 - q^2 = 0$$

Non-linear Consistency



Non-linear Consistency



Non-linear Consistency

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+p^2-q^2=0$$

$$(u-\overline{\widetilde{u}})(\widetilde{u}-\overline{u})+p^2-r^2=0$$

$$(u-\widehat{\overline{u}})(\overline{u}-\widehat{u})+r^2-q^2=0$$

$$(\overline{u}-\overline{\widehat{\widetilde{u}}})(\overline{\widetilde{u}}-\overline{\widehat{u}})+p^2-q^2=0$$

$$(\widehat{u}-\widehat{\overline{\widetilde{u}}})(\widehat{\widetilde{u}}-\widehat{\overline{u}})+p^2-r^2=0$$

$$(\widetilde{u}-\widetilde{\widehat{\overline{u}}})(\widetilde{\overline{u}}-\widetilde{\widehat{u}})+r^2-q^2=0$$

$$\Downarrow$$

$$\widetilde{\widehat{\overline{u}}}=\frac{p^2\overline{u}\widetilde{u}-p^2\widehat{u}\widetilde{u}-q^2\overline{u}\widehat{u}+q^2\widehat{u}\widetilde{u}+r^2\overline{u}\widehat{u}-r^2\overline{u}\widetilde{u}}{p^2\overline{u}-p^2\widehat{u}-q^2\overline{u}+q^2\widetilde{u}+r^2\widehat{u}-r^2\widetilde{u}}$$

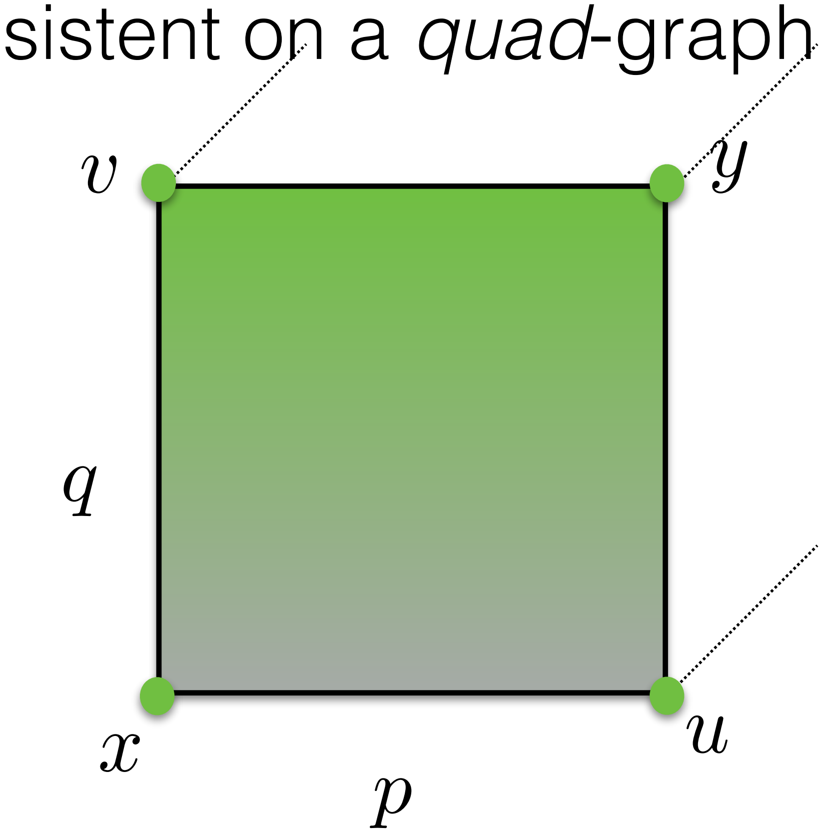
Classification

- Motivated by work of Nijhoff, Capel *et al (1983—'01)* Adler, Bobenko & Suris (2003,2009) classified all affine linear equations

$$Q(w, \widetilde{w}, \widehat{w}, \widehat{\widetilde{w}}; p, q) = 0$$

which are multi-dimensionally consistent on a *quad*-graph

$$Q(x, u, v, y; p, q) = 0$$



CAC Equations

$$
\begin{aligned}
Q4: \quad & a_0 xuvy + a_1(xuv + uvy + vyx + yxu) + a_2(xy + uv) \\
& + \overline{a}_2(xu + vy) + \widetilde{a}_2(xv + uy) \\
& + a_3(x + u + v + y) + a_4 = 0
\end{aligned}
$$

CAC Equations

- ABS: Three classes of equations
- The "mistress" equation:

$$
\begin{aligned}
Q4: \quad & a_0 xuvy + a_1(xuv + uvy + vyx + yxu) + a_2(xy + uv) \\
& + \overline{a}_2(xu + vy) + \widetilde{a}_2(xv + uy) \\
& + a_3(x + u + v + y) + a_4 = 0
\end{aligned}
$$

where the coefficients lie on an elliptic curve.

- The two other classes are labelled H and A.

Some ABS Equations

- H1:

$$(x-y)(u-v)+p^2-q^2=0$$

- H3:

$$\mathcal{Q}(xu+vy)-\mathcal{P}(uv+uy)+\frac{p^2-q^2}{\mathcal{P}\mathcal{Q}}=0$$

where $\mathcal{P}^2=a^2-p^2, \mathcal{Q}^2=a^2-q^2$

- Q3:

$$\mathcal{P}(uv+uy)-\mathcal{Q}(xu+vy)-(p^2-q^2)\left(uv+xy+\frac{\delta^2}{4\mathcal{P}\mathcal{Q}}\right)=0$$

