Geometry and Integrability

How I learnt to love exceptional Lie groups

Nalini Joshi

@monsoon0

THE UNIVERSITY OF SYDNEY

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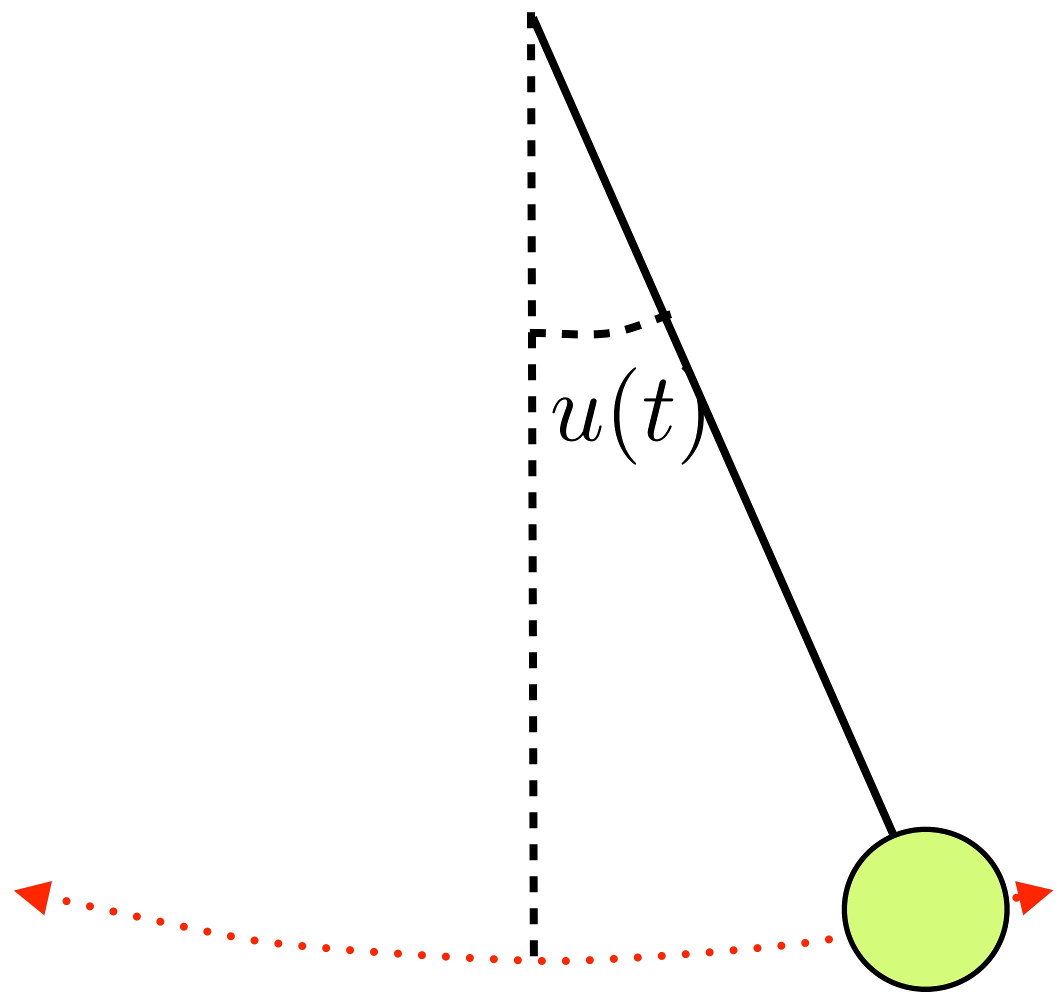
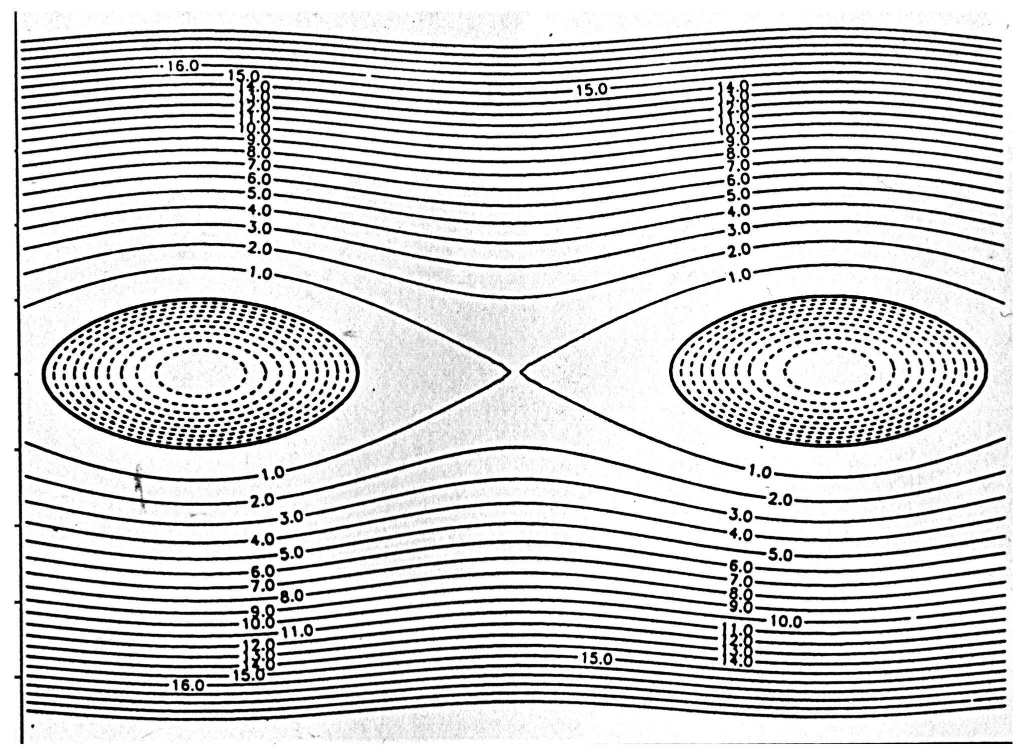
Dynamical Systems



$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -\sin\big(u(t)\big) \end{cases}$$

Phase Space



$$H(u,v) = \frac{v^2}{2} - \cos\big(u(t)\big)$$

Another View

$$f(t) = e^{iu(t)}$$

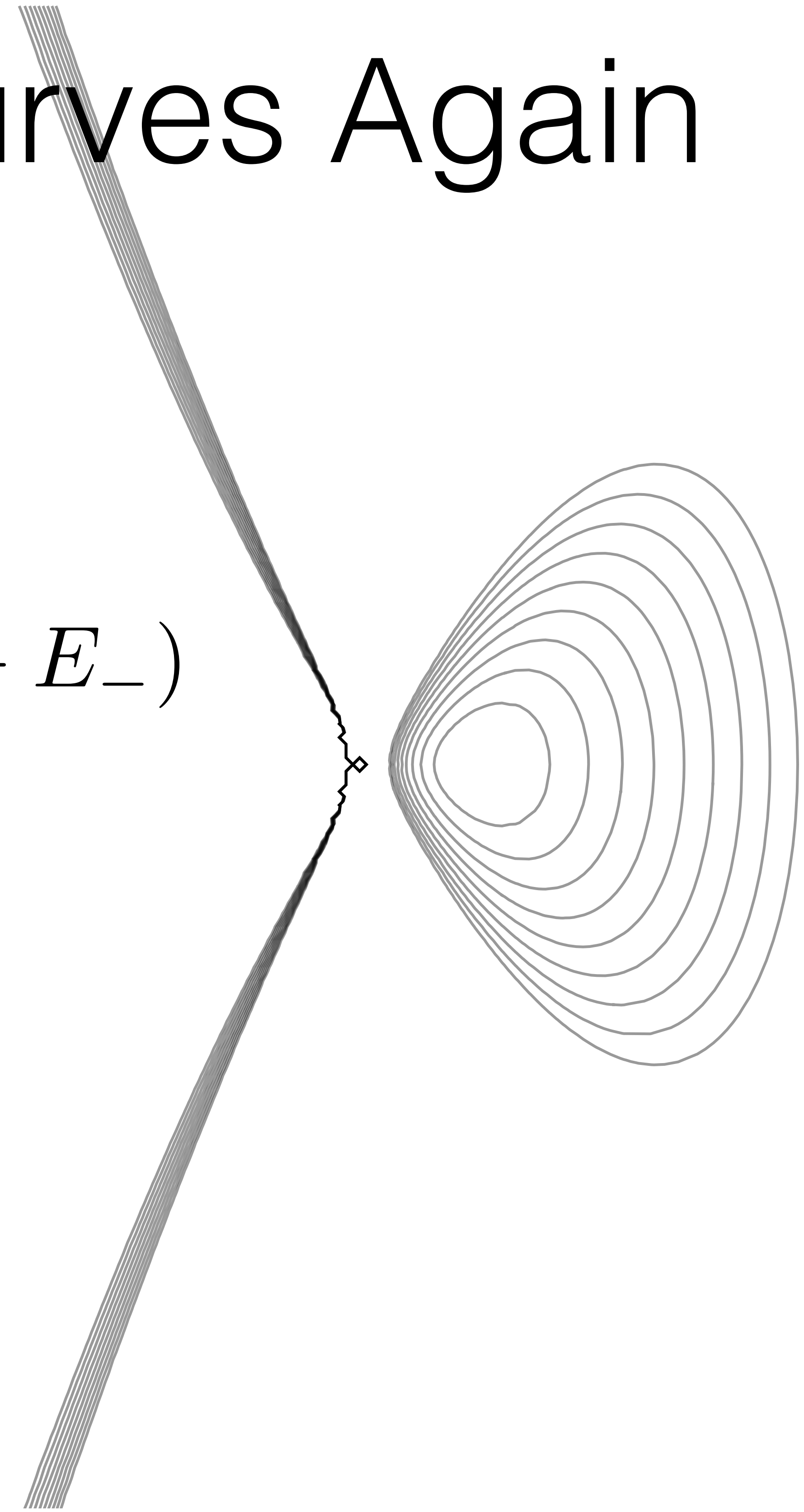
$$\Rightarrow \begin{cases} \ddot{f} &= \frac{\dot{f}^2}{f} - \frac{1}{2}\left(f^2 - 1\right) \\ E &= \frac{\dot{f}^2}{2f^2} + \frac{1}{2}\left(f + \frac{1}{f}\right) \end{cases}$$

$$\Rightarrow \dot{f}^2 = -f^3 - 2\,E\,f^2 - 1$$

Phase Curves Again

$$y^2 = -\,x\,(x - E_+)\,(x - E_-)$$

$$E_\pm = E \pm \sqrt{E^2 - 1}$$



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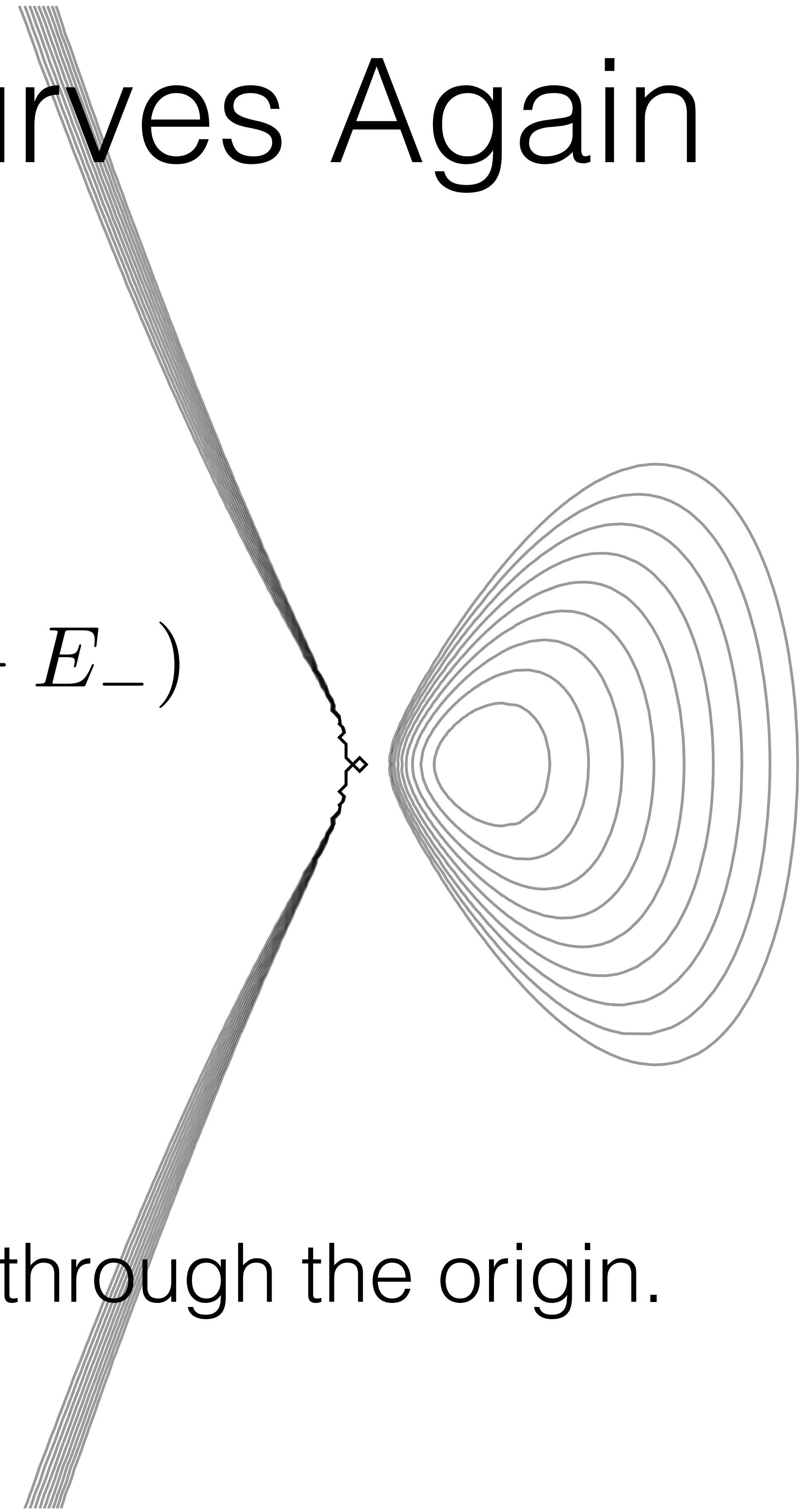


The trajectories all go through the origin.

Two Problems

- The trajectories are indistinguishable as they pass through the origin.
- The phase space is no longer compact; Liouville's theorem* does not necessarily hold.

* Liouville's thm gives the solution by quadratures.

Elliptic Curves

- These properties are shared by many nonlinear mathematical models.
- A prototypical model:

$$\ddot{w} = 6\, w^2 - \frac{g_2}{2}$$

$$\Rightarrow \qquad \frac{\dot{w}^2}{2} = 2\, w^3 - \frac{g_2}{2}\, w - \frac{g_3}{2}$$

$$\Rightarrow \qquad w(t) = \wp\big(t - t_0; g_2, g_3\big)$$

g_2 is given, g_3 is free

Weierstrass Cubics

- In phase space, $\dot{w} = y, w = x,$ the conserved quantity becomes

$$y^2 = 4\,x^3 - g_2\,x - g_3$$

- Initial values determine g_3

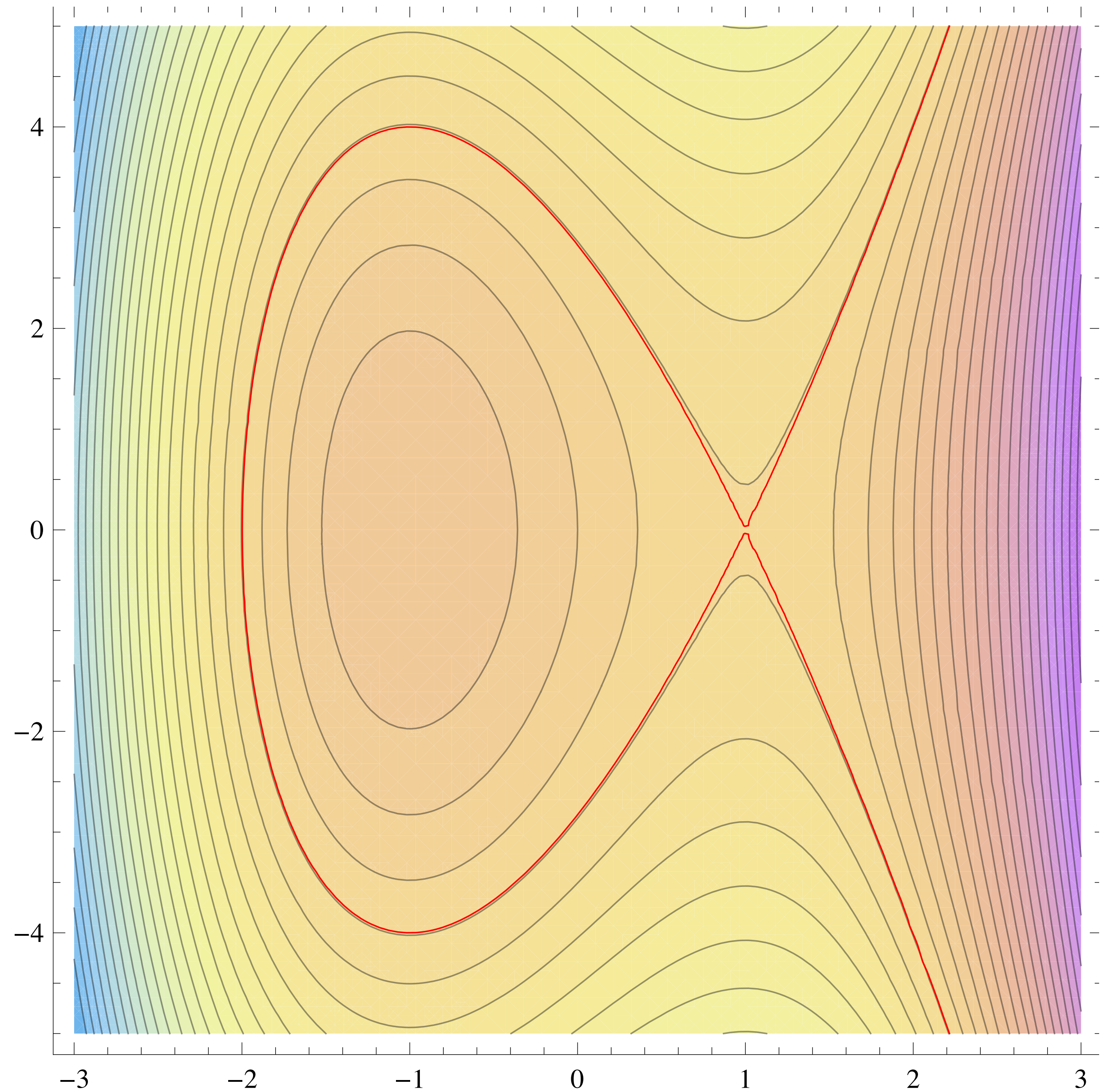
- Each value of g_3 defines a level curve of

$$f(x,y) = y^2 - 4\,x^3 + g_2\,x$$

Cubic Pencil

A Weierstrass cubic pencil:

$$y^2 - 4x^3 + g_2 x + g_3 = 0, \quad g_2 = 2, g_3 = -E$$



Projective Space

- What if *x*, *y* become unbounded?
- Use projective geometry: $x = \frac{u}{w}, y = \frac{v}{w}$

$$[x, y, 1] = [u, v, w] \ \in \mathbb{CP}^2$$

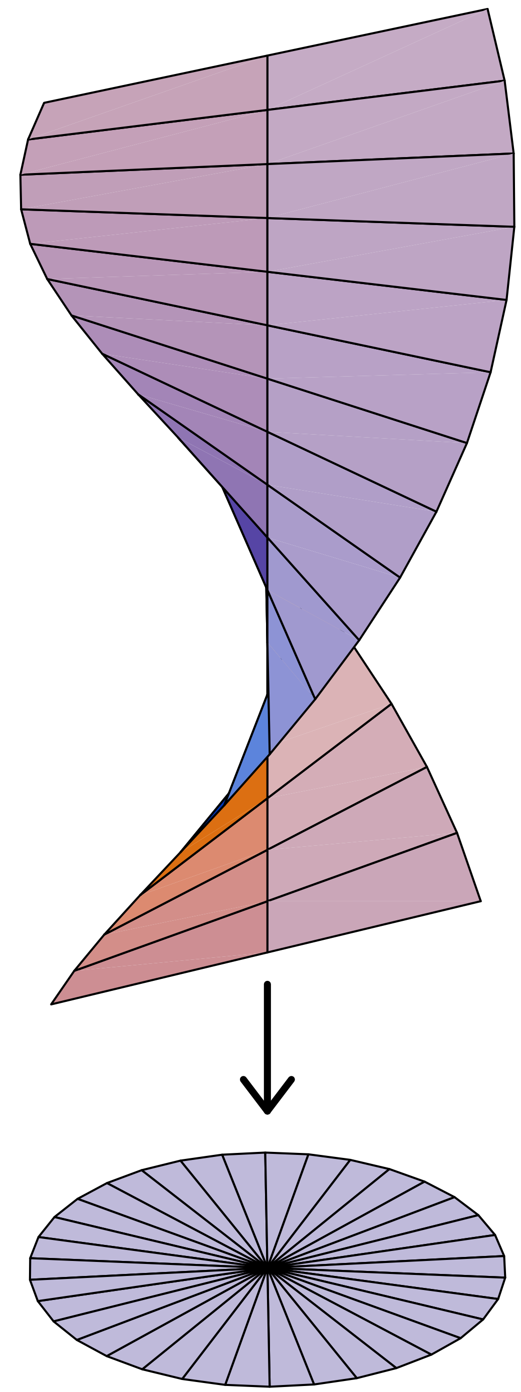
- The level curves are now

$$F_{\mathrm{I}} = wv^2 - 4u^3 + g_2 uw^2 + g_3 w^3$$

all intersecting at the base point [0, 1, 0].

