Symmetry through Geometry

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@monsoon0

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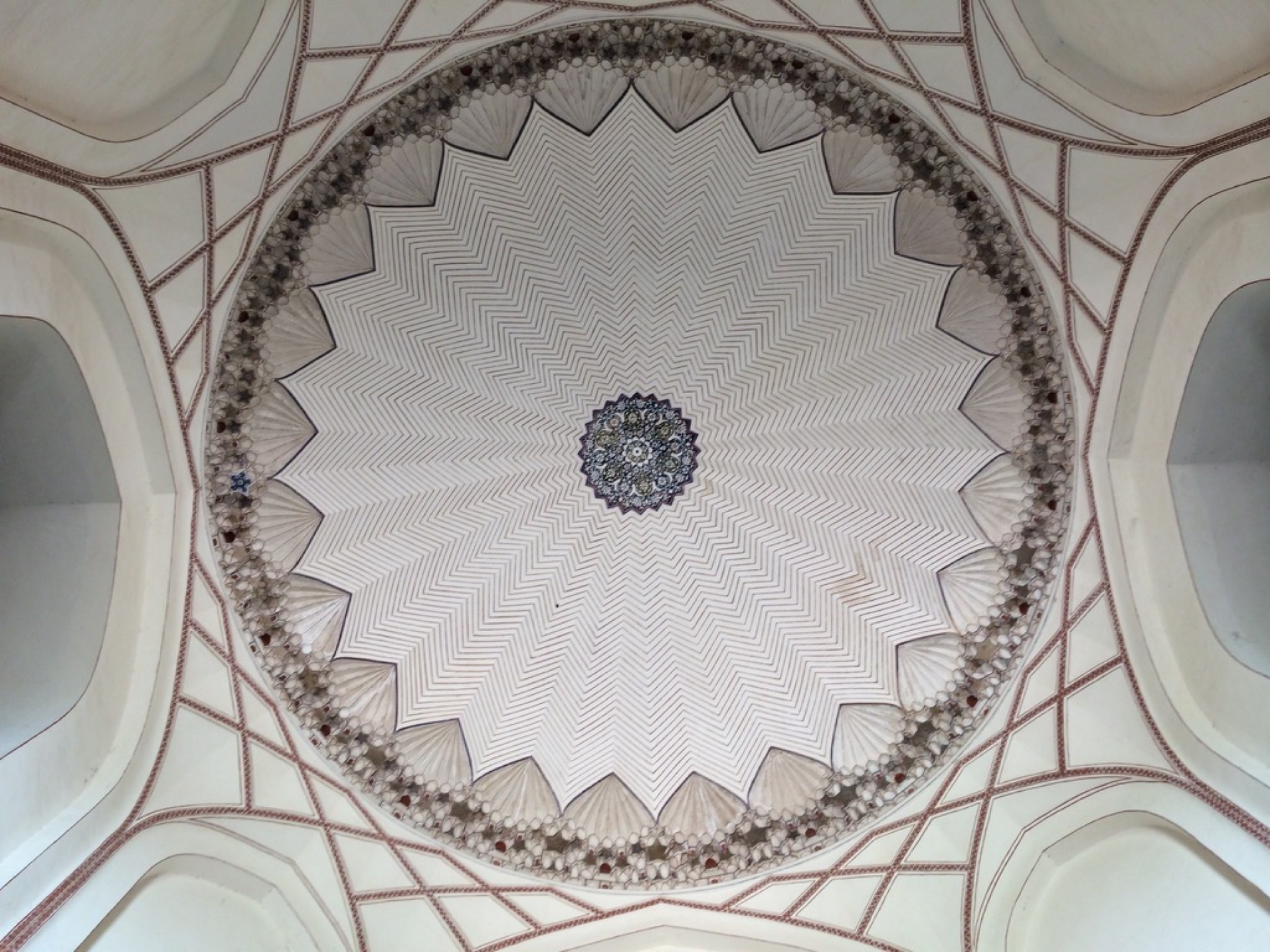
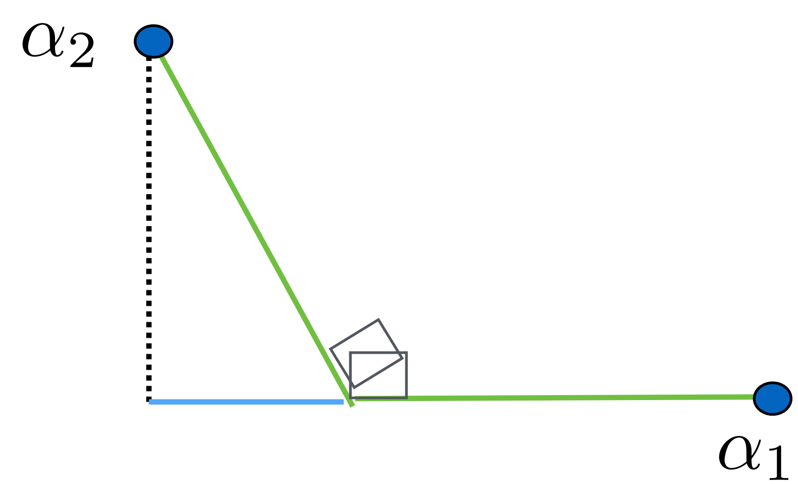
Belur, Karnataka, India



Humayun's Tomb, Delhi, India

A Reflection



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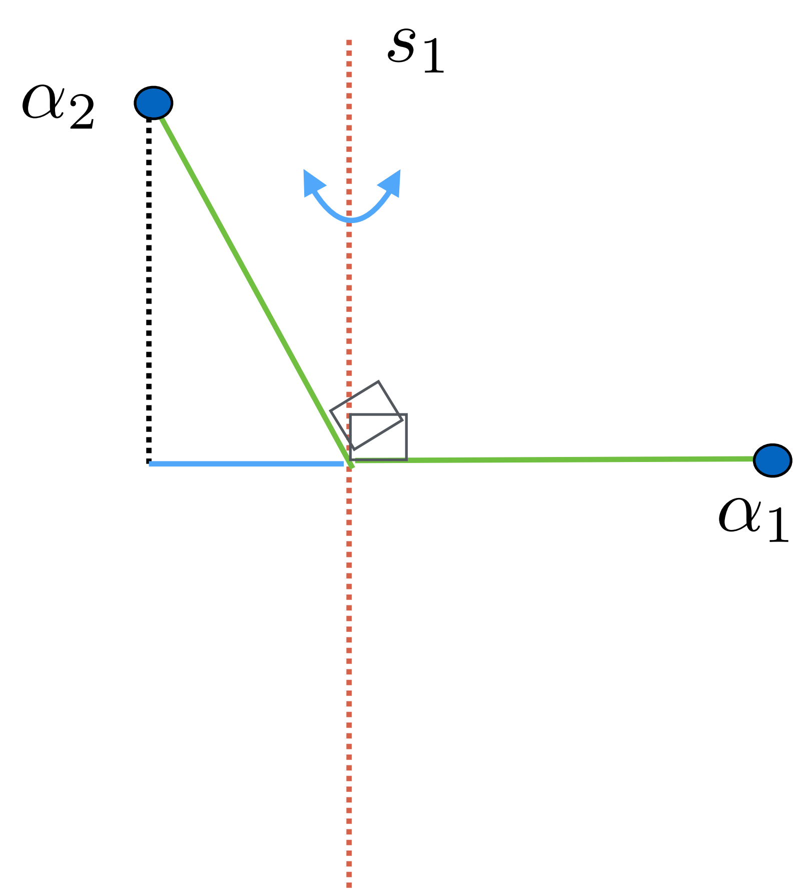

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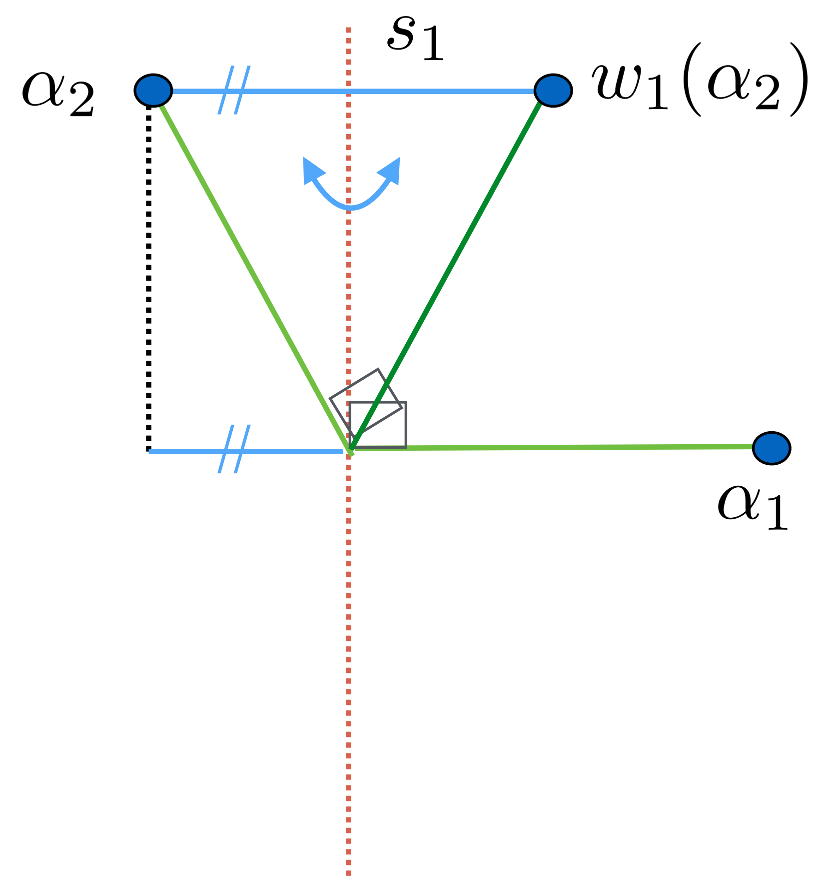
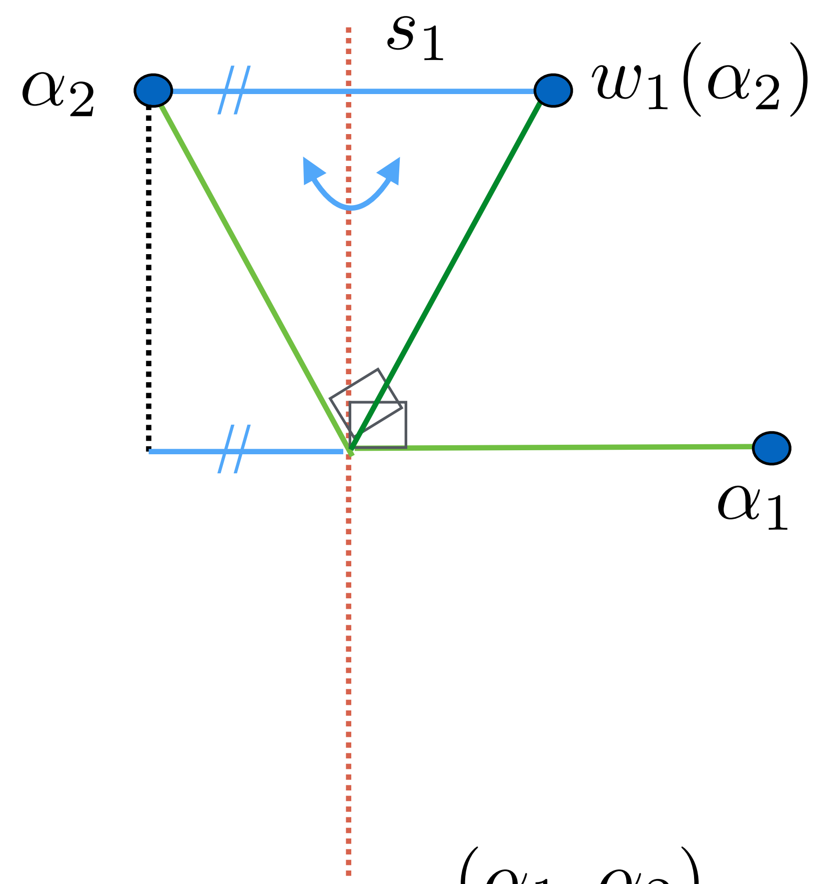


A Reflection

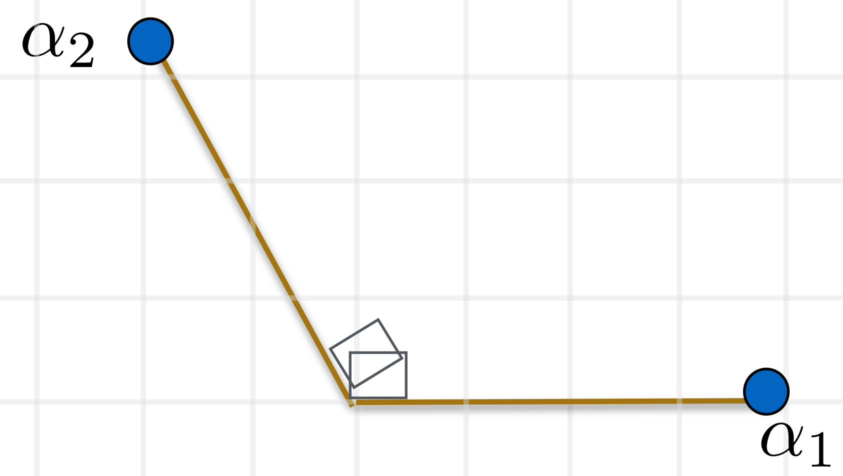


$$w_1(\alpha_2) = \alpha_2 - 2\,\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\,\alpha_1$$

$$= (-1, \sqrt{3}) + (2, 0)$$

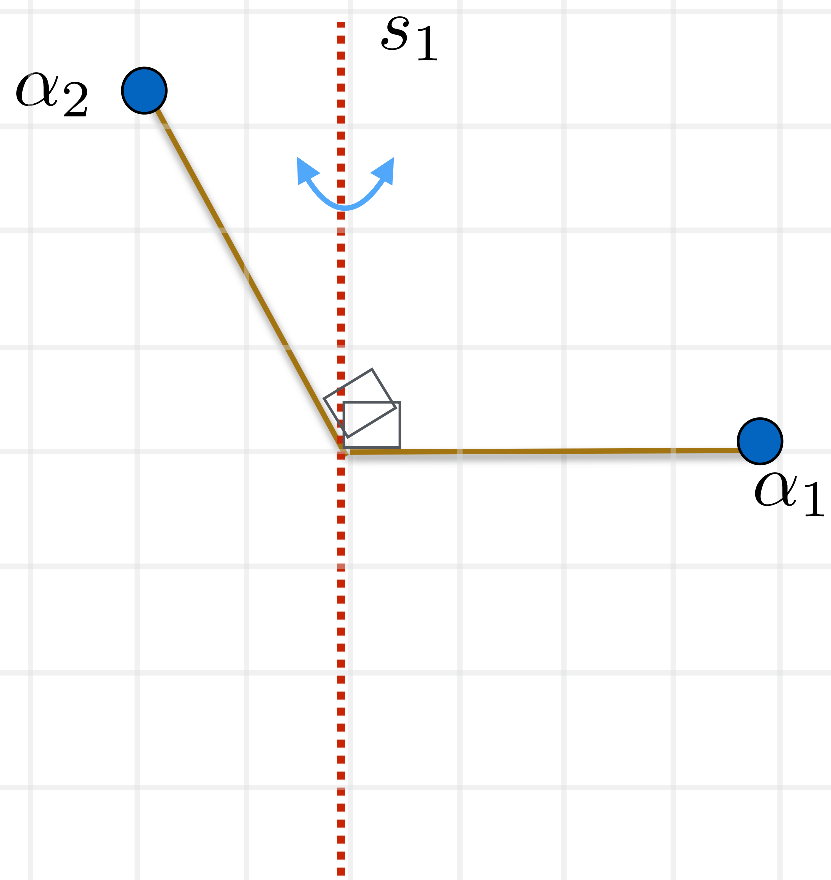
$$= (1, \sqrt{3})$$

Root System



α_1 and α_2 are “simple” roots

Root System



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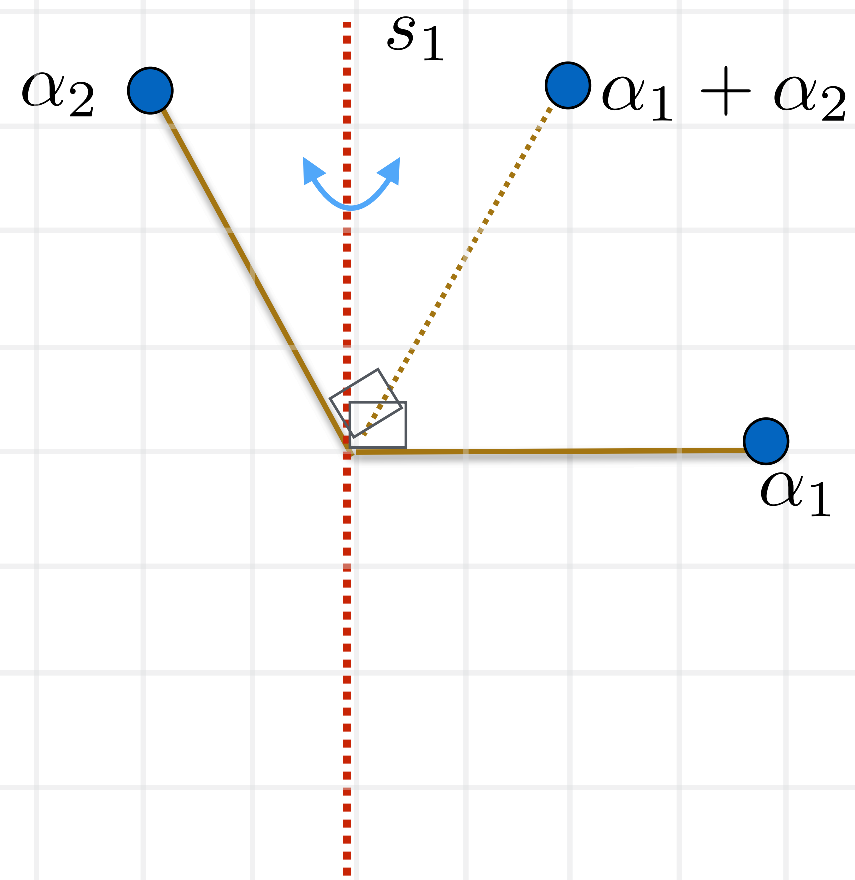

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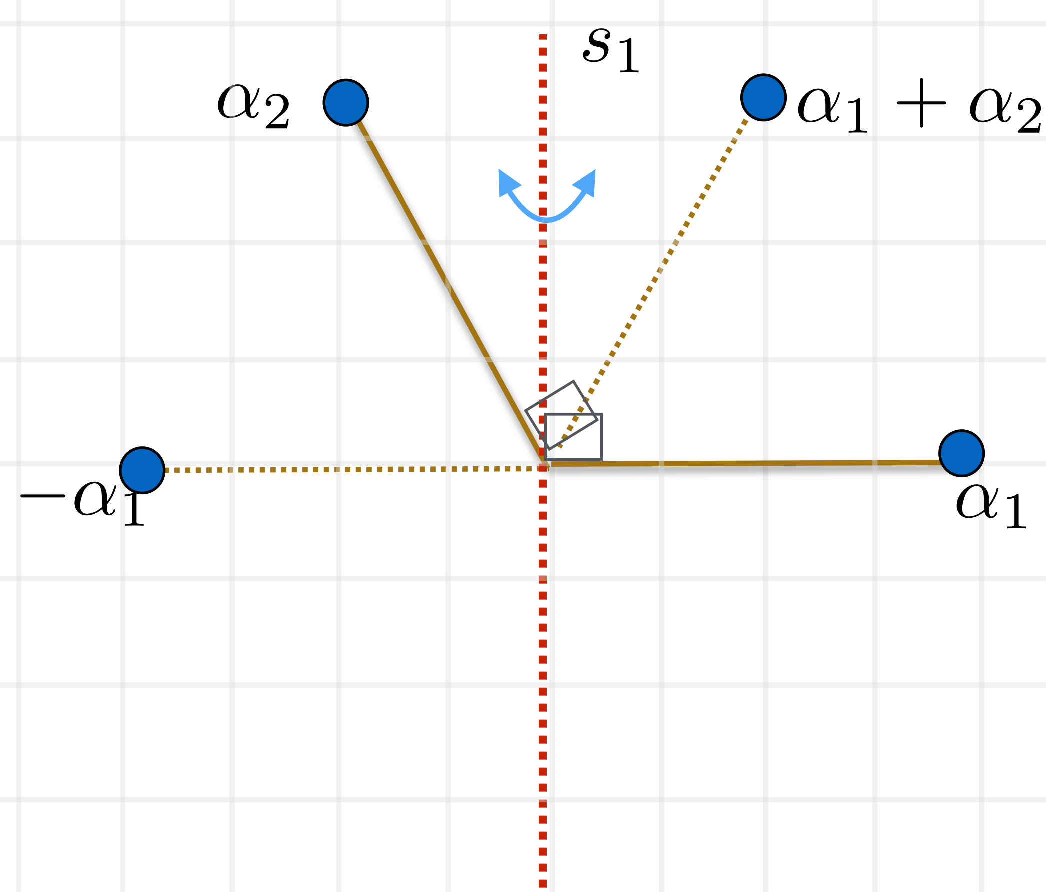

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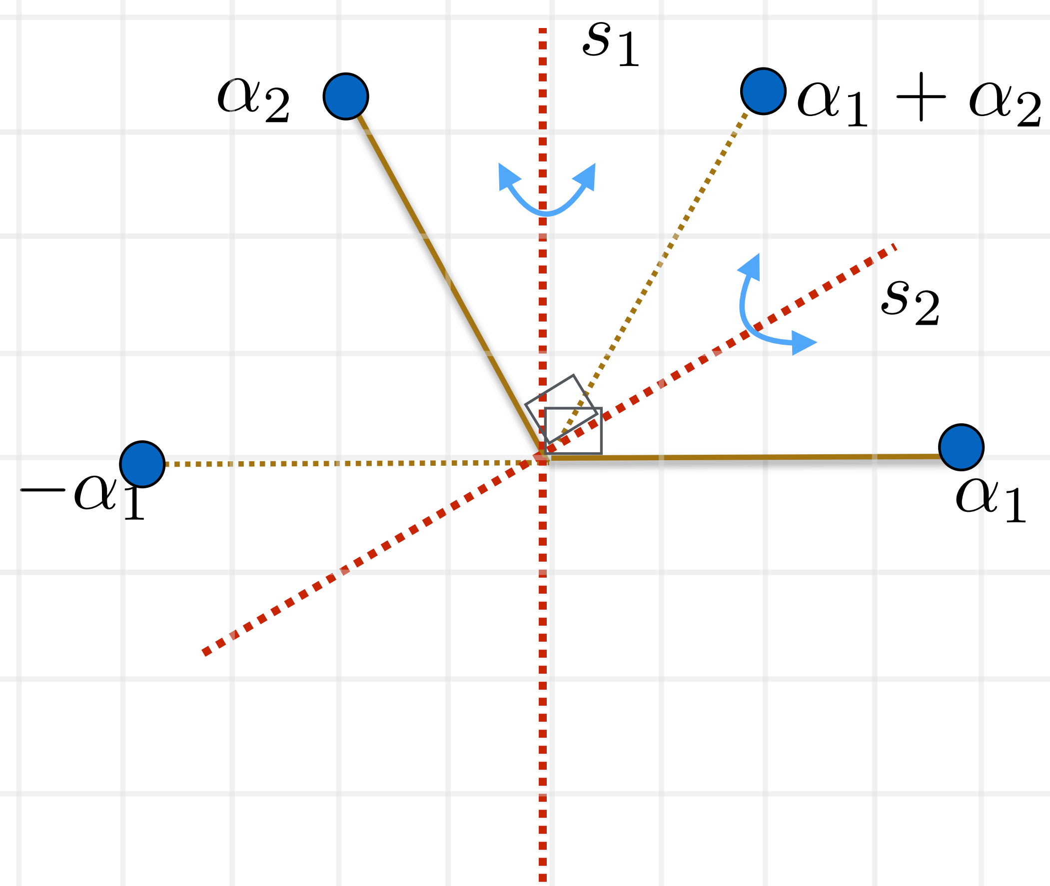

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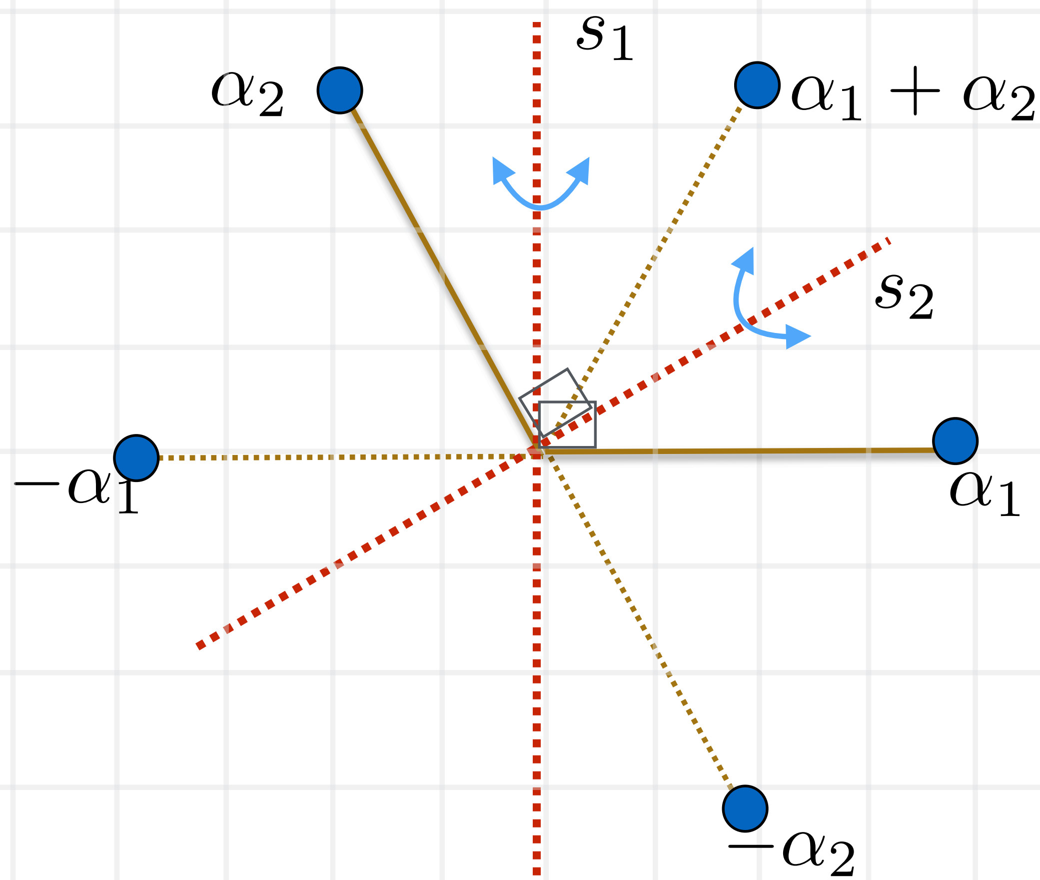

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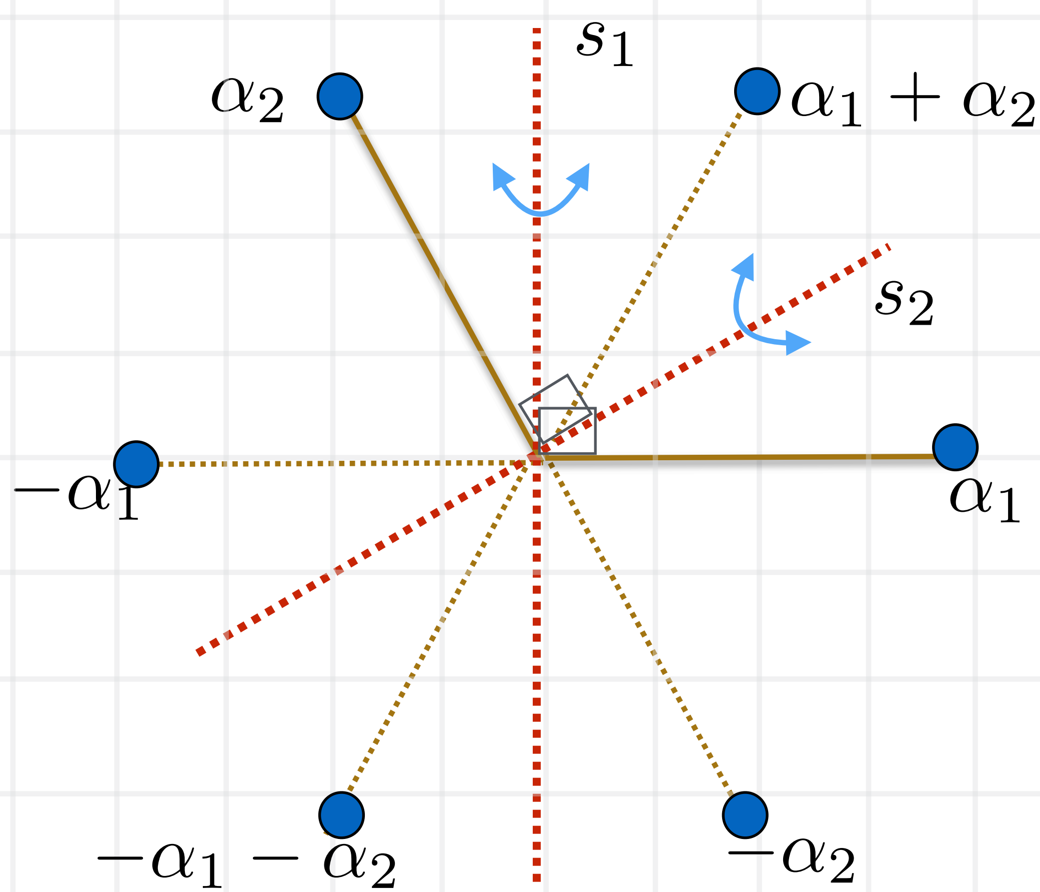
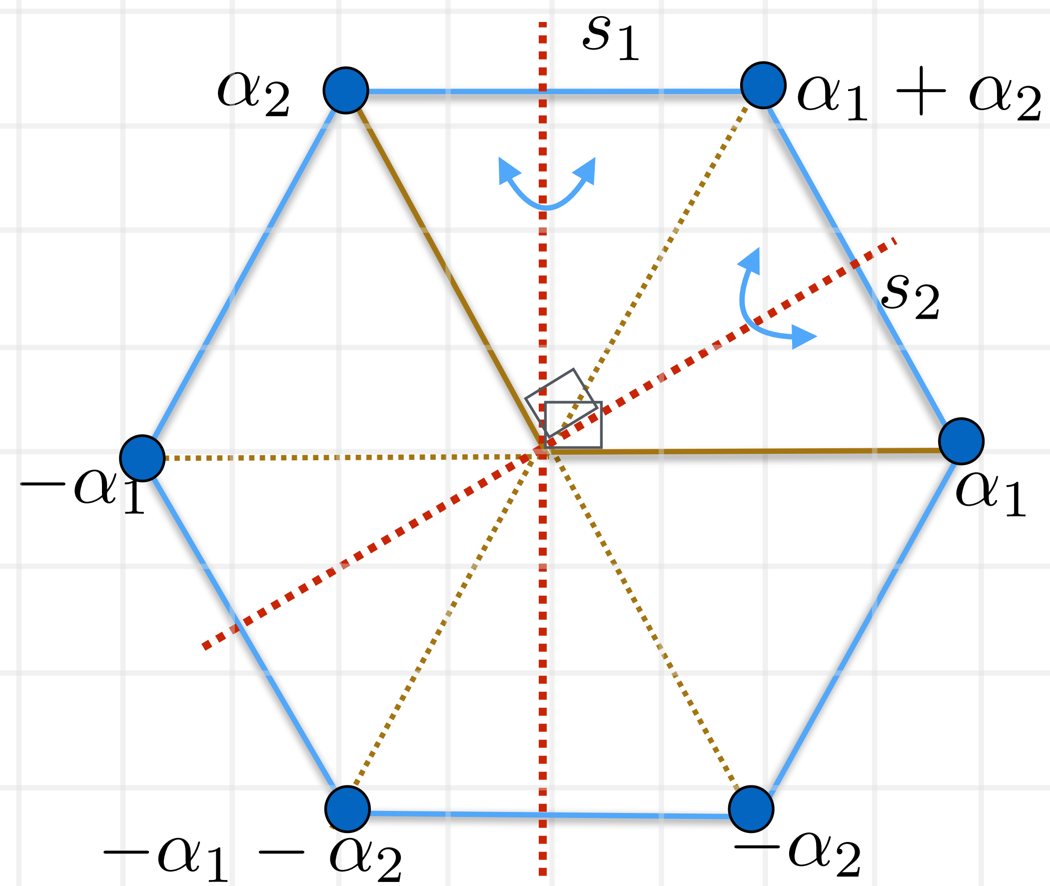