where

$$\mathcal{P}^2=(p^2-a^2)(p^2-b^2)$$

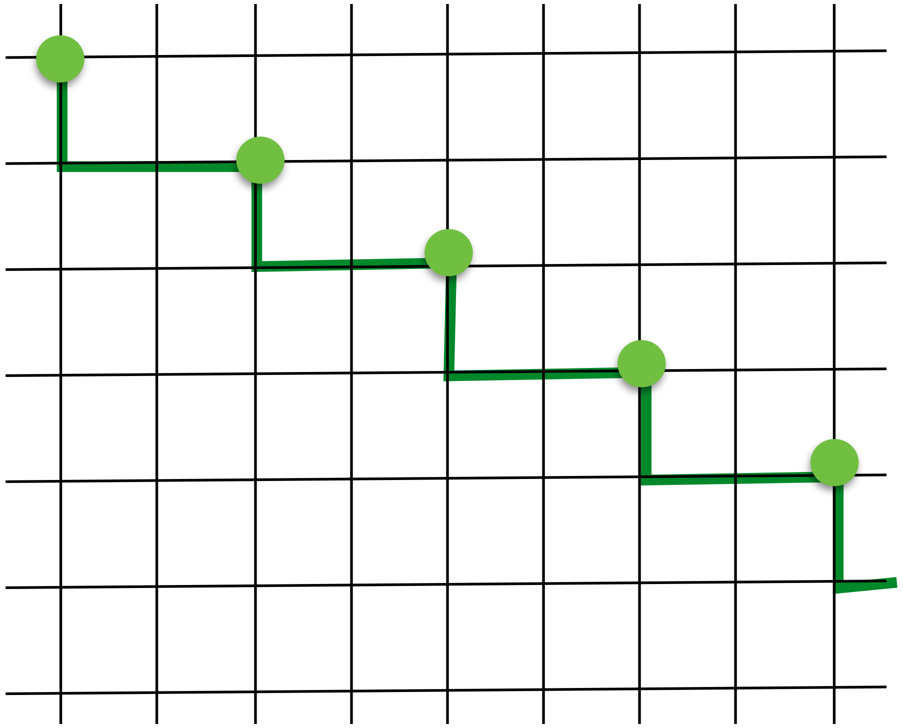
$$\mathcal{Q}^2=(q^2-a^2)(q^2-b^2)$$

These results also arise as integrable systems.

Part III

- Lattices
- Dynamics on *N*-cubes
- Symmetry reductions

Discrete Staircases



$$w(l+2,k) = w(l,k+1)$$

Reductions

- Grammaticos *et al* 2005 showed for $\mathrm{H3}_{\delta=0}$

$$\frac{\widehat{\overline{w}}}{w} = \frac{\alpha\overline{w} - \beta\,\widehat{w}}{\alpha\,\widehat{w} - \beta\,\overline{w}}$$

- $r = \dfrac{\beta}{\alpha}$ and $\widehat{w} = \overline{\overline{w}} \;\Rightarrow\; \overline{\overline{\overline{r}}}\,r = \overline{r}\,\overline{\overline{r}}$

- $$h = \frac{\overline{\overline{w}}}{\overline{w}} \;\Rightarrow\; \overline{h}\,h\,\underline{h} = \frac{1 - r\,h}{r - h}$$

  a discrete third *q*-Painlevé equation (qP_3)

- Other examples of reductions now known, but no systematic approach.

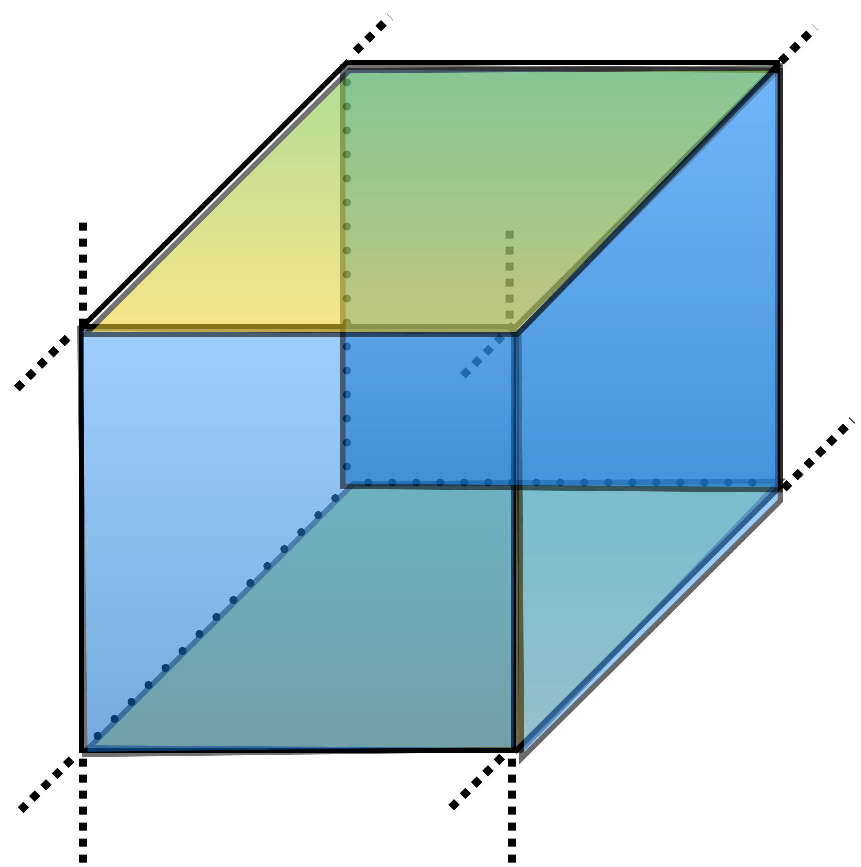
Different Equations on Faces

Boll (20011, 2012) showed that combinations of H3 and H6 provide new consistent systems on the 3-cube, where