$\Rightarrow$ To describe solutions, resolve the flow through this point

Resolving a base pt



From JJ Duistermaat, QRT Maps and Elliptic Surfaces, Springer Verlag, 2010

Resolution

- "Blow up" the singularity or base point:

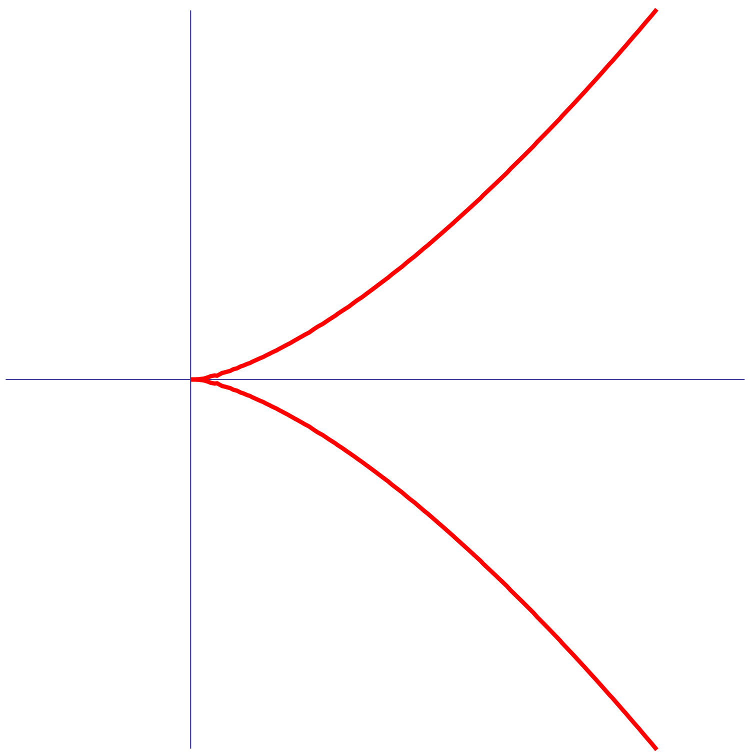
$$f(x, y) = y^2 - x^3$$
$$(x, y) = (x_1, x_1\, y_1)$$
$$\Rightarrow x_1^2\, y_1^2 - x_1^3 = 0$$
$$\Leftrightarrow x_1^2\, (y_1^2 - x_1) = 0$$

- Note that

$$x_1 = x, y_1 = y/x$$

Method

$$y^2 = x^3$$



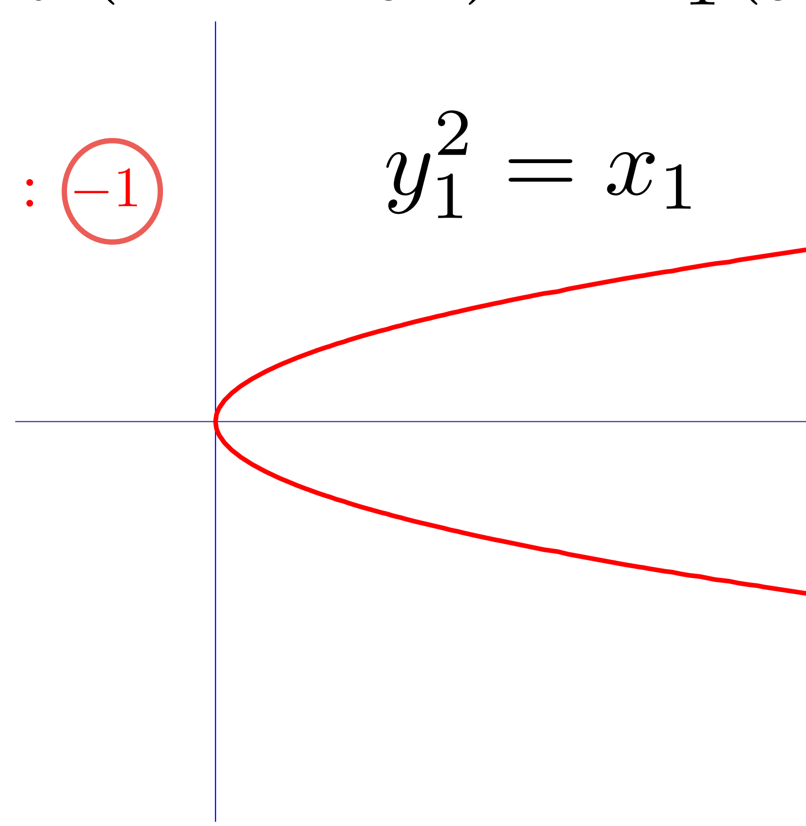
$$f(x, y) = y^2 - x^3$$

$$(x, y) = (x_1, x_1\, y_1)$$

$$f(x_1, x_1\, y_1) = x_1^2 (y_1^2 - x_1)$$

$L_1 : -1$

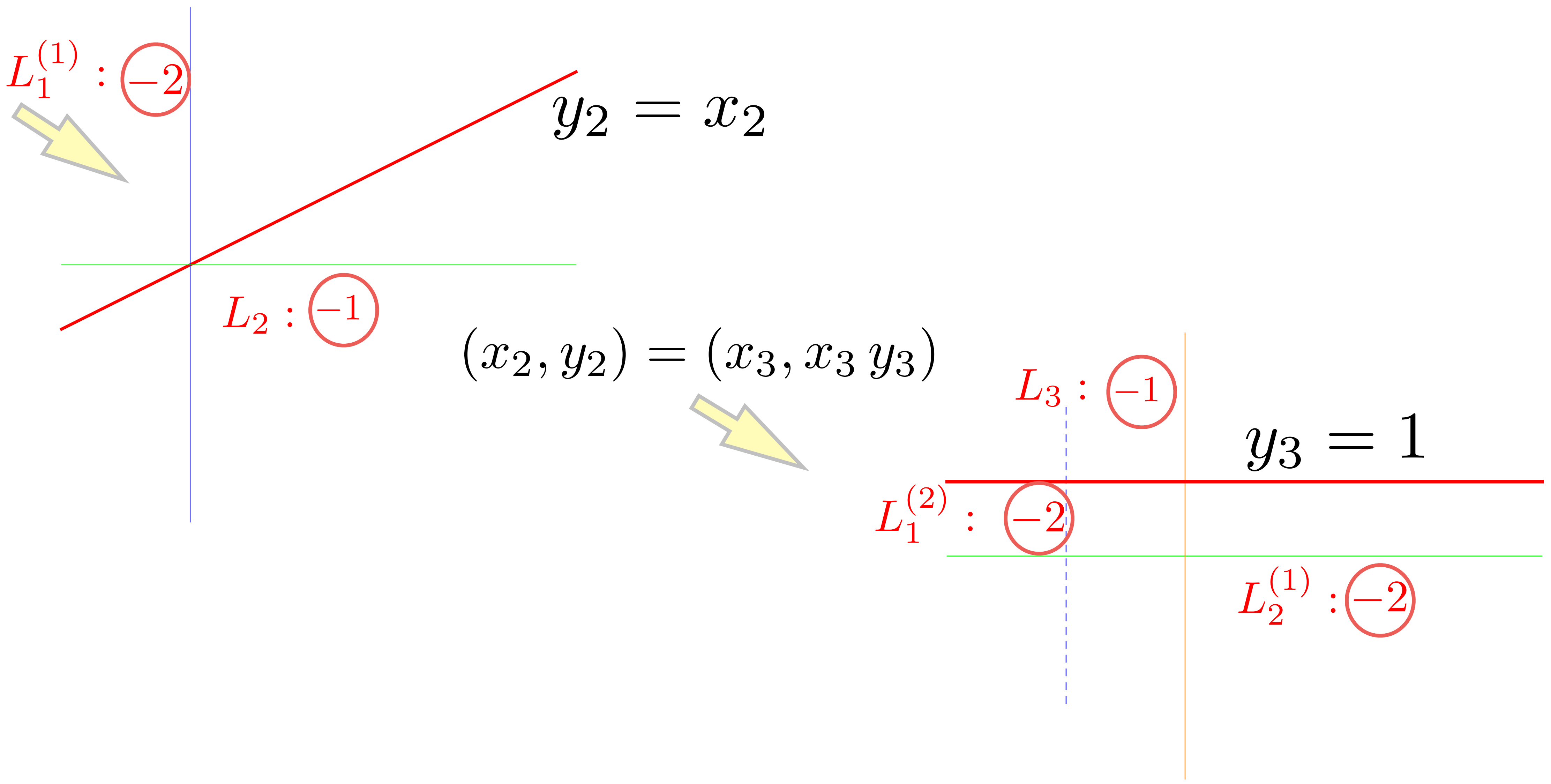
$$y_1^2 = x_1$$



$$f_1(x_1, y_1) = y_1^2 - x_1$$

$$f_1(x_2\, y_2, y_2) = y_2(y_2 - x_2)$$

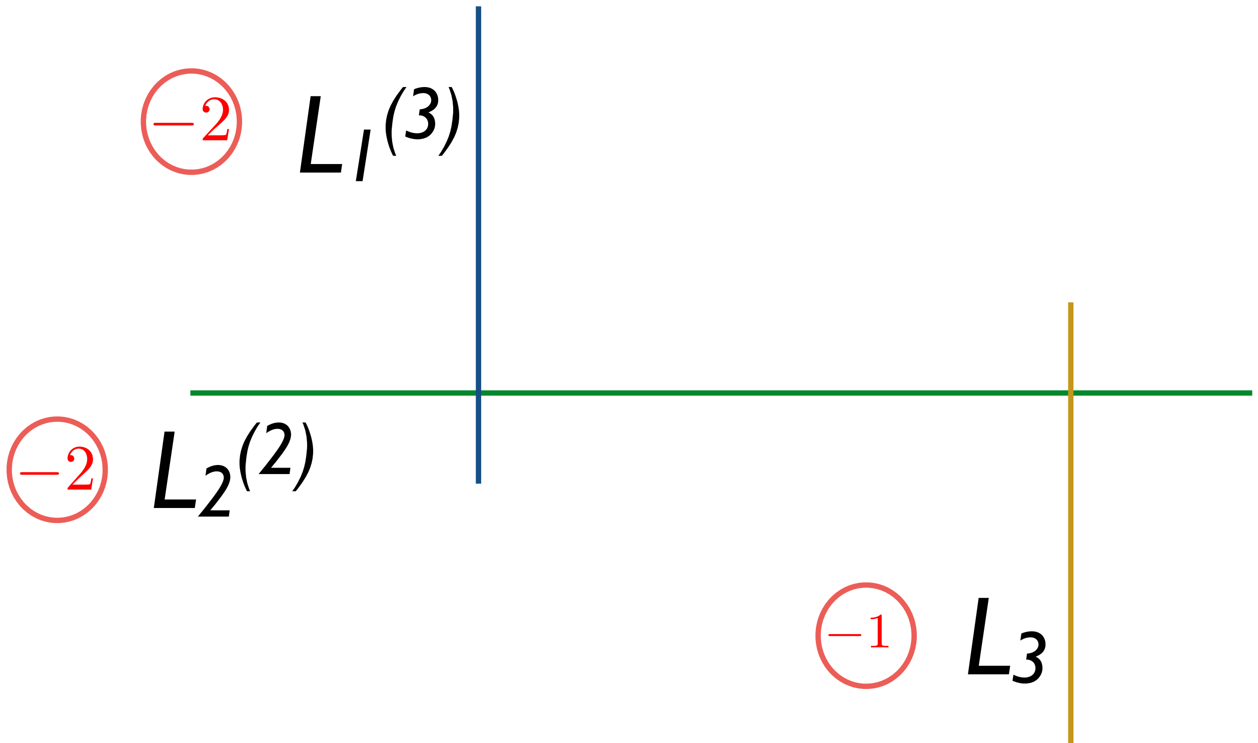
$$(x_1, y_1) = (x_2\, y_2, y_2)$$



$$f_2(x_2, y_2) = y_2 - x_2$$

$$f_2(x_3, x_3\, y_3) = x_3(y_3 - 1)$$

Initial-Value Space



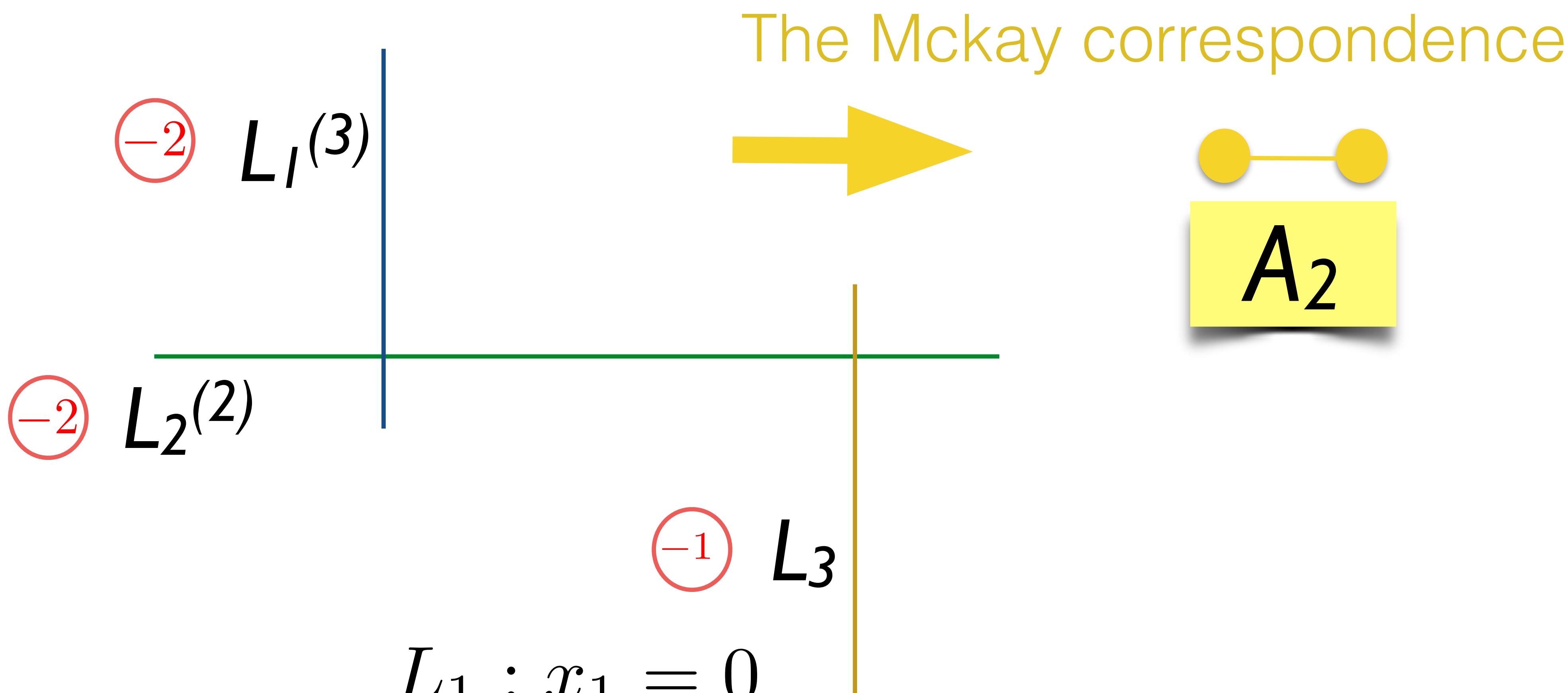
$$L_1 : x_1 = 0$$

$$L_2 : y_2 = 0$$

$$L_3 : x_3 = 0$$

Now the space is compactified and regularised.

Initial-Value Space



$$L_1 : x_1 = 0$$

$$L_2 : y_2 = 0$$

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Now the space is compactified and regularised.

Good Resolution

- When all curves intersect each other transversally at distinct points, the result is called a “good resolution”.
- Hironaka’s theorem guarantees this in complex projective space.
- Note: each transformation had the form

$$x_1 = x, y_1 = y/x$$

Why should we care?

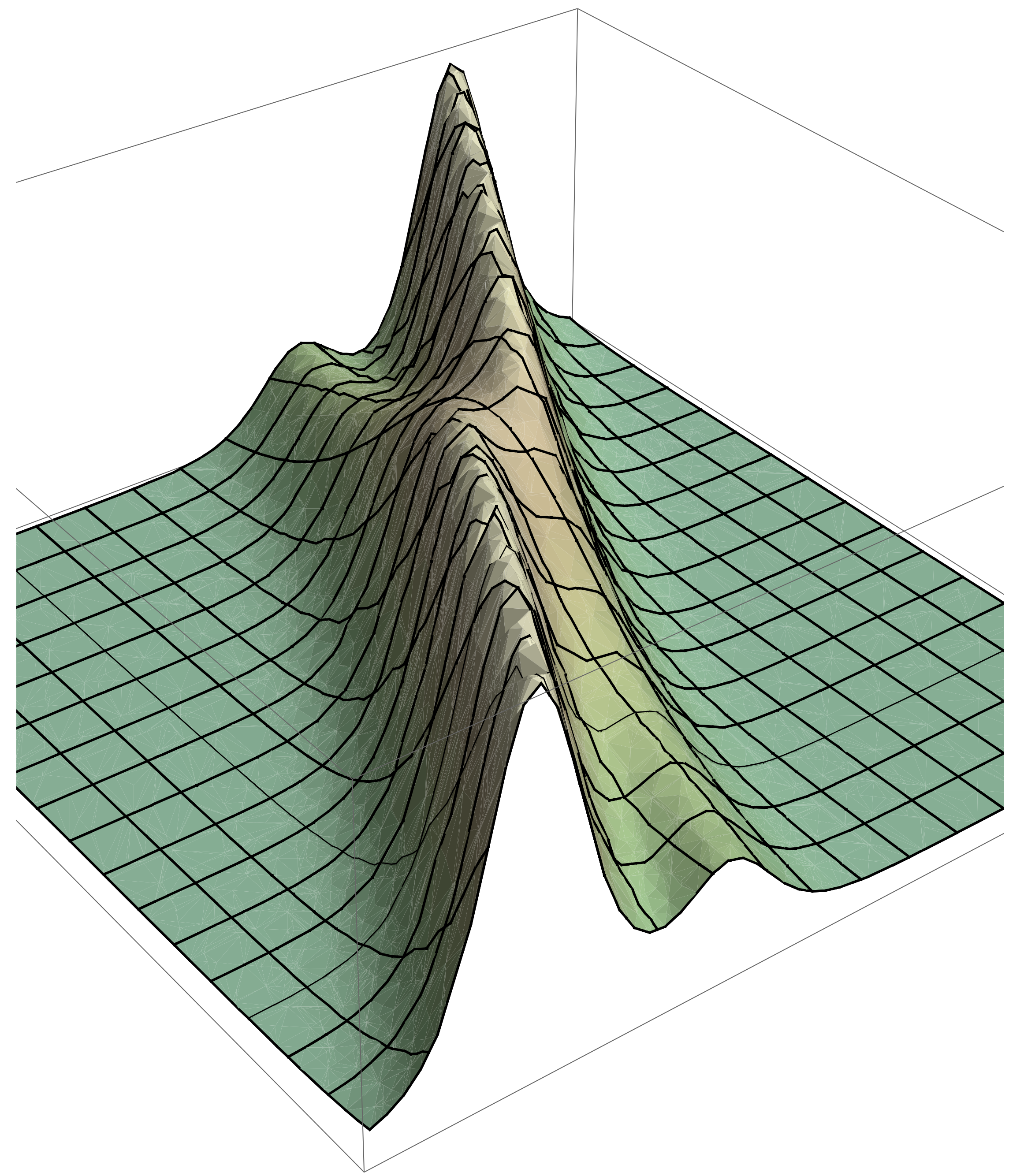
Motivation

- Korteweg-de Vries equation

$$w_\tau + 6\,w\,w_\xi + w_{\xi\xi\xi} = 0$$

$$\begin{cases} w = -2\,y(x) - 2\,\tau \\ x = \xi + 6\,\tau^2 \end{cases}$$

$$\Rightarrow \begin{cases} w_\tau & = -24\,\tau\,y_x - 2 \\ w_\xi & = -2\,y_x \\ w_{\xi\xi\xi} & = -2\,y_{xxx} \end{cases}$$