α_1 and α_2 are “simple” roots

Root System



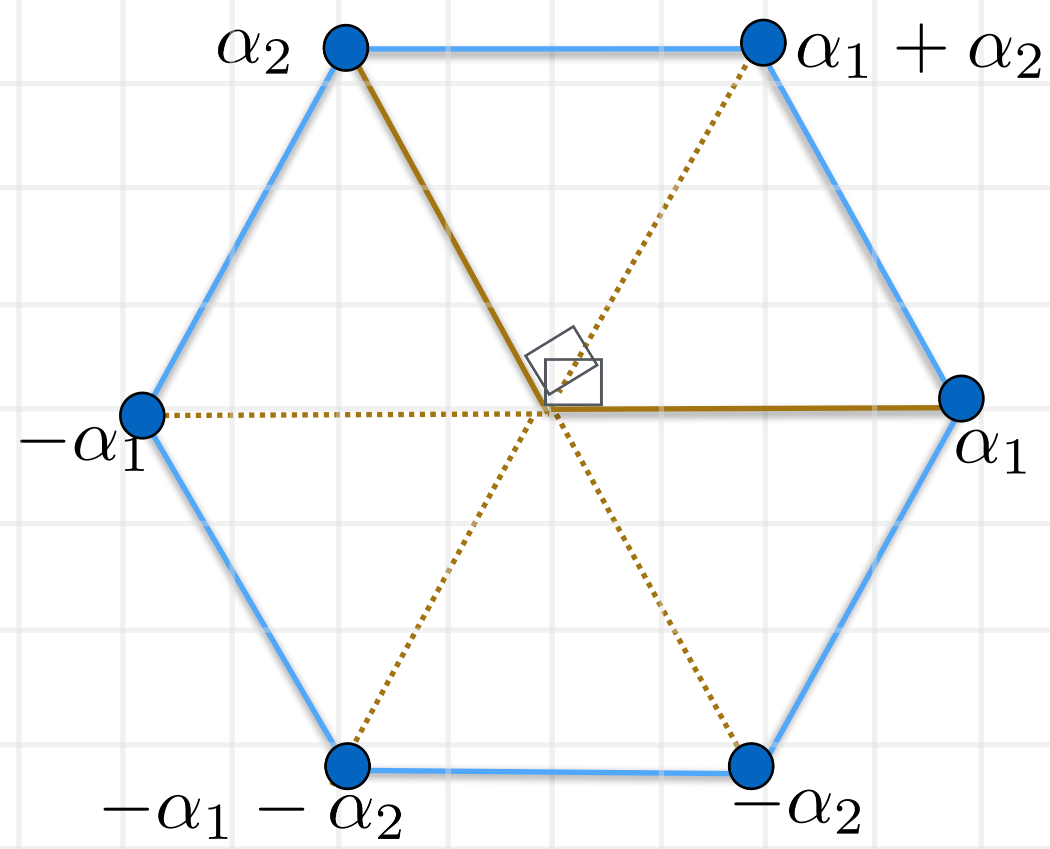
α_1 and α_2 are “simple” roots

Reflection Groups

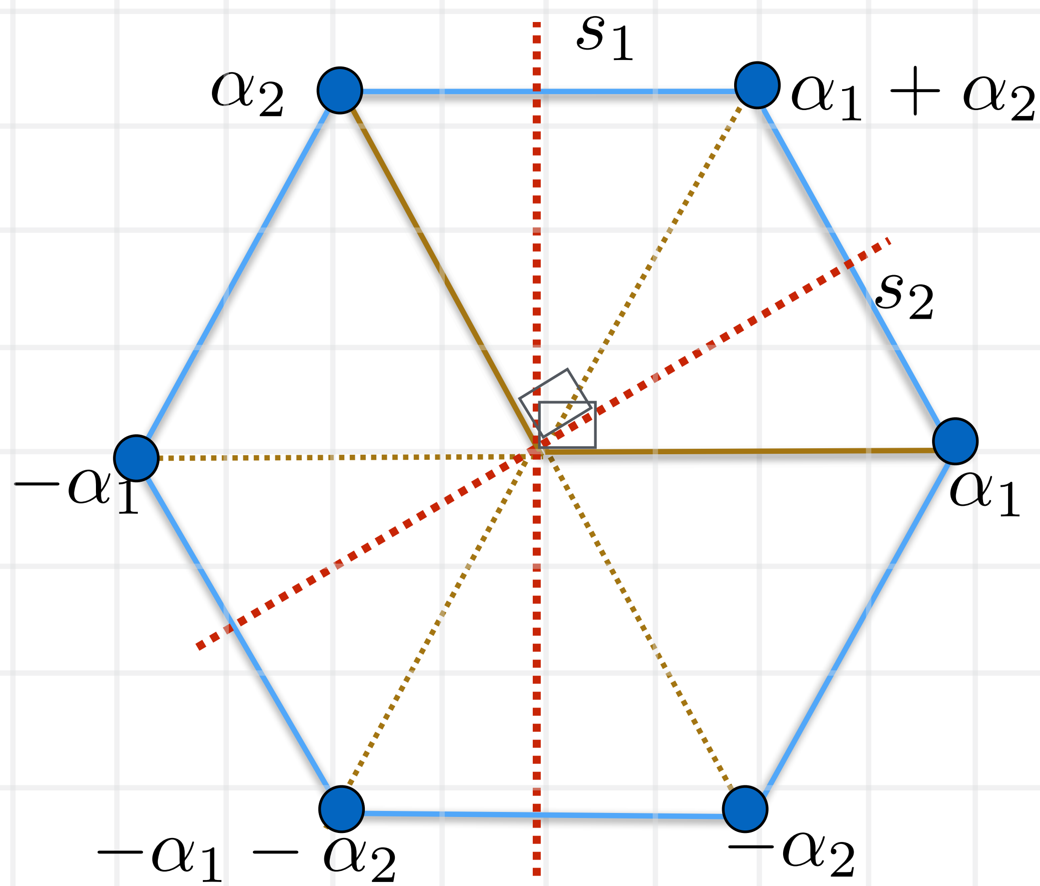
- Roots: $\alpha_1, \alpha_2, \ldots, \alpha_n$
- Reflections: $w_i(\alpha_j) = \alpha_j - \; 2 \, \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \, \alpha_i$
- Co-roots: $\check{\alpha}_i = \; 2 \, \frac{\alpha_i}{(\alpha_i, \alpha_i)}$
- Weights: $h_1, h_2, \ldots, h_n$

$$(h_i, \check{\alpha}_i) = \delta_{ij}$$

A_2

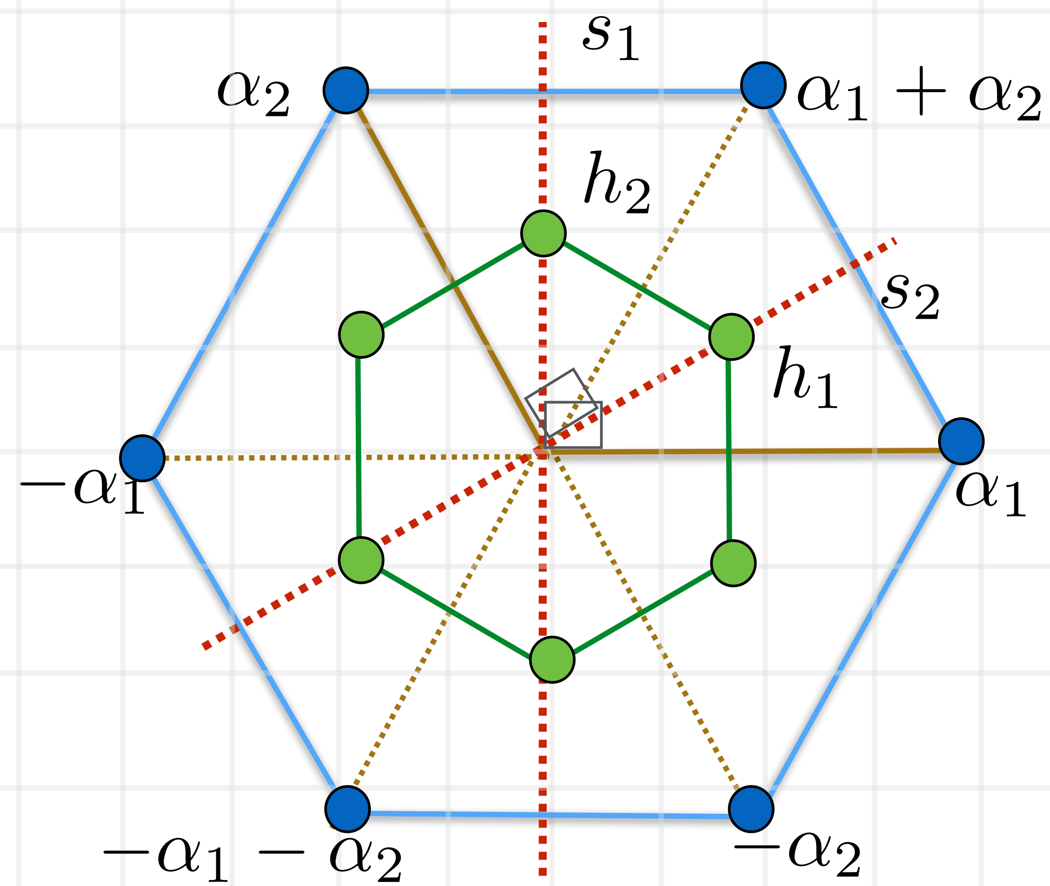


A_2

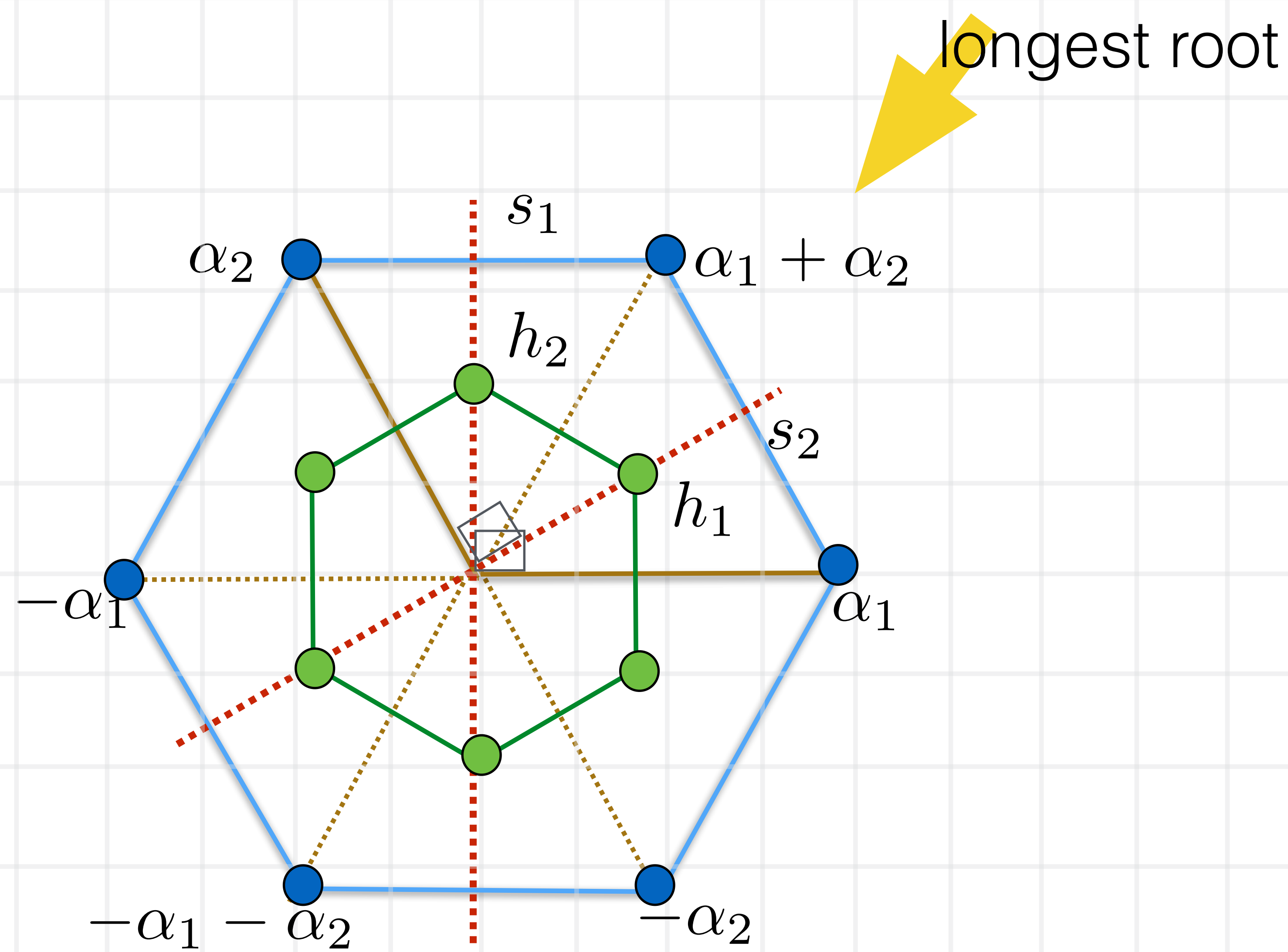


s_1
α_2
$\alpha_1 + \alpha_2$
s_2
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A2

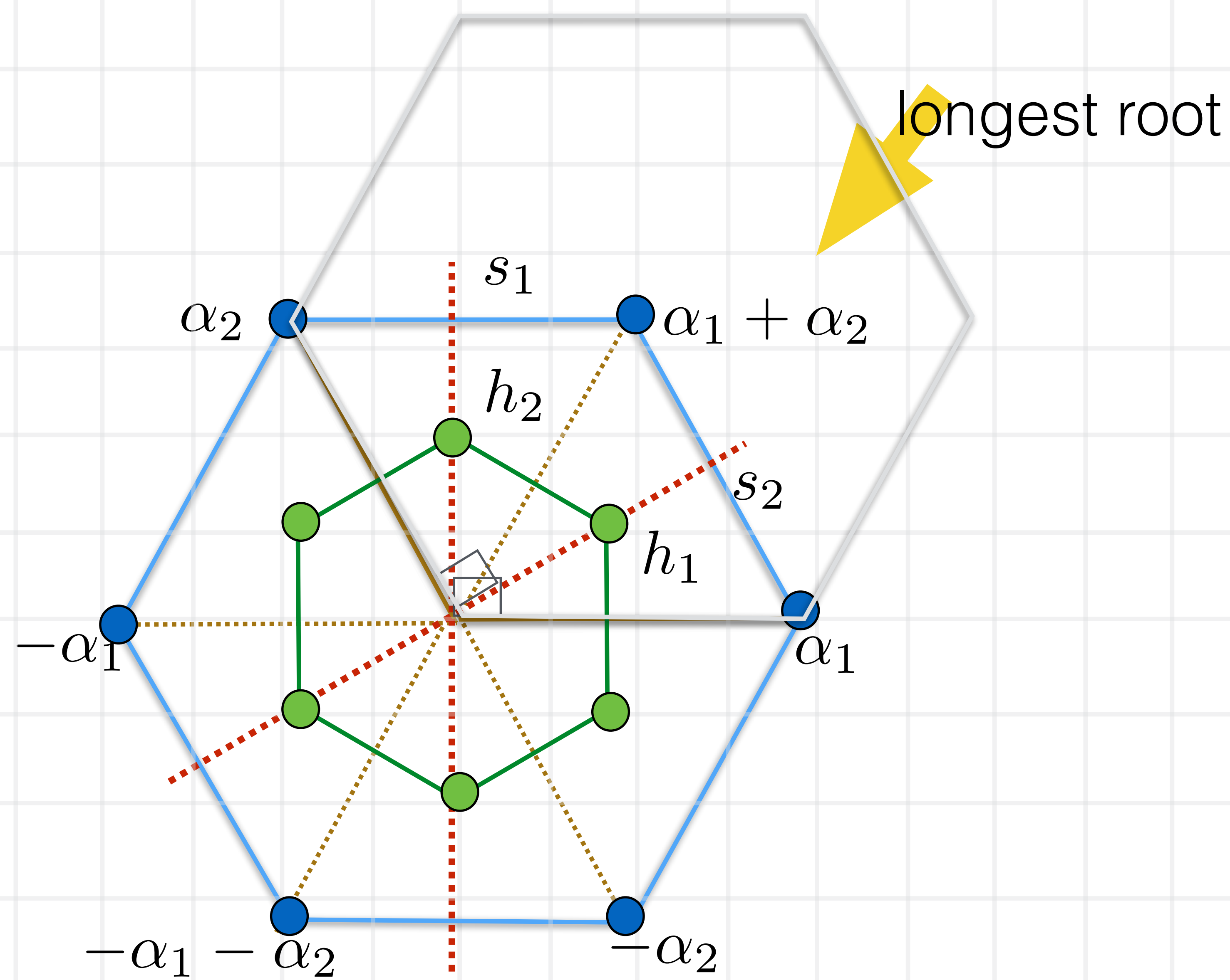

s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2



longest root
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

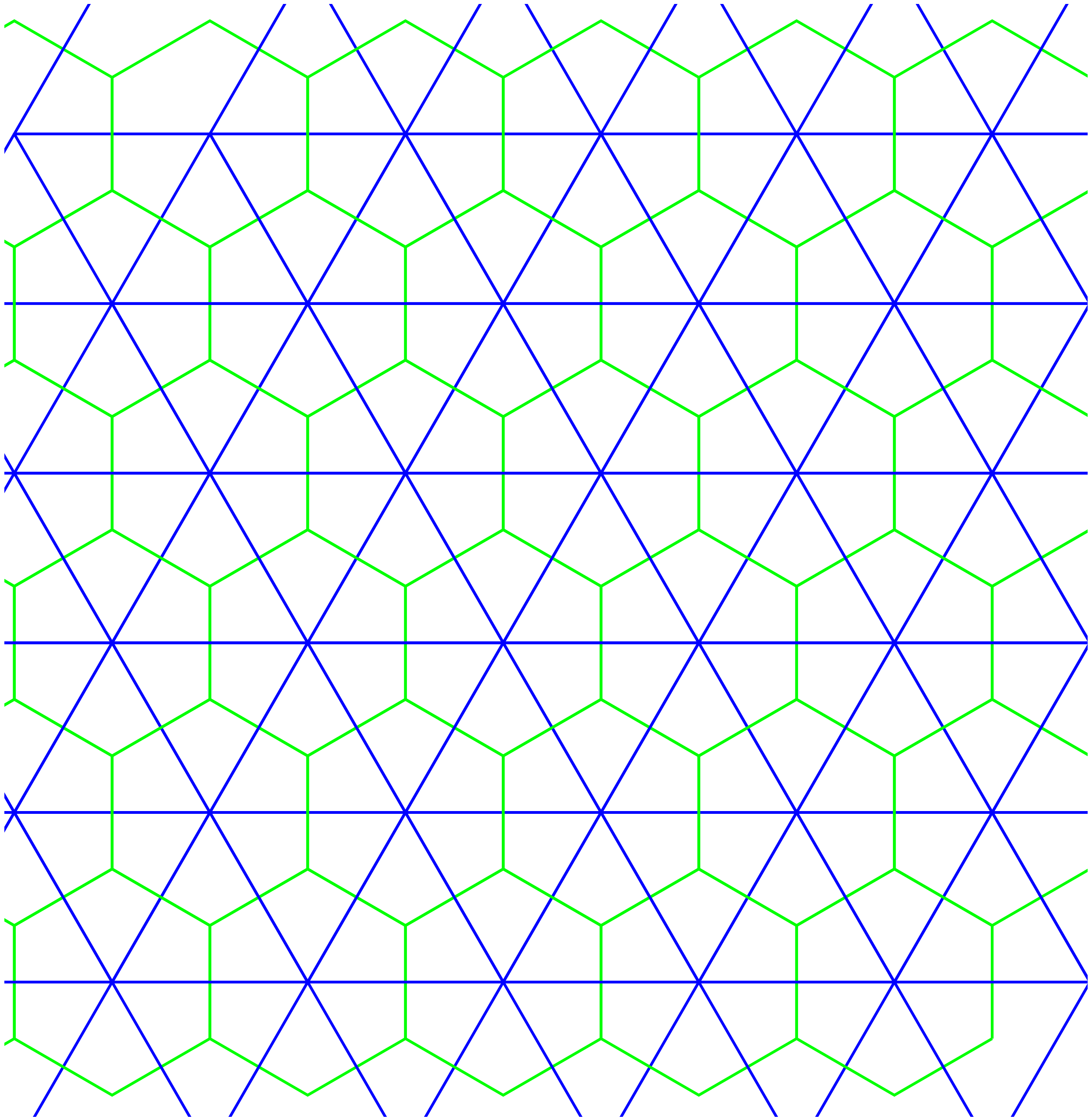
A$_2$



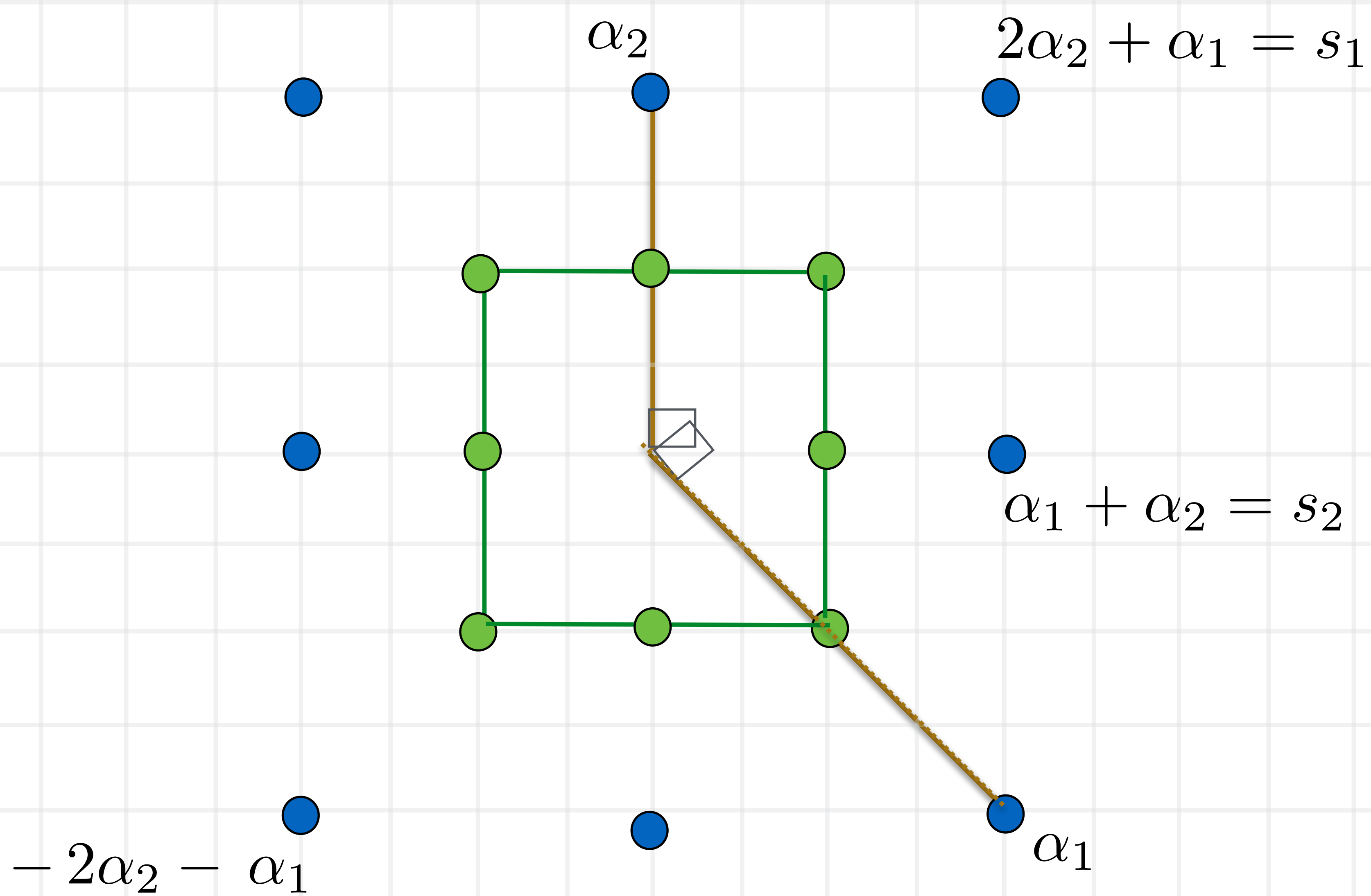
longest root
s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

Translation by longest root

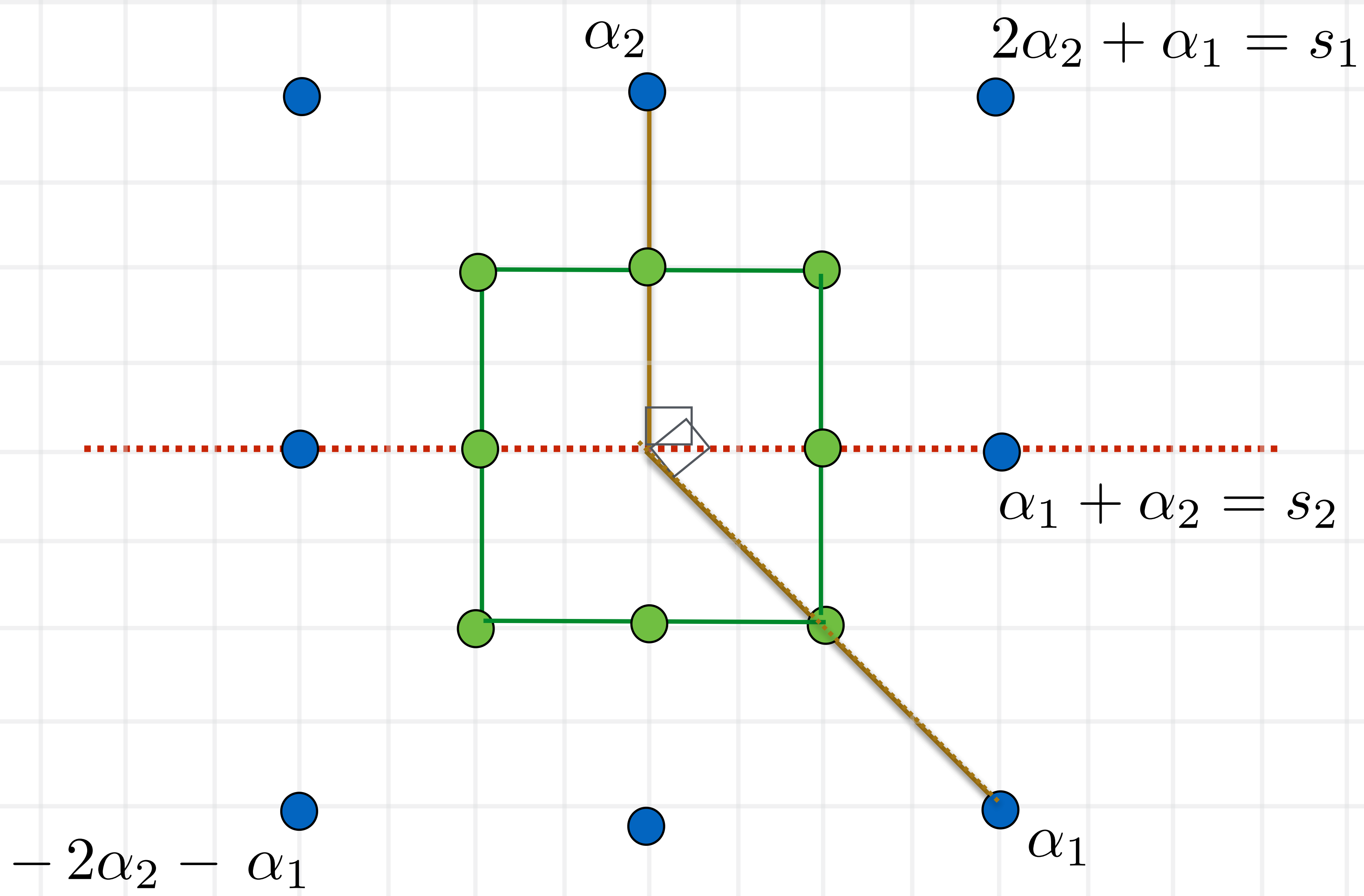
$A_2^{(1)}$



B_2


α_2
$2\alpha_2 + \alpha_1 = s_1$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B2