$$H6: \quad xy + uv + \delta_1 xu + \delta_2 vy = 0$$

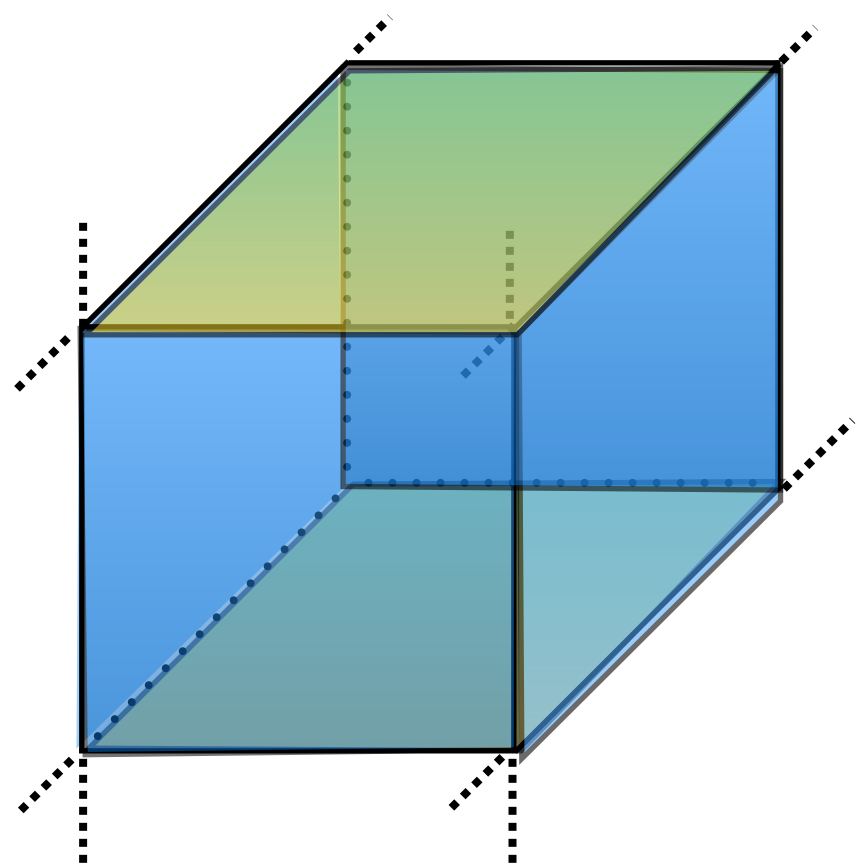
We place H3 ($\delta=0$) on two faces and H6 ($\delta_2=0$) on four faces.

H3 & H6 on 3-cube



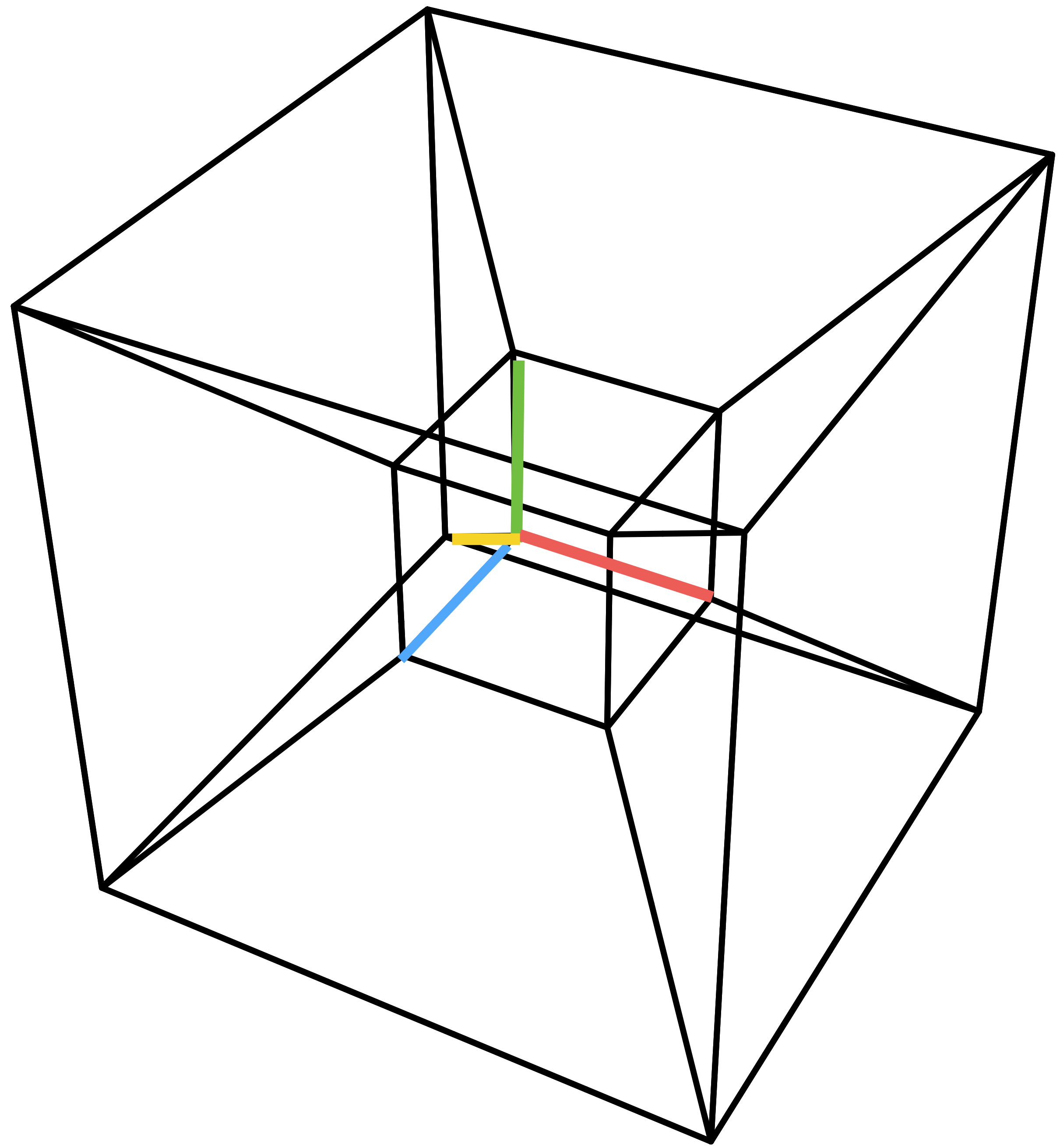
- H3 is on top and bottom faces
- H6 is on the front, right, back and left faces.
- Consistency imposes conditions on parameters.