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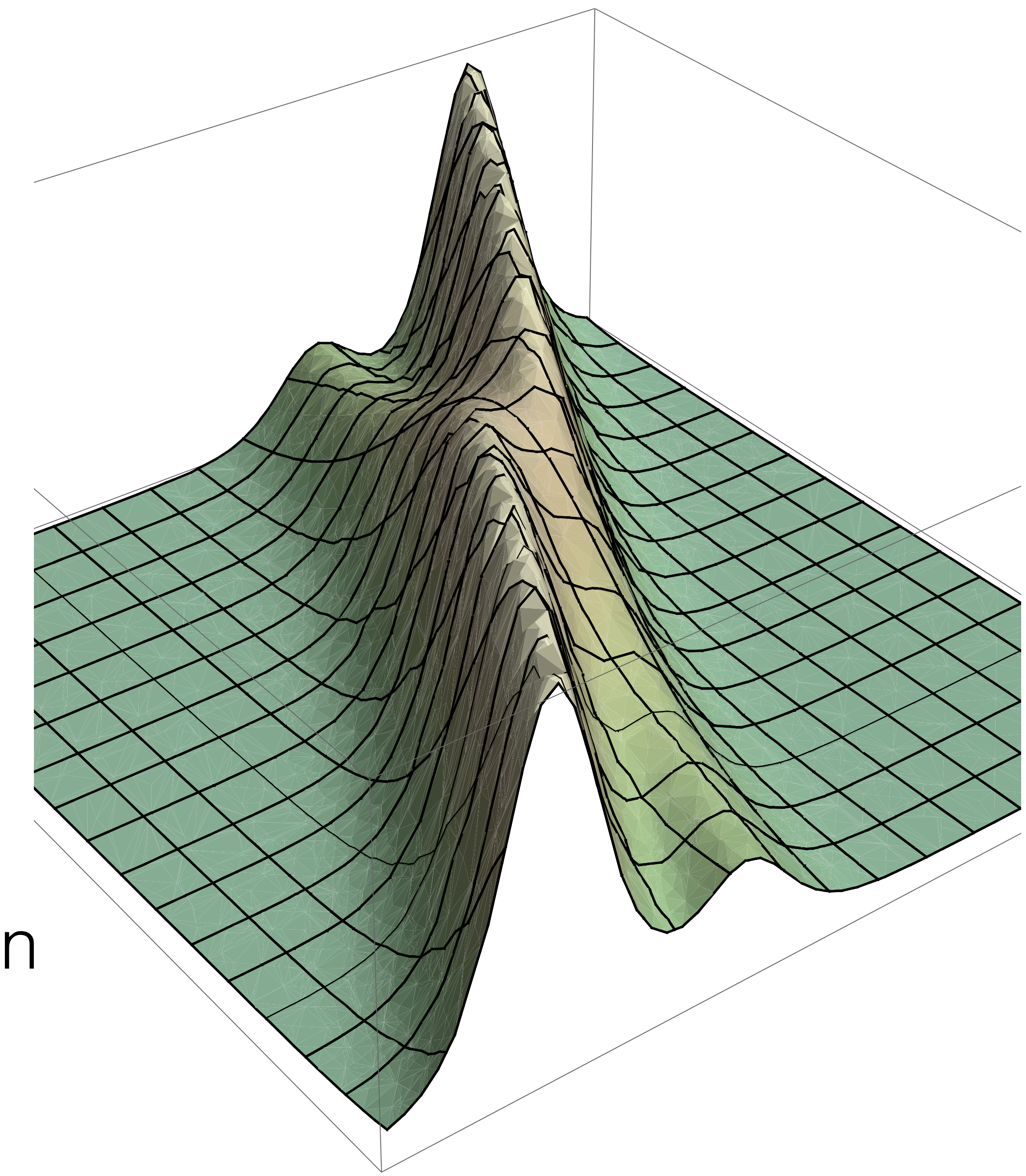
The first Painlevé equation

$$y'' = 6\,y^2 - x$$



Applications

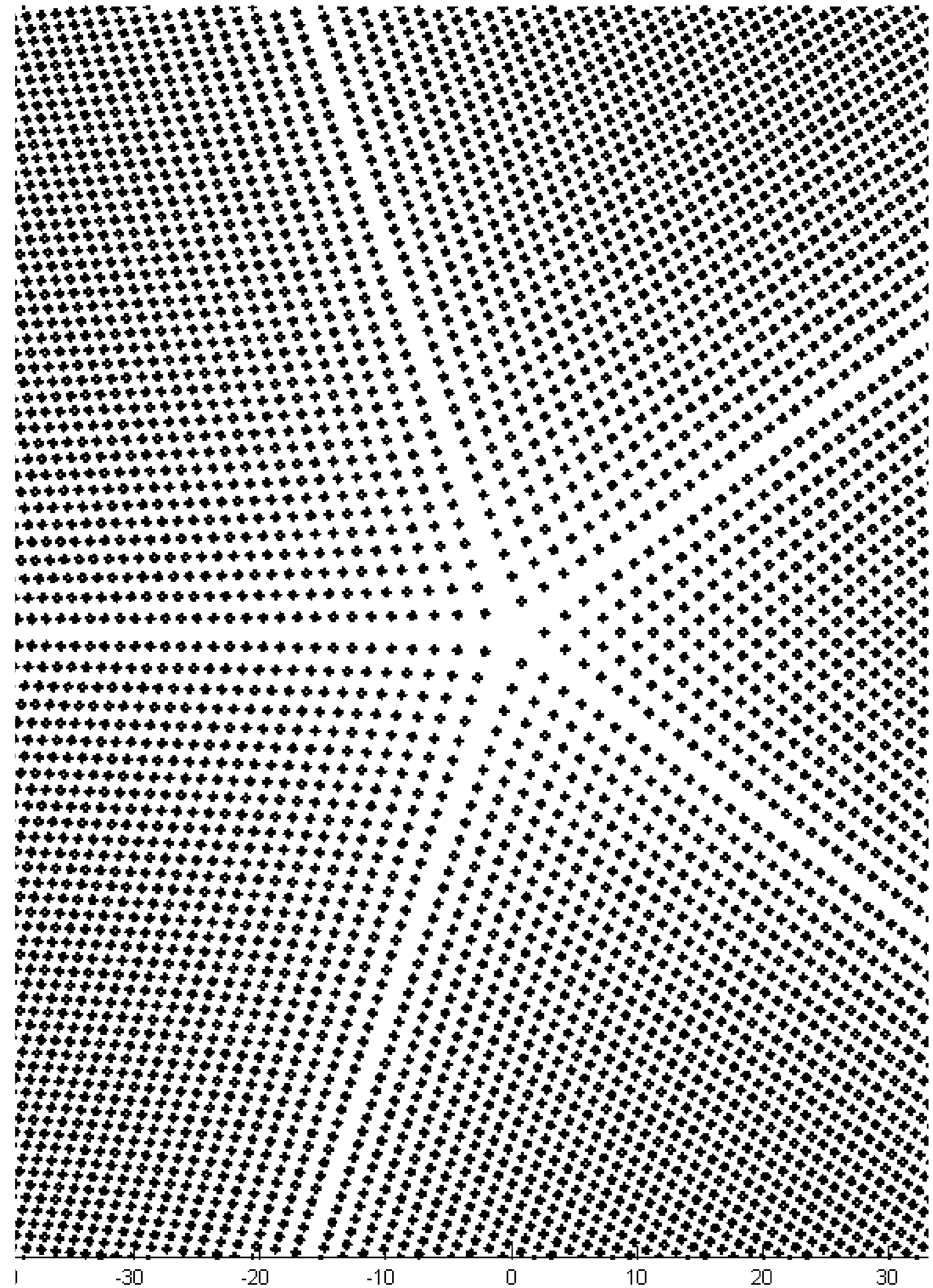
- Electrical structures of interfaces in steady electrolysis *L. Bass, Trans Faraday Soc 60 (1964)1656–1663*
- Spin-spin correlation functions for the 2D Ising model *TT Wu, BM McCoy, CA Tracy, E Barouch Phys Rev B13 (1976) 316–374*
- Spherical electric probe in a continuum gas *PCT de Boer, GSS Ludford, Plasma Phys 17 (1975) 29–41*
- Cylindrical Waves in General Relativity *S Chandrashekar, Proc. R. Soc. Lond. A 408 (1986) 209–232*
- Non-perturbative 2D quantum gravity *Gross & Migdal PRL 64(1990) 127-130*
- Orthogonal polynomials with non-classical weight function *AP Magnus J. Comput Appl. Anal. 57 (1995) 215–237*
- Level spacing distributions and the Airy kernel *CA Tracy, H Widom CMP 159 (1994) 151–174*
- Spatially dependent ecological models: *J & Morrison Anal Appl 6 (2008) 371-381*
- Gradient catastrophe in fluids: *Dubrovin, Grava & Klein J. Nonlin. Sci 19 (2009) 57-94*

What do we know about the solutions of these equations?

Complex Solutions

- Movable poles
- Transcendentality of general solutions
- Special solutions
- Asymptotic behaviours

u(0) = 0, u'(0) = 0



Fornberg & Weideman 2009

$\mathrm{P_I}$

PI General Solutions

- In Boutroux's coordinates:

$$w_1 = t^{1/2}\, u_1(z),\ \ w_2 = t^{3/4} u_2(z),\ \ z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6u_1^2 - 1 \end{pmatrix} - \frac{1}{5z} \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix}$$

- a perturbation of an elliptic curve as $|z| \to \infty$

$$E = \frac{u_2^2}{2} - 2u_1^3 + u_1 \ \Rightarrow\ \frac{dE}{dz} = \frac{1}{5z}\ (6E + 4u_1)$$

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P_l in $\mathbb{CP}^2$

$\mathrm{P_I}$ in $\mathbb{CP}^2$

First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021}$$

$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1}u_{022}$$

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Second chart: $[u_2^{-1} : u_1\, u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$

$$\dot{u}_{031} = -u_{031}^2 - 6u_{032}^2 + 3(5z)^{-1}u_{031}$$

$$\dot{u}_{032} = -u_{031}u_{032} - 6u_{031}^{-1}u_{032}^3 + 1 + (5z)^{-1}u_{032}$$

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base pt $\quad b_0 : \; u_{031} = 0, u_{032} = 0$

First Blow-up

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- Chart (1,1): $[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$

$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$

$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

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- Chart (1,2): $[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$

$$\dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\,(5z)^{-1}\,u_{121}$$
$$\dot{u}_{122} = u_{121}^{-1} - 2\,(5z)^{-1}\,u_{122}$$

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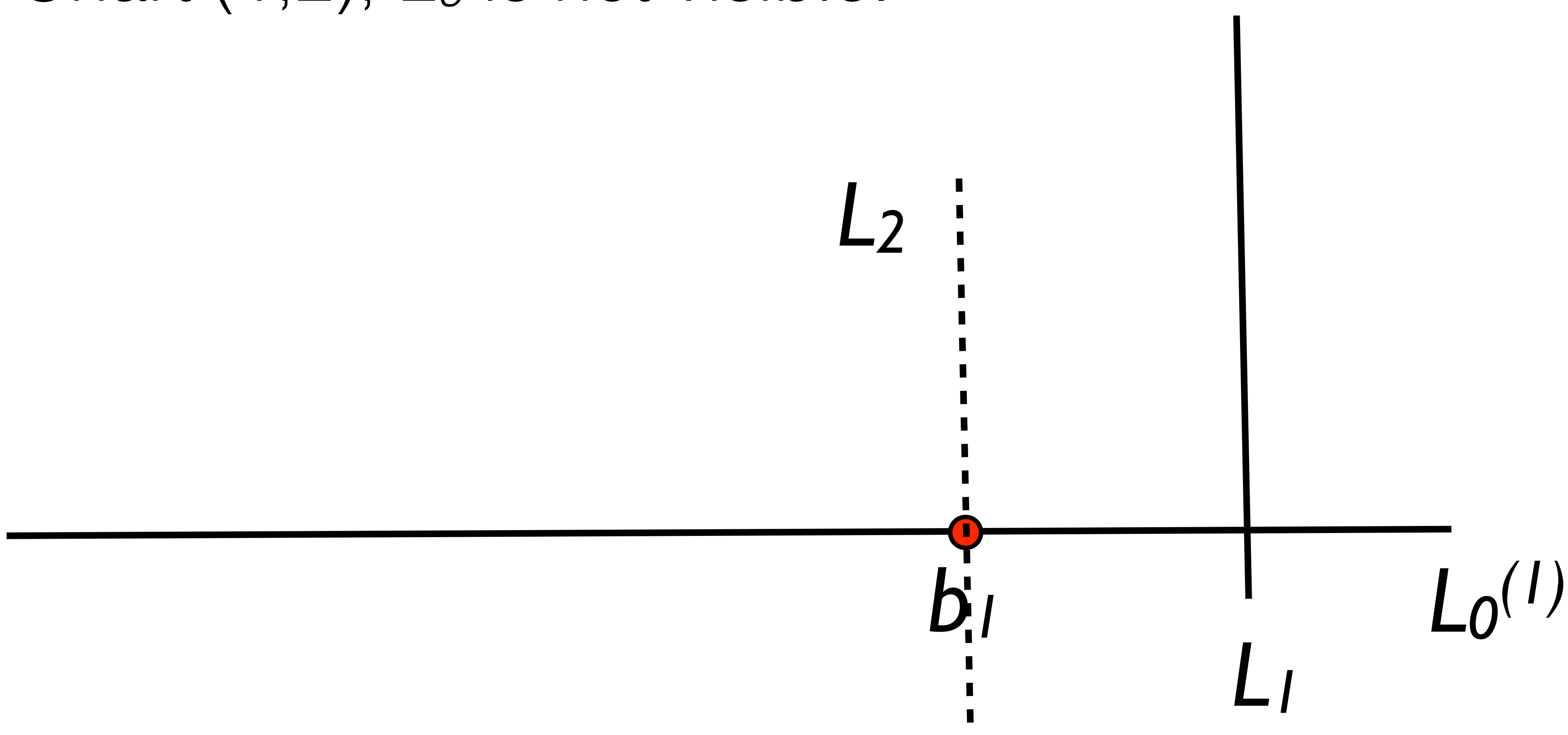
$$\dot{u}_{122} = u_{121}^{-1} - 2\,(5z)^{-1}\,u_{122}$$

→ base pt $b_1 : \; u_{111} = 0, u_{112} = 0$

Exceptional Lines

- In Chart (1, 1), $u_{111}=0$ defines the proper transform $L_0^{(1)}$, while $u_{112}=0$ is L_1.
- In Chart (1,2), L_0 is not visible.



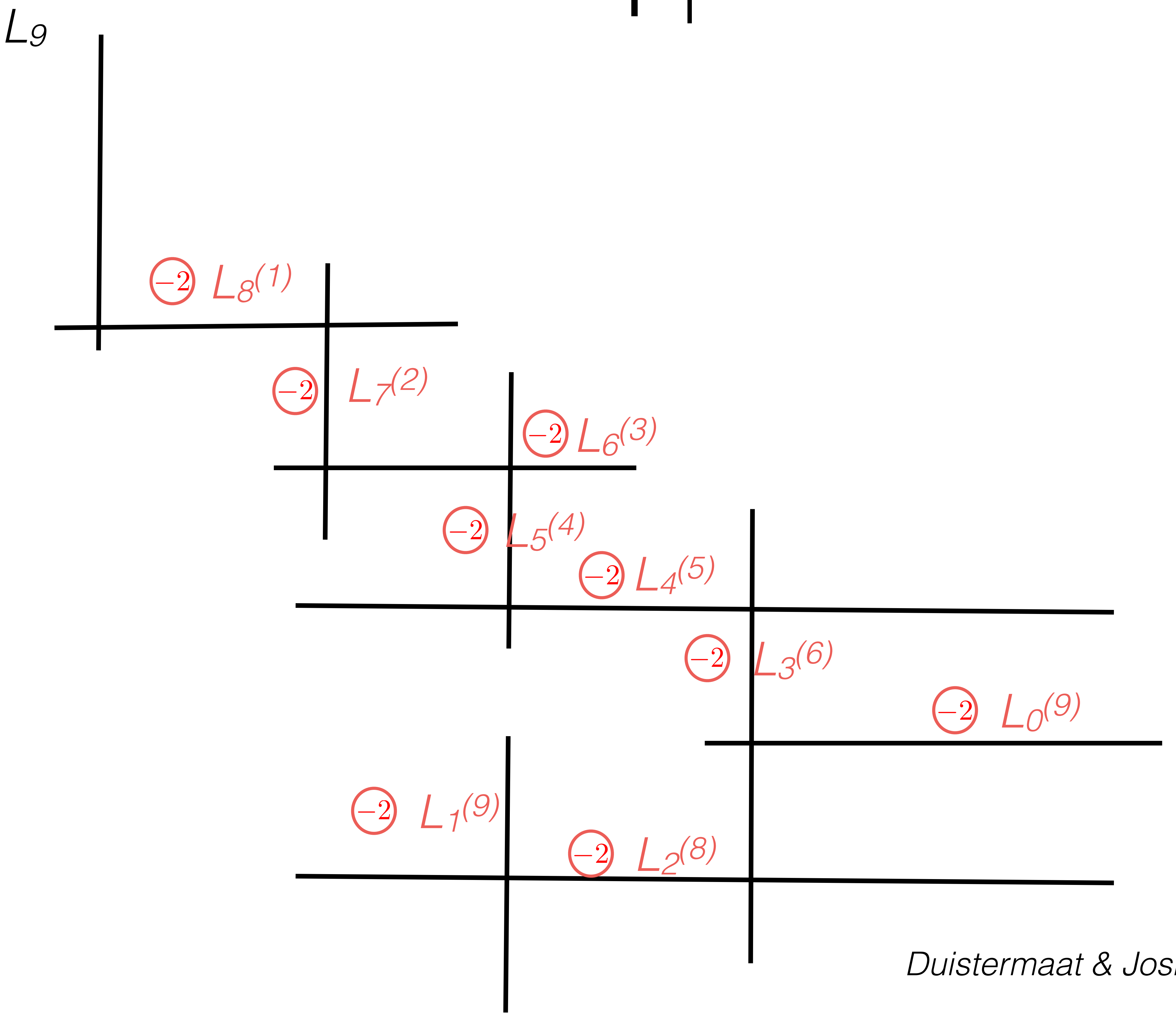
Initial-Value Space of P_I

- There are nine base points:

$$b_0 : u_{031} = 0, u_{032} = 0$$
$$b_1 : u_{111} = 0, u_{112} = 0$$
$$b_2 : u_{211} = 0, u_{212} = 0$$
$$b_3 : u_{311} = 4, u_{312} = 0$$
$$b_4 : u_{411} = 4, u_{412} = 0$$
$$b_5 : u_{511} = 0, u_{512} = 0$$
$$b_6 : u_{611} = 0, u_{612} = 0$$
$$b_7 : u_{711} = 32, u_{712} = 0$$
$$b_8 : u_{811} = -\frac{2^8}{(5\,z)}, u_{812} = 0$$

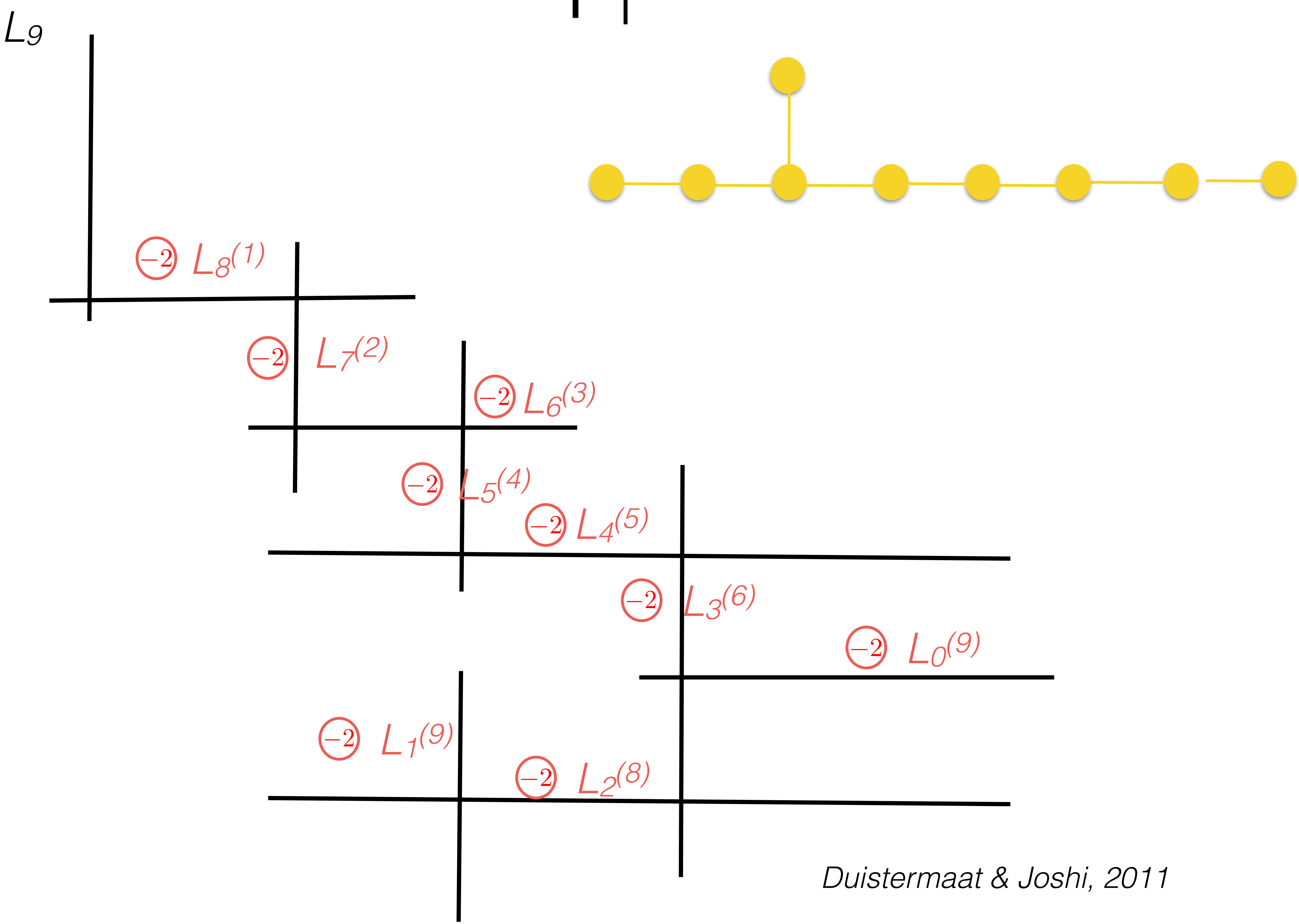
- Only the last one differs from the elliptic case.