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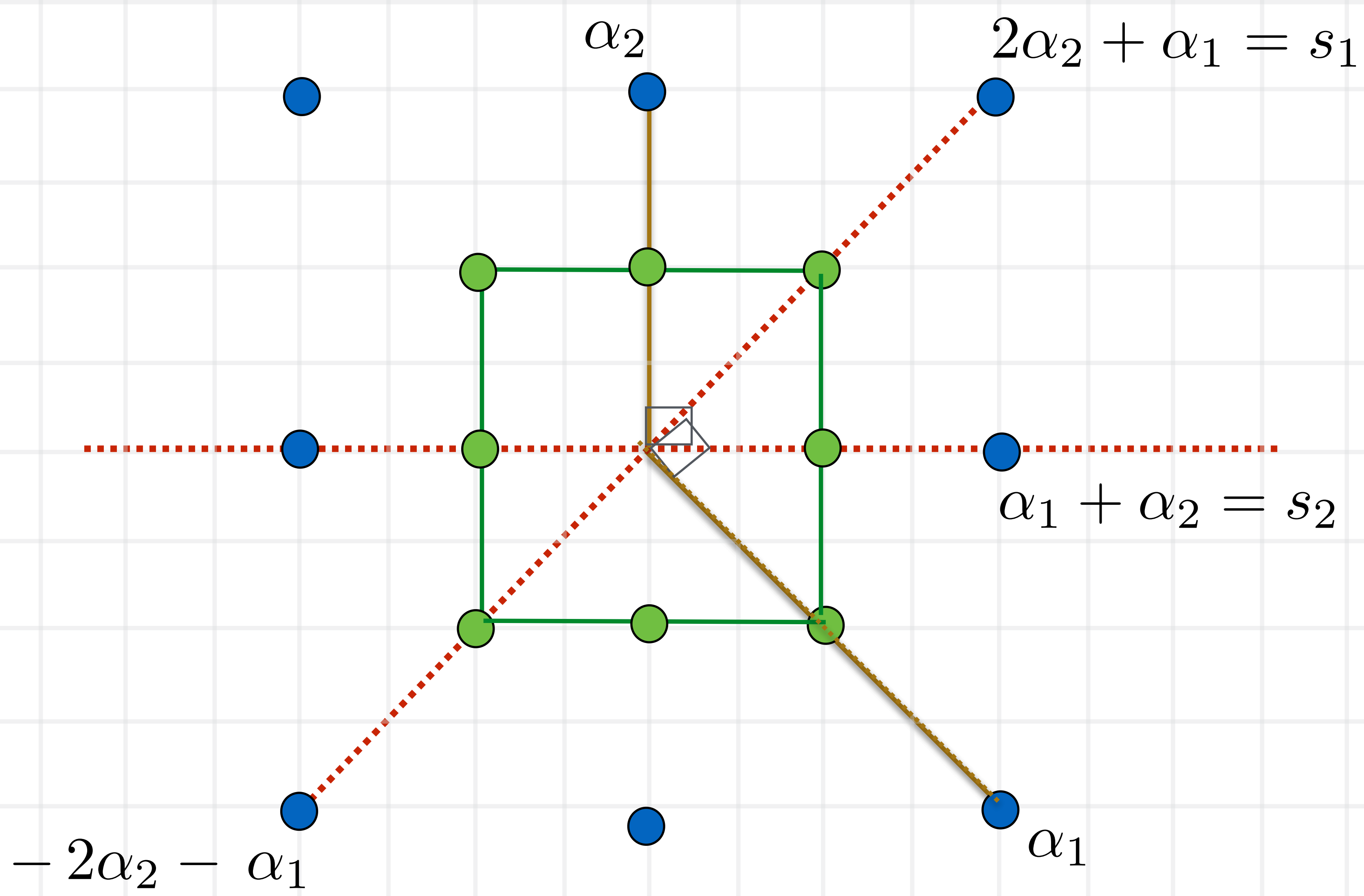
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
α_1

B_2



α_2
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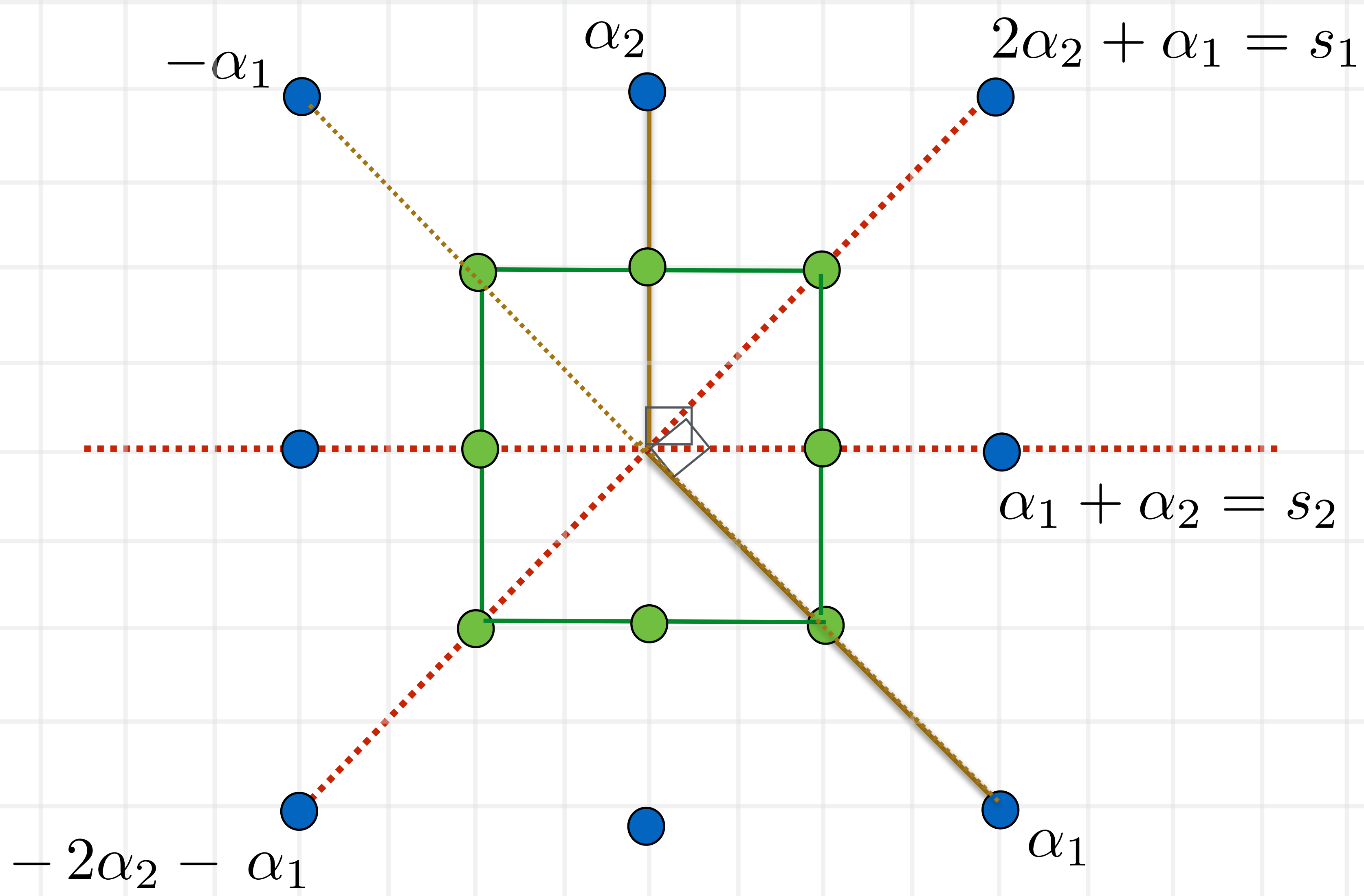
α_1

B2


α_2
$2\alpha_2 + \alpha_1 = s_1$
$-\alpha_1$
$\alpha_1 + \alpha_2 = s_2$
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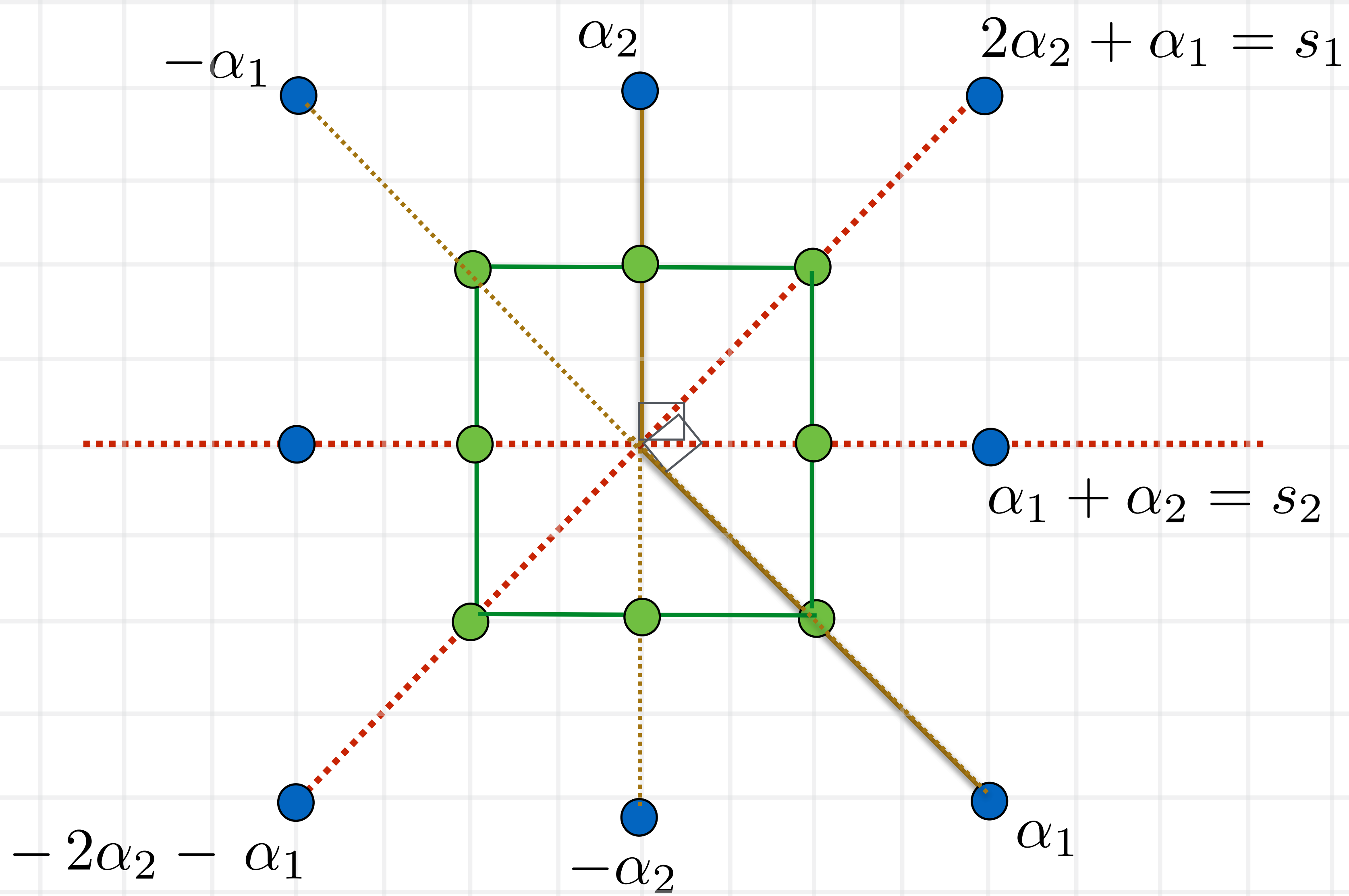
α_1

B_2


α_2
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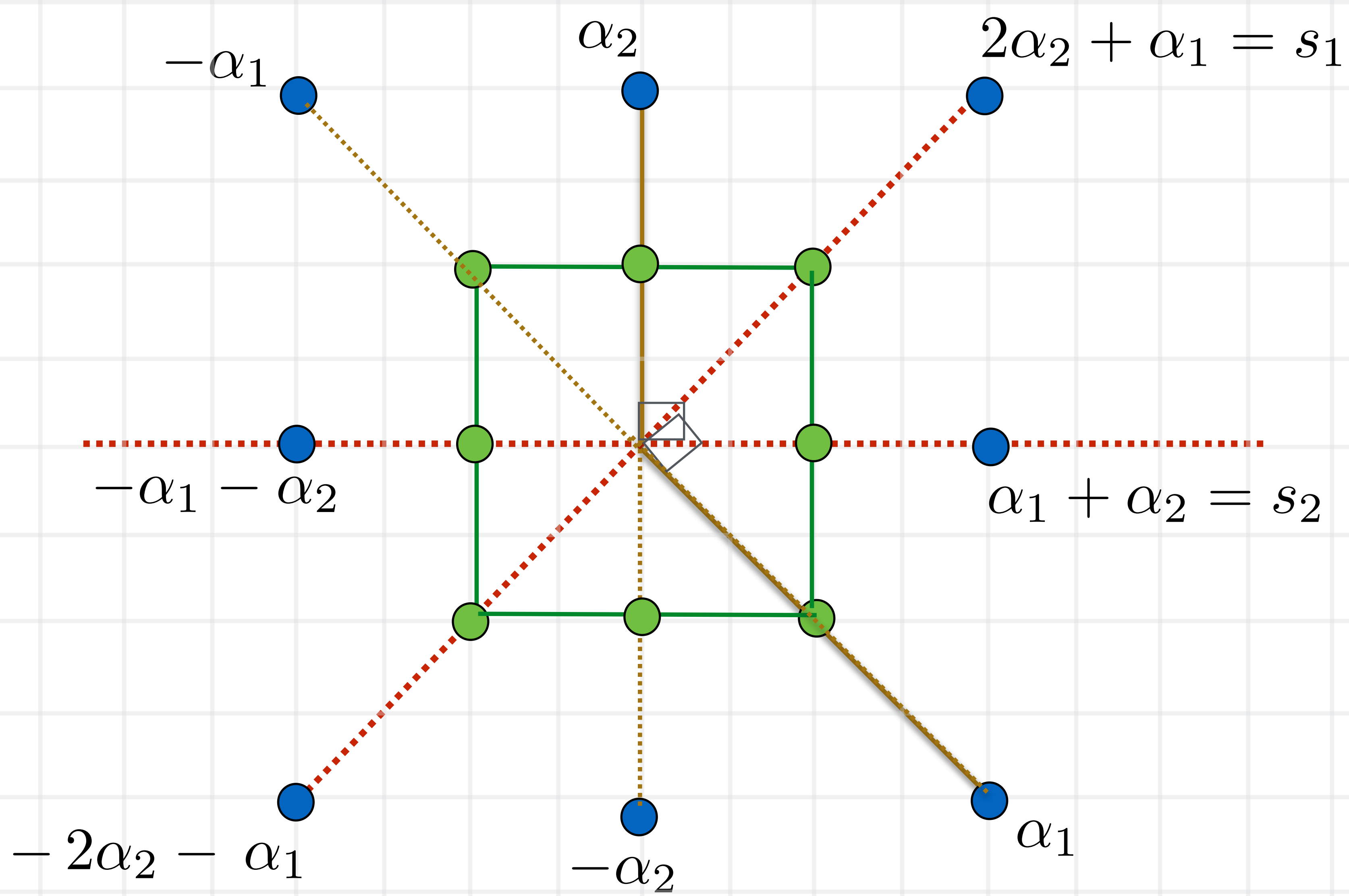
B2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α_2
$2\alpha_2 + \alpha_1 = s_1$
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$\alpha_1 + \alpha_2 = s_2$
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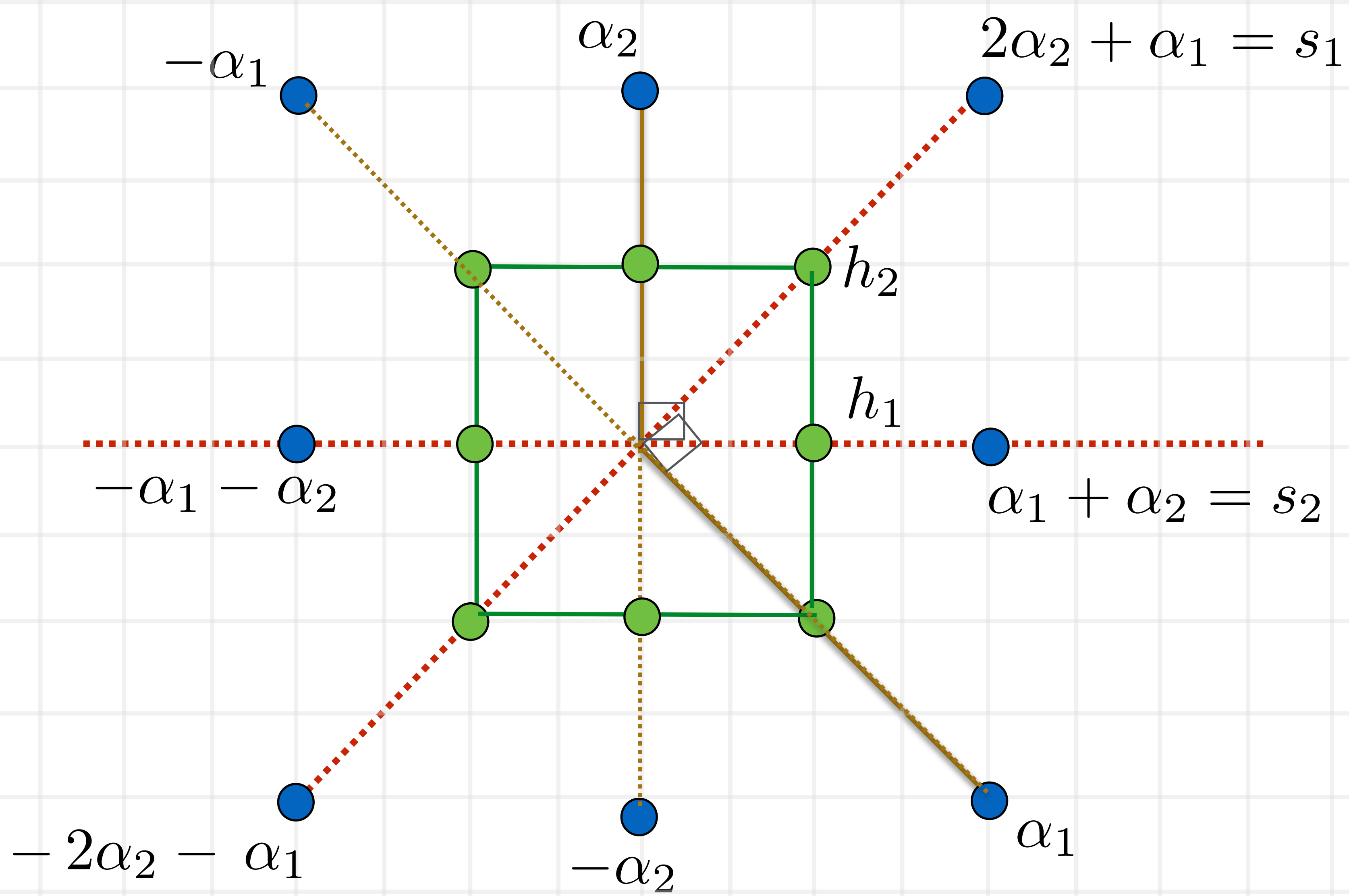
α_1

B$_2$



$-\alpha_1$
α_2
$2\alpha_2 + \alpha_1 = s_1$
h_2
h_1
$-\alpha_1 - \alpha_2$
$\alpha_1 + \alpha_2 = s_2$
$-2\alpha_2 - \alpha_1$
$-\alpha_2$
α_1

Crystallographic Property

$$(\alpha_i, \check{\alpha}_j) \in \mathbb{Z}$$

$$\Rightarrow \quad (\alpha_i, \check{\alpha}_j)(\check{\alpha}_i, \alpha_j) = 4\cos^2(\theta_{\alpha_i \alpha_j}) \in \mathbb{N}$$

$$\Rightarrow \quad \cos(\theta_{\alpha_i \alpha_j}) = 0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}, \pm 1$$

$$\Rightarrow \quad \theta_{\alpha_i \alpha_j} = \pi - \theta_{s_i s_j} = \pi - \ \frac{\pi}{m_{ij}}$$

$$\Rightarrow \quad m_{ij} = 2, 3, 4, 6$$

Part I

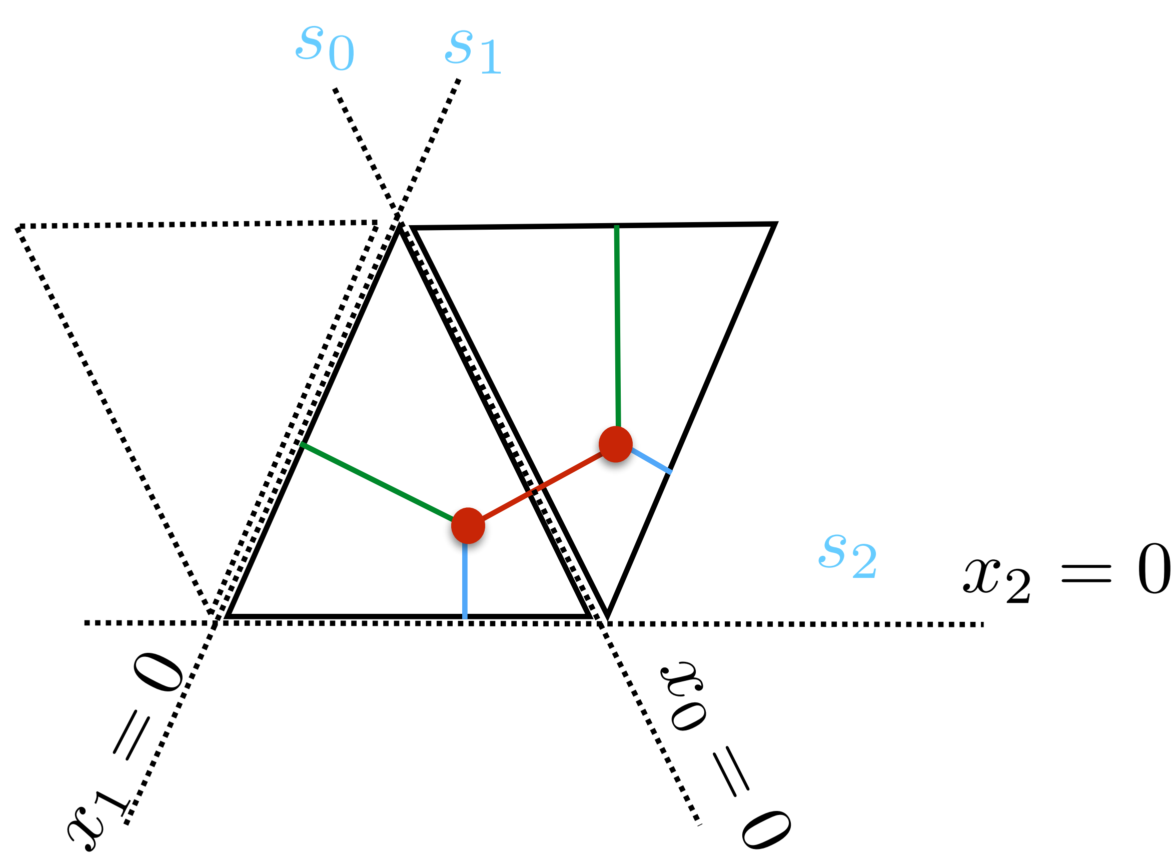
- Lattices
- Dynamics on *N*-cubes
- Symmetry reductions

On the Lattice

$$\widetilde{\mathcal{W}}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$$

$$s_j^2 = 1, \quad (s_j s_{j+1})^3 = 1, \quad (j = 0, 1, 2)$$

$$\pi^3 = 1, \quad \pi s_j = s_{j+1}\pi$$



$$s_0(x_0, x_1, x_2) = (-x_0, x_1 + x_0, x_2 + x_0)$$

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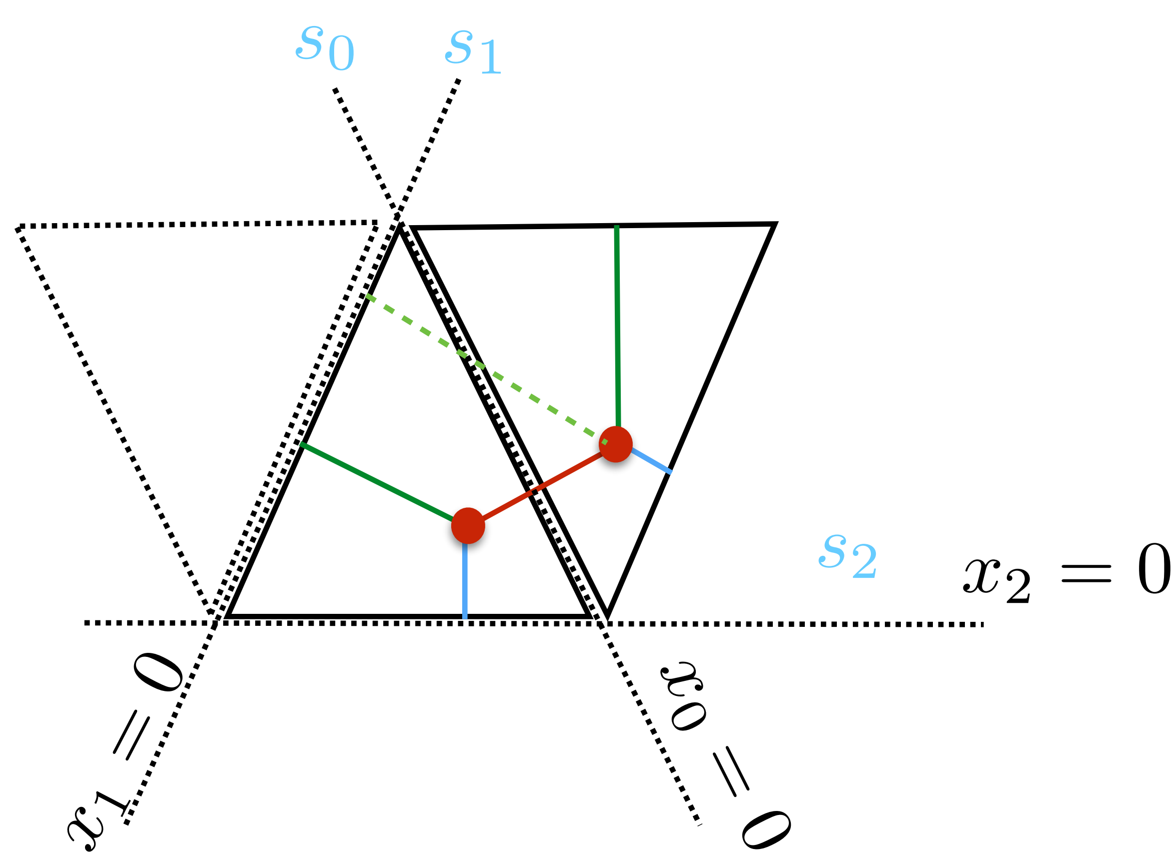

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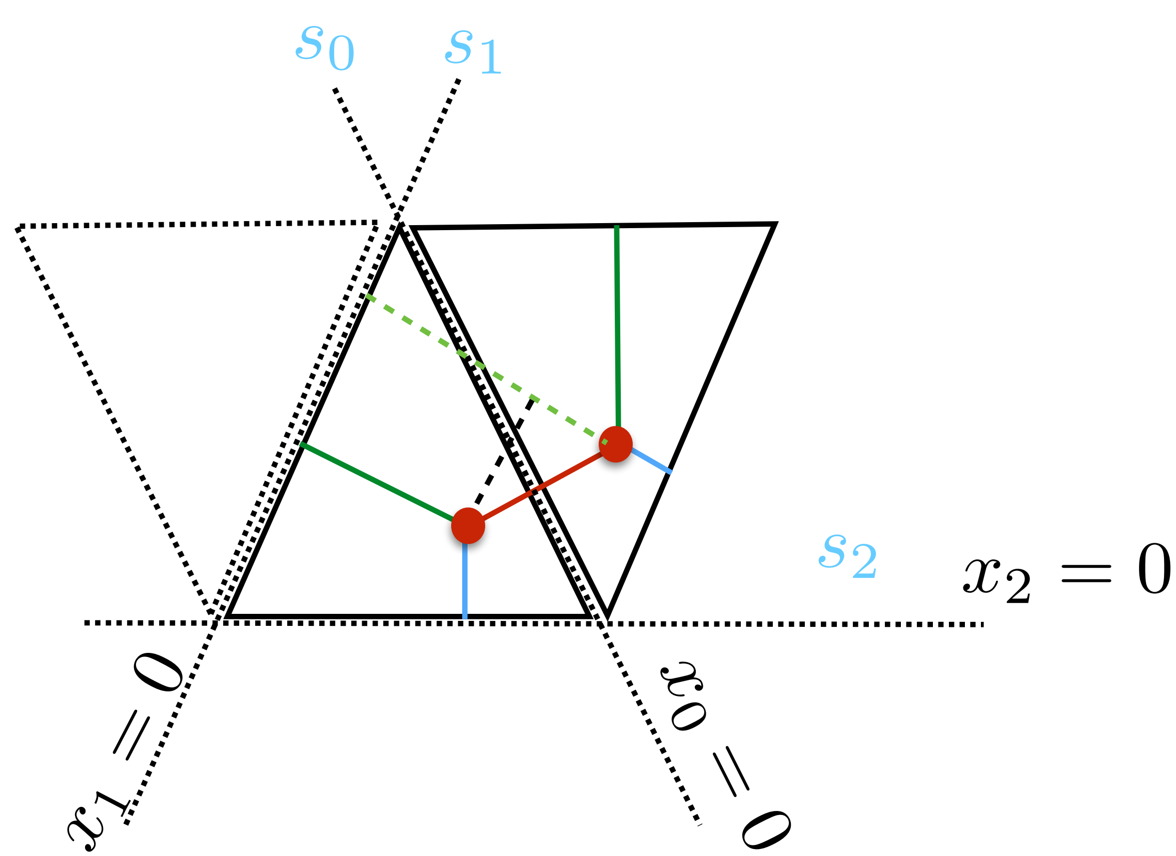