H3 & H6 on 3-cube



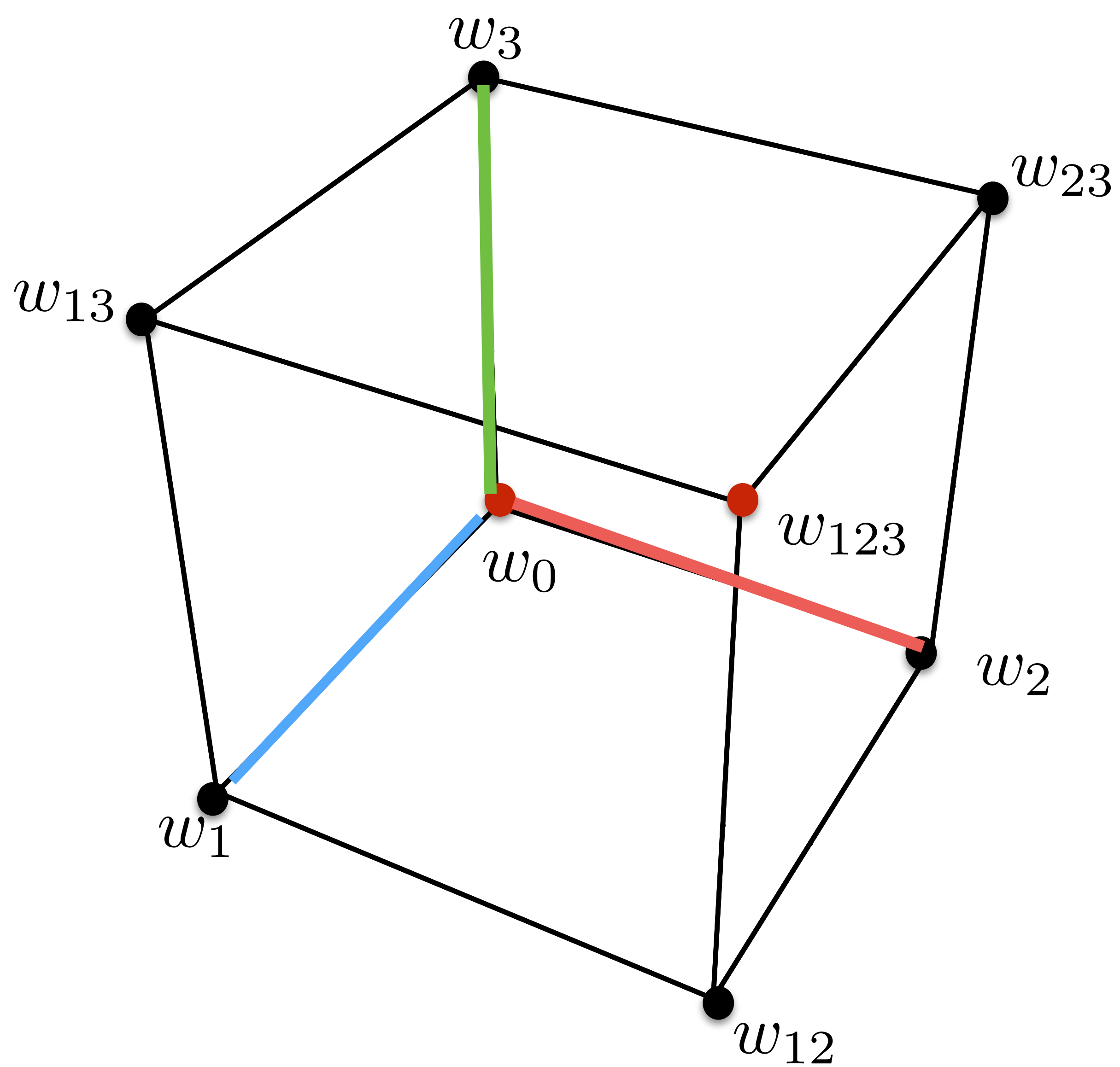
- H3 is on top and bottom faces
- H6 is on the front, right, back and left faces.
- Consistency imposes conditions on parameters.