$\mathrm{P_I}$



Duistermaat & Joshi, 2011

P_I



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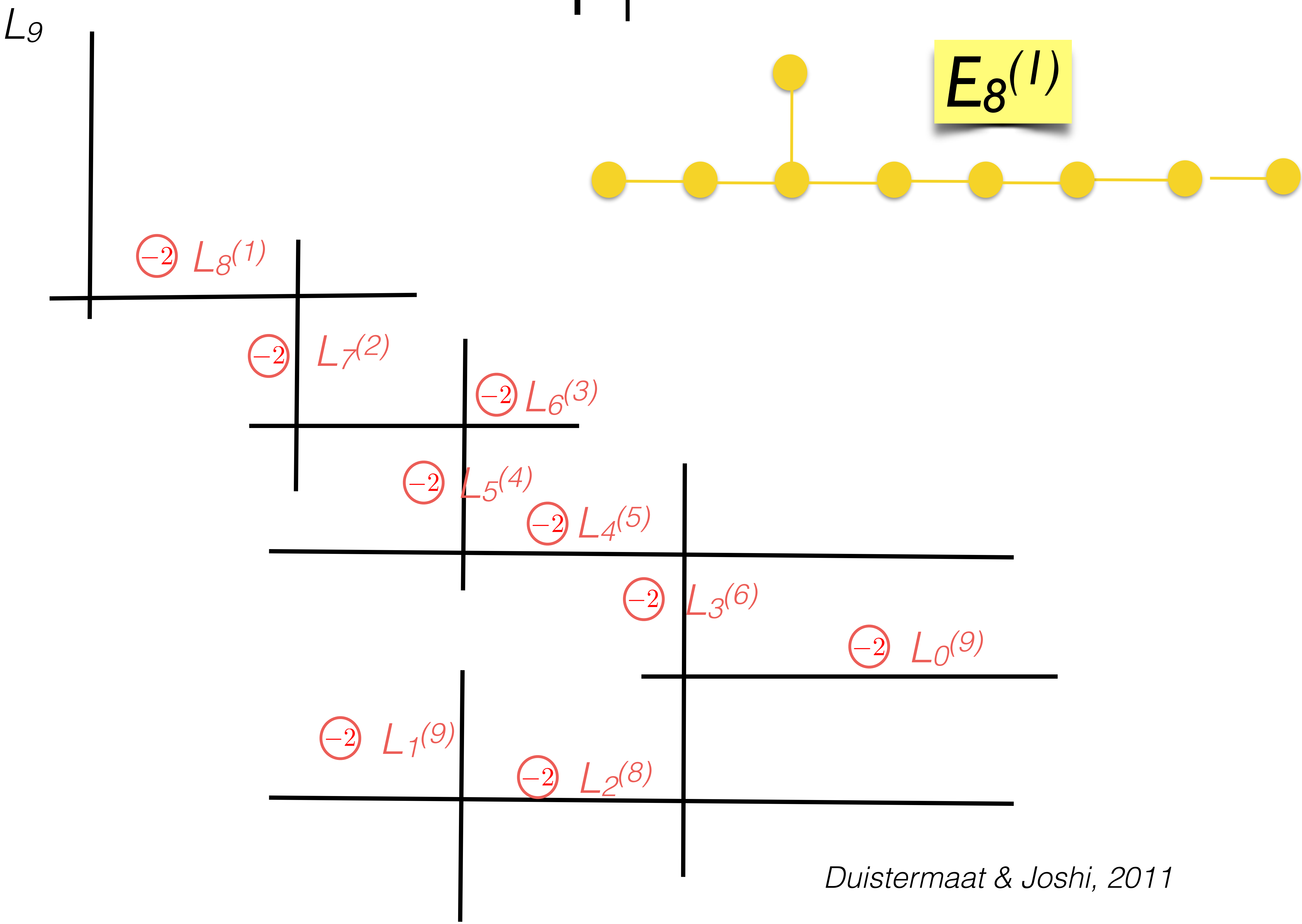

Duistermaat & Joshi, 2011

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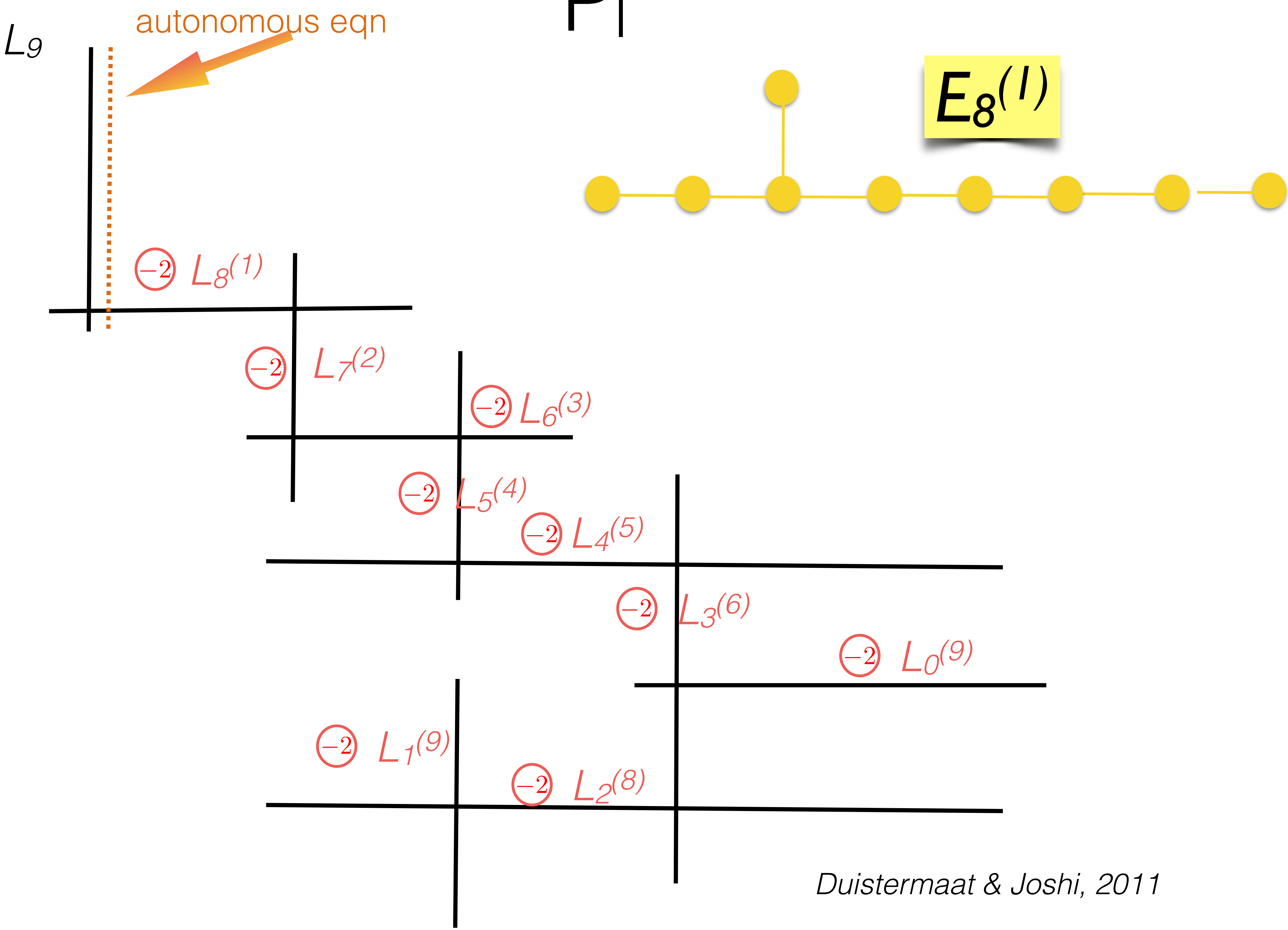
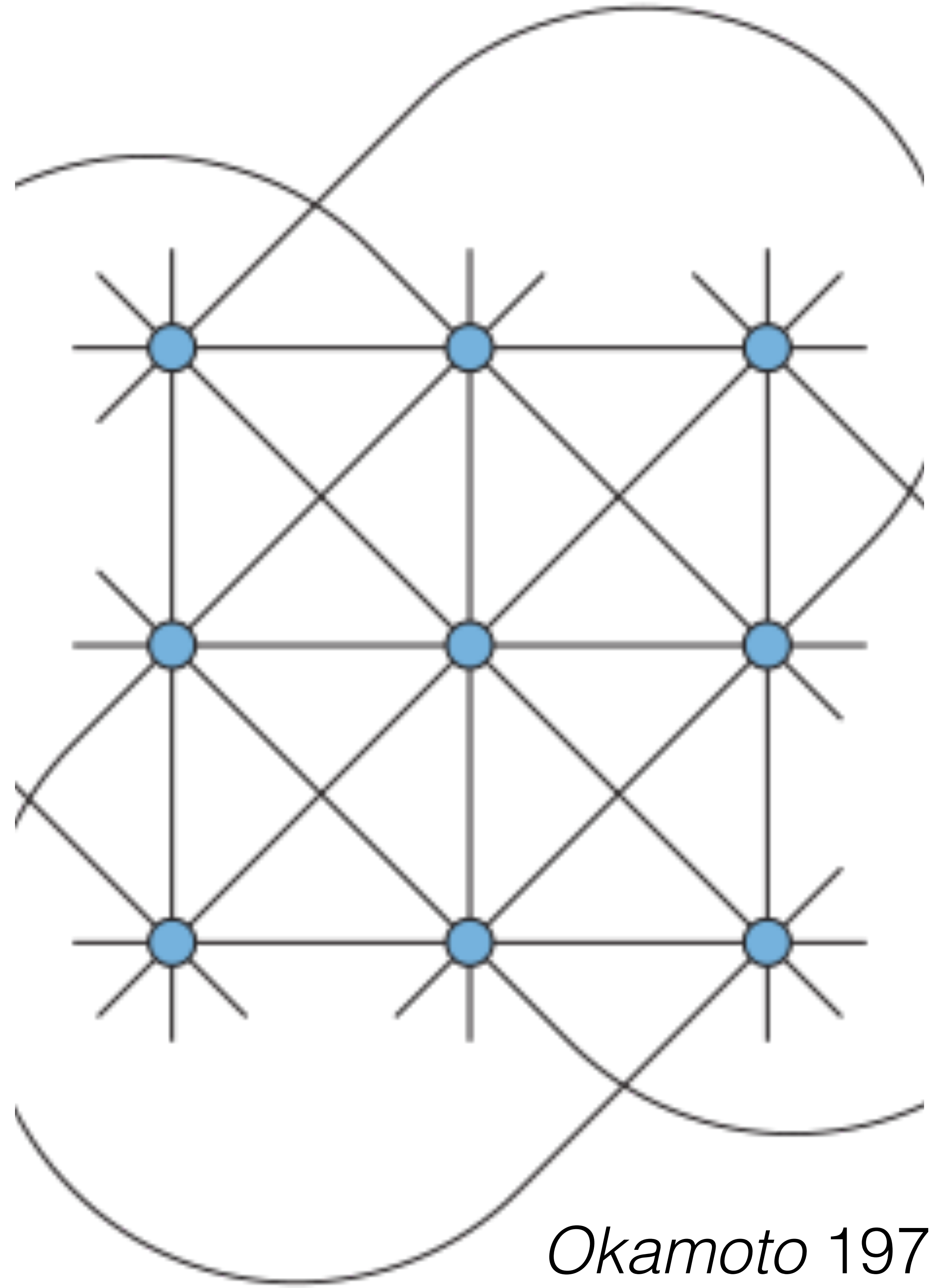


Duistermaat & Joshi, 2011

Unifying Property

The space of initial values of a Painlevé system is resolved by “blowing up” 9 points in $\mathbb{CP}^2$
(or 8 points in $\mathbb{P}^1$x$\mathbb{P}^1$)



Okamoto 1979

Sakai 2001

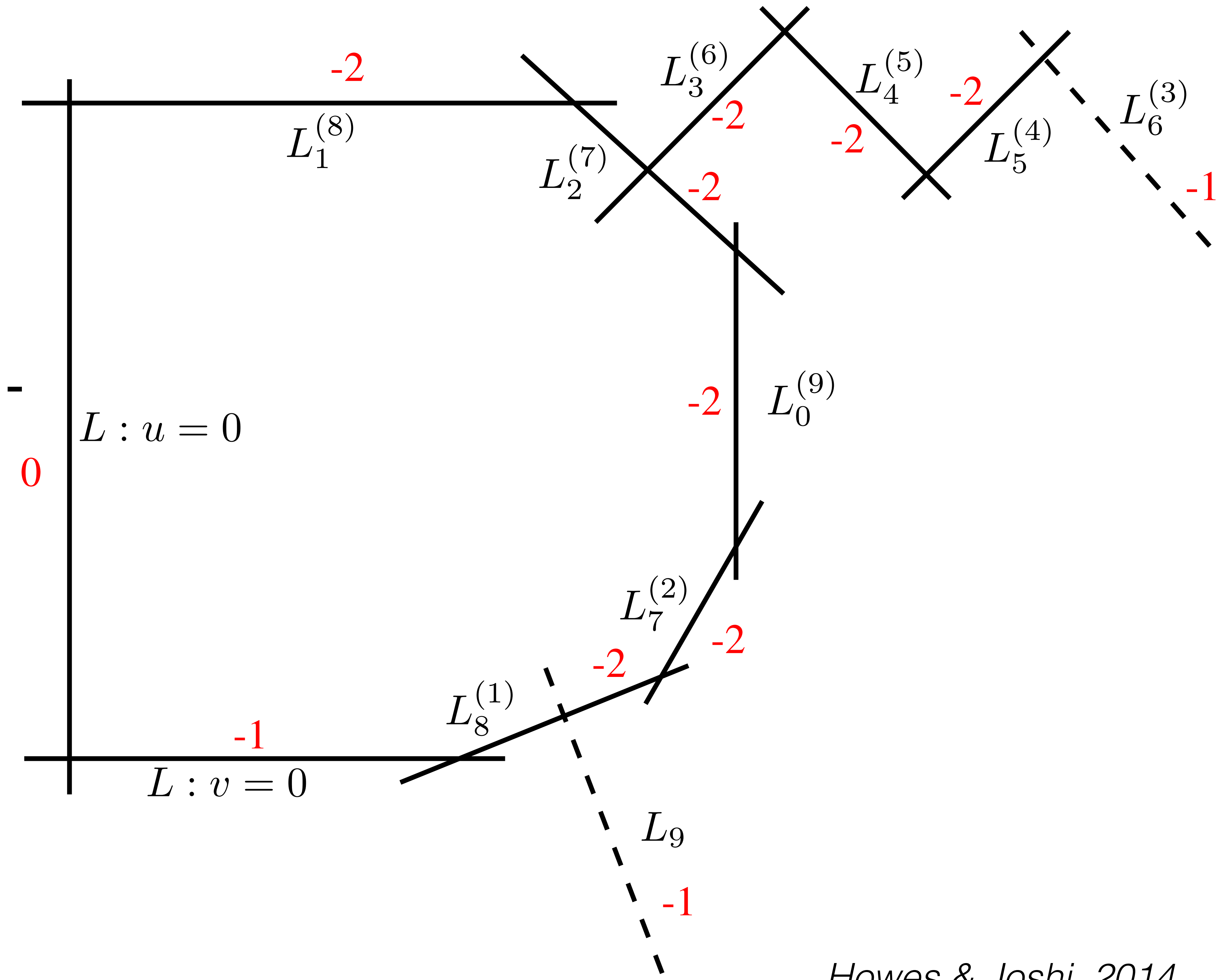
Similarly

- P$_{\text{II}}$: $w_{tt} = 2w^3 + tw + \alpha$

- P$_{\text{IV}}$: $$w_{tt} = \frac{{w_t}^2}{2w} + \frac{3w^3}{2} + 4tw^2 + 2(t^2 - 1 + \alpha_1 + 2\,\alpha_2)w - \frac{2\alpha_1^2}{w}$$

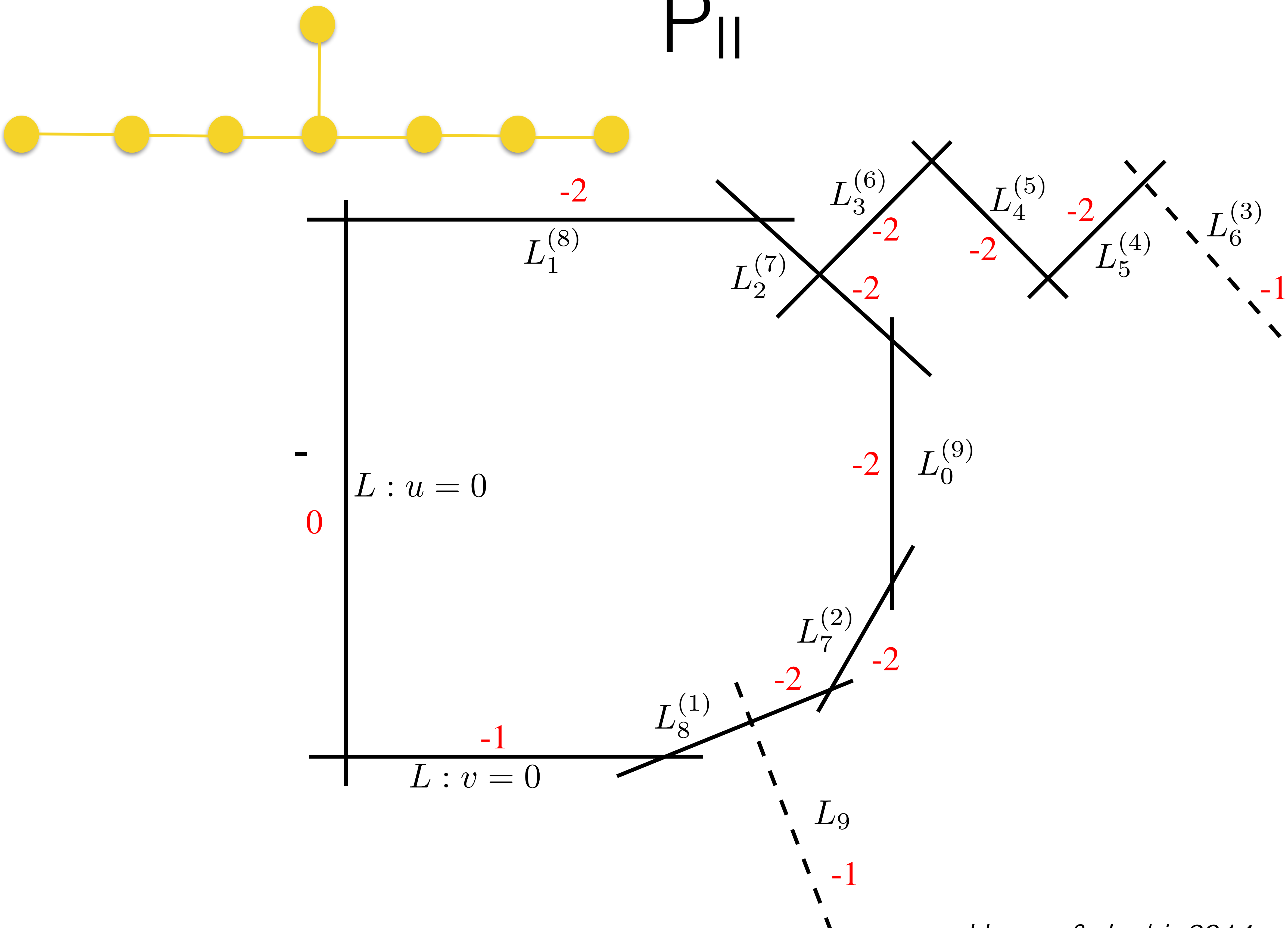
have system forms that are perturbations of autonomous systems in the limit $|t| \to \infty$