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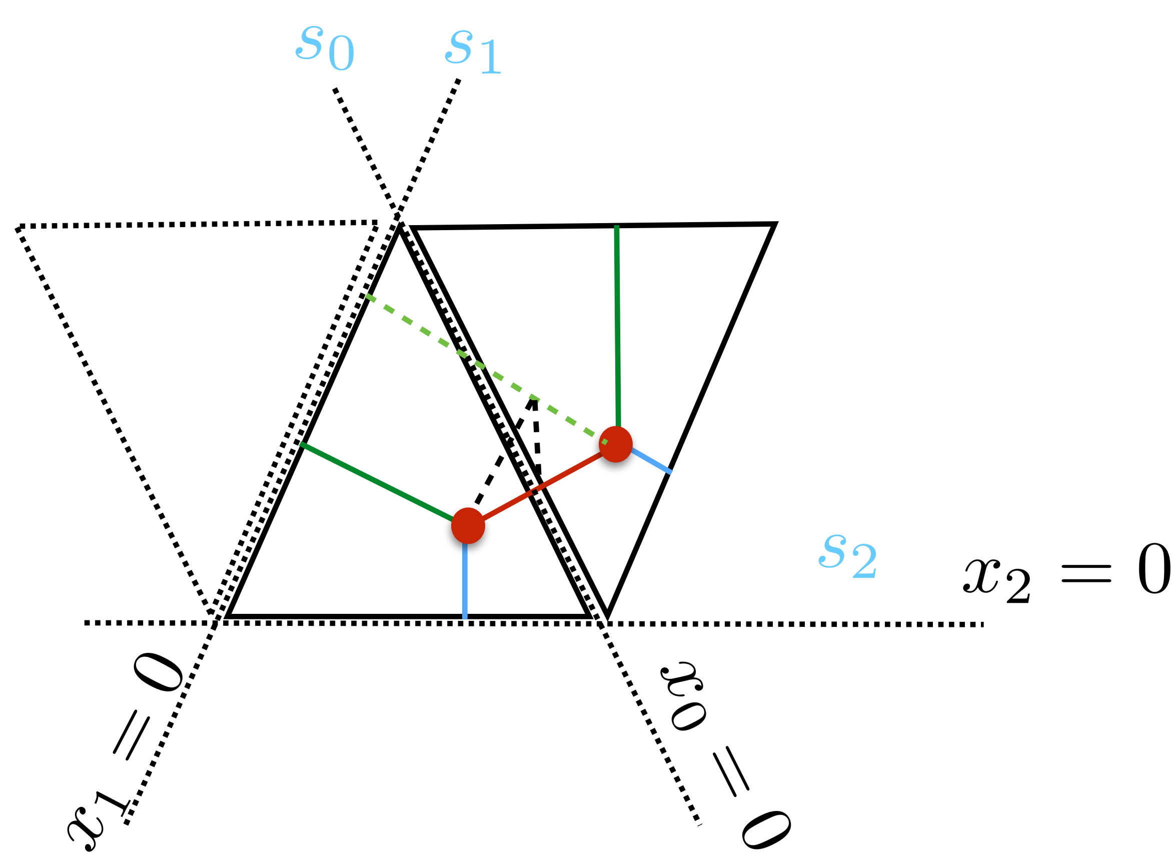

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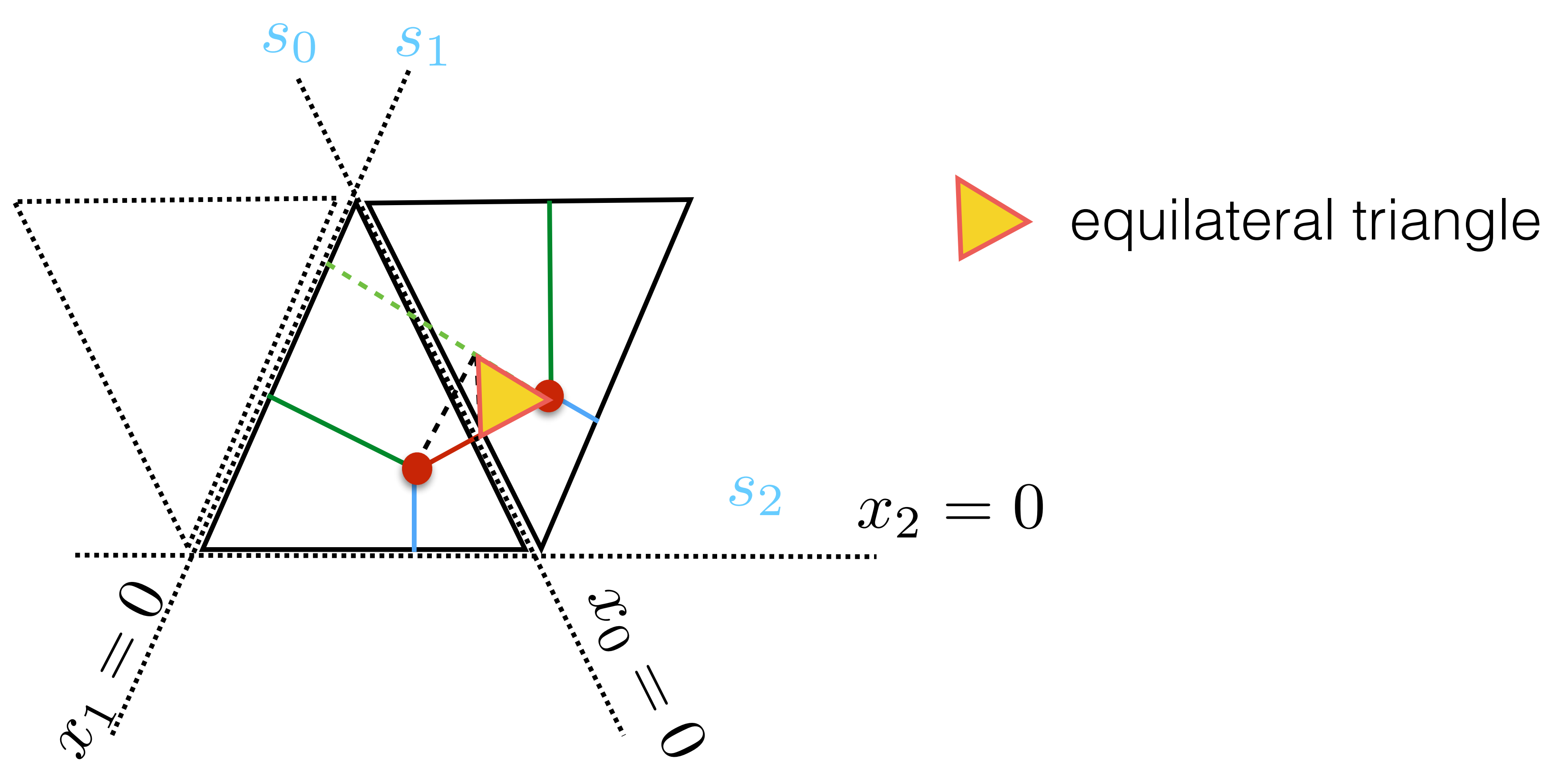

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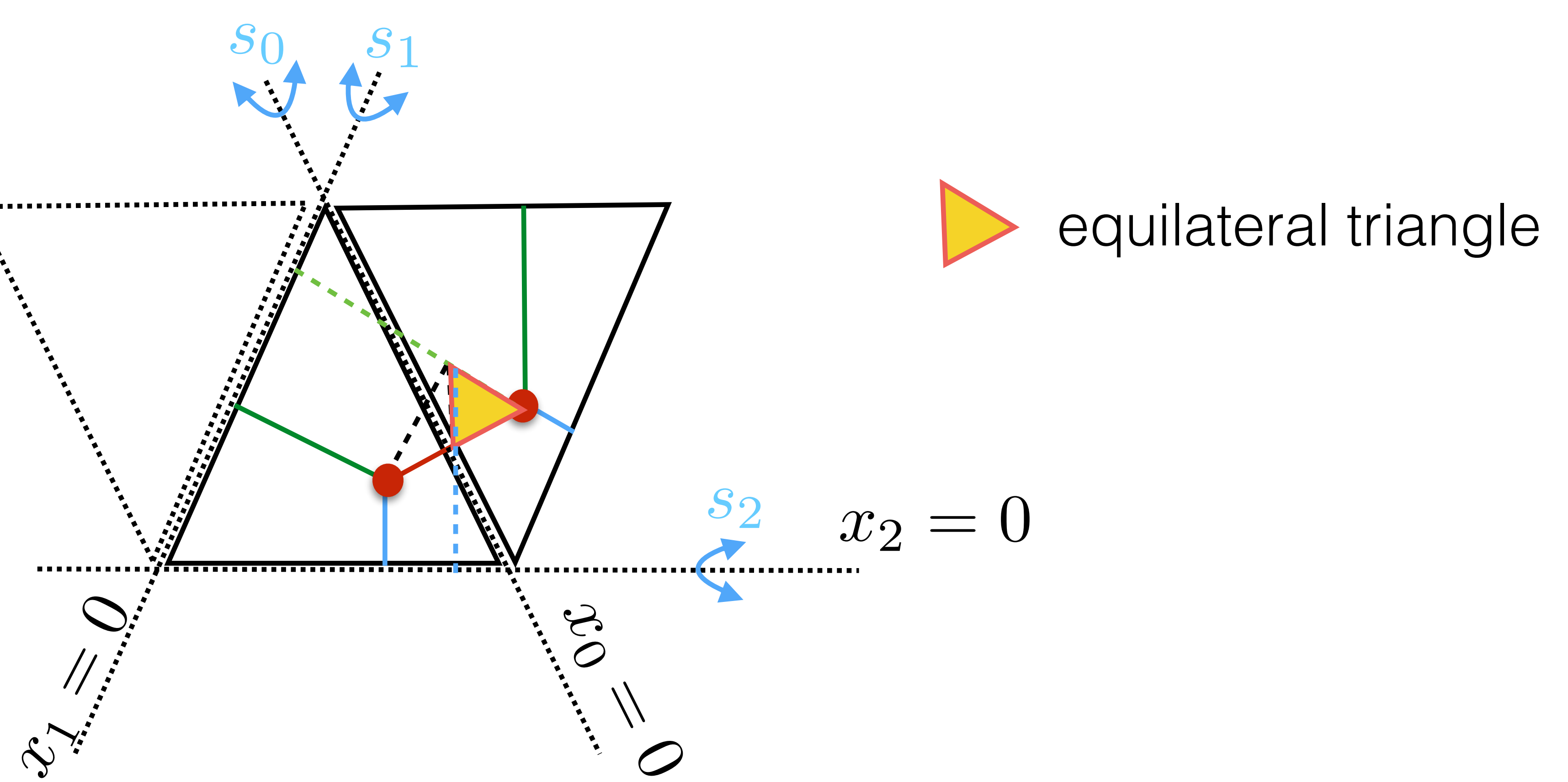

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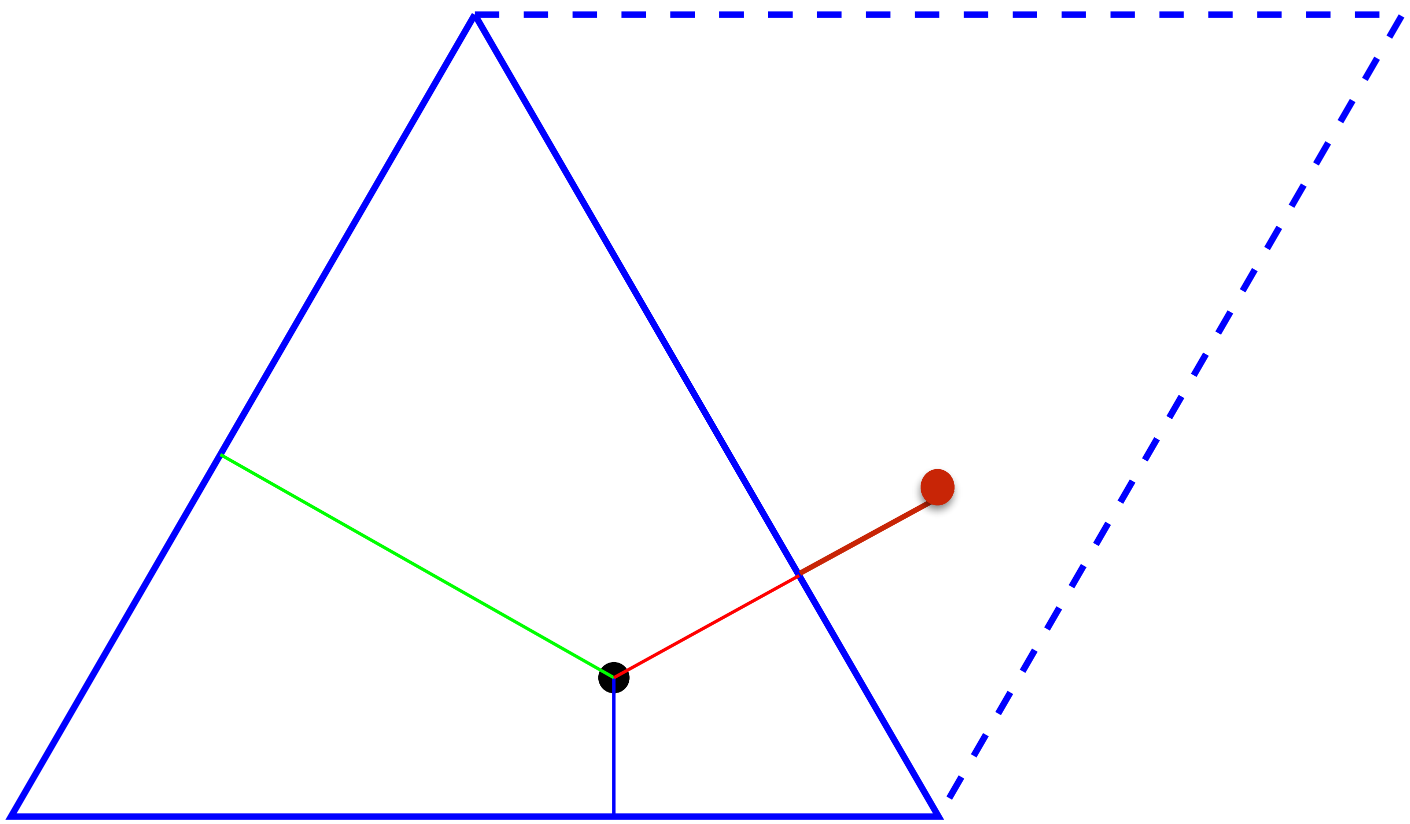
Constancy of coordinates



$$x_0 + x_1 + x_2 = k$$

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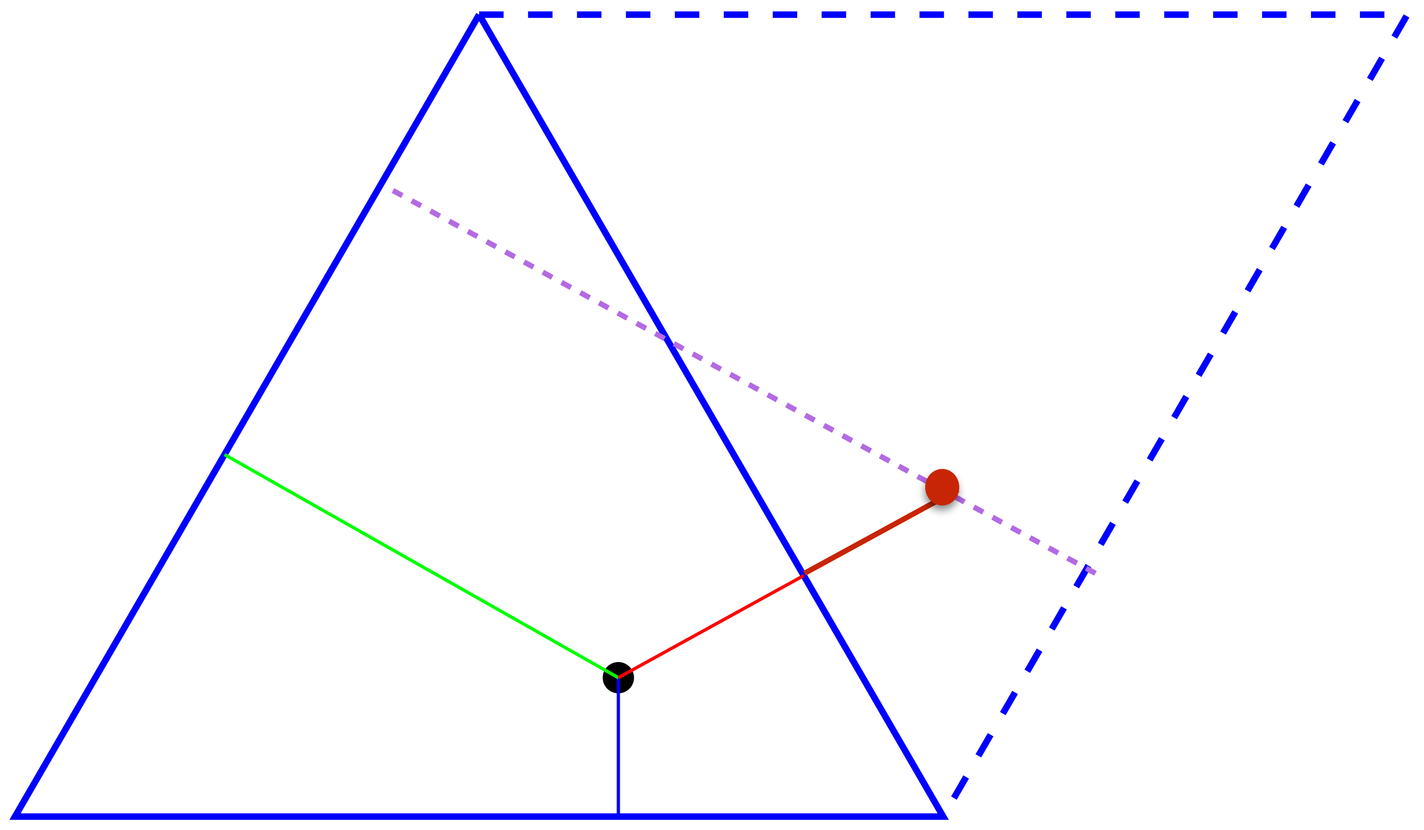

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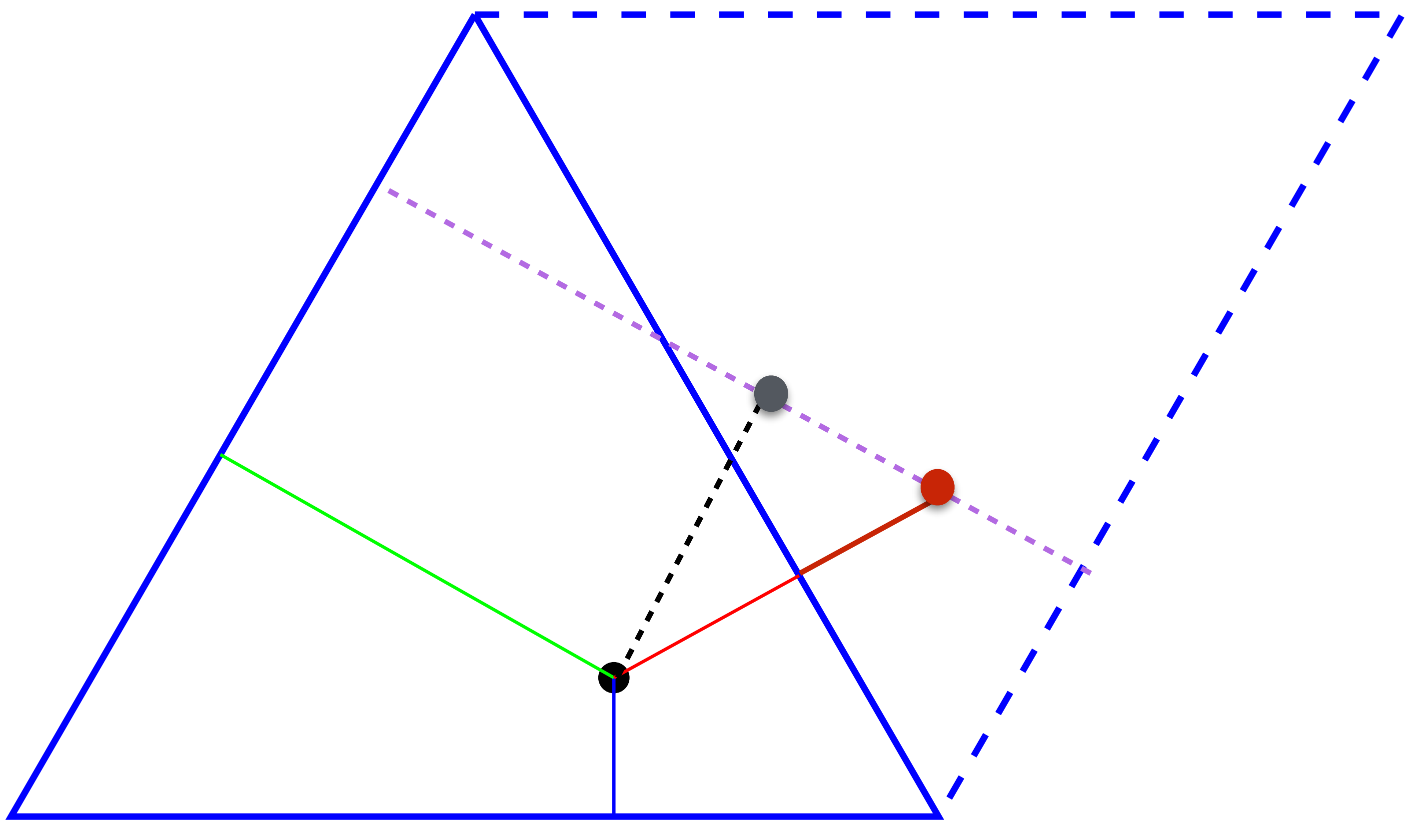

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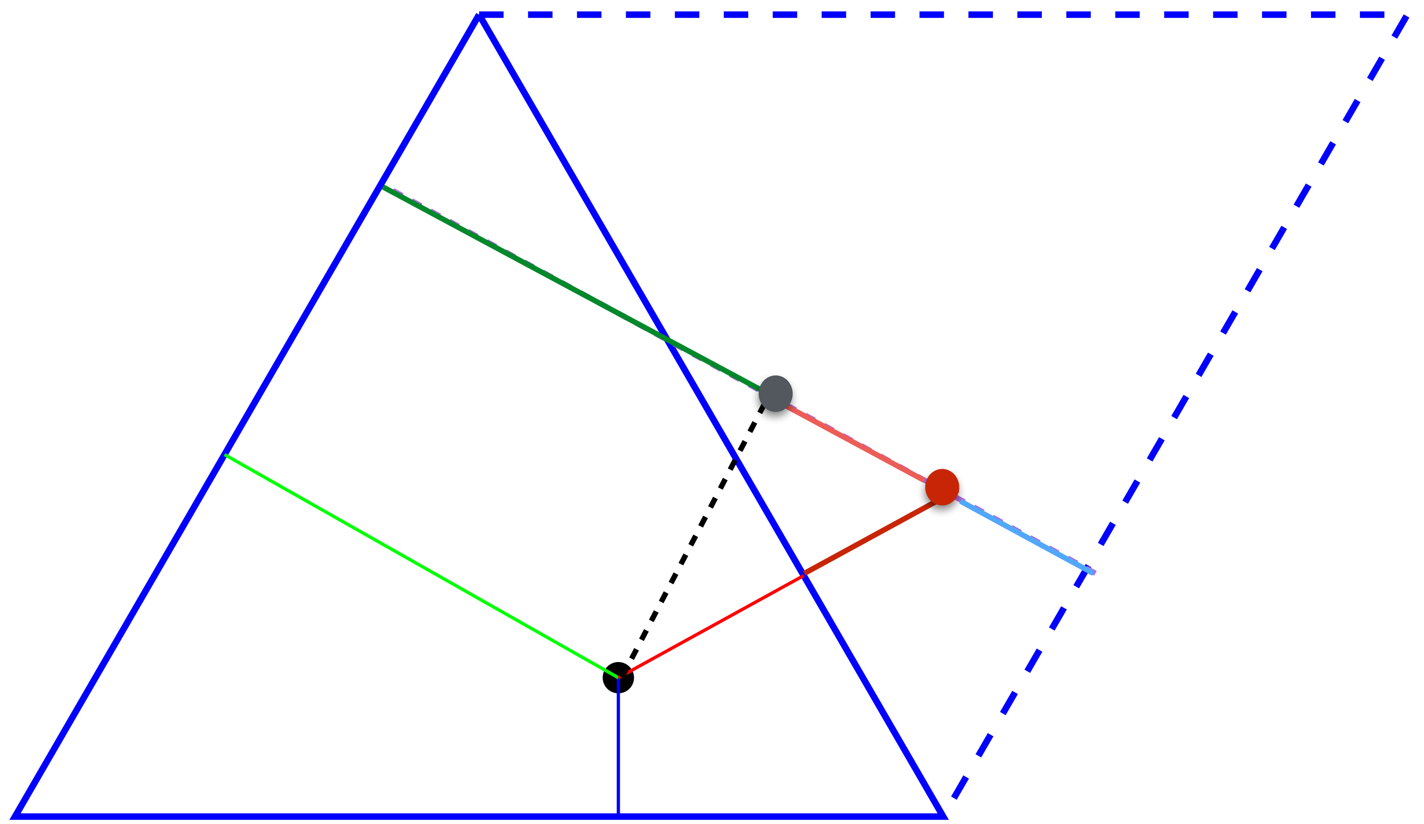
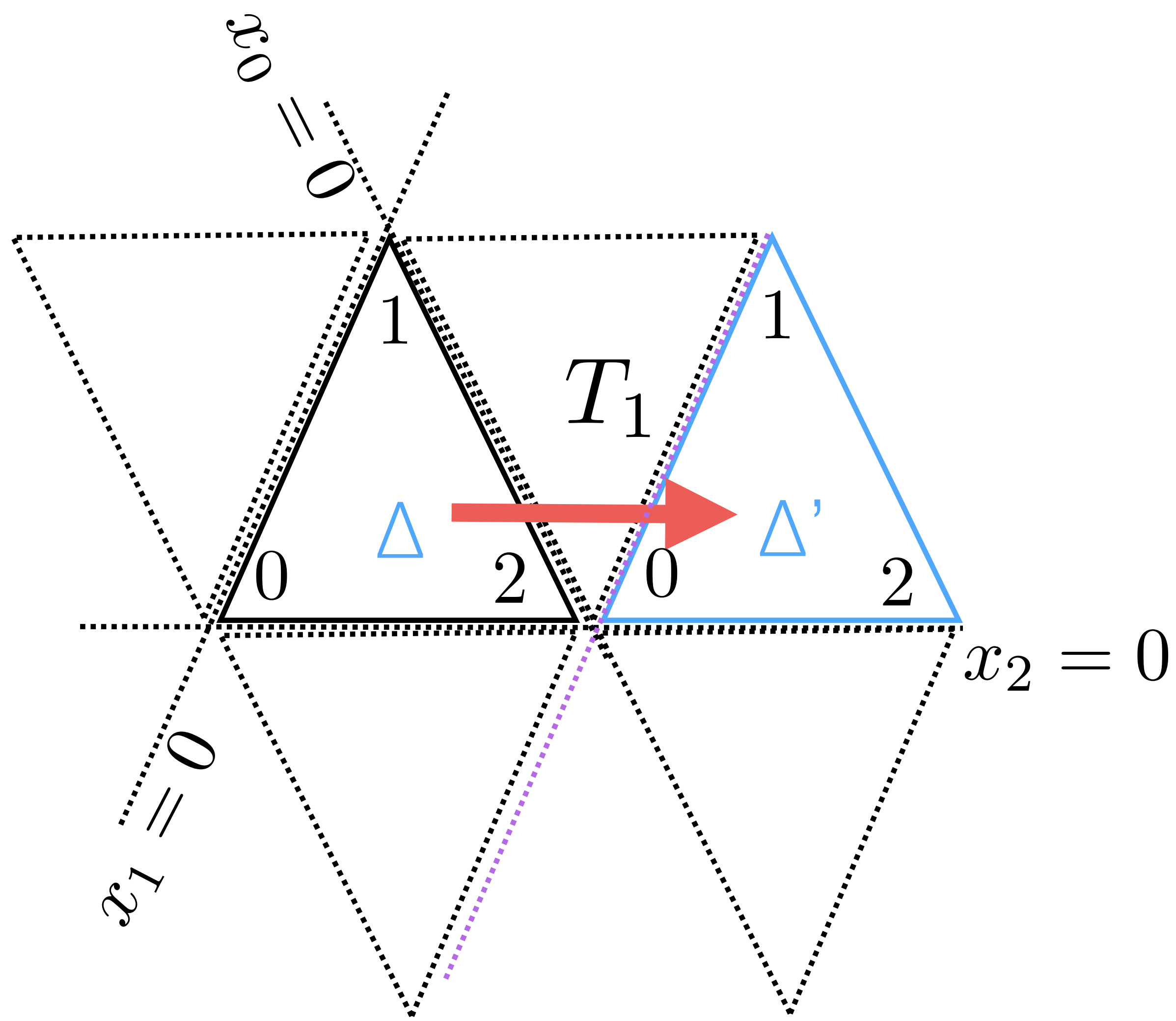


$$x_0 + x_1 + x_2 = k$$

Translations I



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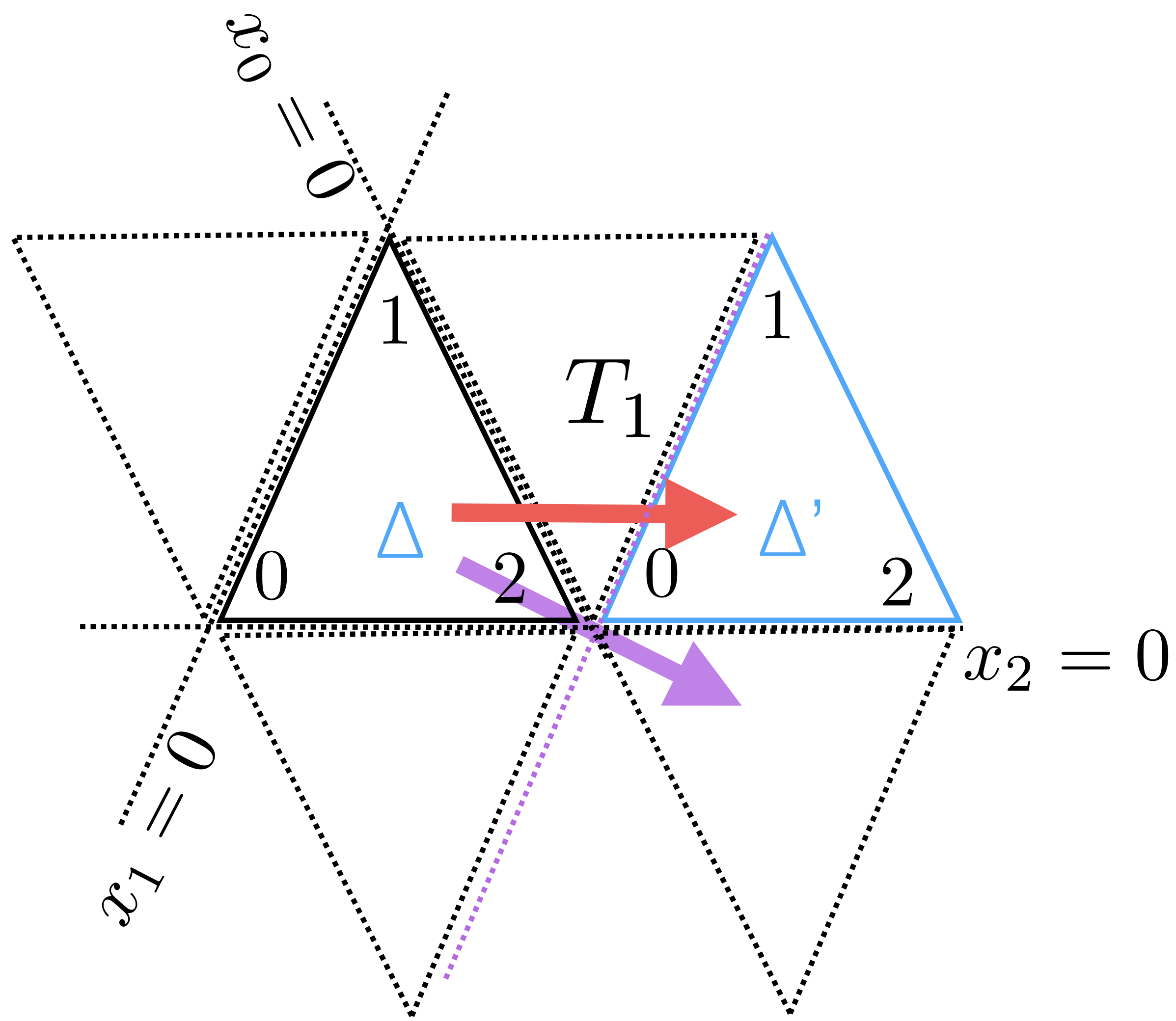