H3 & H6 on 4-cube

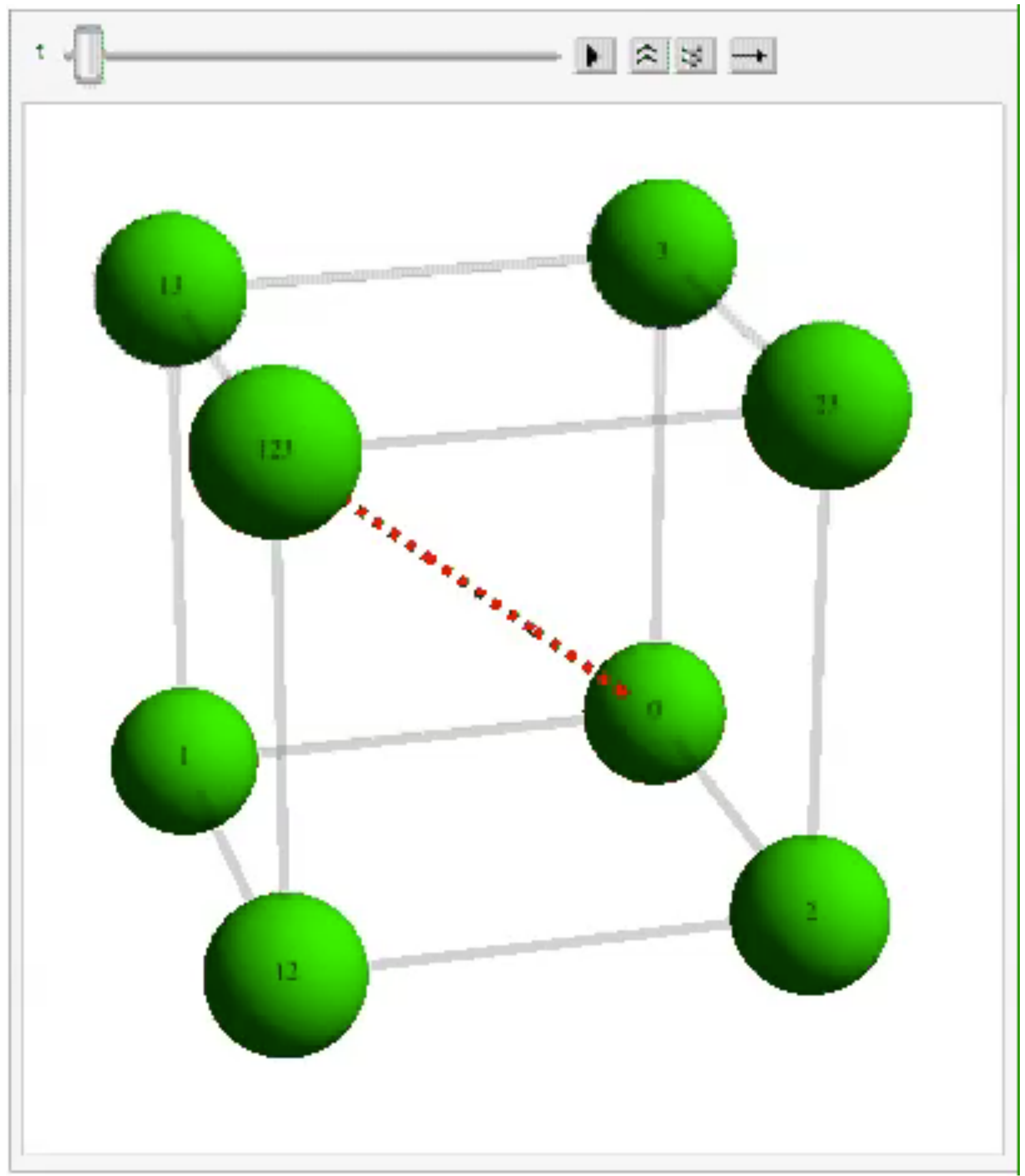


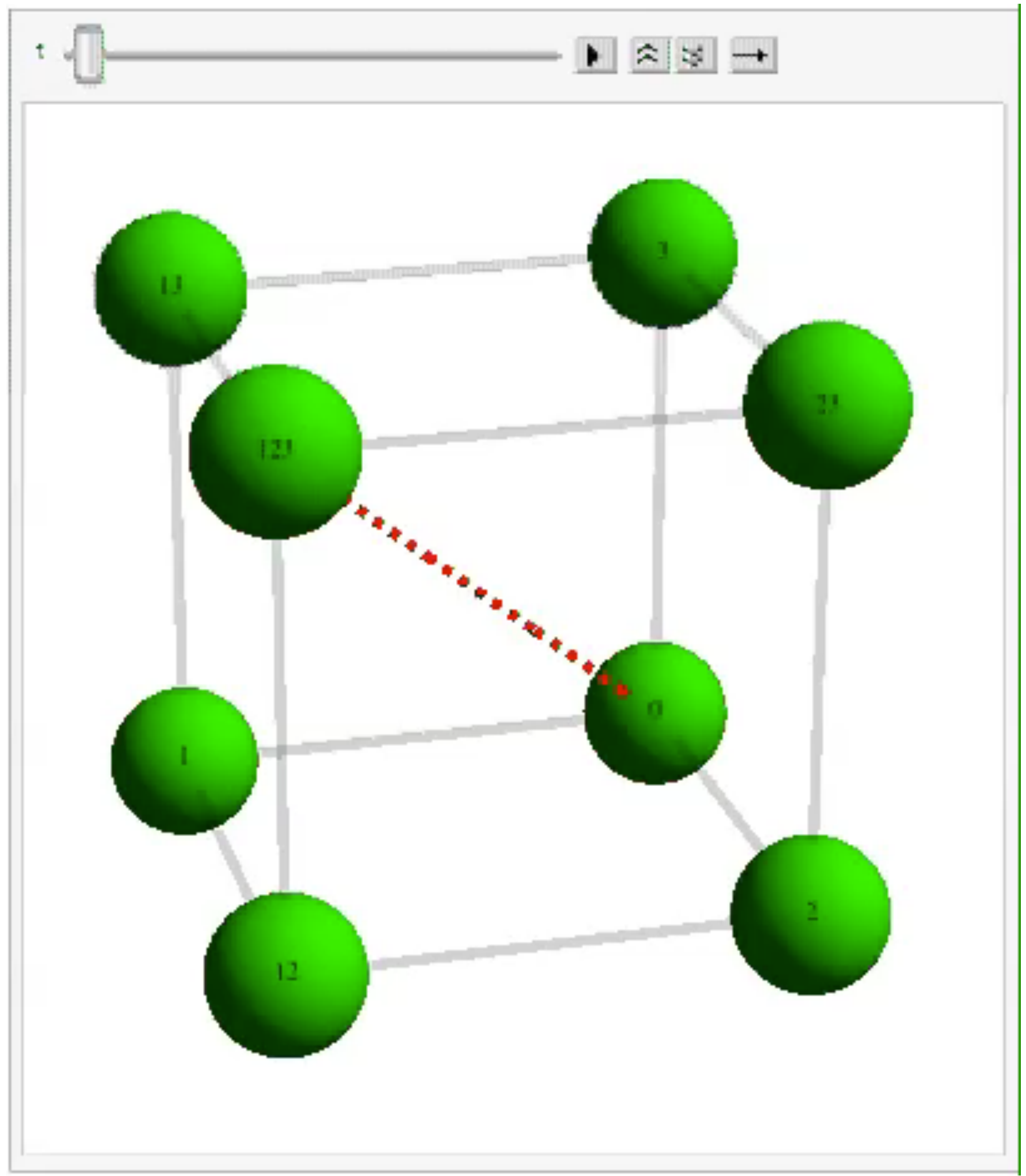
Each sub 3-cube in this 4-cube has 2 copies of H3 and 4 copies of H6 associated to its faces.