P_{II}



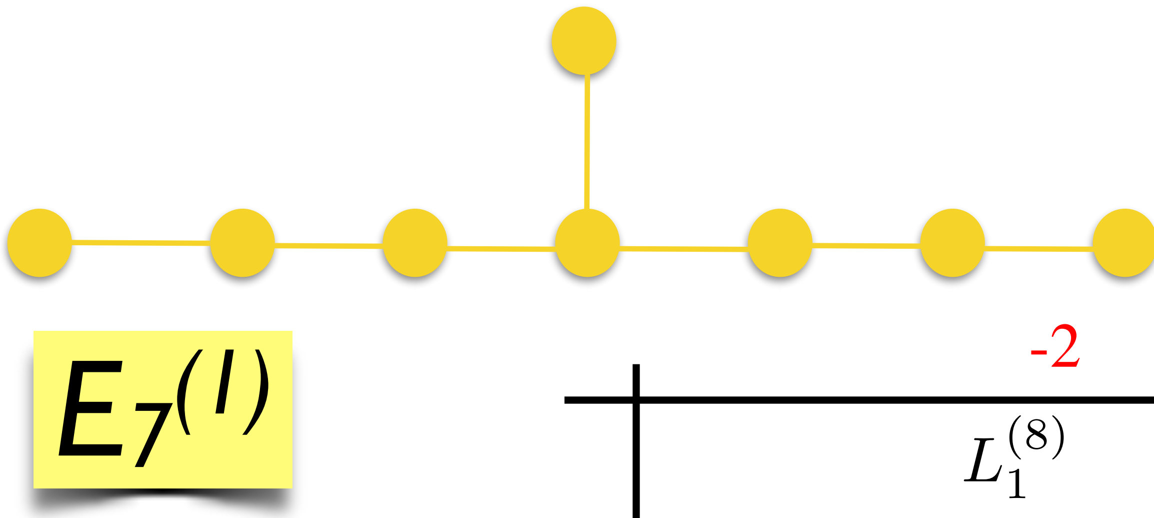
Howes & Joshi, 2014

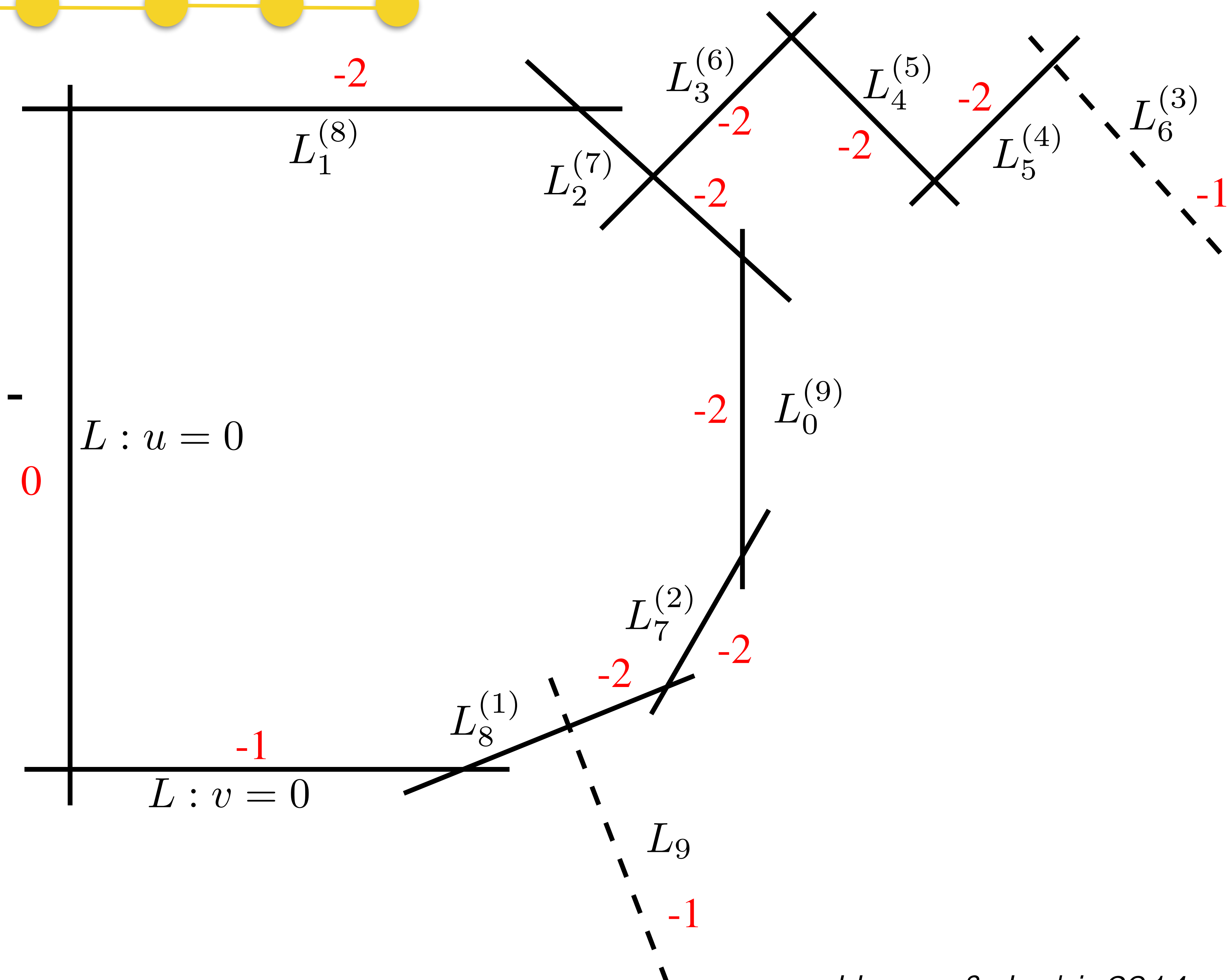
P$_{II}$



Howes & Joshi, 2014

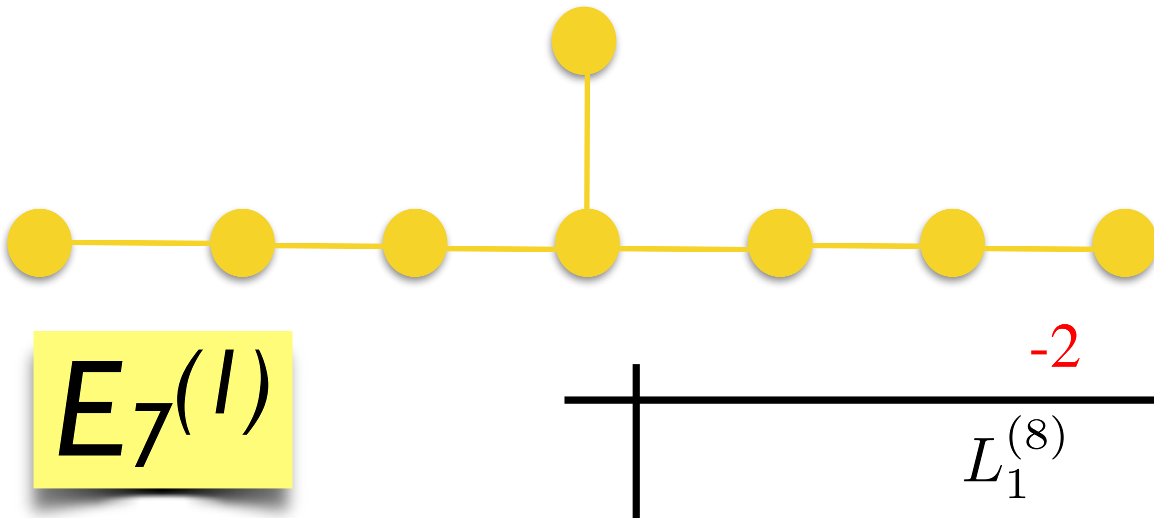
P_{II}

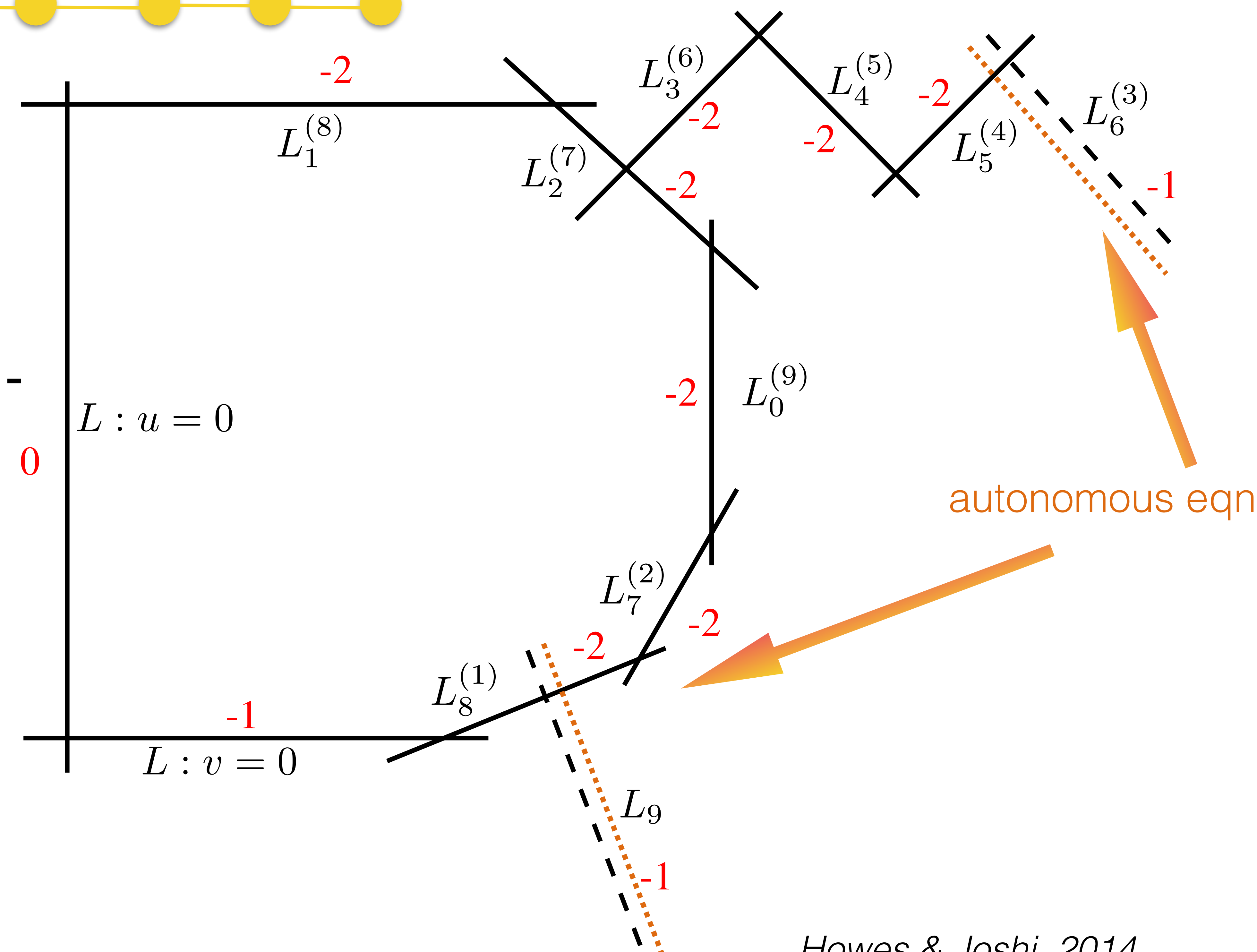




Howes & Joshi, 2014

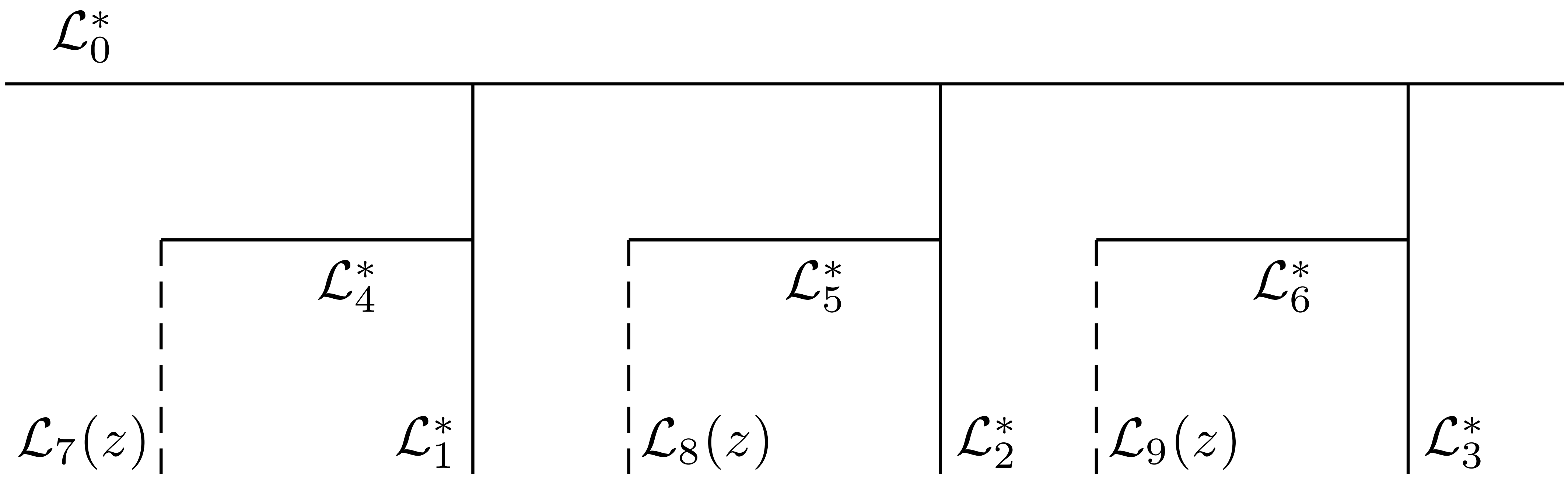
P_{II}





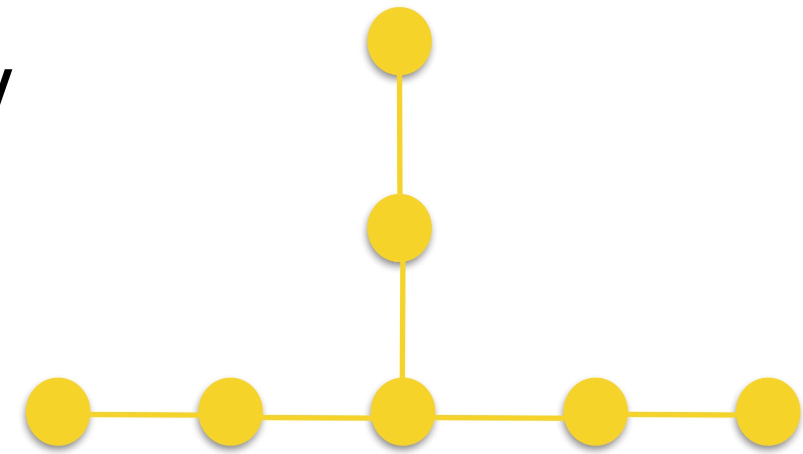
Howes & Joshi, 2014

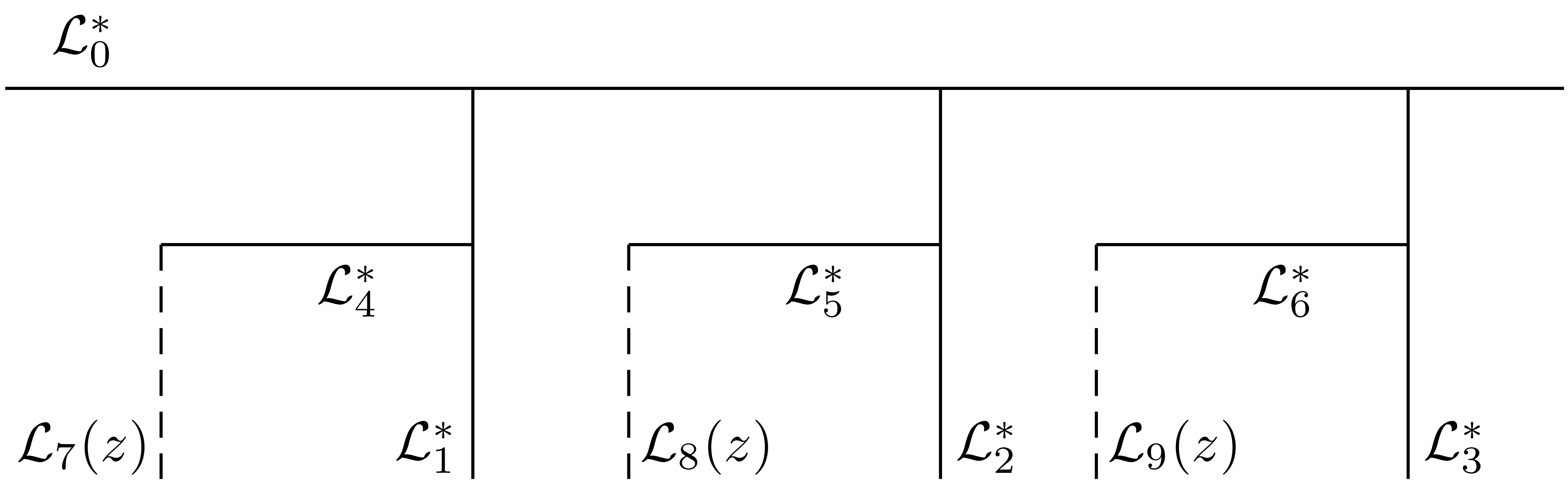
P_{IV}



Joshi & Radnovic, 2015

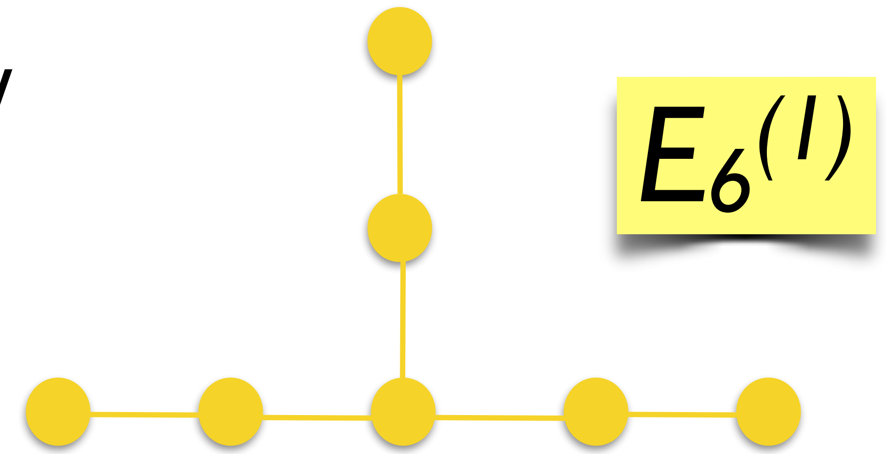
P_{IV}

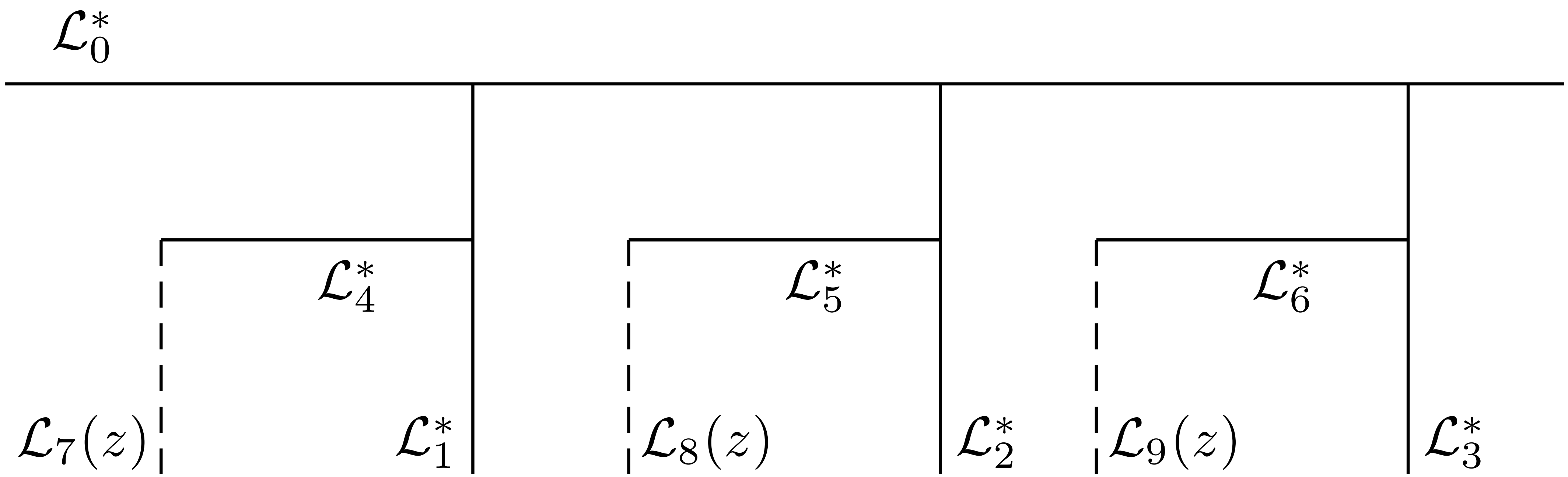




Joshi & Radnovic, 2015

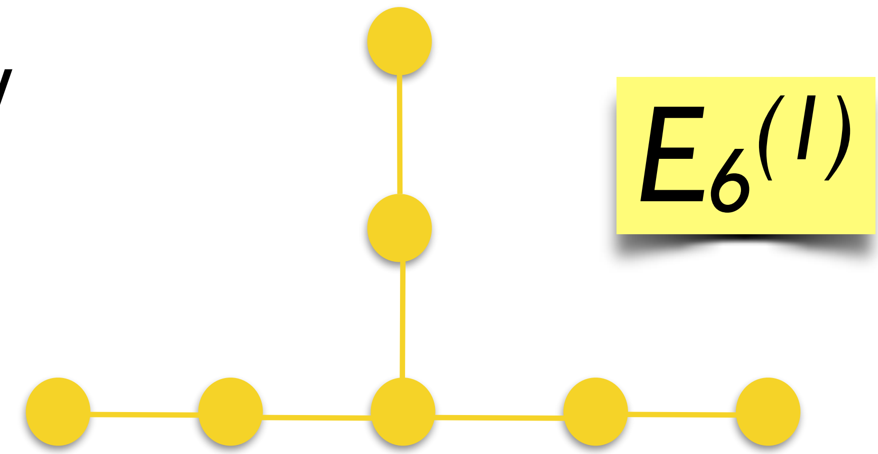
P_{IV}

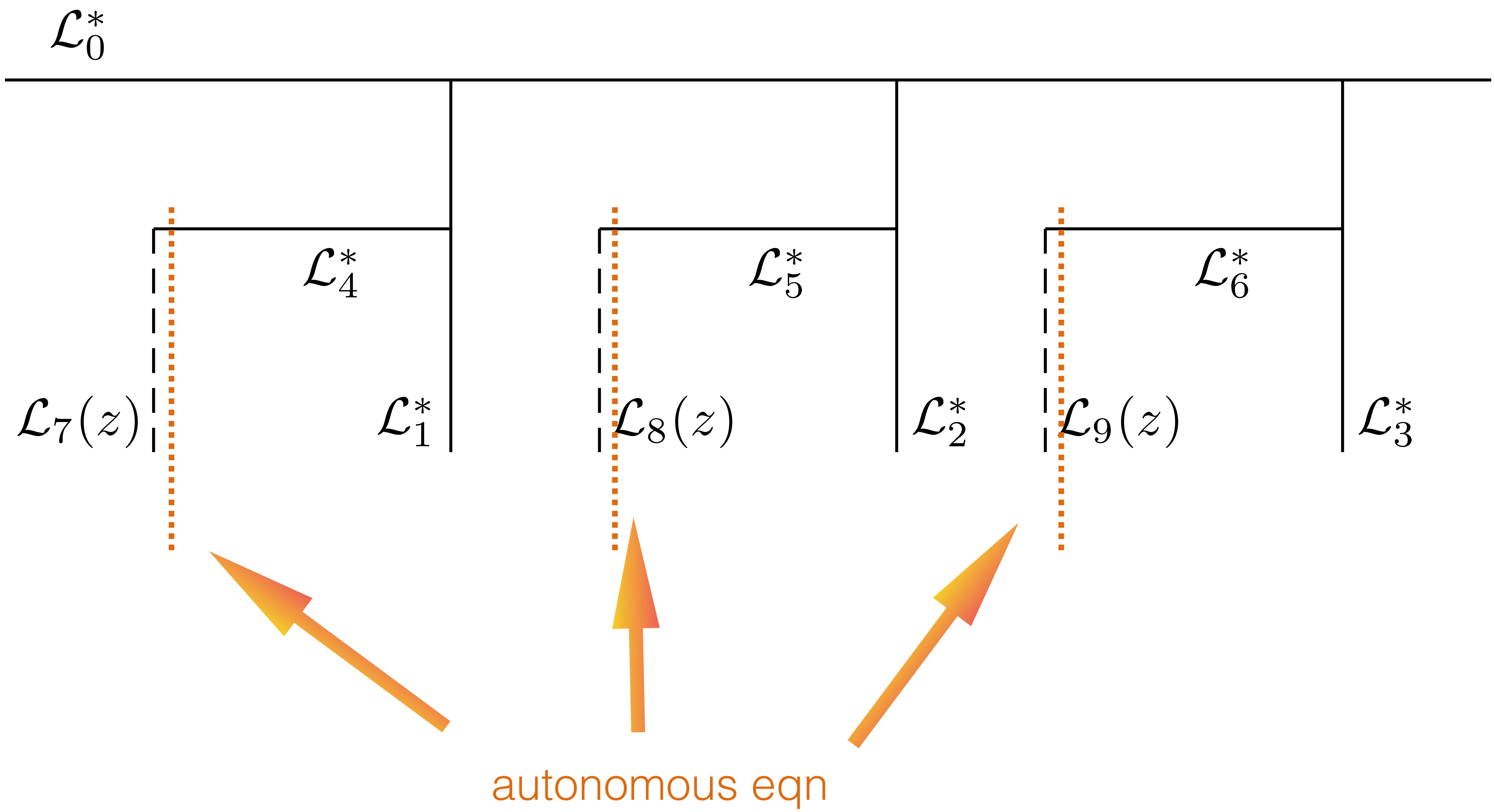




Joshi & Radnovic, 2015

P_{IV}





Joshi & Radnovic, 2015

Global results for P_I , P_{II} , P_{IV}

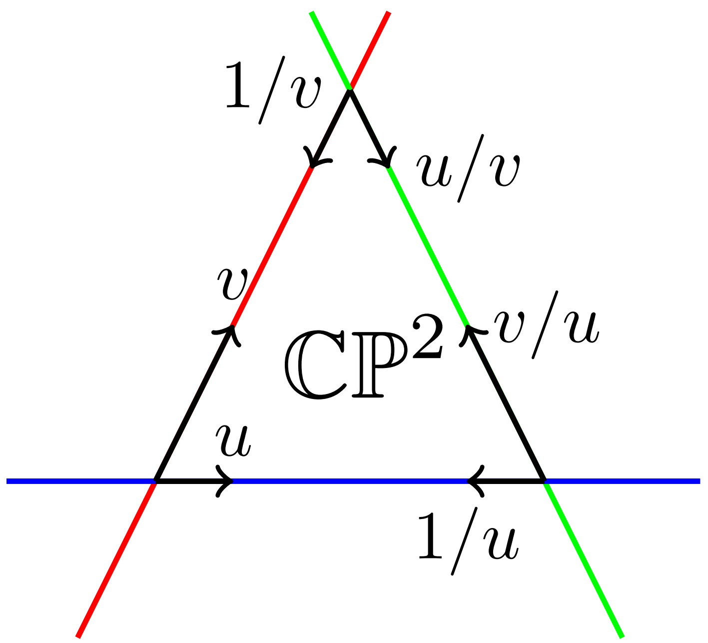
- The union of exceptional lines is a repeller for the flow.
- There exists a complex limit set, which is non-empty, connected and compact.
- Every solution of P_I , every solution of P_{II} whose limit set is not {0}, and every non-rational solution of P_{IV} intersects the last exceptional line(s) infinitely many times => infinite number of movable poles and movable zeroes.

Duistermaat & J (2011); Howes & J (2014); J & Radnovic (2014)

But, there is more…

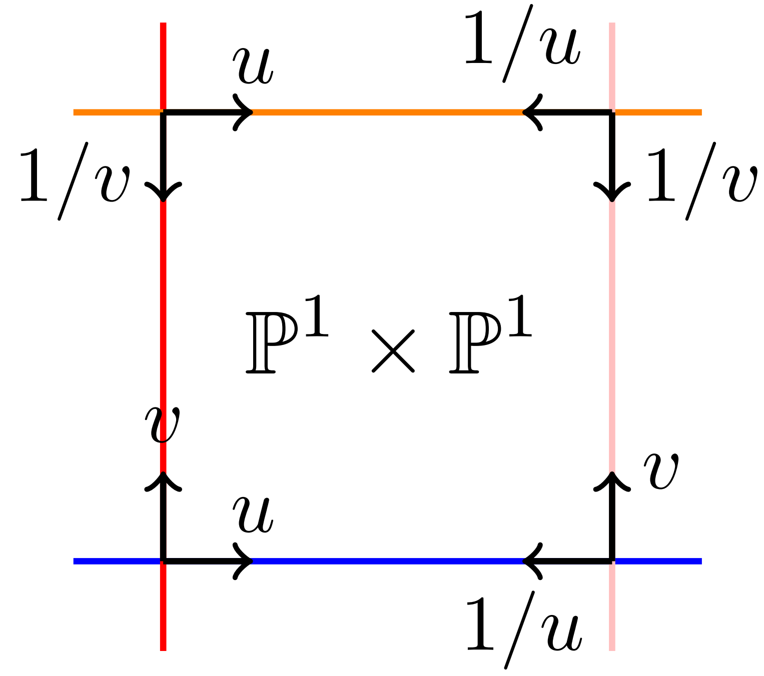
For Discrete Equations

Instead of



3 coordinate charts

we use



4 coordinate charts

discrete PI

$$\begin{cases} \overline{u} + \overline{v} + u &= \frac{2\alpha n + \alpha + \beta - c}{\overline{v}} + \gamma \\ \overline{v} + u + v &= \frac{2\alpha n + \alpha + \beta}{v} + \gamma \end{cases}$$

$$\begin{cases} u + \underline{u} + v &= \frac{2\alpha n - \alpha + \beta - c}{v} + \gamma \\ \underline{u} + v + \underline{v} &= \frac{2\alpha n - \alpha + \beta}{\underline{v}} + \gamma \end{cases}$$

where

$$\overline{u} = u(n+1), u = u(n), \underline{u} = u(n-1)$$

Charts

- We use the coordinate charts

$$
\begin{aligned}
(u, v)& \\
(u_{01}, v_{01}) &= (1/u, v) \\
(u_{02}, v_{02}) &= (u, 1/v) \\
(u_{03}, v_{03}) &= (1/u, 1/v)
\end{aligned}
$$

For example,