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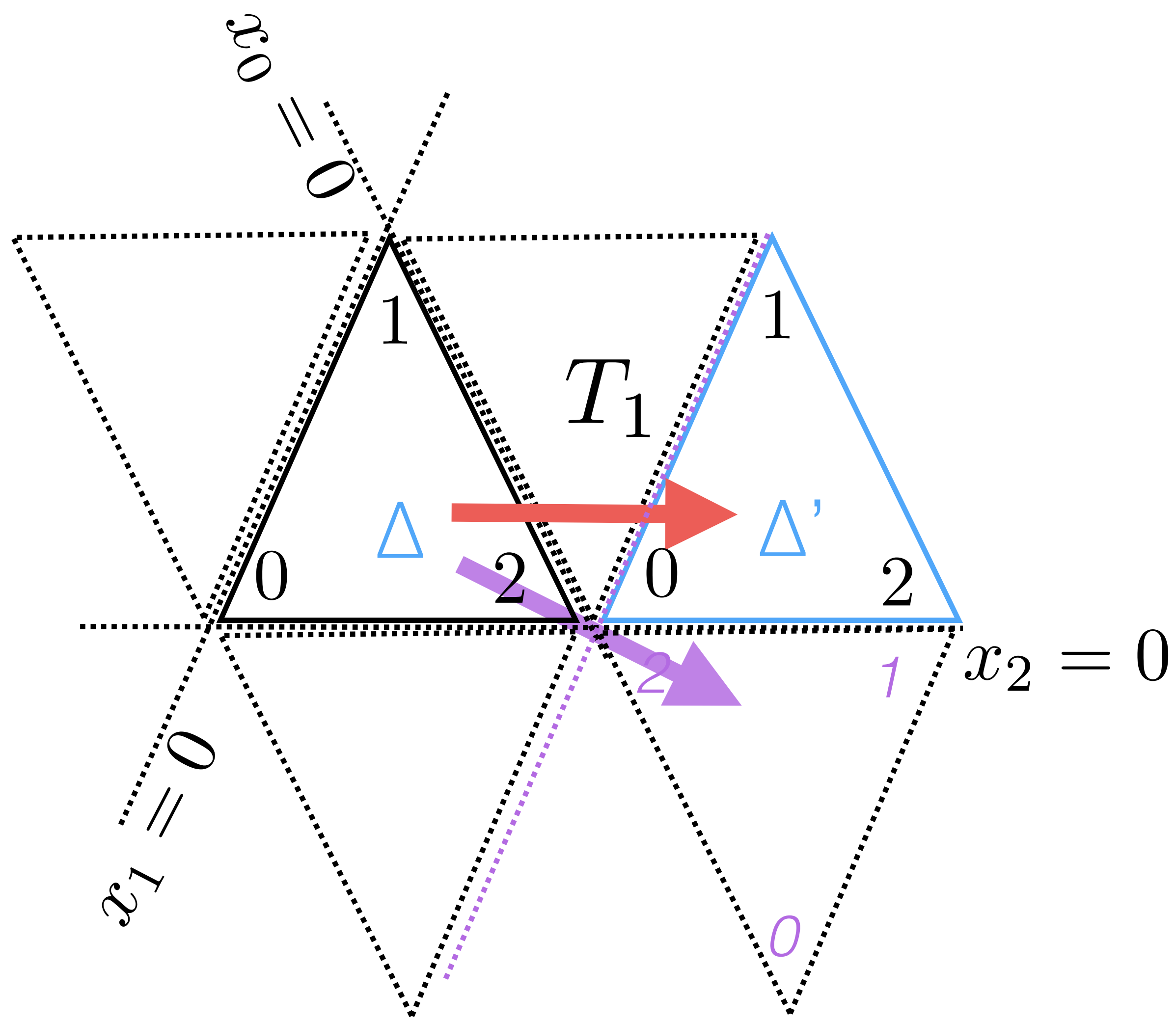

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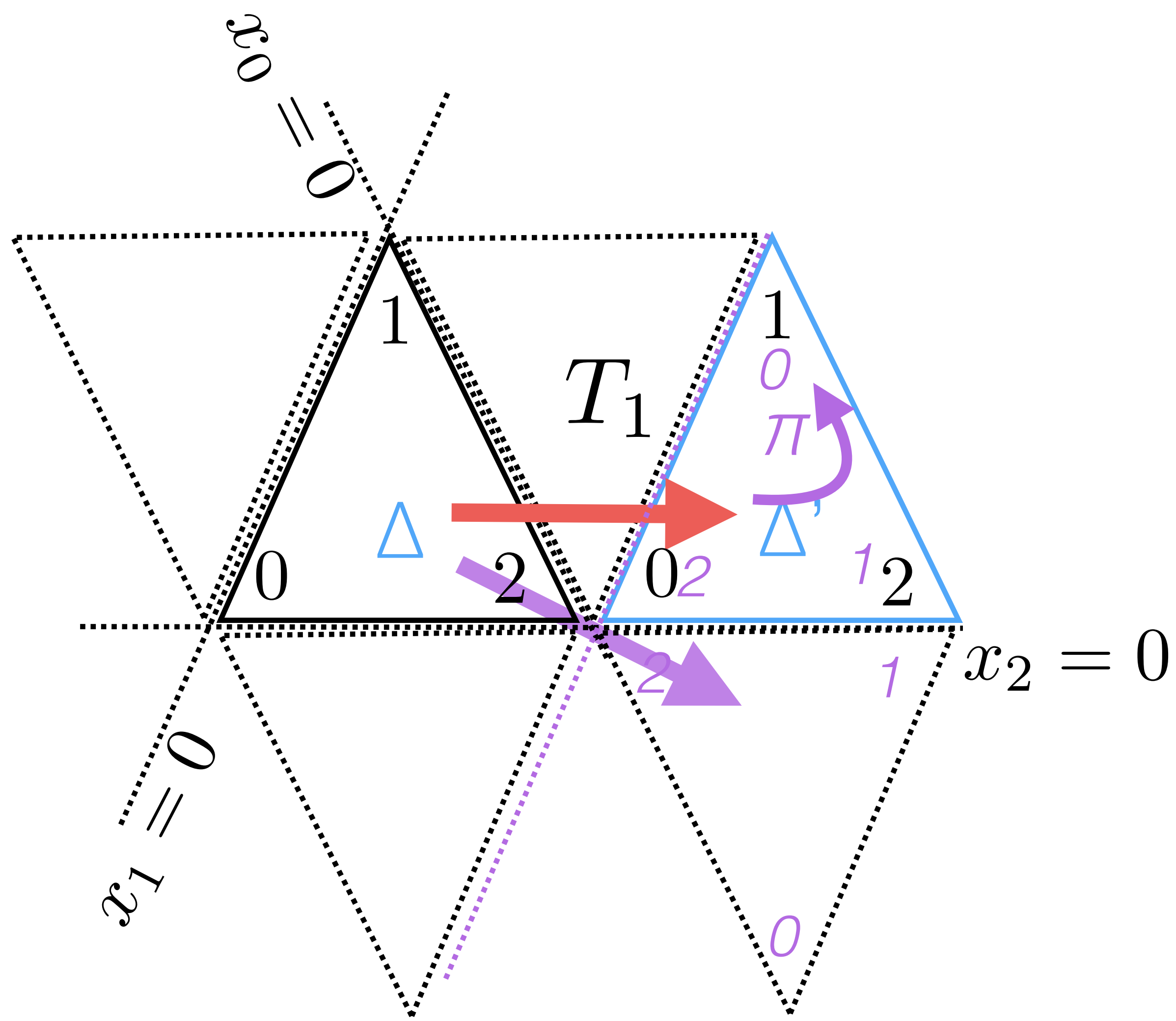
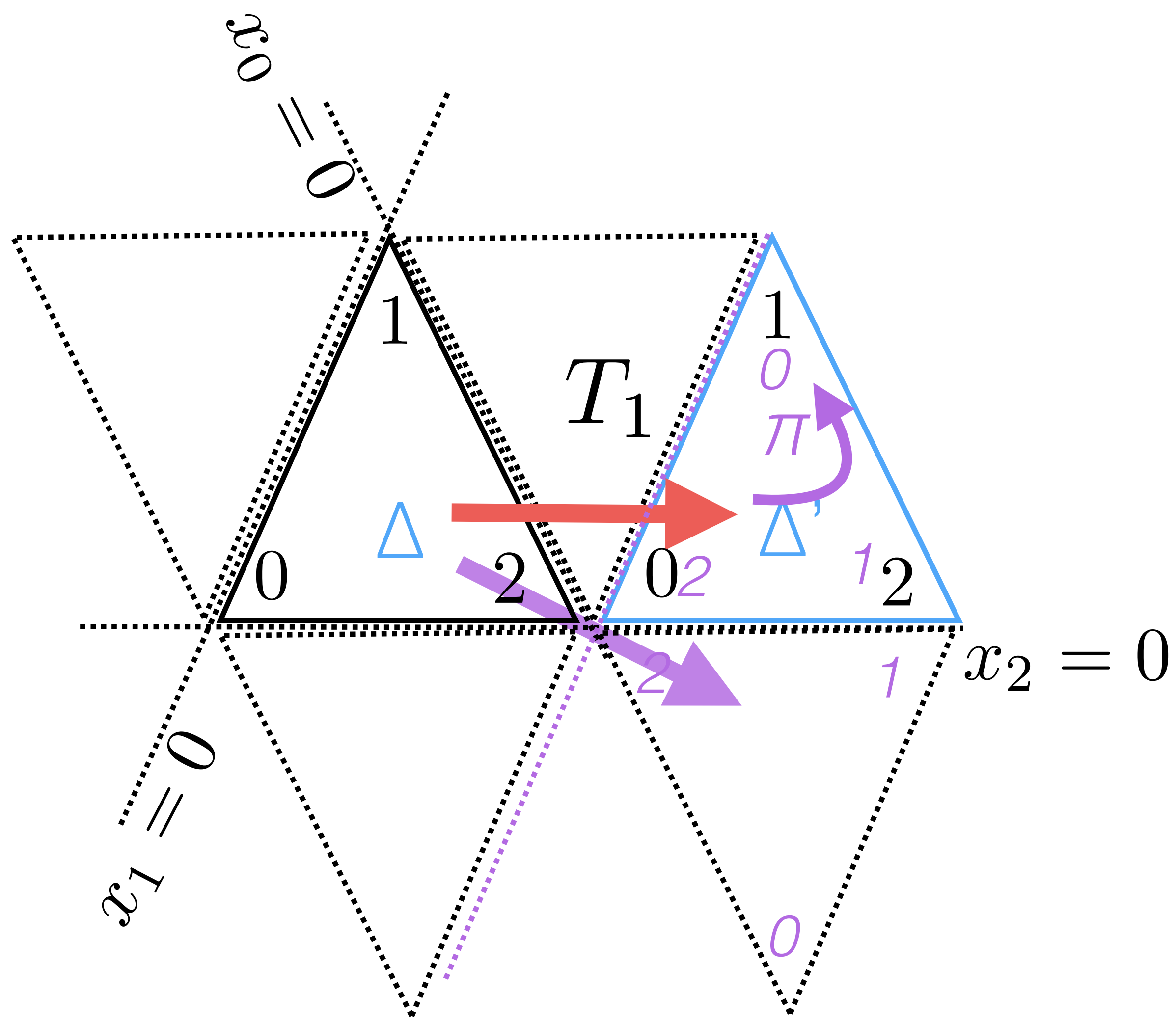


Translations I



$$T_1 = \pi\, s_2\, s_1$$

Translations II

We have

$$
\begin{aligned}
T_1(x_0) &= \pi\, s_2\, s_1(x_0) \\
&= \pi\, s_2\, (x_0 + x_1) \\
&= \pi\, (x_0 + x_1 + 2x_2) \\
&= x_1 + x_2 + 2\, x_0 = x_0 + k
\end{aligned}
$$

Translations II

We have

$$
\begin{aligned}
T_1(x_0) &= \pi\, s_2\, s_1(x_0) \\
&= \pi\, s_2\, (x_0 + x_1) \\
&= \pi\, (x_0 + x_1 + 2x_2) \\
&= x_1 + x_2 + 2\, x_0 = x_0 + k
\end{aligned}
$$

$$
\Rightarrow \quad T_1(x_0) = x_0 + k, \;\; T_1(x_1) = x_1 - k, \;\; T_1(x_2) = x_2
$$

Cremona Isometries

	x_0	x_1	x_2	f_0	f_1	f_2
s_0	$-x_0$	$x_1 + x_0$	$x_2 + x_0$	f_0	$f_1 + \frac{x_0}{f_0}$	$f_2 - \frac{x_0}{f_0}$
s_1	$x_0 + x_1$	$-x_1$	$x_2 + x_1$	$f_0 - \frac{x_1}{f_1}$	f_1	$f_2 - \frac{x_1}{f_1}$
s_2	$x_0 + x_2$	$x_1 + x_2$	$-x_2$	$f_0 + \frac{x_2}{f_2}$	$f_1 - \frac{x_2}{f_2}$	f_2

Noumi 2004

Cremona Isometries

	x_0	x_1	x_2	f_0	f_1	f_2
s_0	$-x_0$	$x_1 + x_0$	$x_2 + x_0$	f_0	$f_1 + \frac{x_0}{f_0}$	$f_2 - \frac{x_0}{f_0}$
s_1	$x_0 + x_1$	$-x_1$	$x_2 + x_1$	$f_0 - \frac{x_1}{f_1}$	f_1	$f_2 - \frac{x_1}{f_1}$
s_2	$x_0 + x_2$	$x_1 + x_2$	$-x_2$	$f_0 + \frac{x_2}{f_2}$	$f_1 - \frac{x_2}{f_2}$	f_2

Noumi 2004

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$\Rightarrow$

$$\begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0 + n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1 - n}{u_n} \end{cases}$$

Translations again

Using

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \qquad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

This is a discrete Painlevé equation.

McKay’s Correspondence

- The Affine Weyl group is associated with a singular space with a canonical divisor
- Translations keep the singular space and its divisor invariant
- Reflections in singular space $\Rightarrow$ Cremona isometries

Dolgachev 1983

McKay's Correspondence

- The Affine Weyl group is associated with a singular space with a canonical divisor
- Translations keep the singular space and its divisor invariant
- Reflections in singular space $\Rightarrow$ Cremona isometries

Dolgachev 1983

The Cremona isometries give rise to discrete Painlevé equations.

Part 2

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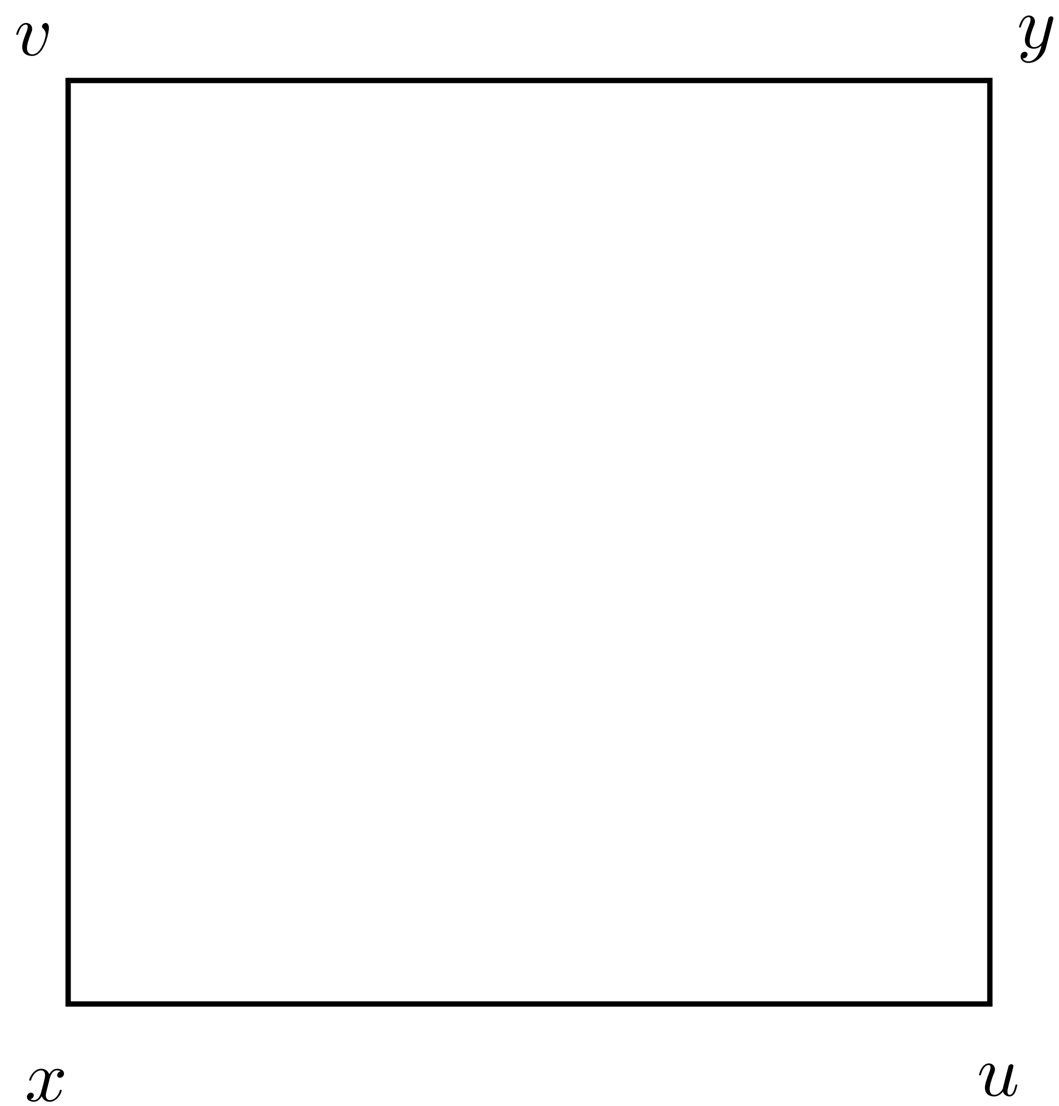
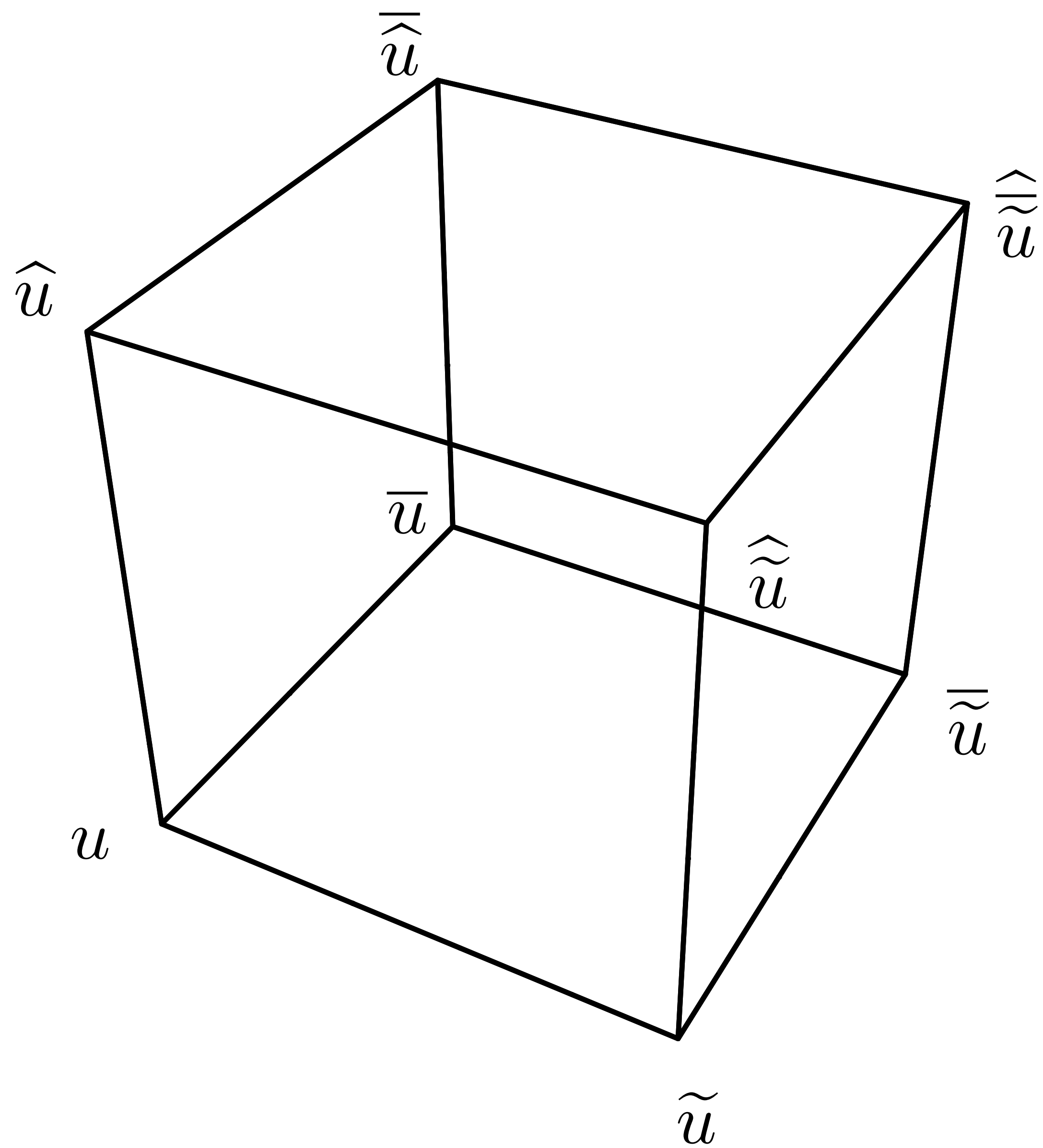
- Dynamics on *N*-cubes
- Symmetry reductions



$$Q(x, u, v, y) = 0$$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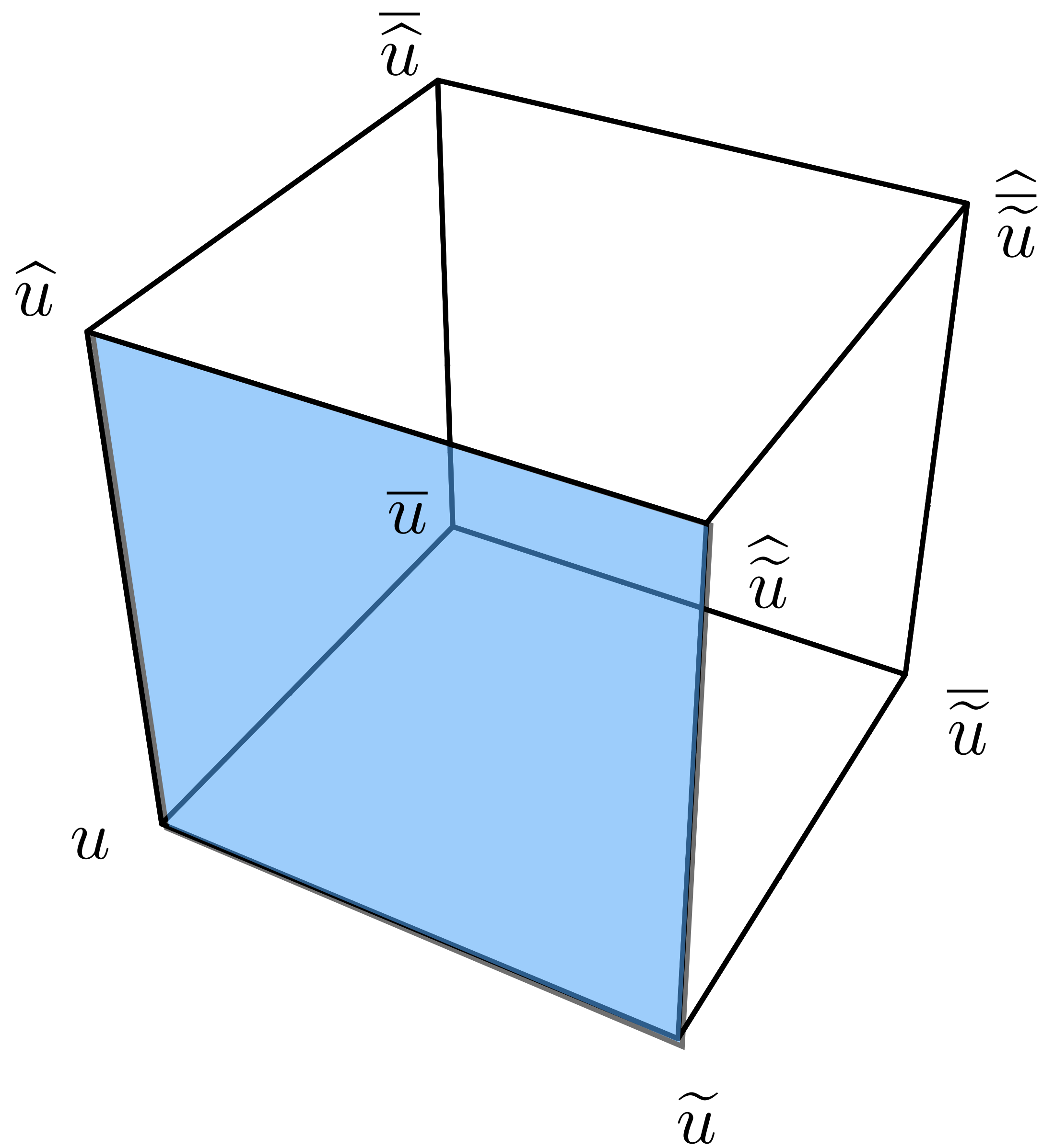
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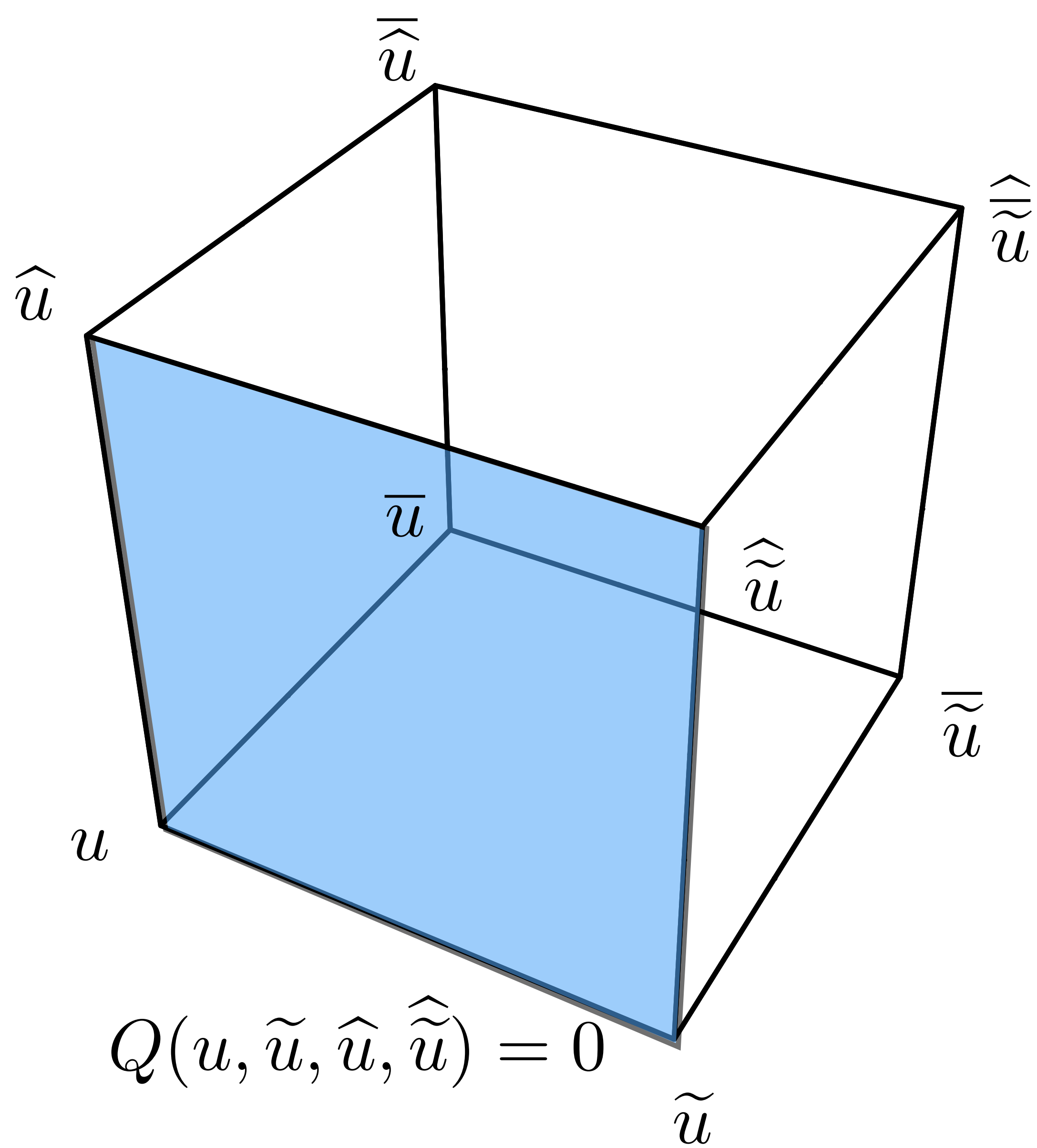
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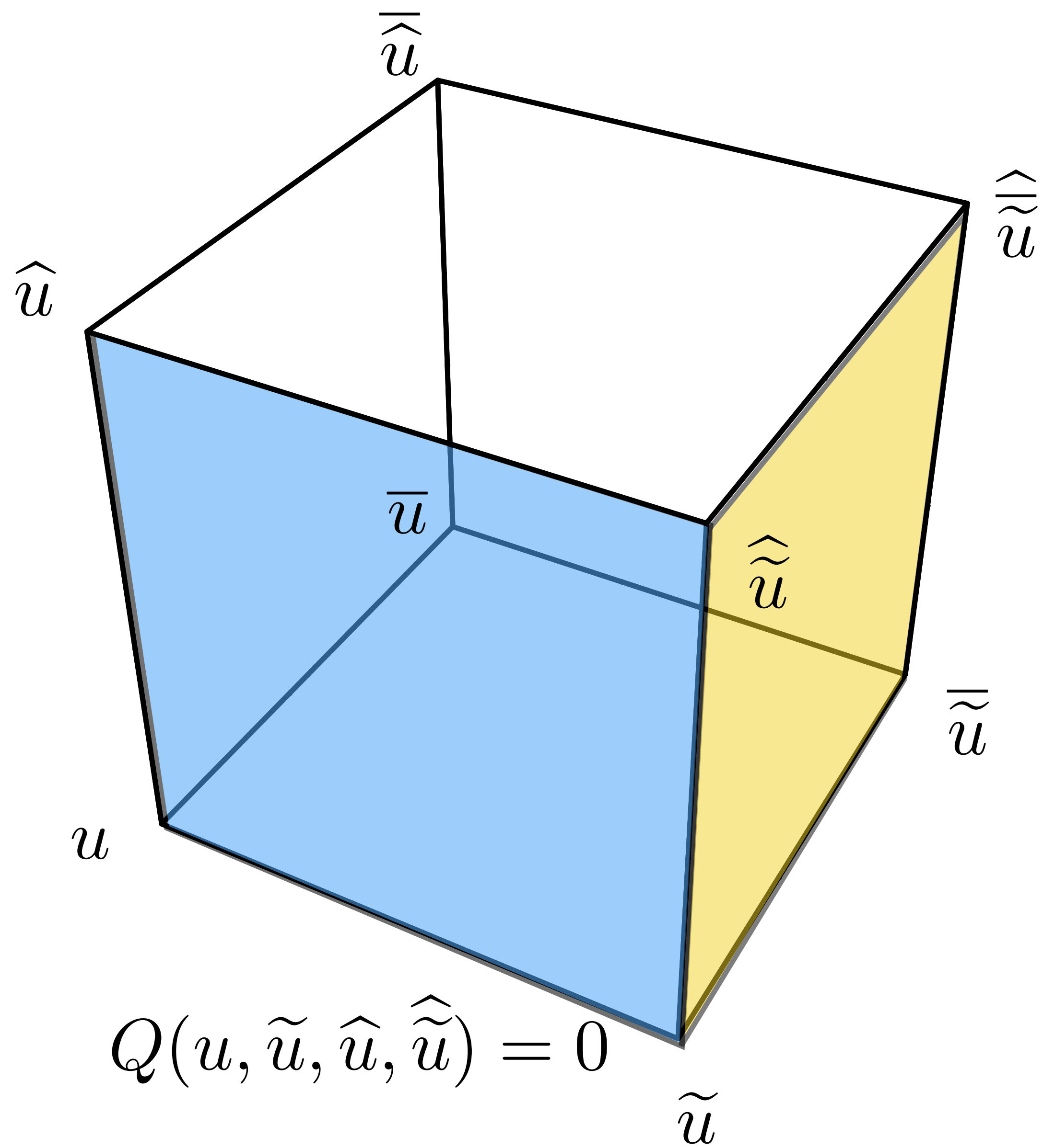