In 3D



Push one corner of the cube to the diagonally opposite corner
$\Rightarrow$ a hexagon

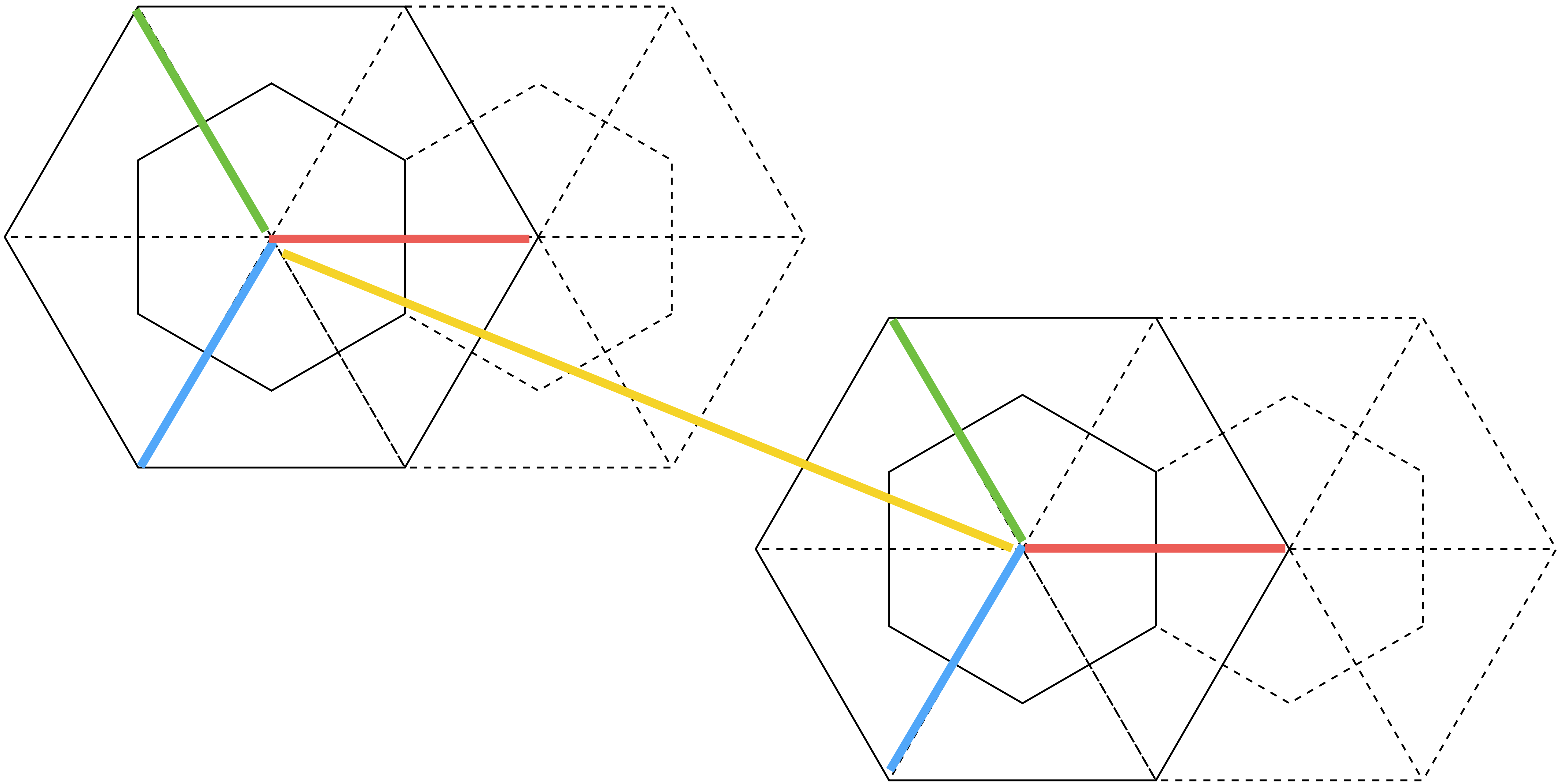

t
1
12
2



t
13
3
123
23
1
0
12
2

Reduction

$\widehat{\overline{\widetilde{w}}} = -i\,\lambda\,w \qquad \mathring{\lambda} = q\,\lambda$



Reductions

- The reductions have symmetry group $\widetilde{\mathcal{W}}\big((A_2+A_1)^{(1)}\big)$
- They are *q*-discrete Painlevé equations