$$\overline{v} = \frac{(2\alpha n + \beta + c + \gamma\, u_{02} - u_{02}^2)v_{02} - u_{02}}{u_{02}v_{02}}$$

First Base Point

In chart (02) there is a base point at

$$(u_{02}, v_{02}) = (0, 0) =: p_1$$

Blow-up:

$$u_{11} = u_{02}/v_{02} \, , v_{11} = v_{02}$$

$$u_{12} = u_{02} \, , v_{12} = v_{02}/u_{02}$$

$$e_1 : v_{11} = 0, u_{12} = 0$$

Blowing up

$$\bar{v}|_{u_{11},v_{11}} = \frac{\bar{u}_{02}}{\bar{v}_{02}}|_{u_{11},v_{11}} = \frac{2\alpha n + \beta + c + \gamma u_{11} v_{11} - u_{11}^2 v_{11}^2 - u_{11}}{u_{11} v_{11}}$$

There is a base point at

$$(u_{11}, v_{11}) \; = \; (2\alpha n + \beta + c, 0) \; =: \; p_2$$

Blow-up:

$$u_{21} = (u_{11} - 2\alpha n - \beta - c)/v_{11} \, , v_{21} = v_{11}$$

$$u_{22} = u_{11} - 2\alpha n - \beta - c \, , v_{22} = v_{11}/(u_{11} - 2\alpha n - \beta - c)$$

No further base points in this chart

Blowing up

$$\bar{v}|_{u_{11},v_{11}} = \frac{\bar{u}_{02}}{\bar{v}_{02}}|_{u_{11},v_{11}} = \frac{2\alpha n + \beta + c + \gamma u_{11}v_{11} - u_{11}^2 v_{11}^2 - u_{11}}{u_{11}v_{11}}$$

There is a base point at

$$(u_{11}, v_{11}) = (2\alpha n + \beta + c, 0) =: p_2$$

Blow-up:

$$u_{21} = (u_{11} - 2\alpha n - \beta - c)/v_{11}\,, v_{21} = v_{11}$$

$$u_{22} = u_{11} - 2\alpha n - \beta - c\,, v_{22} = v_{11}/(u_{11} - 2\alpha n - \beta - c)$$

No further base points in this chart

Blowing up

$$\bar{v}|_{u_{11},v_{11}} = \frac{\bar{u}_{02}}{\bar{v}_{02}}|_{u_{11},v_{11}} = \frac{2\alpha n + \beta + c + \gamma u_{11}v_{11} - u_{11}^2 v_{11}^2 - u_{11}}{u_{11}v_{11}}$$

There is a base point at

$$(u_{11}, v_{11}) = (2\alpha n + \beta + c, 0) =: p_2$$

Blow-up:

$$u_{21} = (u_{11} - 2\alpha n - \beta - c)/v_{11}\,, v_{21} = v_{11}$$

$$u_{22} = u_{11} - 2\alpha n - \beta - c\,, v_{22} = v_{11}/(u_{11} - 2\alpha n - \beta - c)$$

No further base points in this chart

All Base Points

$$p_1 : (u, 1/v) = (0, 0)$$

$$p_2 : (uv, 1/v) = (2\alpha n + \beta + c, 0)$$

$$p_3 : (1/u, 1/v) = (0, 0)$$

$$p_4 : (u/v, 1/v) = (-1, 0)$$

$$p_5 : (1/u, v) = (0, 0)$$

$$p_6 : (u(u+v)/u, 1/v) = (-\gamma, 0)$$

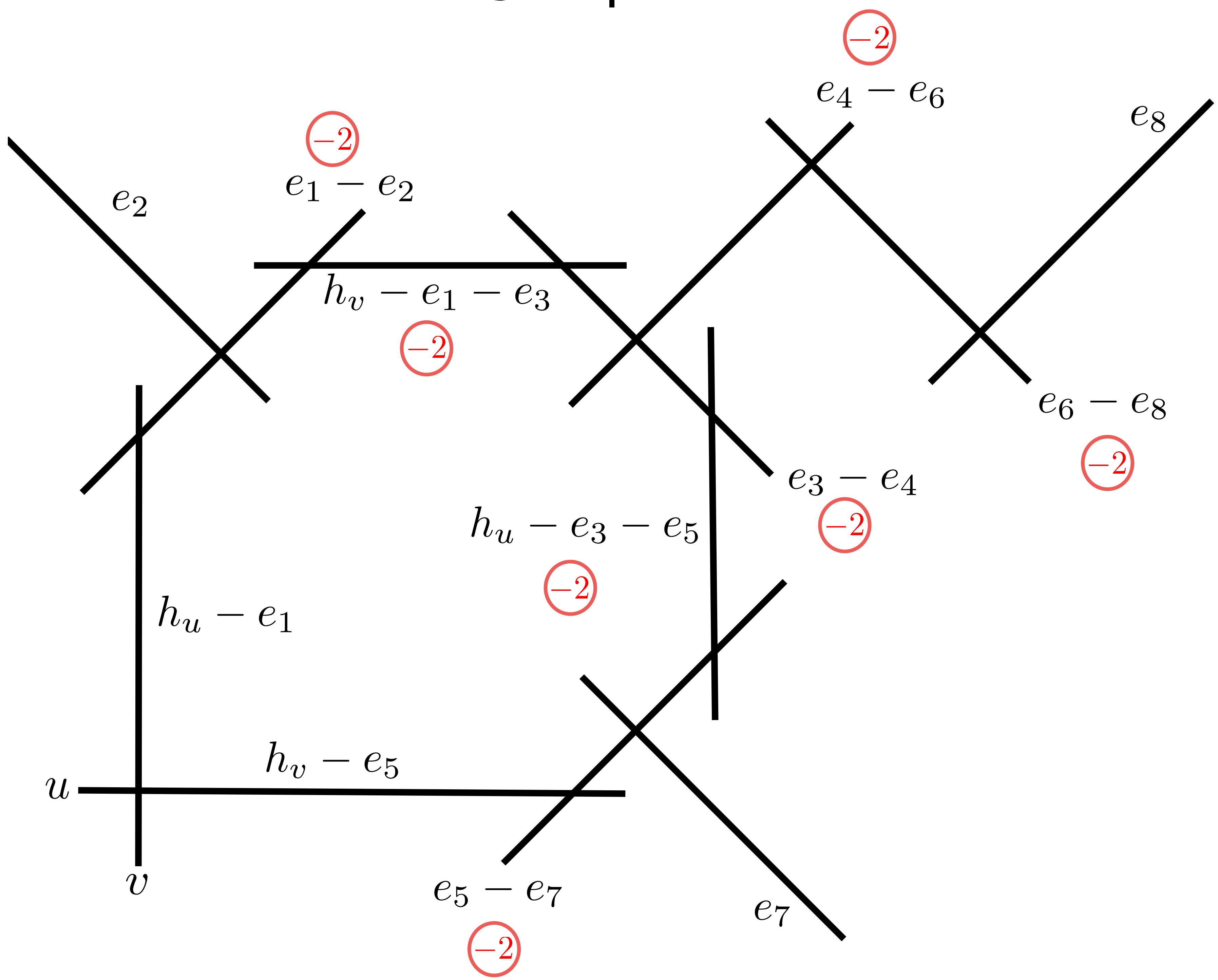
$$p_7 : (1/(uv), v) = ((2\alpha n - \alpha + \beta - c)^{-1}, 0)$$