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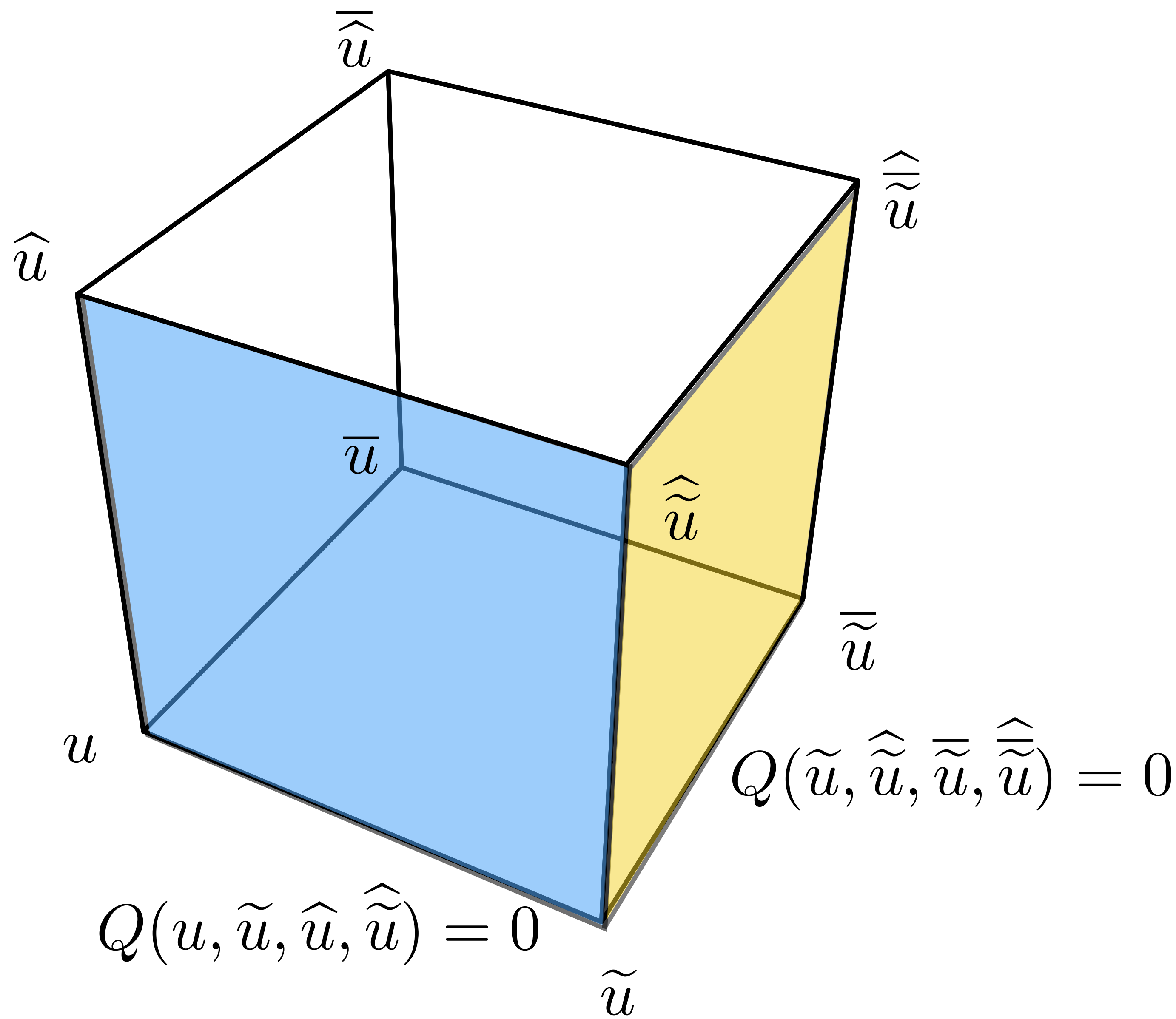

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$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0$
$\widetilde{u}$

Consistency around a Cube

Consider $Q(x, u, v, y) = x + u + v + y$

$$u + \widetilde{u} + \widehat{u} + \widehat{\widetilde{u}} = 0$$

$$u + \widetilde{u} + \overline{u} + \overline{\widetilde{u}} = 0$$

$$u + \overline{u} + \widehat{u} + \widehat{\overline{u}} = 0$$

$$\widetilde{u} + \overline{\widetilde{u}} + \widehat{\widetilde{u}} + \widehat{\overline{\widetilde{u}}} = 0$$

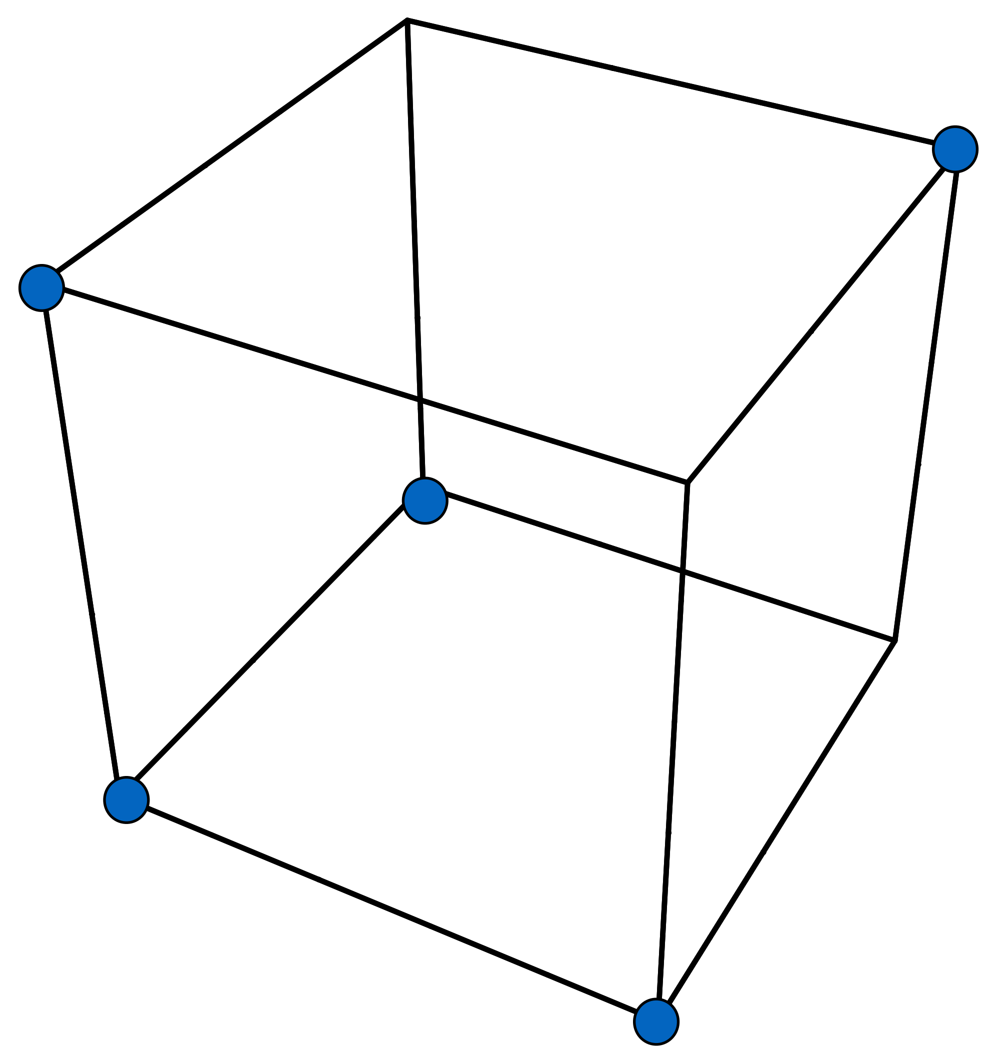
$$\widehat{u} + \widehat{\widetilde{u}} + \widehat{\overline{u}} + \widehat{\overline{\widetilde{u}}} = 0$$

$$\overline{u} + \widetilde{\overline{u}} + \widehat{\overline{u}} + \widehat{\overline{\widetilde{u}}} = 0$$

All 3 paths to the last vertex lead to the same value:

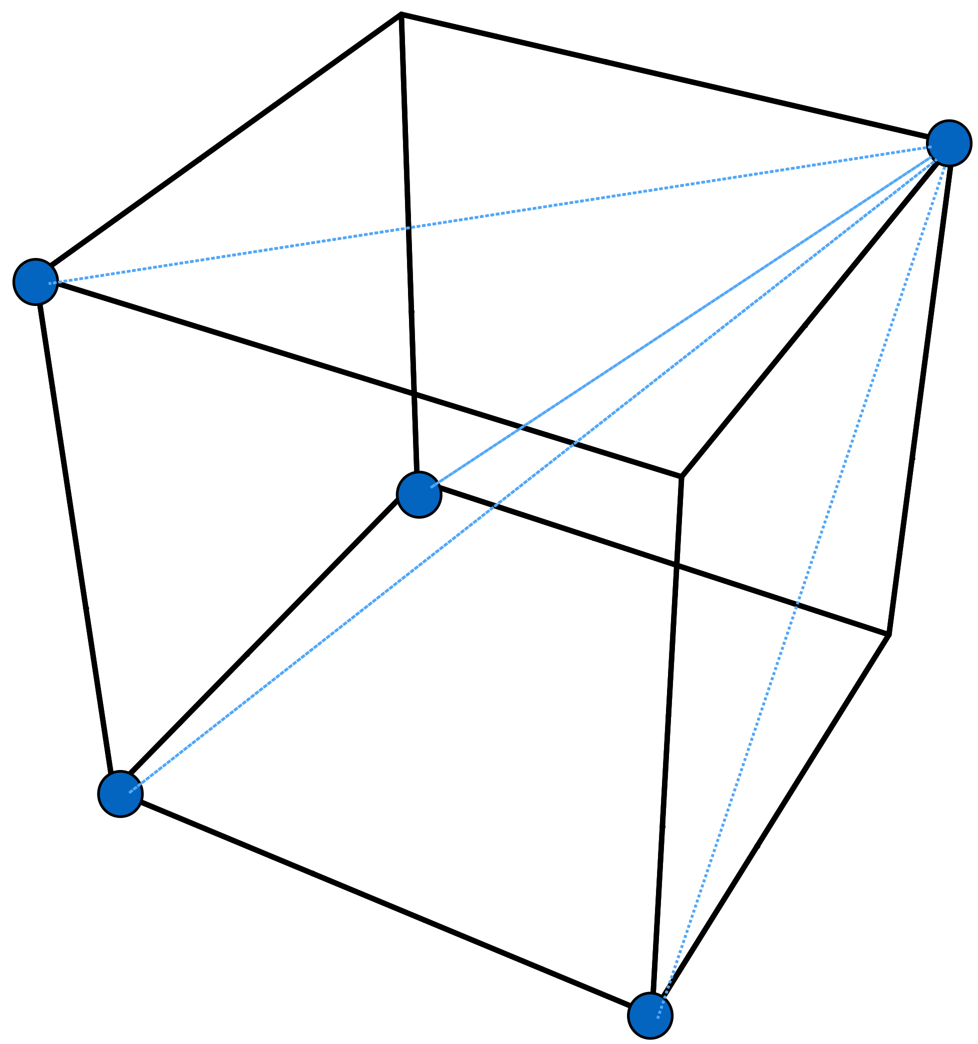
$$\widehat{\overline{\widetilde{u}}} = 2u + \widetilde{u} + \overline{u} + \widehat{u}$$

Tetrahedral Condition



The last vertex depends only on 4 earlier vertices to which it is not connected by an edge.

Tetrahedral Condition



The last vertex depends only on 4 earlier vertices to which it is not connected by an edge.

Are there more examples that are consistent around a cube?

Duality

- The Weber equation:

$$w'' + \left(\alpha + \frac{1}{2} - \frac{1}{4}x^2\right) w = 0$$

has recurrence relations: $w(x) = D_\alpha(x)$

$$D'_\alpha(x) = -\frac{x}{2} D_\alpha(x) + \alpha\, D_{\alpha-1}(x)$$

$$D'_{\alpha-1}(x) = \frac{x}{2} D_{\alpha-1}(x) - D_\alpha(x)$$

which imply

$$D_{\alpha+1}(x) - x\, D_\alpha(x) + \alpha D_{\alpha-1}(x) = 0$$

Duality

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Continuous

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which imply

$$D_{\alpha+1}(x) - x\, D_\alpha(x) + \alpha D_{\alpha-1}(x) = 0$$

Discrete

Dynamics in 2D

- Given a parameter λ, the *Bäcklund transformation*

$$\left(\widetilde{w} + w\right)_x = 2\,\lambda - \tfrac{1}{2}\left(\widetilde{w} - w\right)^2$$

relates two solutions $\widetilde{w}, w$ of the potential KdV equation.

$$w_t = w_{xxx} + 3\,w_x^2$$

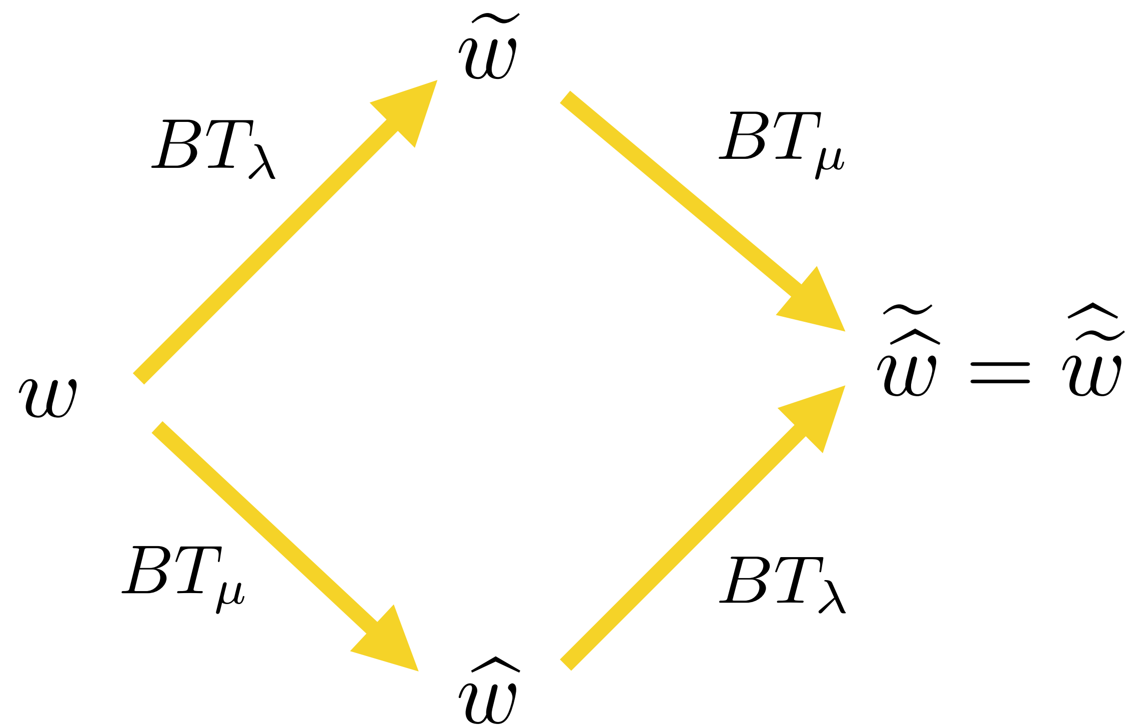
Wahlquist & Estabrook, 1976

- Take two such transformations

$$BT_\lambda :\; w \overset{\lambda}{\mapsto} \widetilde{w}, \quad \left(\widetilde{w} + w\right)_x = 2\lambda - \frac{1}{2}\left(\widetilde{w} - w\right)^2$$

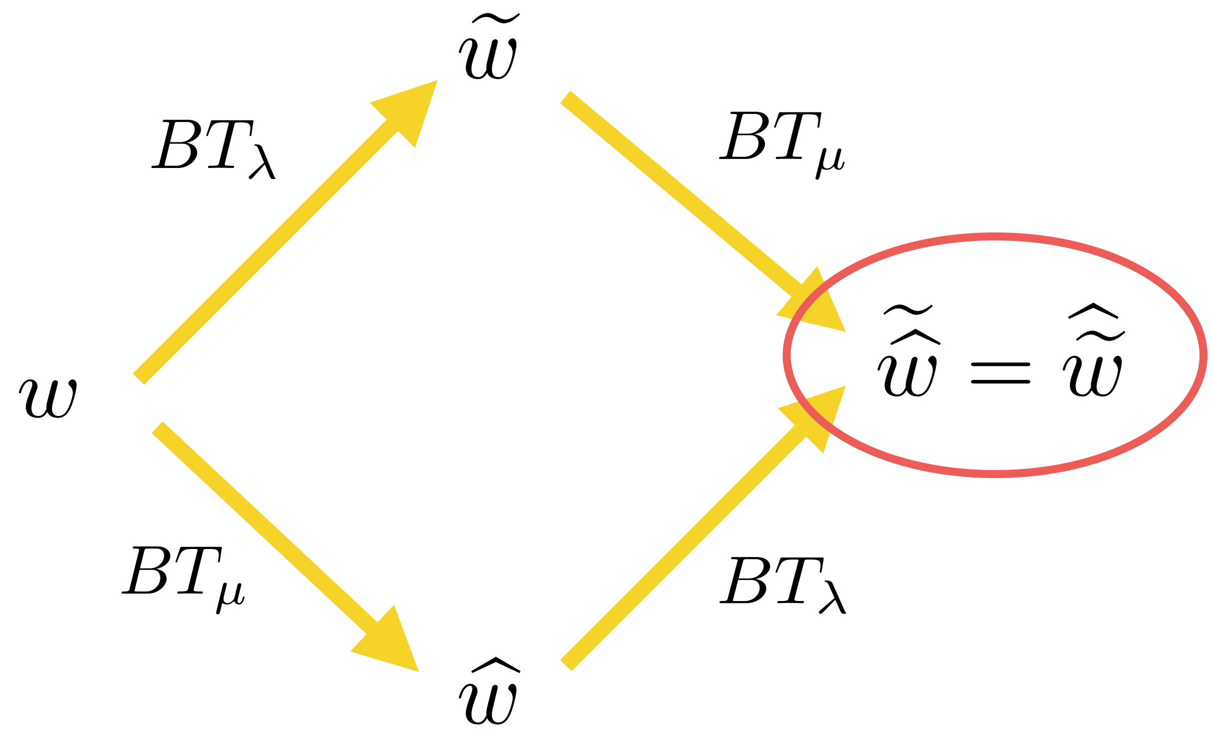
$$BT_\mu :\; w \overset{\mu}{\mapsto} \widehat{w}, \quad \left(\widehat{w} + w\right)_x = 2\mu - \frac{1}{2}\left(\widehat{w} - w\right)^2$$

Permutability



Two different compositions of BTs give the same solution.

Permutability



Two different compositions of BTs give the same solution.

Lattice Equations

- Eliminating derivatives between BT_λ, BT_μ and *their* derivatives $\Rightarrow$

$$(\widehat{\widetilde{w}} - w)(\widehat{w} - \widetilde{w}) = 4(\mu - \lambda)$$

or

$$\big(w_{n+1,m+1} - w_{n,m}\big)\big(w_{n,m+1} - w_{n+1,m}\big) = 4(\mu - \lambda)$$

where $w_{n,m} = BT_\lambda^n \circ BT_\mu^m \, w$

Nijhoff, Quispel, Capel, 1983
Nijhoff, Quispel, van der Linden, Capel, 1983

Lattice Equations

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where $w_{n,m} = BT_\lambda^n \circ BT_\mu^m \, w$

- Affine linear
- In *N*-D

Nijhoff, Quispel, Capel, 1983
Nijhoff, Quispel, van der Linden, Capel, 1983

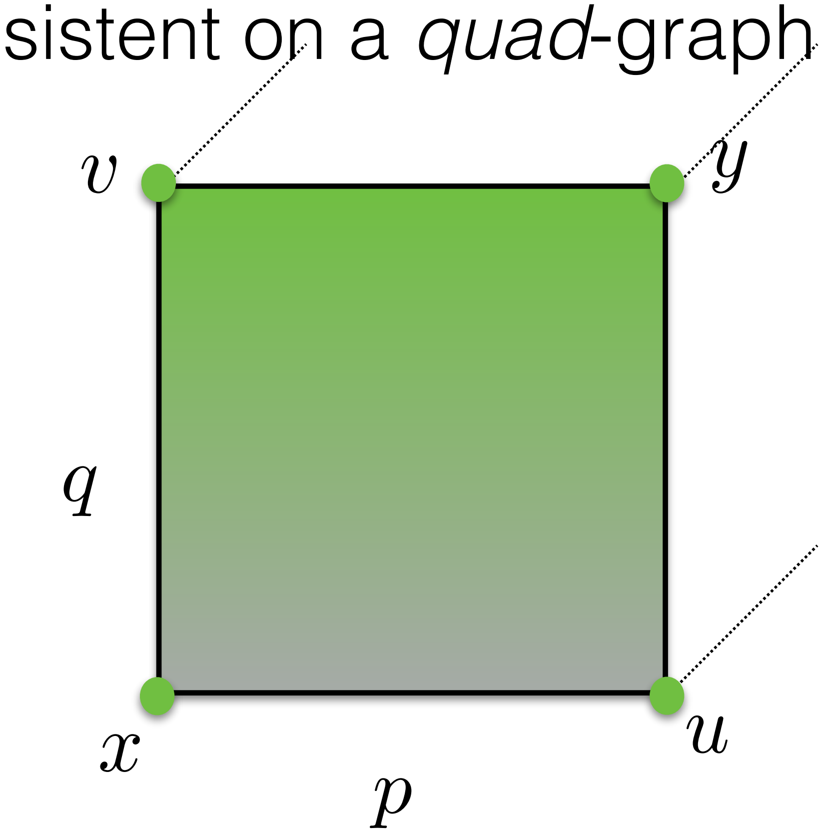
Classification

- Motivated by work of Nijhoff, Capel *et al (1983—'01)* Adler, Bobenko & Suris (2003,2009) classified all affine linear equations

$$Q(w,\widetilde{w},\widehat{w},\widehat{\widetilde{w}};p,q)=0$$

which are multi-dimensionally consistent on a *quad*-graph

$$Q(x,u,v,y;p,q)=0$$



On a 3-cube

$$Q(w, \overline{w}, \widetilde{w}, \widetilde{\overline{w}}; \alpha, \gamma) = 0$$

$$Q(w, \overline{w}, \widehat{w}, \widehat{\overline{w}}; \alpha, \beta) = 0$$

$$Q(w, \widehat{w}, \widetilde{w}, \widehat{\widetilde{w}}; \beta, \gamma) = 0$$

$$Q(\widehat{w}, \widehat{\overline{w}}, \widehat{\widetilde{w}}, \widehat{\overline{\widetilde{w}}}; \alpha, \gamma) = 0$$

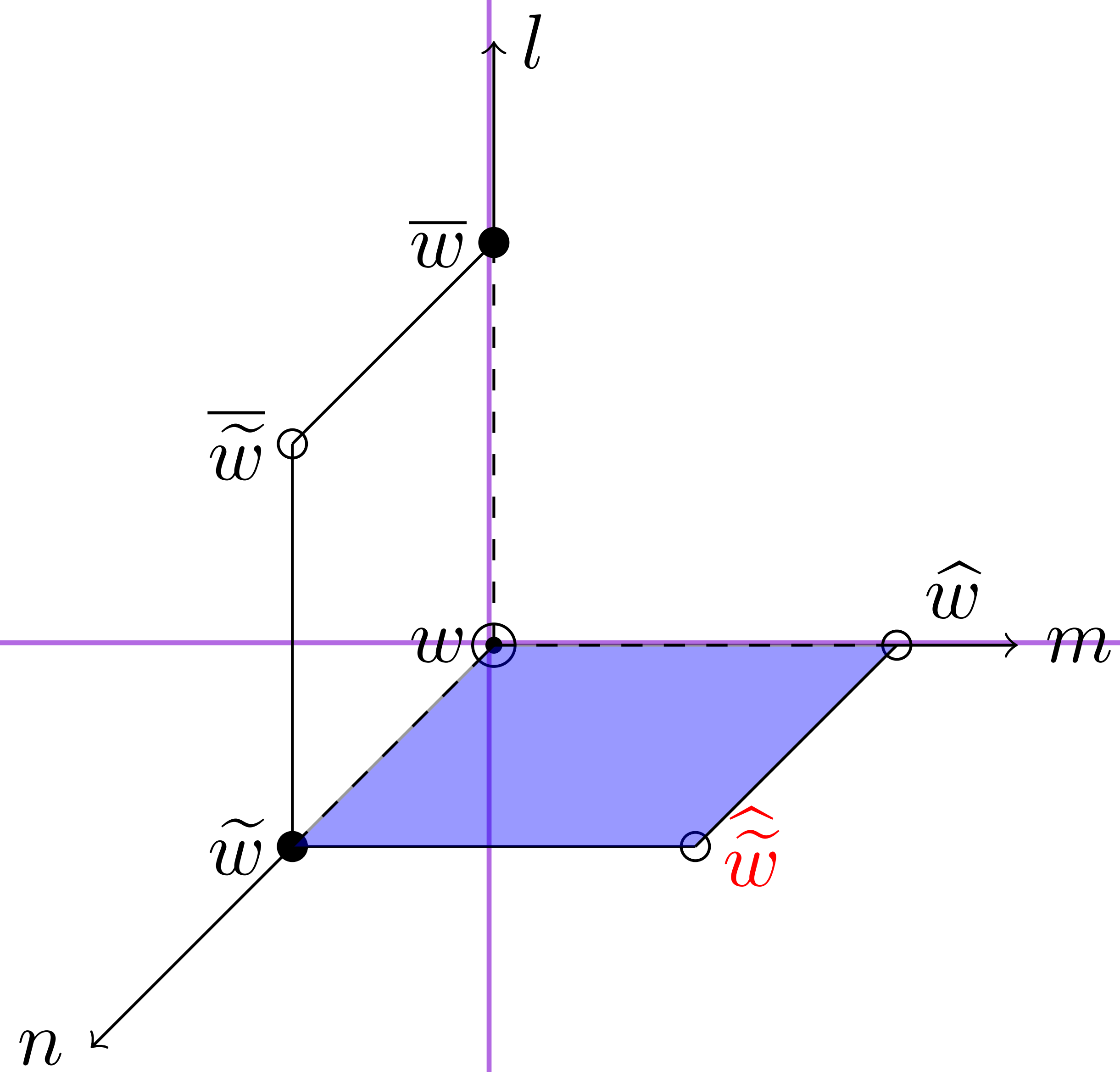
$$Q(\widetilde{w}, \overline{\widetilde{w}}, \widehat{\widetilde{w}}, \widehat{\overline{\widetilde{w}}}; \alpha, \beta) = 0$$

$$Q(\overline{w}, \widehat{\overline{w}}, \overline{\widetilde{w}}, \widehat{\overline{\widetilde{w}}}; \beta, \gamma) = 0$$

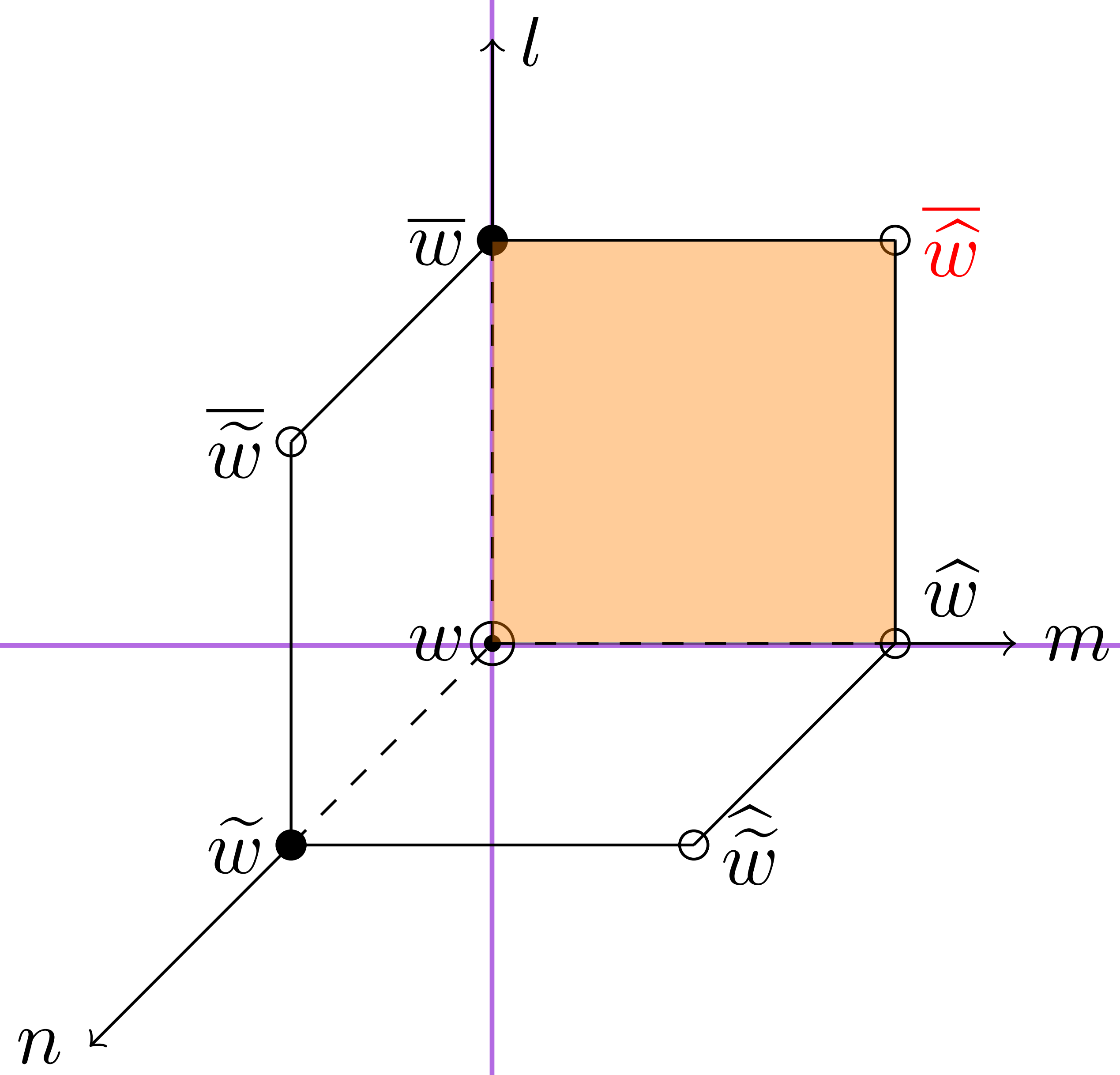
There are 6 equations (one for each face of the cube).