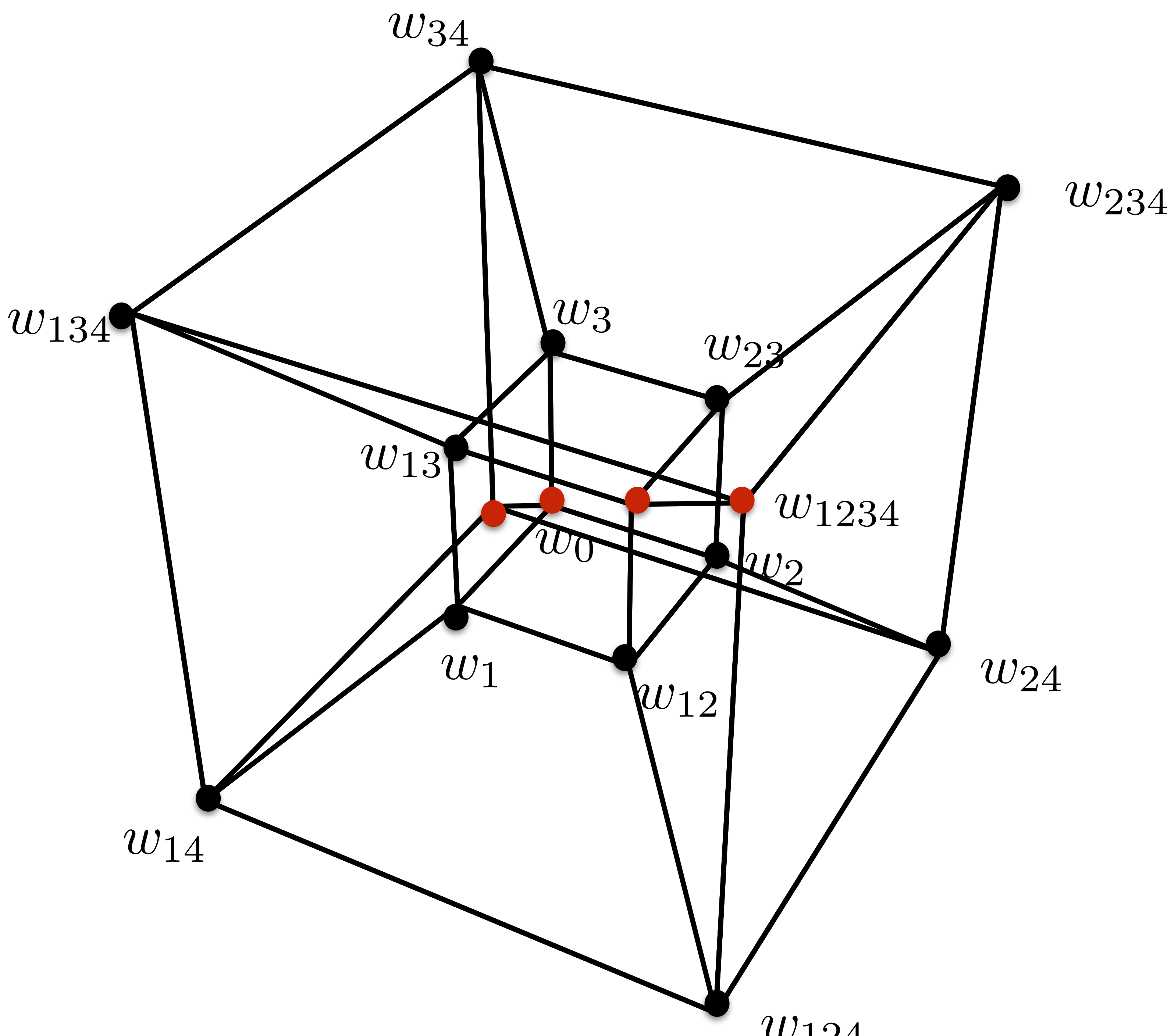
$$q\text{-P}_{\text{IV}}:\begin{cases} f(qt) = ab\,g(t)\,\dfrac{1+c\,h(t)\,(a\,f(t)+1)}{1+a\,f(t)\,(b\,g(t)+1)}, \\ g(qt) = bc\,h(t)\,\dfrac{1+a\,f(t)\,(b\,g(t)+1)}{1+b\,g(t)\,(c\,h(t)+1)}, \\ h(qt) = ca\,f(t)\,\dfrac{1+b\,g(t)\,(c\,h(t)+1)}{1+c\,h(t)\,(a\,f(t)+1)}, \end{cases}$$

$$q\text{-P}_{\text{III}}:\begin{cases} g(qt) = \dfrac{a}{g(t)f(t)}\,\dfrac{1+tf(t)}{t+f(t)}, \\ f(qt) = \dfrac{a}{f(t)g(qt)}\,\dfrac{1+btg(qt)}{bt+g(qt)}, \end{cases}$$

$$q\text{-P}_{\text{II}}:\quad f(pt) = \frac{a}{f(p^{-1}t)f(t)}\,\frac{1+tf(t)}{t+f(t)},$$

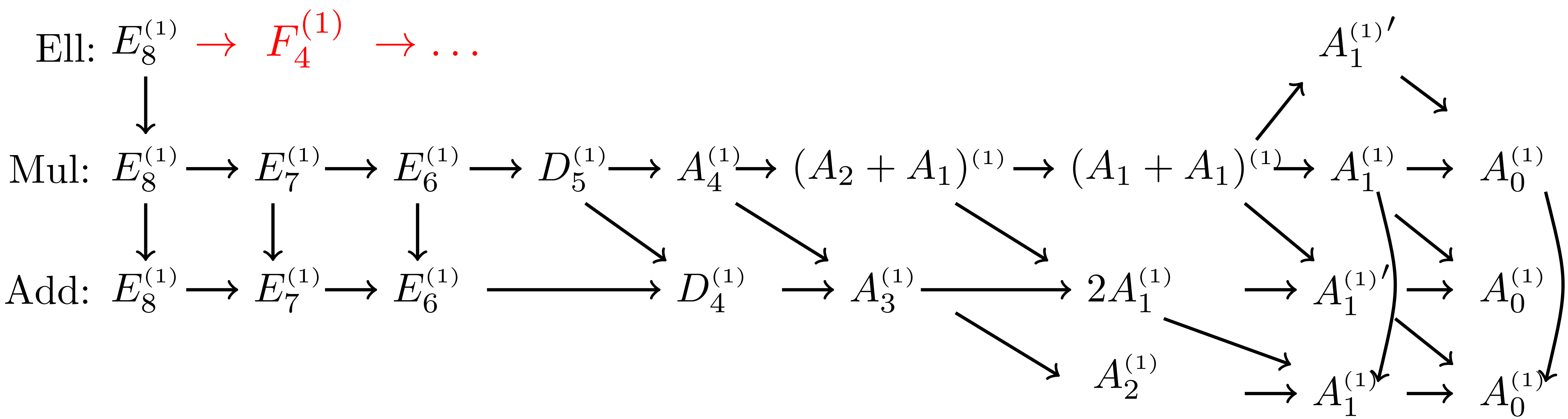
- Reductions also provide linear problems (Lax pairs)