$$p_7 : (v(\gamma u + uv + v^2)/v, 1/v) = (-\gamma^2 - \alpha + 2c, 0)$$

All Base Points

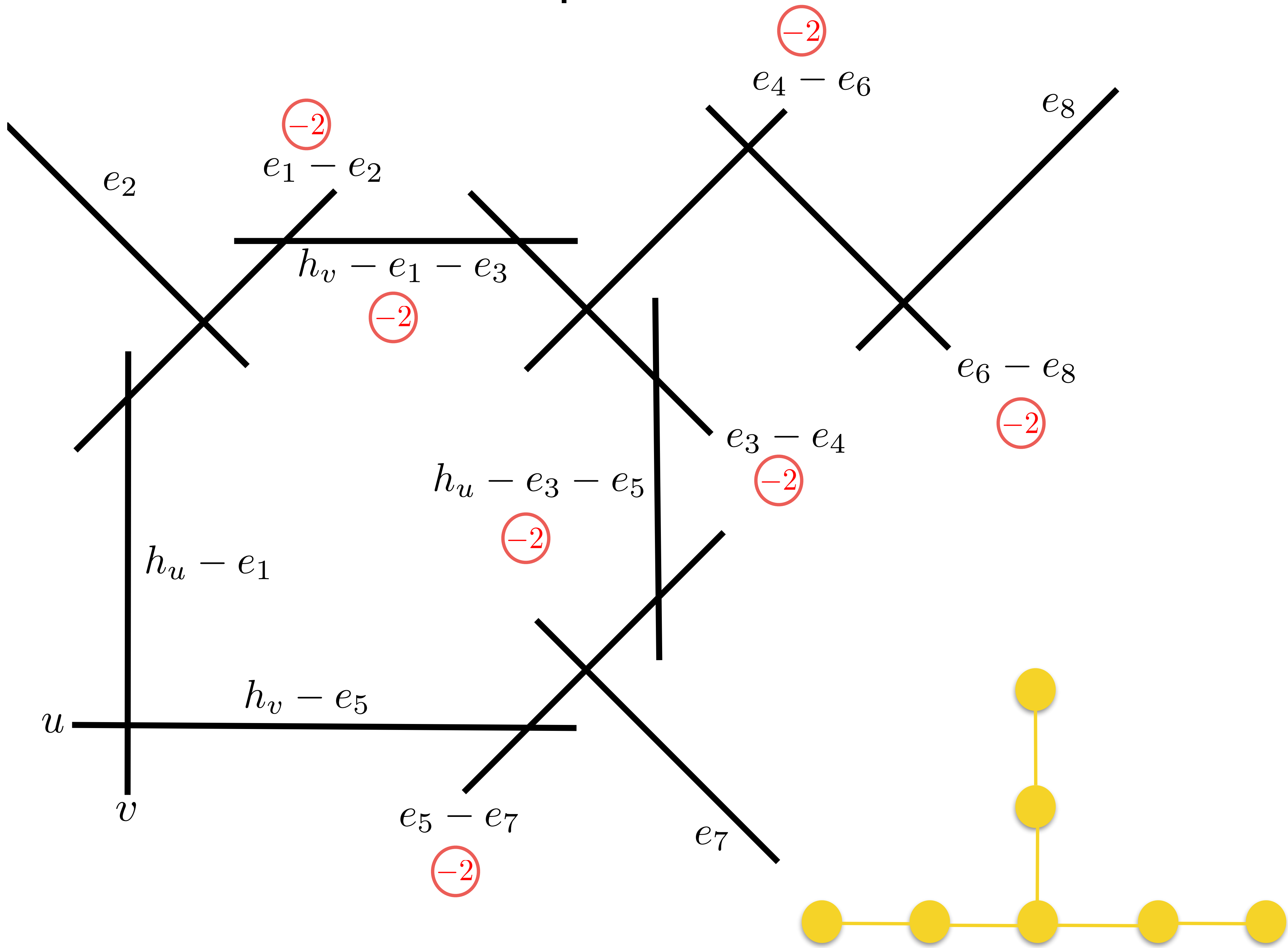
$$
\begin{aligned}
p_1 &:(u, 1/v) = (0,0) \\
p_2 &:(uv, 1/v) = (2\alpha n + \beta + c, 0) \\
p_3 &:(1/u, 1/v) = (0,0) \\
p_4 &:(u/v, 1/v) = (-1, 0) \\
p_5 &:(1/u, v) = (0,0) \\
p_6 &:(u(u+v)/u, 1/v) = (-\gamma, 0) \\
p_7 &:(1/(uv), v) = ((2\alpha n - \alpha + \beta - c)^{-1}, 0) \\
p_7 &:(v(\gamma u + uv + v^2)/v, 1/v) = (-\gamma^2 - \alpha + 2c, 0)
\end{aligned}
$$

dP_I



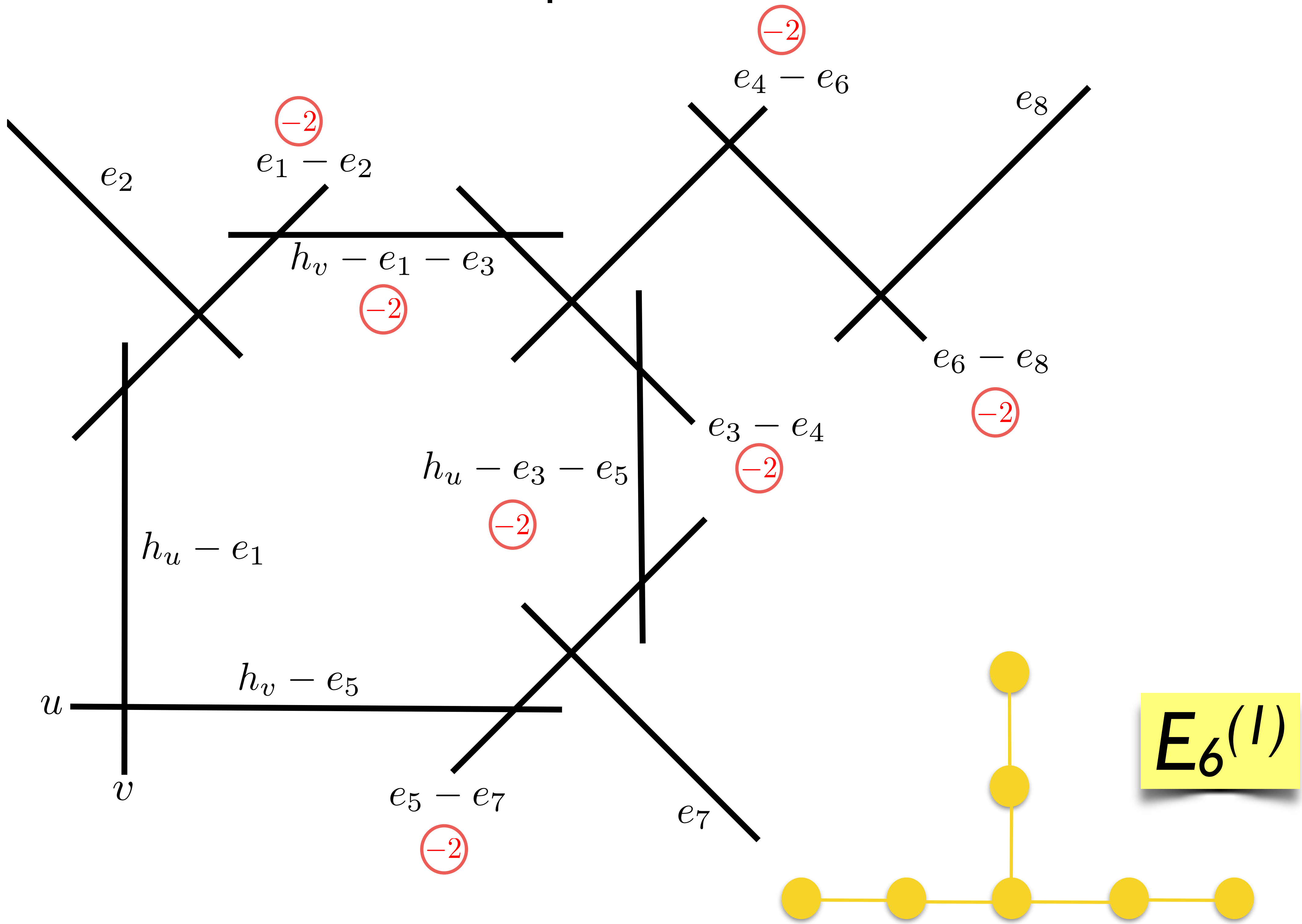
-2
$e_4 - e_6$
e_8
-2
$e_1 - e_2$
e_2
$h_v - e_1 - e_3$
-2
$e_6 - e_8$
-2
$e_3 - e_4$
-2
$h_u - e_3 - e_5$
-2
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
-2
e_7

dP_I



−2
e4 − e6
e8
−2
e1 − e2
e2
hv − e1 − e3
−2
e6 − e8
−2
e3 − e4
−2
hu − e3 − e5
−2
hu − e1
hv − e5
u
v
e5 − e7
e7
−2

dP$_{\mathrm{I}}$



-2
$e_4 - e_6$
e_8
-2
$e_1 - e_2$
e_2
$h_v - e_1 - e_3$
-2
$e_6 - e_8$
-2
$e_3 - e_4$
-2
$h_u - e_3 - e_5$
-2
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
-2
e_7
$E_6^{(I)}$

The Picard Group

The equivalence class of lines

$$\mathrm{Pic}(X) = \mathbb{Z}\, h_u + \mathbb{Z}\, h_v + \mathbb{Z}\, e_1 + \ldots + \mathbb{Z}\, e_k$$

equipped with intersection form

$$(h_u, h_u) = (h_v, h_v) = (h_u, e_i) = (h_v, e_j) = 0$$

$$(h_u, h_v) = 1, (e_i, e_j) = -\,\delta_{ij}$$

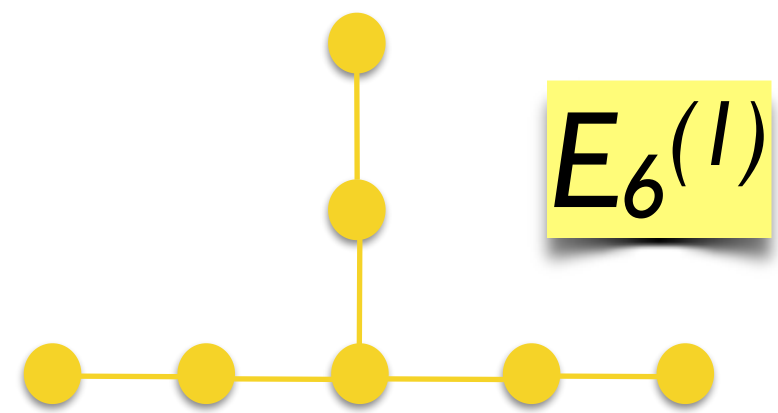
The initial value space and the symmetry group of the equation are orthogonal to each other in the Picard lattice.

Symmetry Group

Symmetry Group

The curves of self-intersection -2:

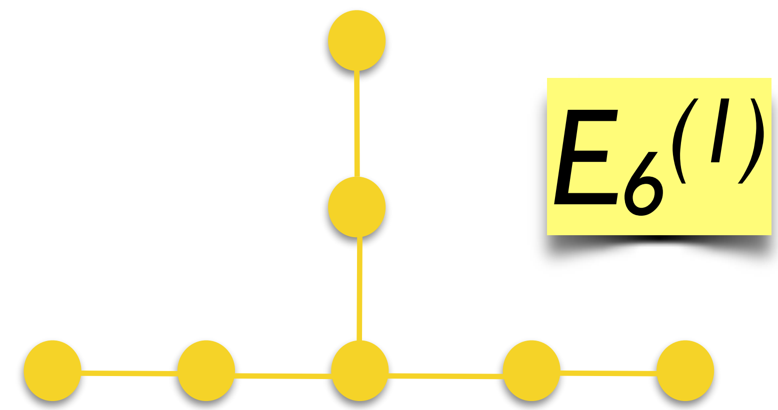
$$D_1 = e_1 - e_2, D_2 = h_v - e_1 - e_3,$$
$$D_3 = e_3 - e_4, D_4 = h_u - e_3 - e_5,$$
$$D_5 = e_5 - e_7, D_6 = e_4 - e_6, D_7 = e_6 - e_8$$



Symmetry Group

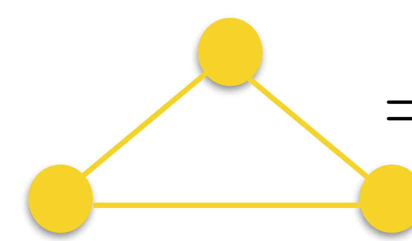
The curves of self-intersection -2:

$$D_1 = e_1 - e_2, D_2 = h_v - e_1 - e_3,$$
$$D_3 = e_3 - e_4, D_4 = h_u - e_3 - e_5,$$
$$D_5 = e_5 - e_7, D_6 = e_4 - e_6, D_7 = e_6 - e_8$$



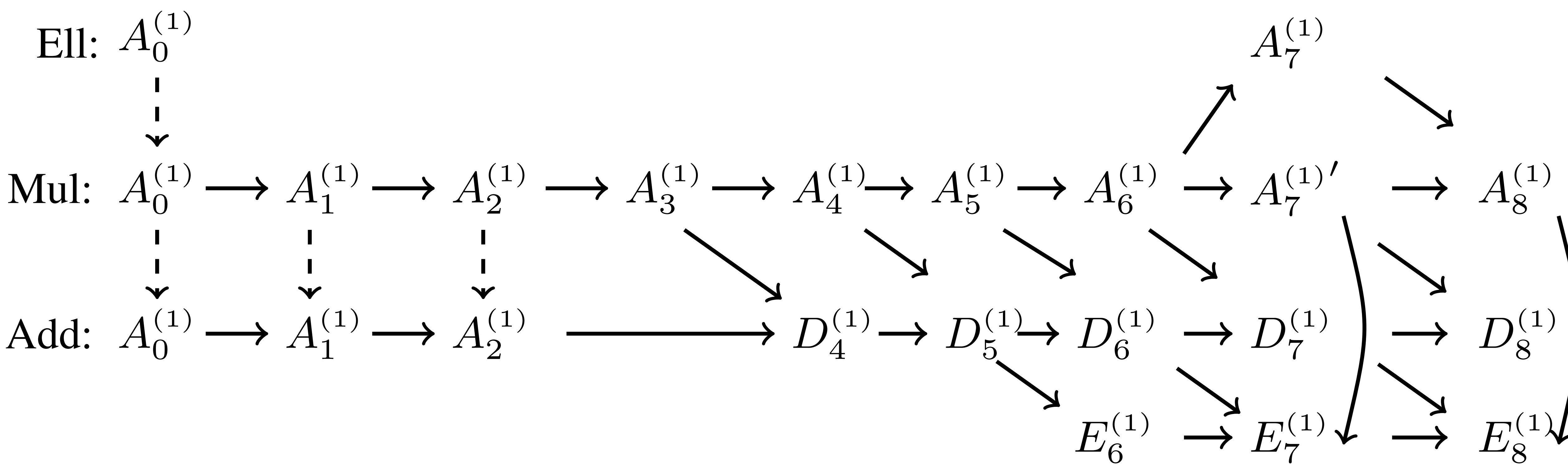
are orthogonal to

$$\alpha_1 = h_u - e_1 - e_2$$
$$\alpha_2 = h_v - e_5 - e_7$$
$$\alpha_0 = h_u + h_v - e_3 - e_4 - e_6 - e_8$$