Multi-dimensional consistency

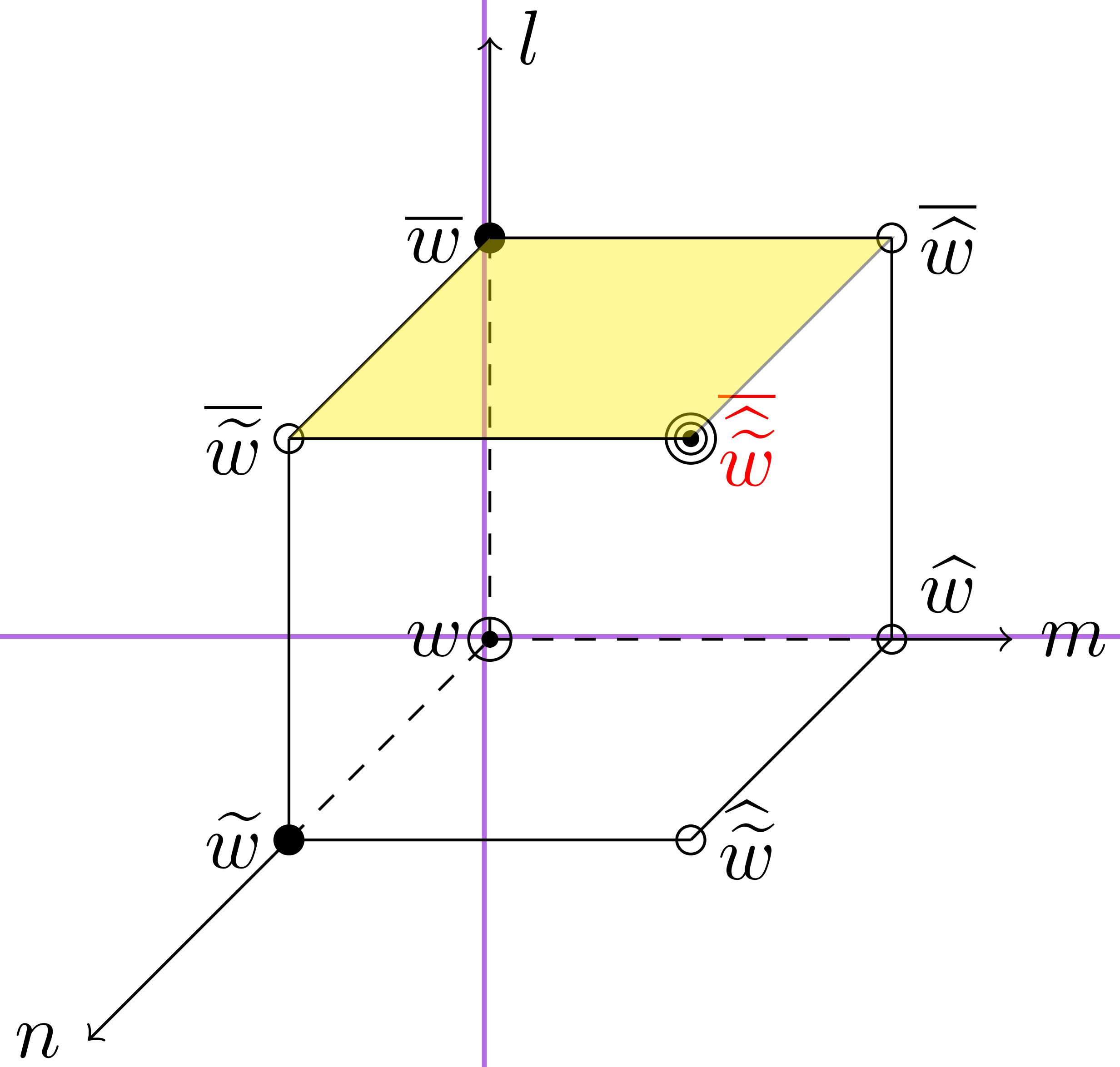
Multi-dimensional consistency



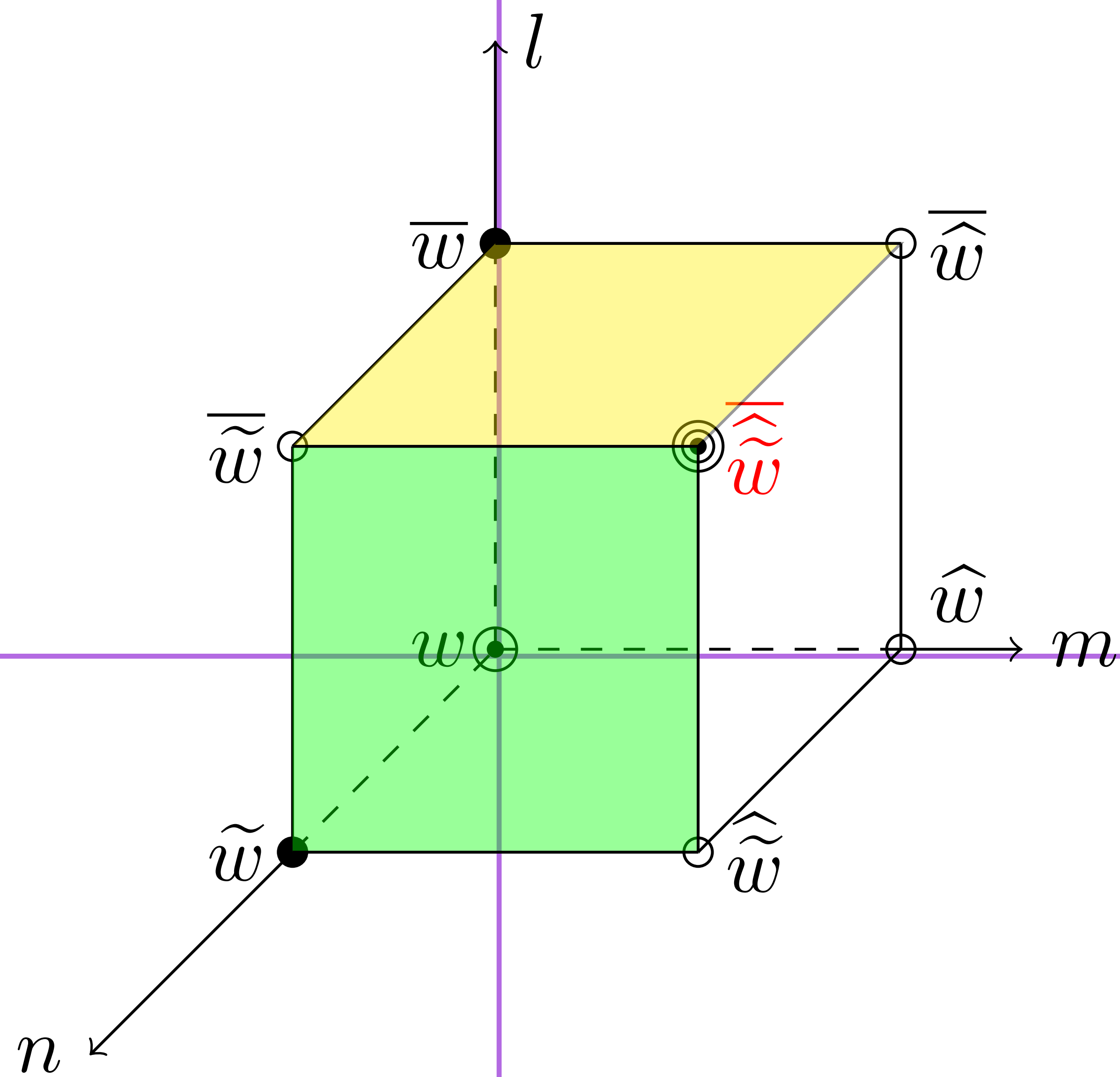
Multi-dimensional consistency



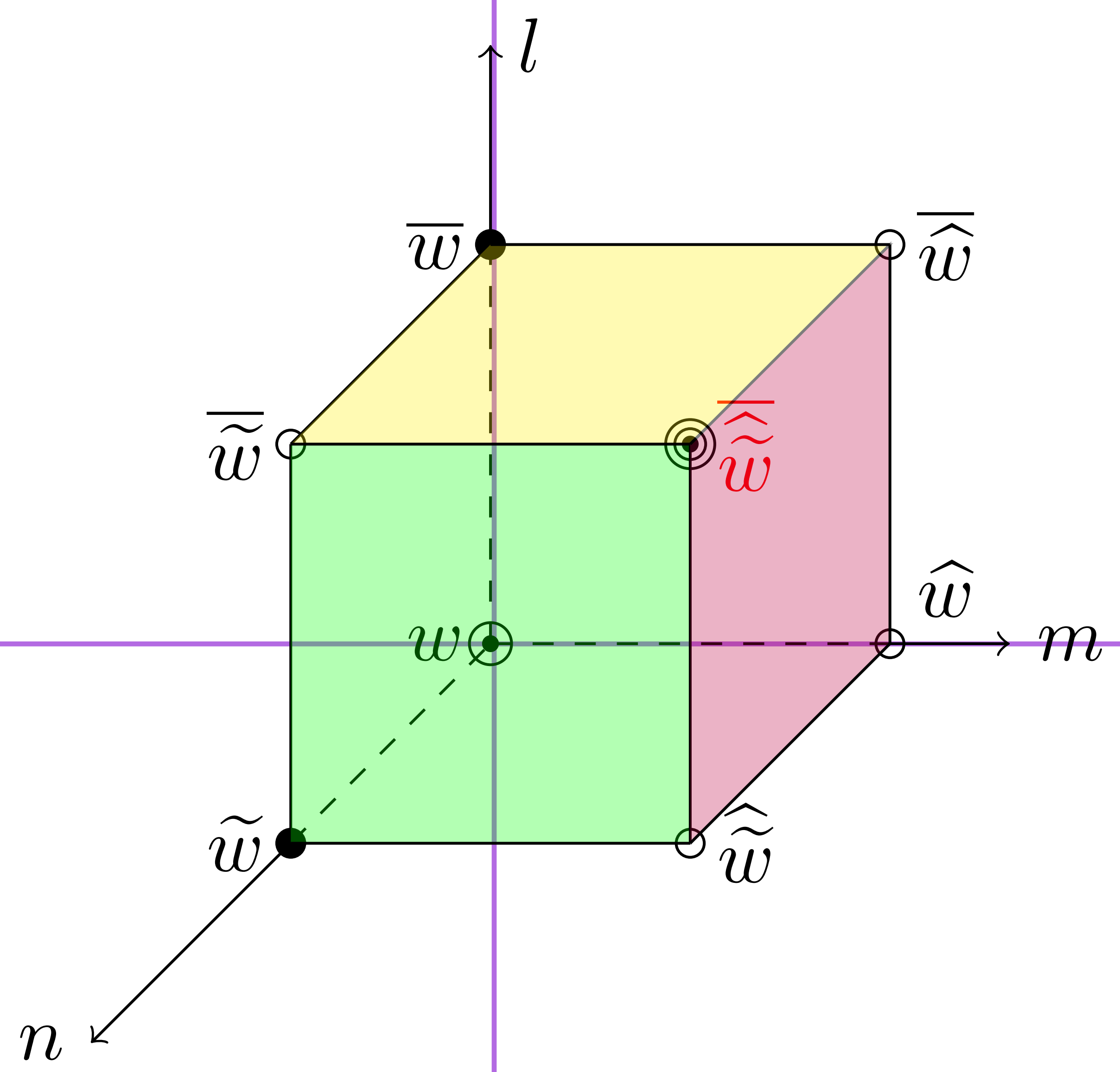
Multi-dimensional consistency



Multi-dimensional consistency



Multi-dimensional consistency



CAC Equations

$$
\begin{aligned}
Q4: \quad & a_0 xuvy + a_1(xuv + uvy + vyx + yxu) + a_2(xy + uv) \\
& + \overline{a}_2(xu + vy) + \widetilde{a}_2(xv + uy) \\
& + a_3(x + u + v + y) + a_4 = 0
\end{aligned}
$$

CAC Equations

- Three classes of equations all obtained from

$$Q4: \quad a_0xuvy + a_1(xuv + uvy + vyx + yxu) + a_2(xy + uv) + \overline{a}_2(xu + vy) + \widetilde{a}_2(xv + uy) + a_3(x + u + v + y) + a_4 = 0$$

  where the coefficients lie on an elliptic curve.

- The two other classes are labelled H and A.

Some ABS Equations

- H1:

$$(x-y)(u-v)+p^2-q^2=0$$

- H3:

$$\mathcal{Q}(xu+vy)-\mathcal{P}(uv+uy)+\frac{p^2-q^2}{\mathcal{P}\mathcal{Q}}=0$$

where $\mathcal{P}^2=a^2-p^2, \mathcal{Q}^2=a^2-q^2$

- Q3:

$$\mathcal{P}(uv+uy)-\mathcal{Q}(xu+vy)-(p^2-q^2)\left(uv+xy+\frac{\delta^2}{4\mathcal{P}\mathcal{Q}}\right)=0$$

where

$$\mathcal{P}^2=(p^2-a^2)(p^2-b^2)$$
$$\mathcal{Q}^2=(q^2-a^2)(q^2-b^2)$$

Generalizations

- In the ABS classification, the same equation is placed on each face of the N-cube.
- Different equations can be placed on each face, so long as consistency is maintained.
- Checkerboard and other patterns arise.

Partial Difference Equations

We consider a 4-cube and identify each translation along an edge on as an iteration

$$\overline{u} = u(l+1, m, n, k)$$

$$\widehat{u} = u(l, m+1, n, k)$$

$$\widetilde{u} = u(l, m, n+1, k)$$

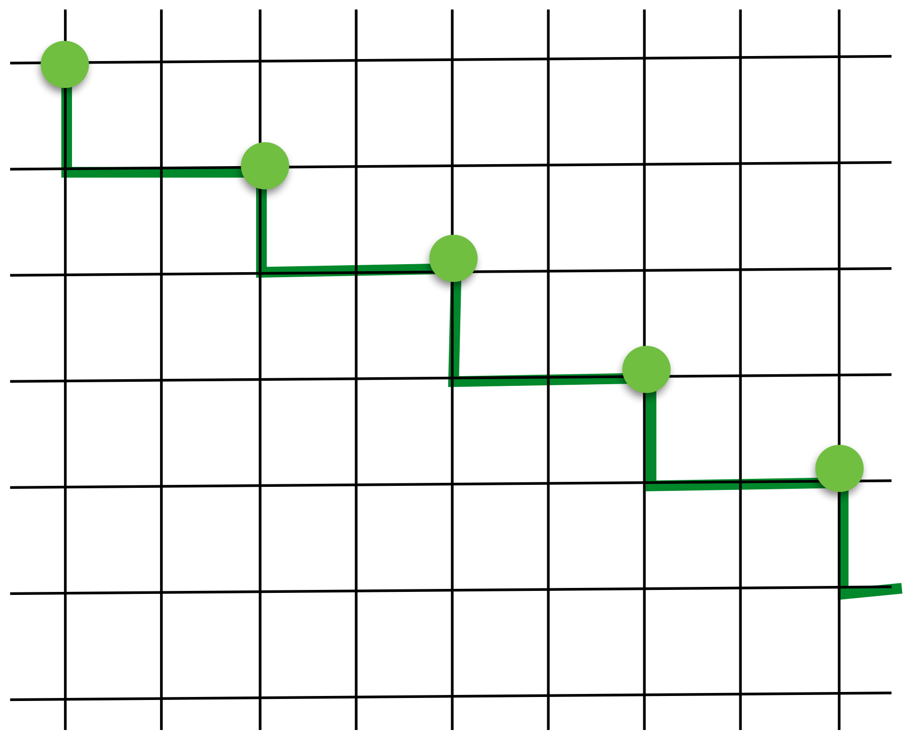
$$\mathring{u} = u(l, m, n, k+1)$$

How are any of these partial difference equations related to Cremona isometries?

Part III

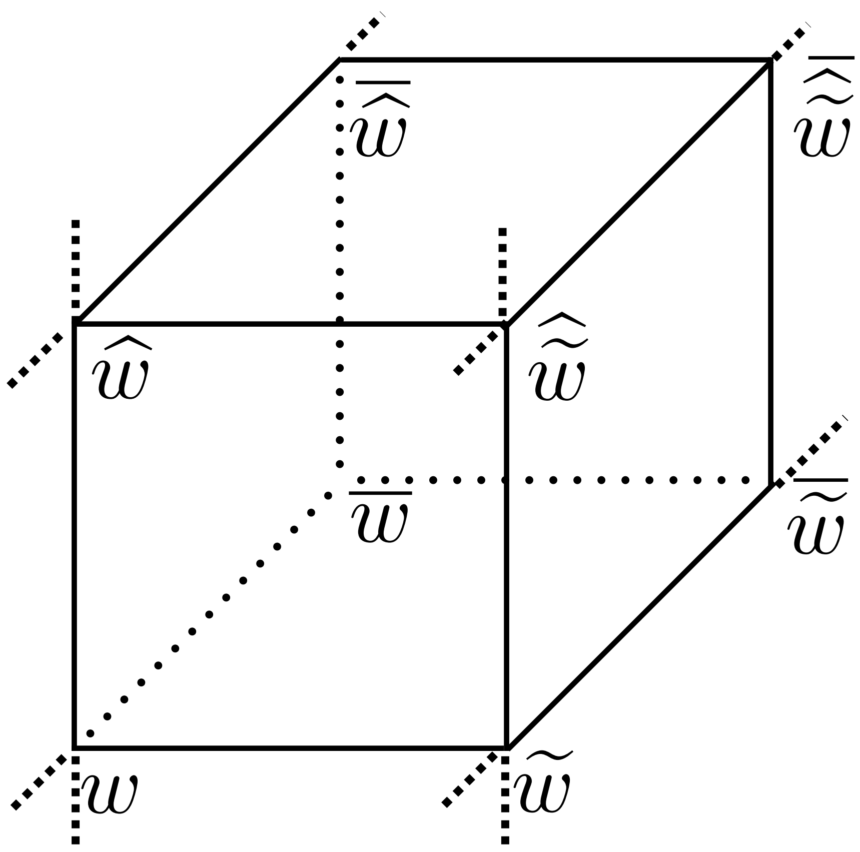
- Lattices
- Dynamics on *N*-cubes
- Symmetry reductions

Discrete Staircases



$$w(l+2, k) = w(l, k+1)$$

H3$_{\delta=0}$

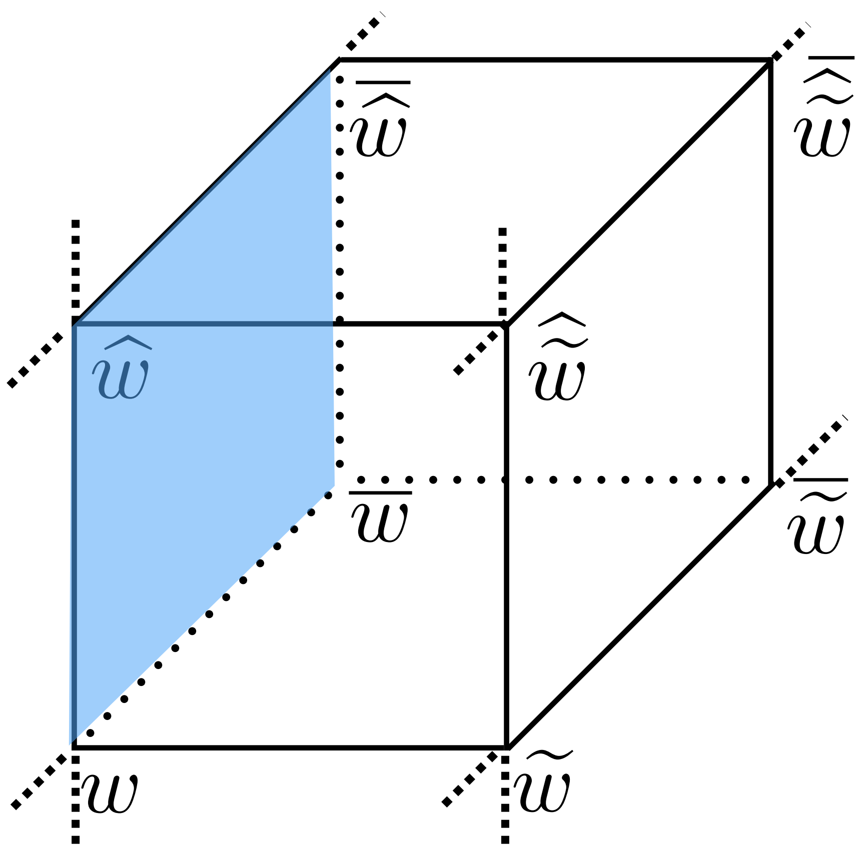


$$\frac{\widehat{\overline{w}}}{w} = \frac{\alpha\overline{w} - \beta\,\widehat{w}}{\alpha\,\widehat{w} - \beta\,\overline{w}}$$

$$w(l,m,k), \alpha = \alpha(l), \beta = \beta(m),$$

$$\bar{\ } : l \mapsto l+1, \hat{\ } : m \mapsto m+1$$

$\mathrm{H3}_{\delta=0}$



$$\frac{\overline{\widehat{w}}}{w} = \frac{\alpha\overline{w} - \beta\,\widehat{w}}{\alpha\,\widehat{w} - \beta\,\overline{w}}$$

$$w(l,m,k), \alpha = \alpha(l), \beta = \beta(m),$$

$$\overline{} : l \mapsto l+1, \widehat{} : m \mapsto m+1$$

Reductions

$$r = \frac{\beta}{\alpha}$$

$$\widehat{w} = \overline{\overline{w}} \;\Rightarrow\; \overline{\overline{\overline{r}}}\, r = \overline{r}\, \overline{\overline{r}}$$

- Grammaticos *et al* 2005 showed

$$h = \frac{\overline{\overline{w}}}{\overline{w}} \;\Rightarrow\; \overline{h}\, h\, \underline{h} = \frac{1 - r\, h}{r - h}$$

- This is the discrete third *q*-Painlevé equation (qP_3)
- Many other examples are now known, starting with ad hoc assumptions, such as specific staircases.

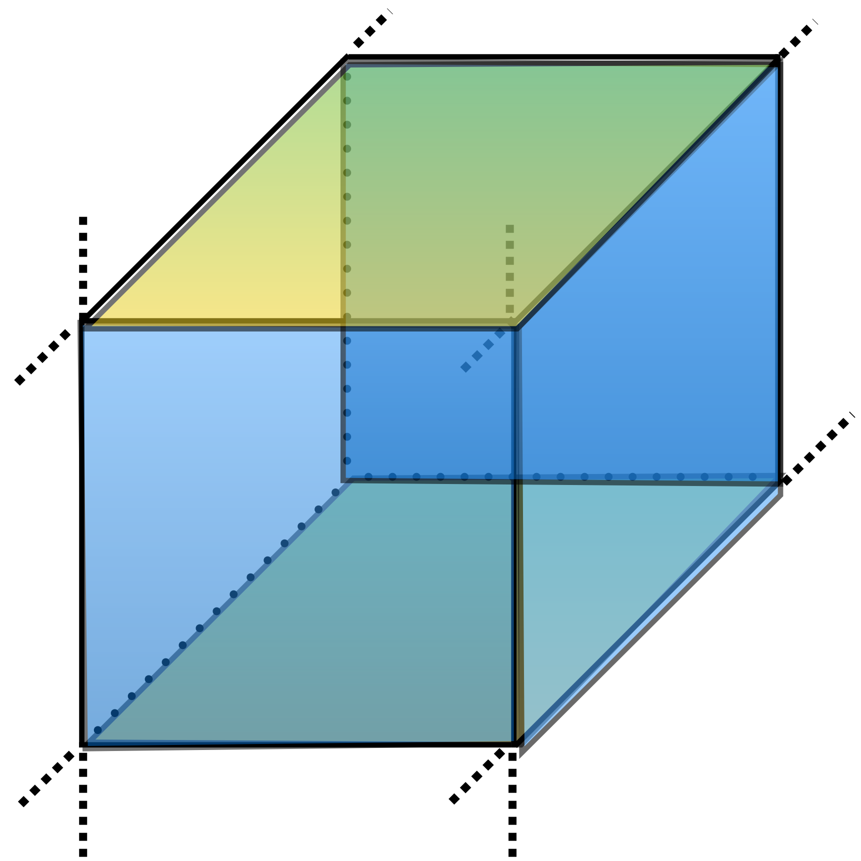
Different Equations on Faces

Boll (20011, 2012) showed that combinations of H3 and H6 provide new consistent systems on the 3-cube, where

$$H6: \quad xy + uv + \delta_1 xu + \delta_2 vy = 0$$

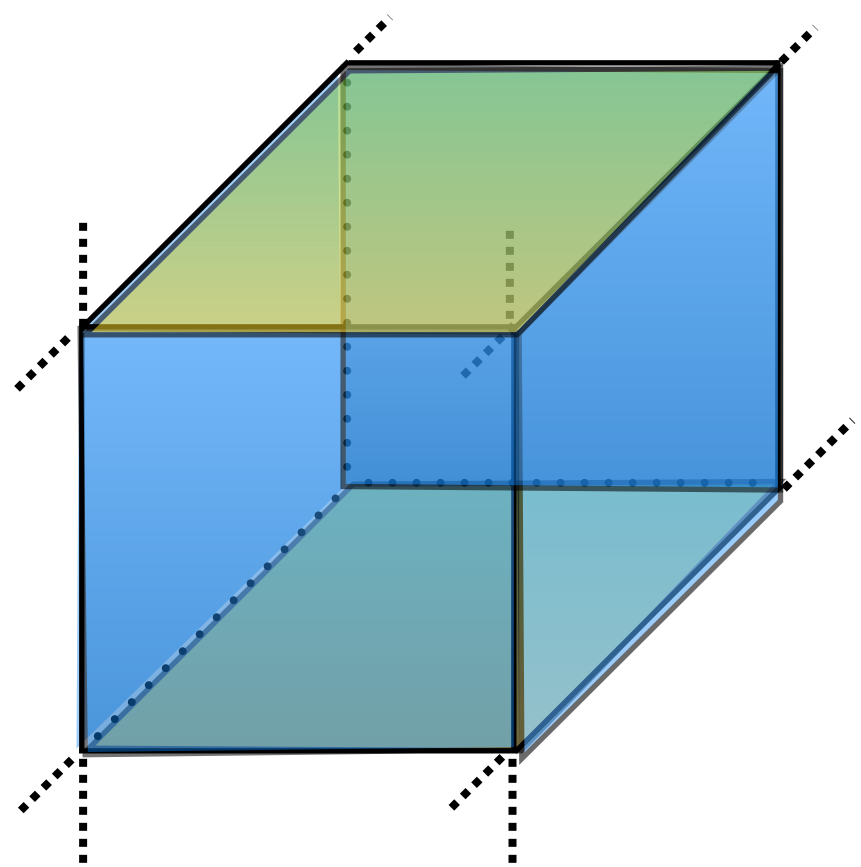
We place H3 ($\delta = 0$) on two faces and H6 ($\delta_2 = 0$) on four faces.