Generalization



Generalizable to *n* dimensions $\Rightarrow \widetilde{\mathcal{W}}\big((A_{n-1} + A_1)^{(1)}\big)$

More Steps in Sakai's Description



Symmetry groups of Painlevé equations

Sakai 2001

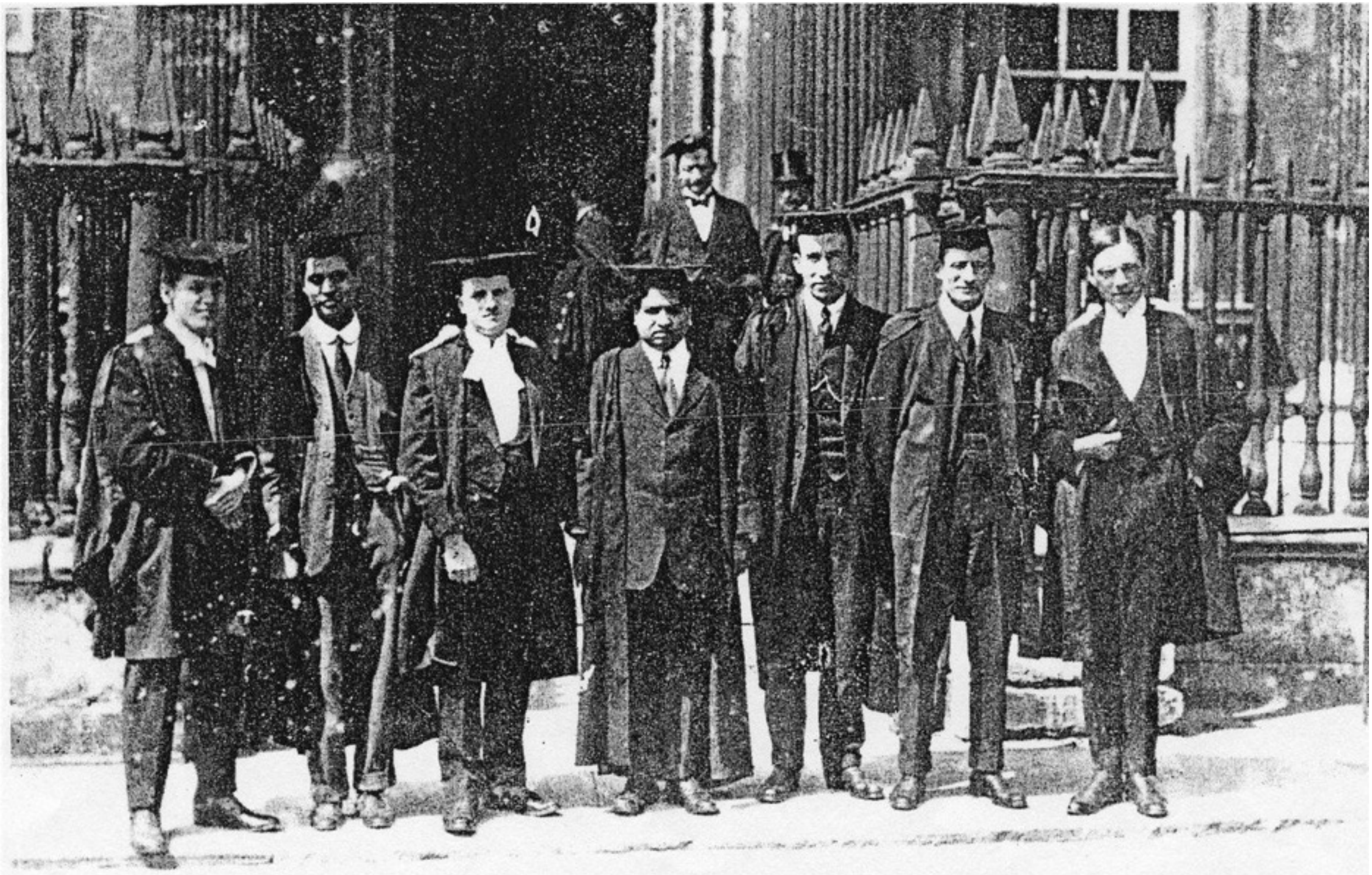
Atkinson, Howes, Joshi, Nakazono 2015

Summary

- Geometry provides a systematic method of finding reductions of partial difference equations.
- Reduction of the n-cube leads to *q*-discrete Painlevé equations of higher dimensions, with symmetry group $\widetilde{\mathcal{W}}\big((A_{n-1}+A_1)^{(1)}\big)$
- The symmetry lattice is realised as tessellations of the Voronoi cell of A_{n-1}.
- The lattice equations are found through ω-lattices, related to tau functions of discrete Painlevé equations.
- Other symmetry groups also arise.

Joshi, Nakazono & Shi 2014,2015

Atkinson, Howes, Joshi, Nakazono 2015



The mathematician's patterns, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. *GH Hardy, A Mathematician's Apology, 1940*