H3 & H6 on 3-cube



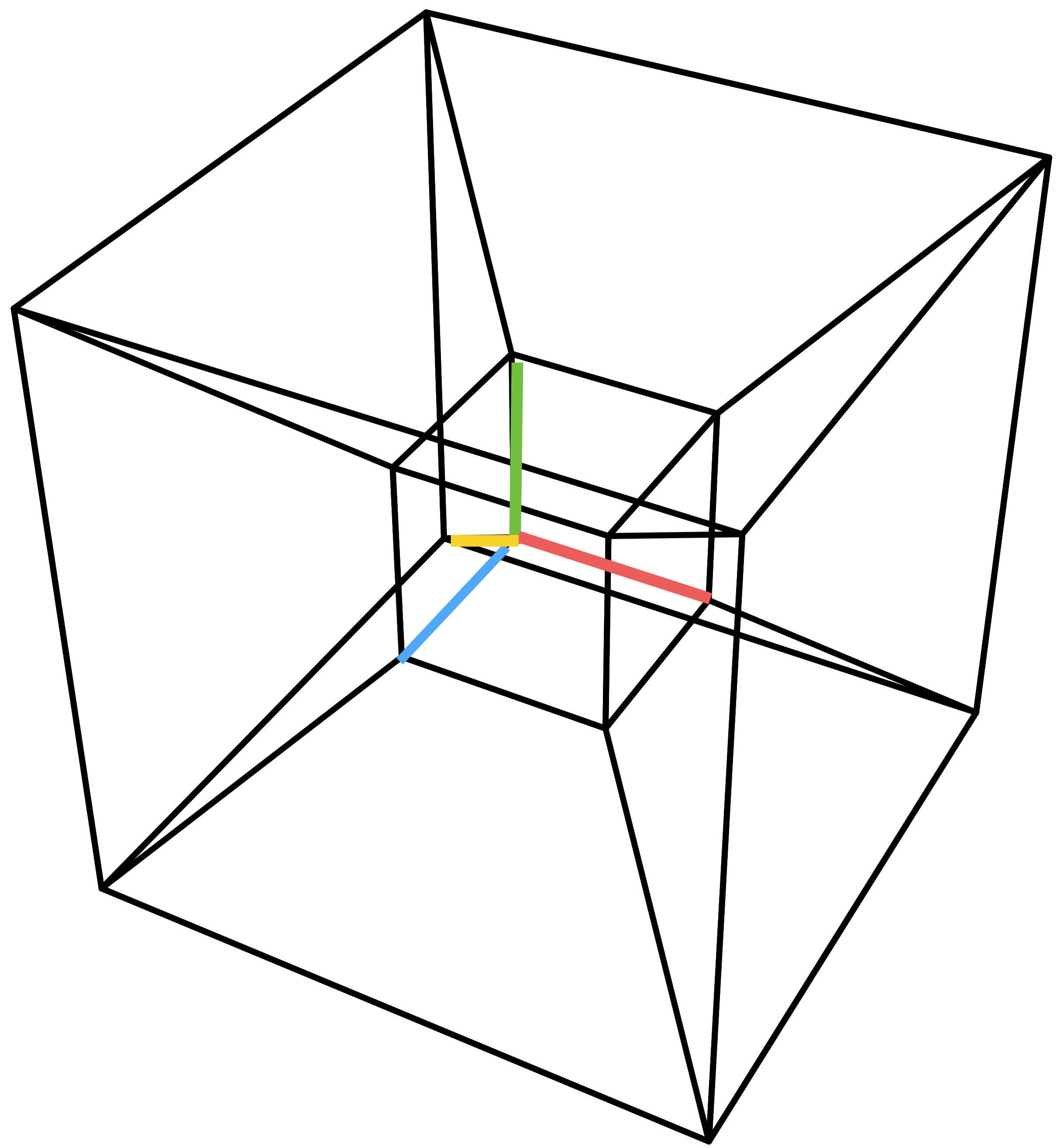
- H3 is on top and bottom faces
- H6 is on the front, right, back and left faces.
- Consistency imposes conditions on parameters.

H3 & H6 on 3-cube



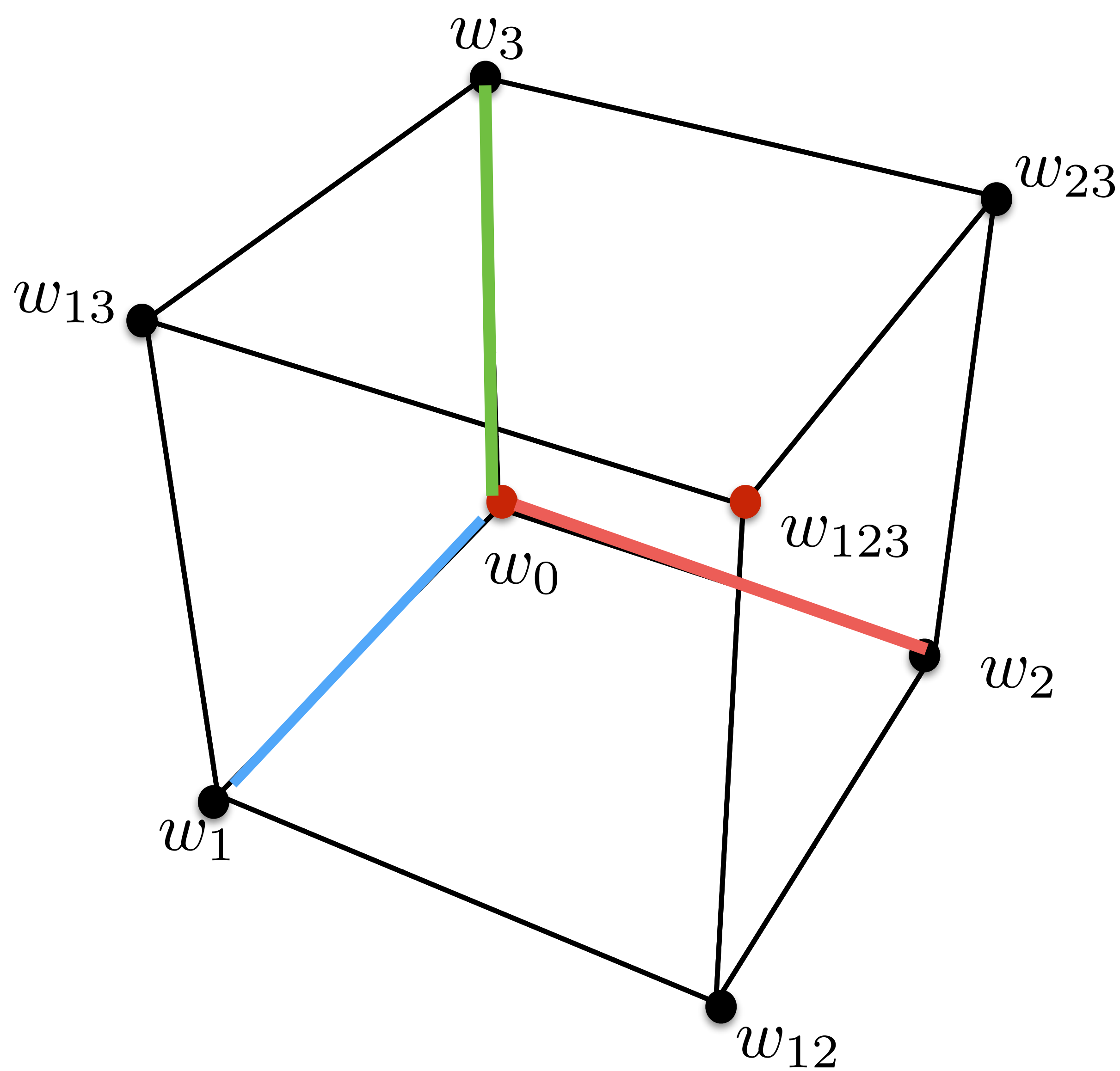
- H3 is on top and bottom faces
- H6 is on the front, right, back and left faces.
- Consistency imposes conditions on parameters.

H3 & H6 on 4-cube



Each sub 3-cube in this 4-cube has 2 copies of H3 and 4 copies of H6 associated to its faces.

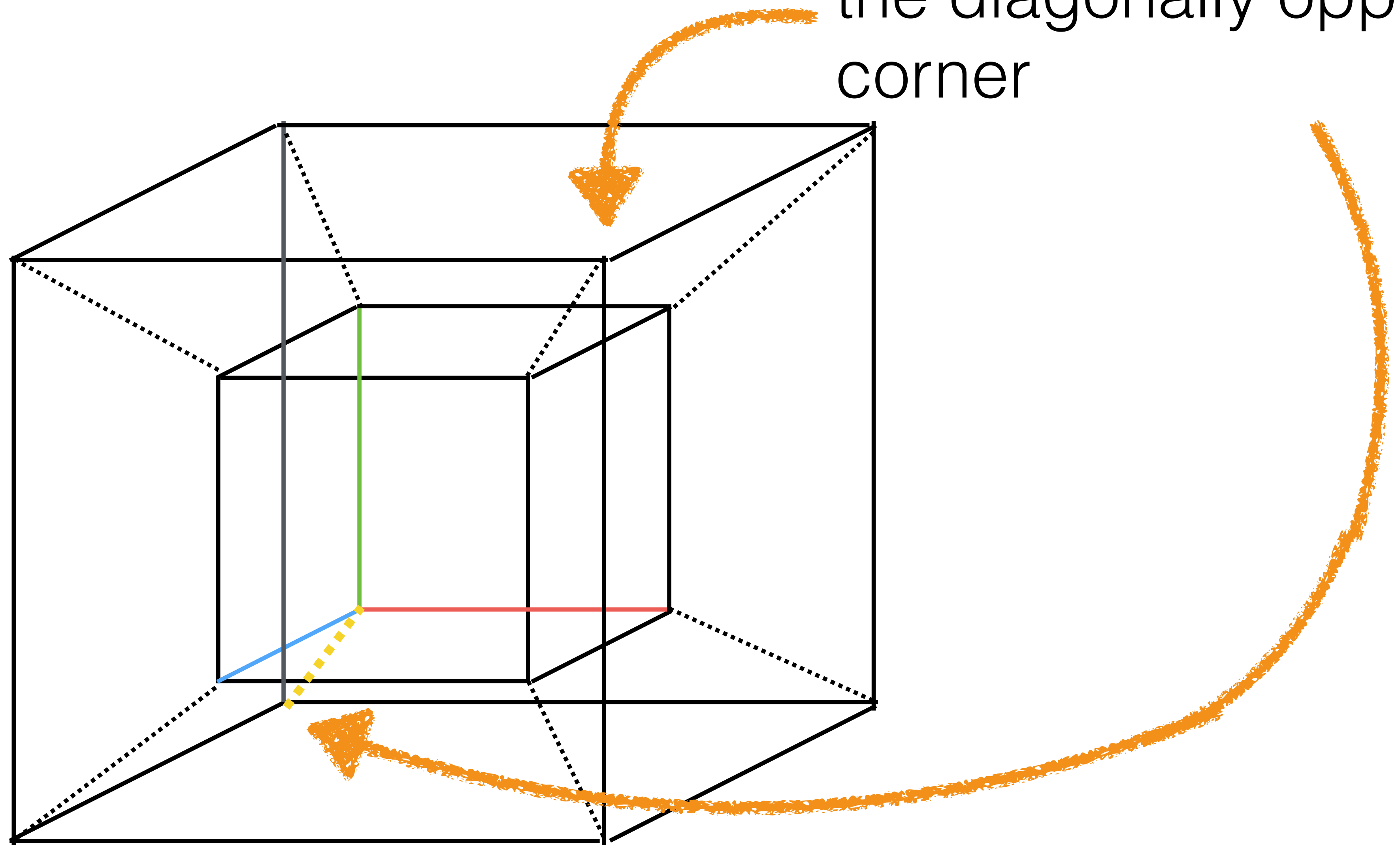
In 3D



Push one corner of the cube to the diagonally opposite corner
$\Rightarrow$ a hexagon

Reduction

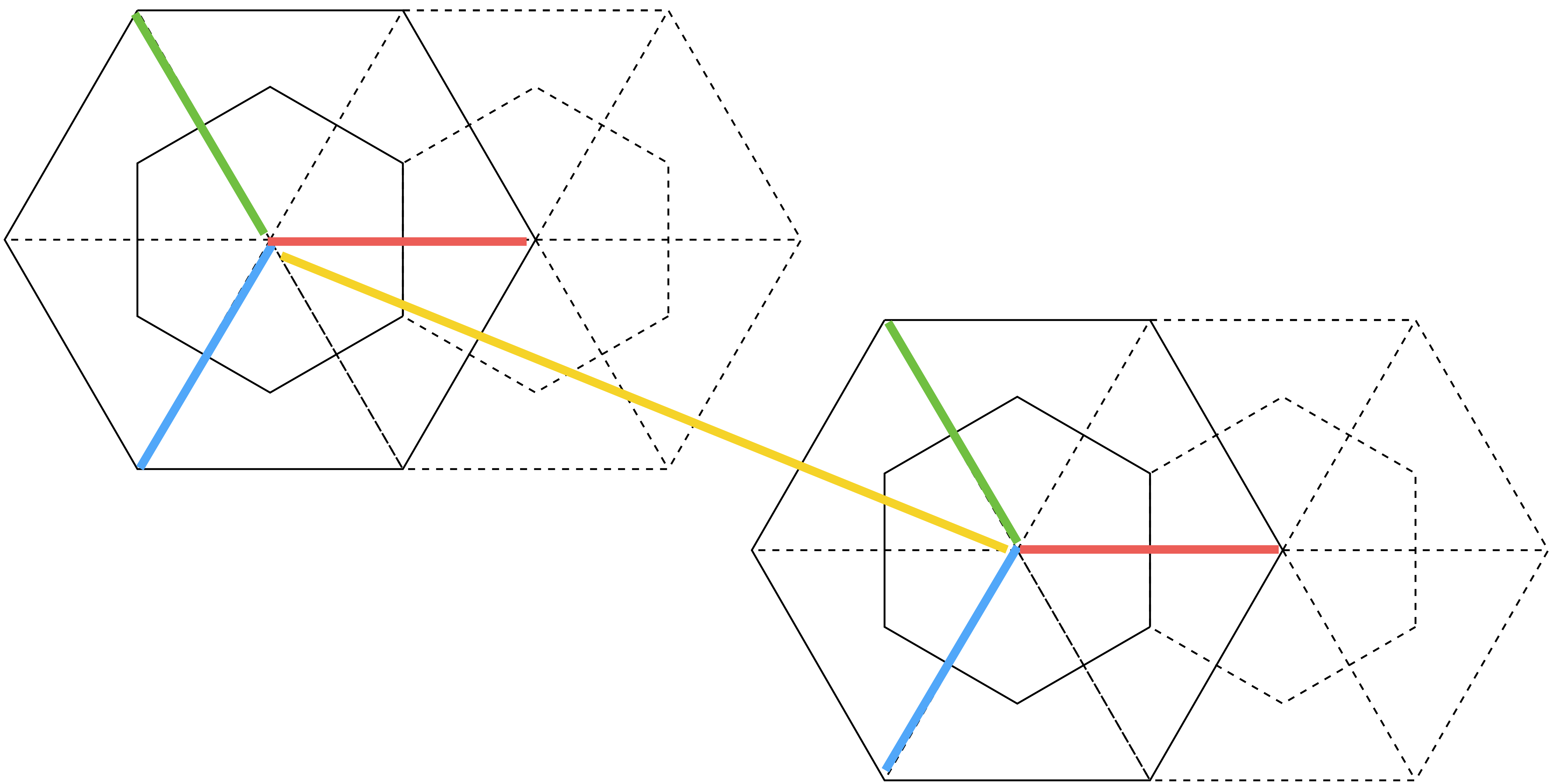
Push one corner to the diagonally opposite corner



For both inner and outer corners.

Reduction

$\widehat{\overline{\widetilde{w}}} = -i\,\lambda\,w \qquad \mathring{\lambda} = q\,\lambda$



Reductions to *q*-discrete Painlevé equations

$$q\text{-P}_{\mathrm{IV}}:\begin{cases} f(qt) = ab\,g(t)\,\dfrac{1+c\,h(t)\,(a\,f(t)+1)}{1+a\,f(t)\,(b\,g(t)+1)}, \\[2ex] g(qt) = bc\,h(t)\,\dfrac{1+a\,f(t)\,(b\,g(t)+1)}{1+b\,g(t)\,(c\,h(t)+1)}, \\[2ex] h(qt) = ca\,f(t)\,\dfrac{1+b\,g(t)\,(c\,h(t)+1)}{1+c\,h(t)\,(a\,f(t)+1)}, \end{cases}$$

$$q\text{-P}_{\mathrm{III}}:\begin{cases} g(qt) = \dfrac{a}{g(t)f(t)}\,\dfrac{1+tf(t)}{t+f(t)}, \\[2ex] f(qt) = \dfrac{a}{f(t)g(qt)}\,\dfrac{1+btg(qt)}{bt+g(qt)}, \end{cases}$$

$$q\text{-P}_{\mathrm{II}}:\quad f(pt) = \frac{a}{f(p^{-1}t)f(t)}\,\frac{1+tf(t)}{t+f(t)},$$

Discrete Monodromy Problems

Reductions also provide linear problems, e.g.

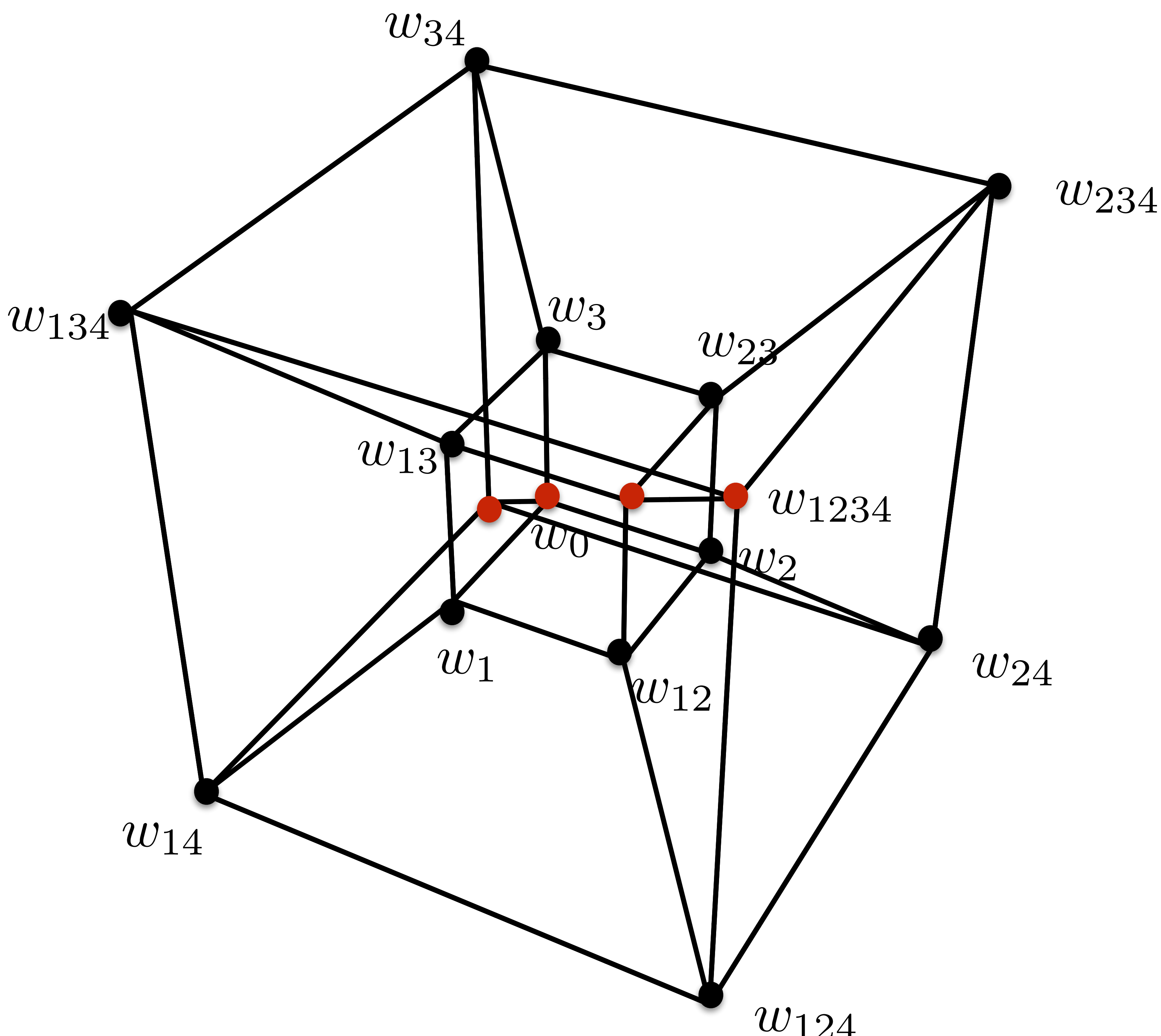
$$\phi(qx,t)=\begin{pmatrix}\frac{qt}{h(t)}x & 1\\ -1 & \frac{qh(t)}{t}x\end{pmatrix}\cdot\begin{pmatrix}\frac{act}{f(t)}x & 1\\ -1 & \frac{acf(t)}{t}x\end{pmatrix}\cdot\begin{pmatrix}\frac{at}{g(t)}x & 1\\ -1 & \frac{ag(t)}{t}x\end{pmatrix}.\phi(x,t),$$

$$\phi(x,qt)=\begin{pmatrix}-\frac{(qt^2-1)h(t)}{(1+b+bch(t))tg(t)}x & -1\\ 1 & 0\end{pmatrix}.\phi(x,t).$$

whose compatibility condition is qP_{IV}

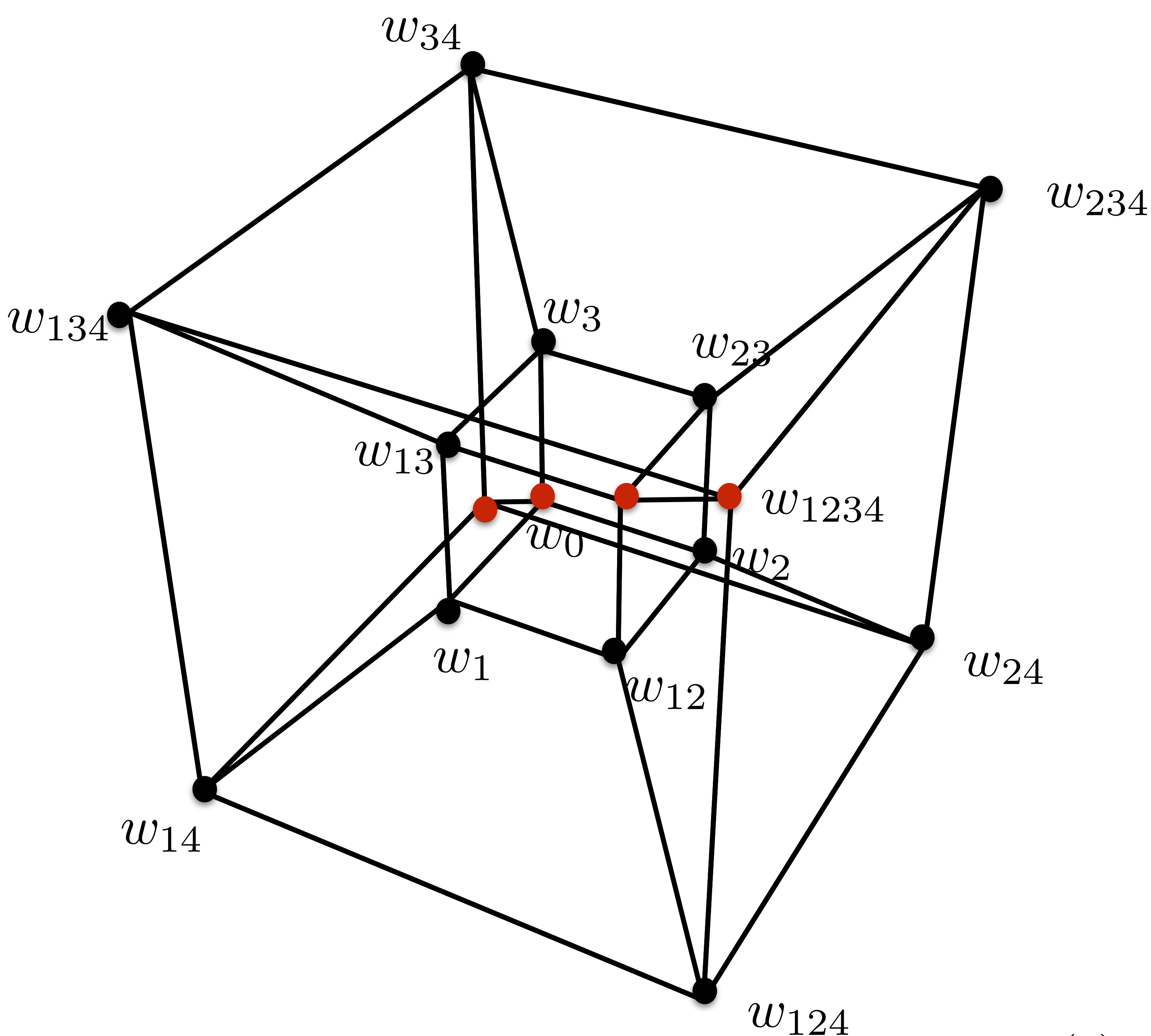
Joshi, Nakazono & Shi, 2015

Generalization



$\Rightarrow \quad \widetilde{\mathcal{W}}(A_{n-1}^{(1)})$

Generalization

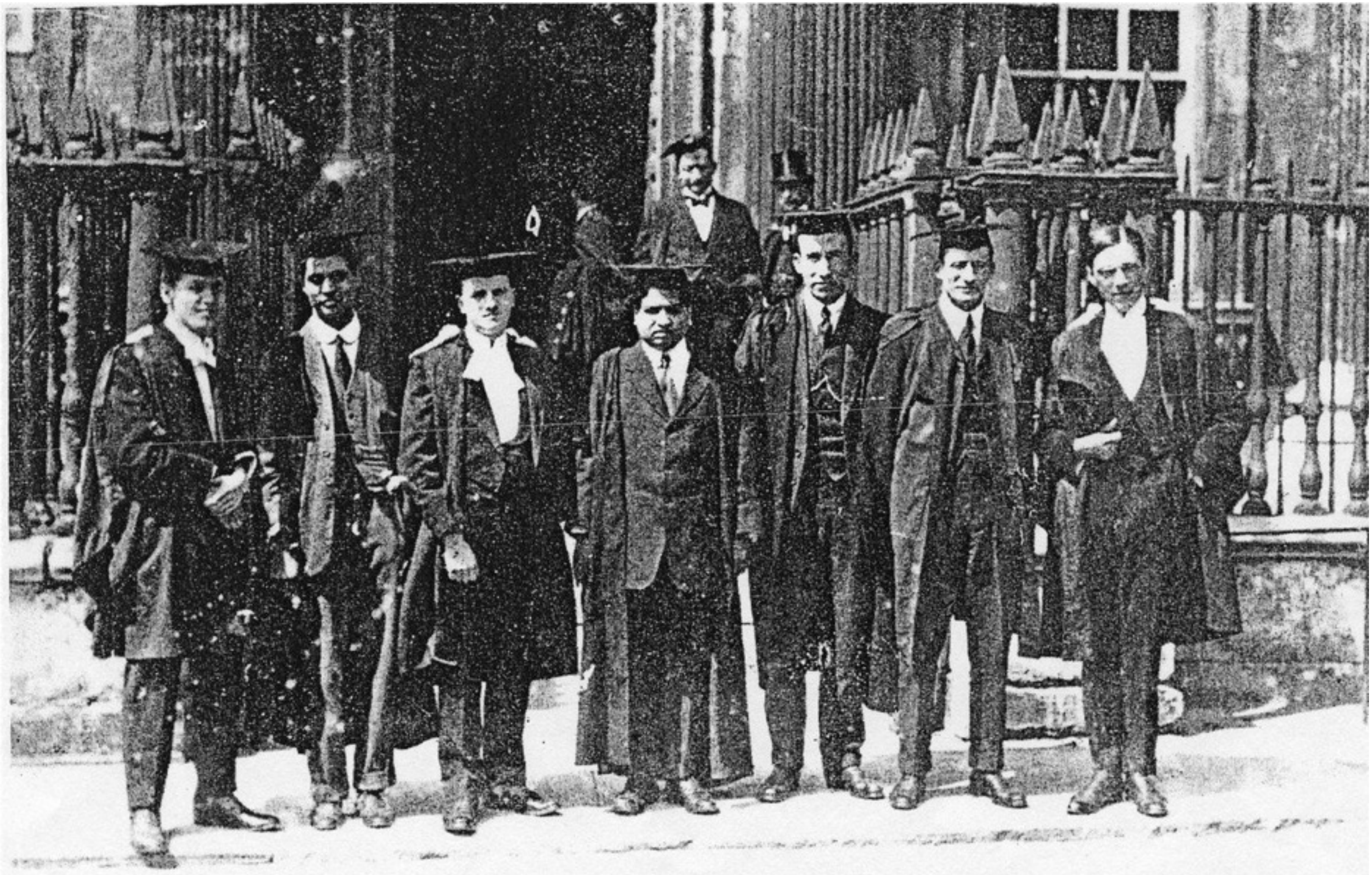


Generalizable to *n* dimensions $\Rightarrow \widetilde{\mathcal{W}}(A_{n-1}^{(1)})$

Summary

- Reduction of the n-cube leads to q-discrete Painlevé equations of higher dimensions, with symmetry group $W(A_{n-1}^{(1)}+A_1^{(1)})$.
- The symmetry lattice is realised as tessellations of the Voronoi cell of A_{n-1}.
- The lattice equations are found through ω-lattices, related to tau functions of discrete Painlevé equations.
- Other symmetry groups also possible.

Joshi, Nakazono & Shi 2014,2015



The mathematician's patterns, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. *GH Hardy, A Mathematician’s Apology, 1940*