$\Rightarrow \; A_2^{(1)}$

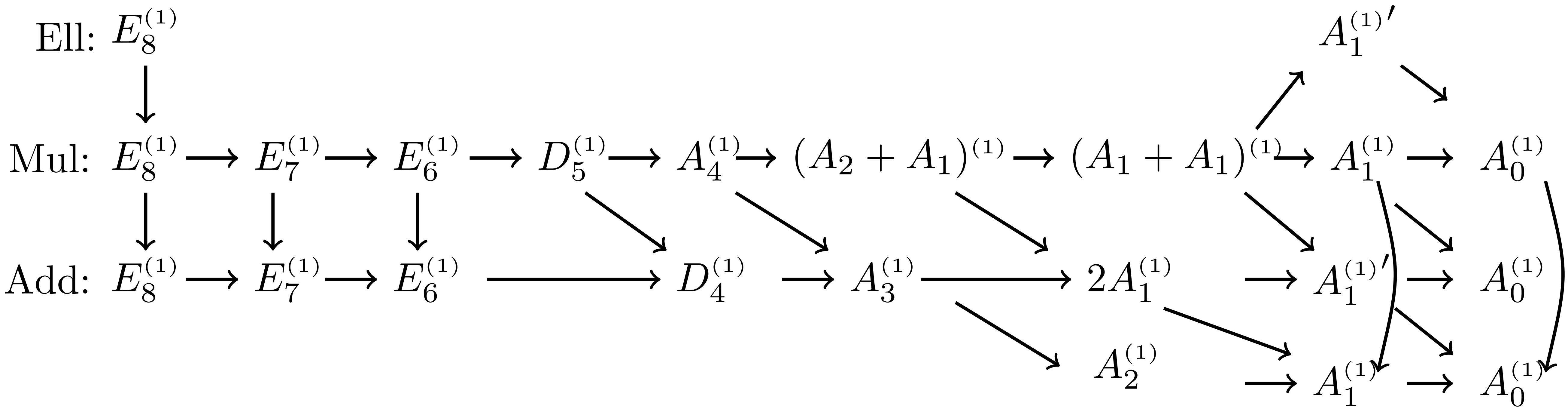
Sakai's Description I



Initial-value spaces of all continuous and discrete Painlevé equations

Sakai 2001

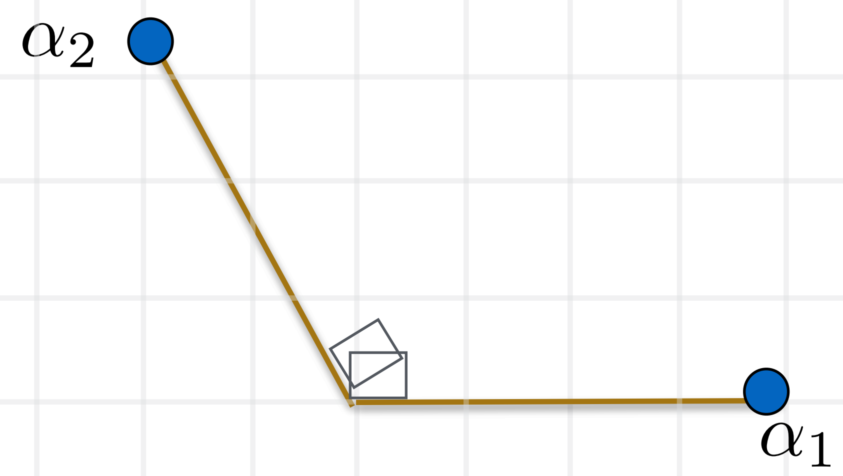
Sakai’s Description II



Symmetry groups of Painlevé equations

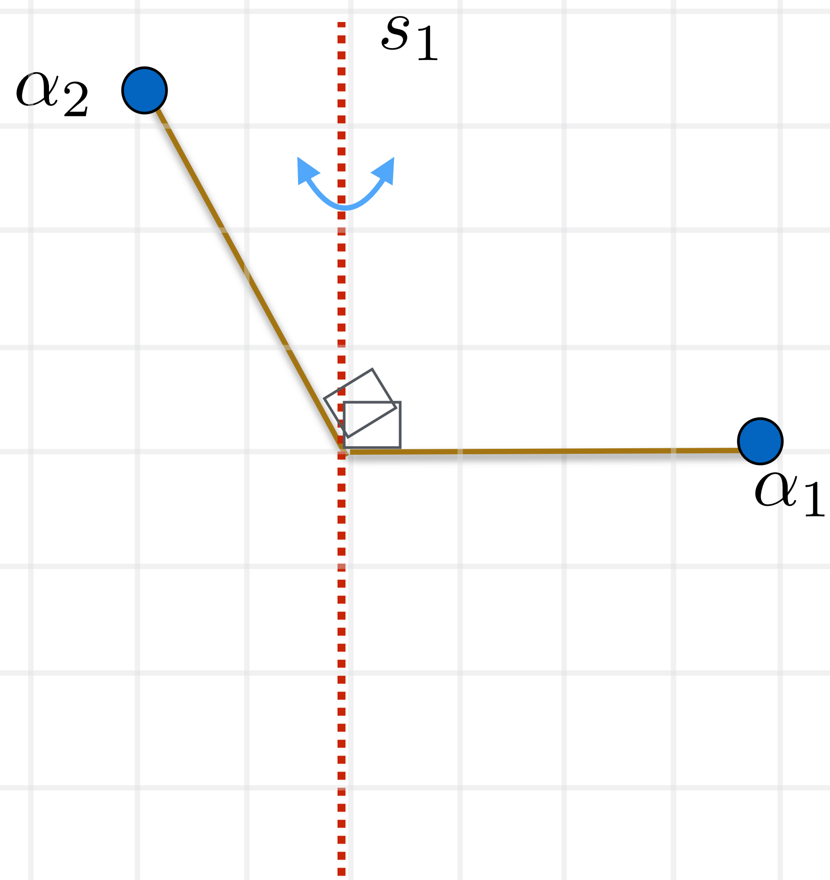
Sakai 2001

A_2 Root System



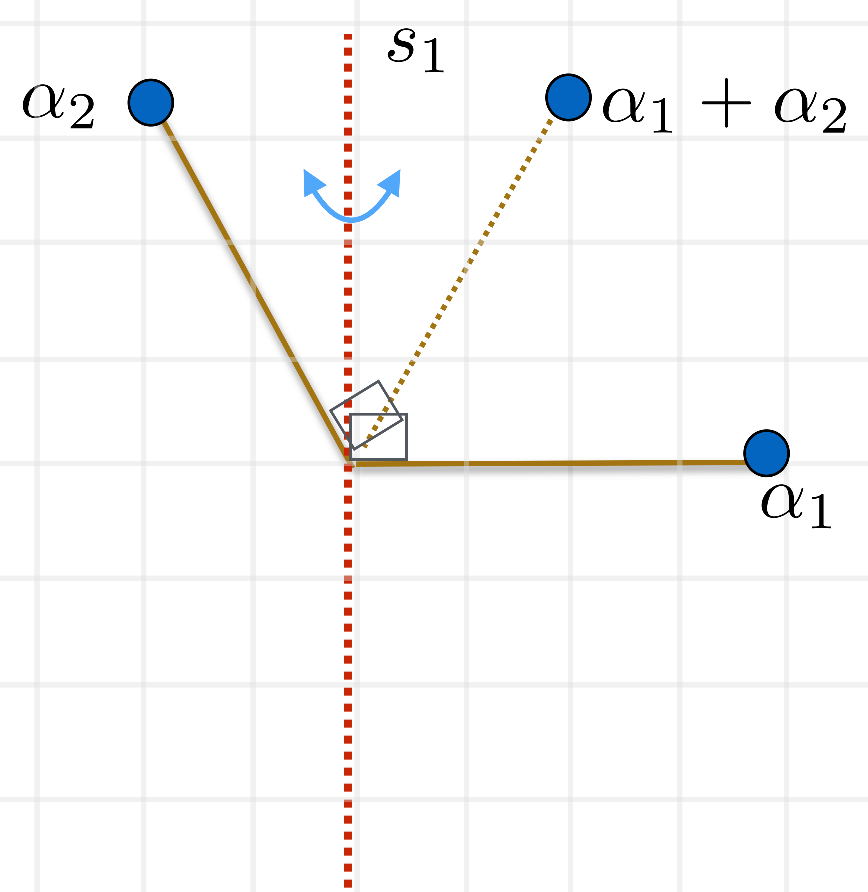
α_1 and α_2 are “simple” roots

A_2 Root System



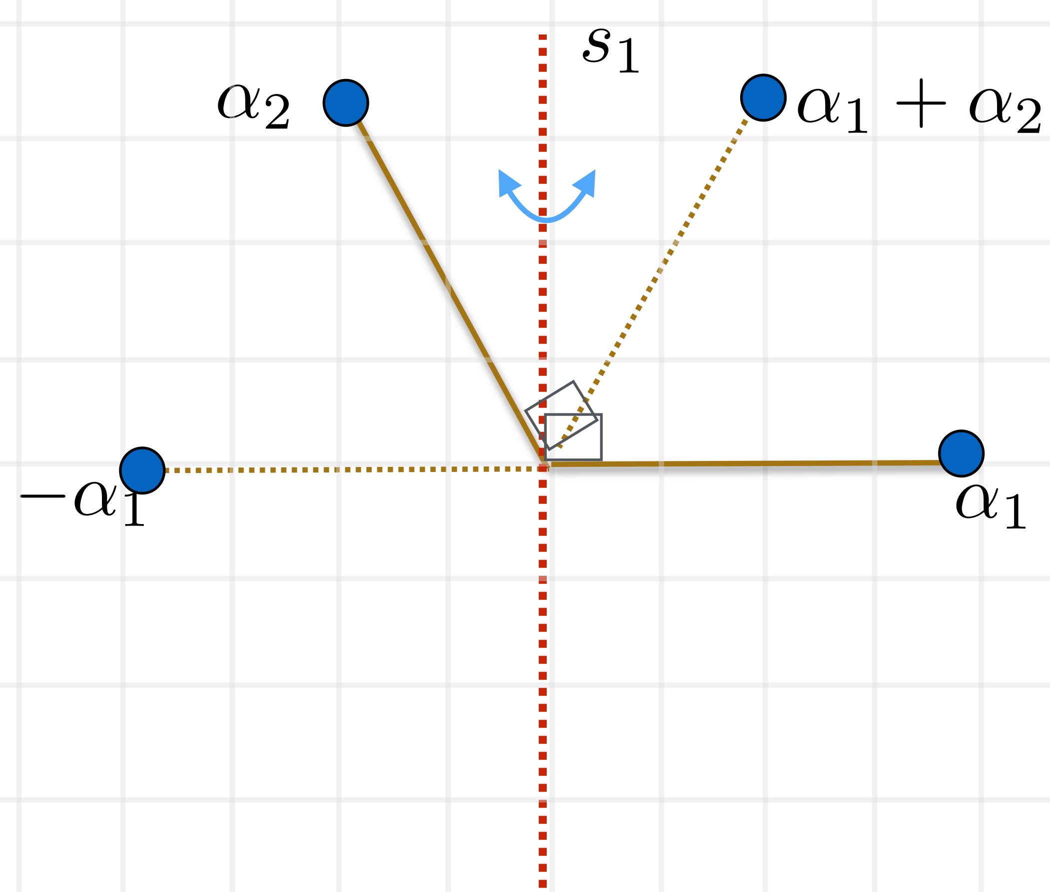
α_1 and α_2 are "simple" roots

A_2 Root System



α_1 and α_2 are “simple” roots

A_2 Root System



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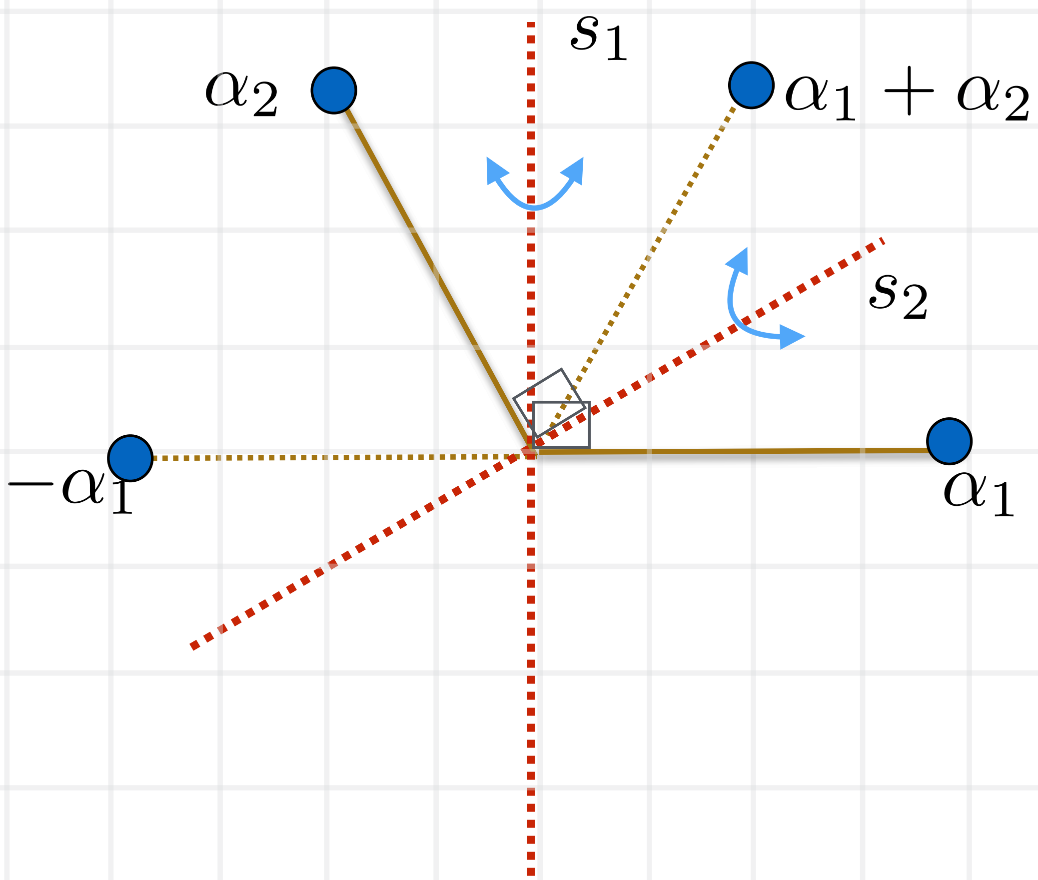
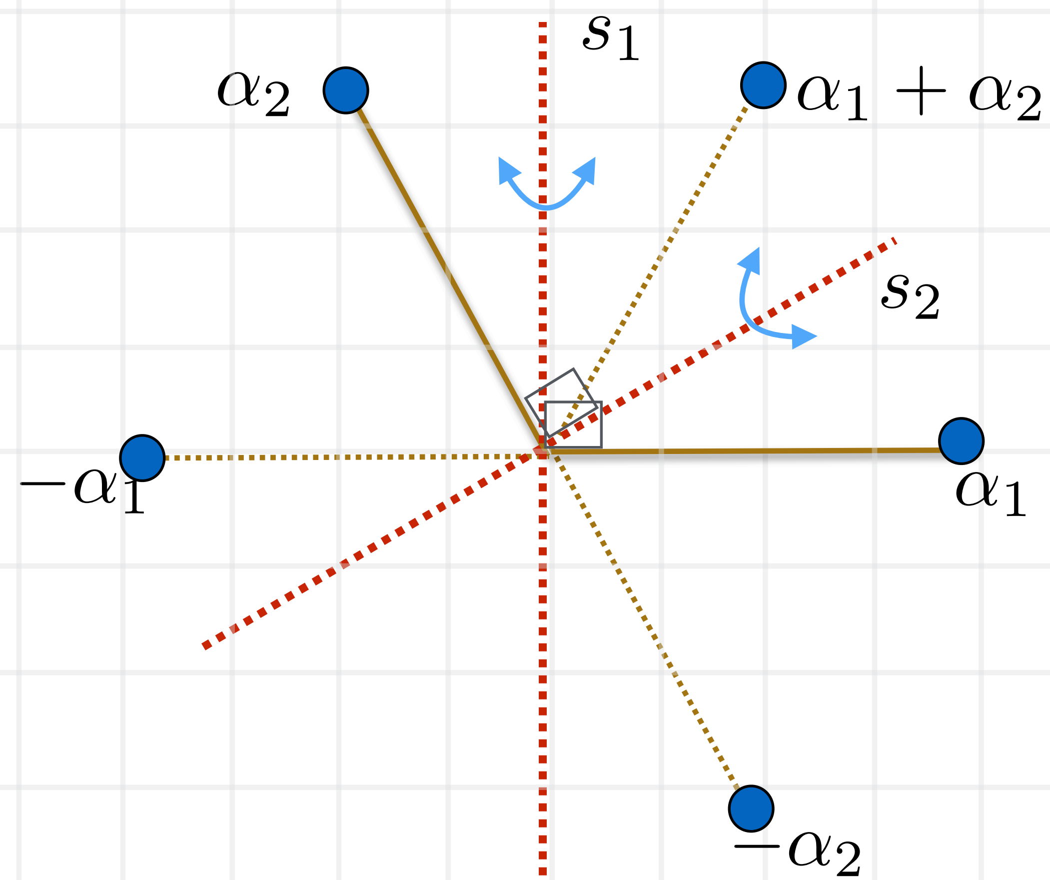
A_2 Root System



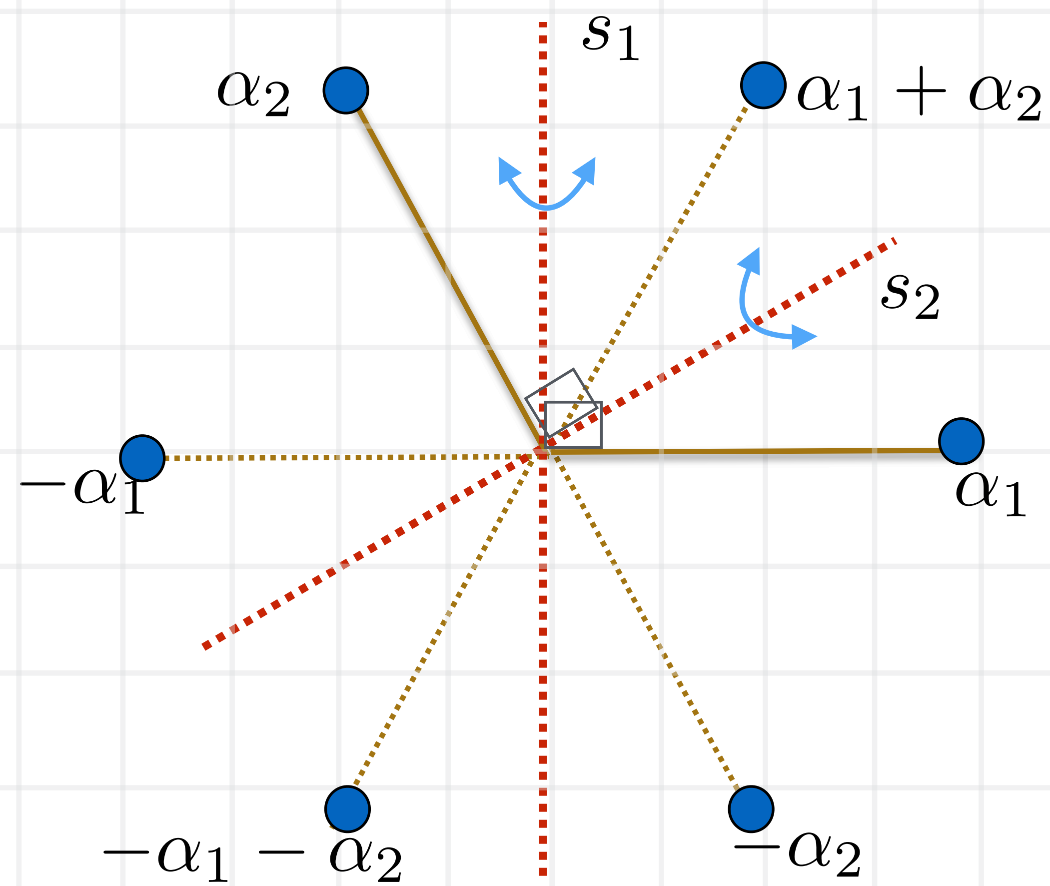
α_1 and α_2 are "simple" roots

A_2 Root System



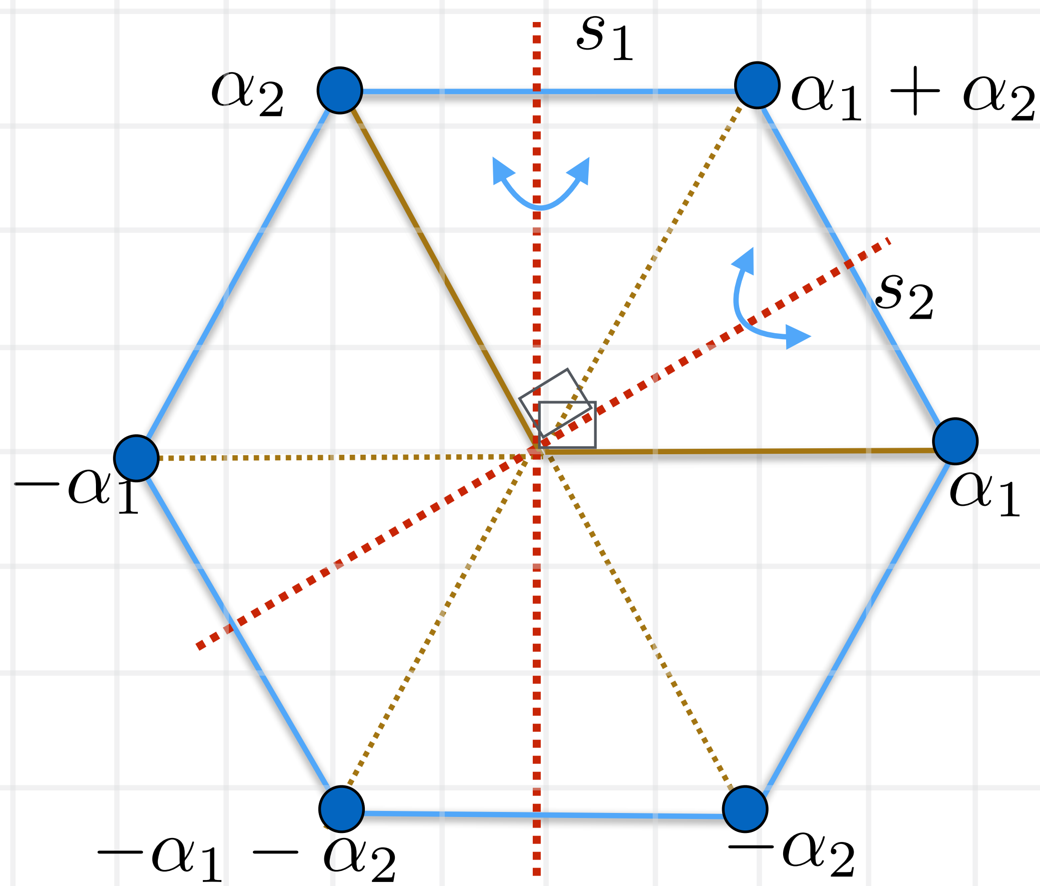
α_1 and α_2 are “simple” roots

A_2 Root System



α_1 and α_2 are "simple" roots

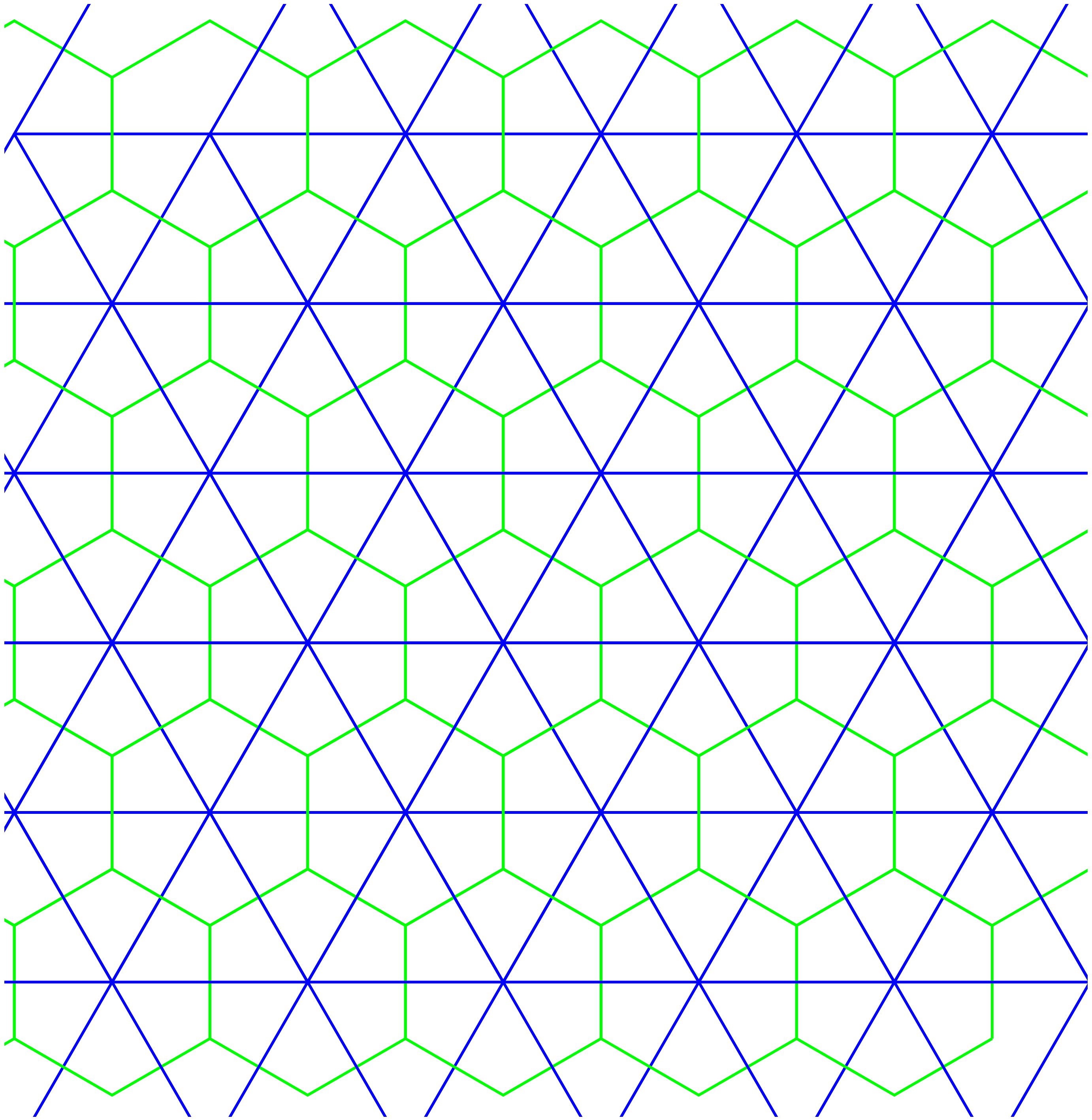
A_2 Root System



α_1 and α_2 are “simple” roots

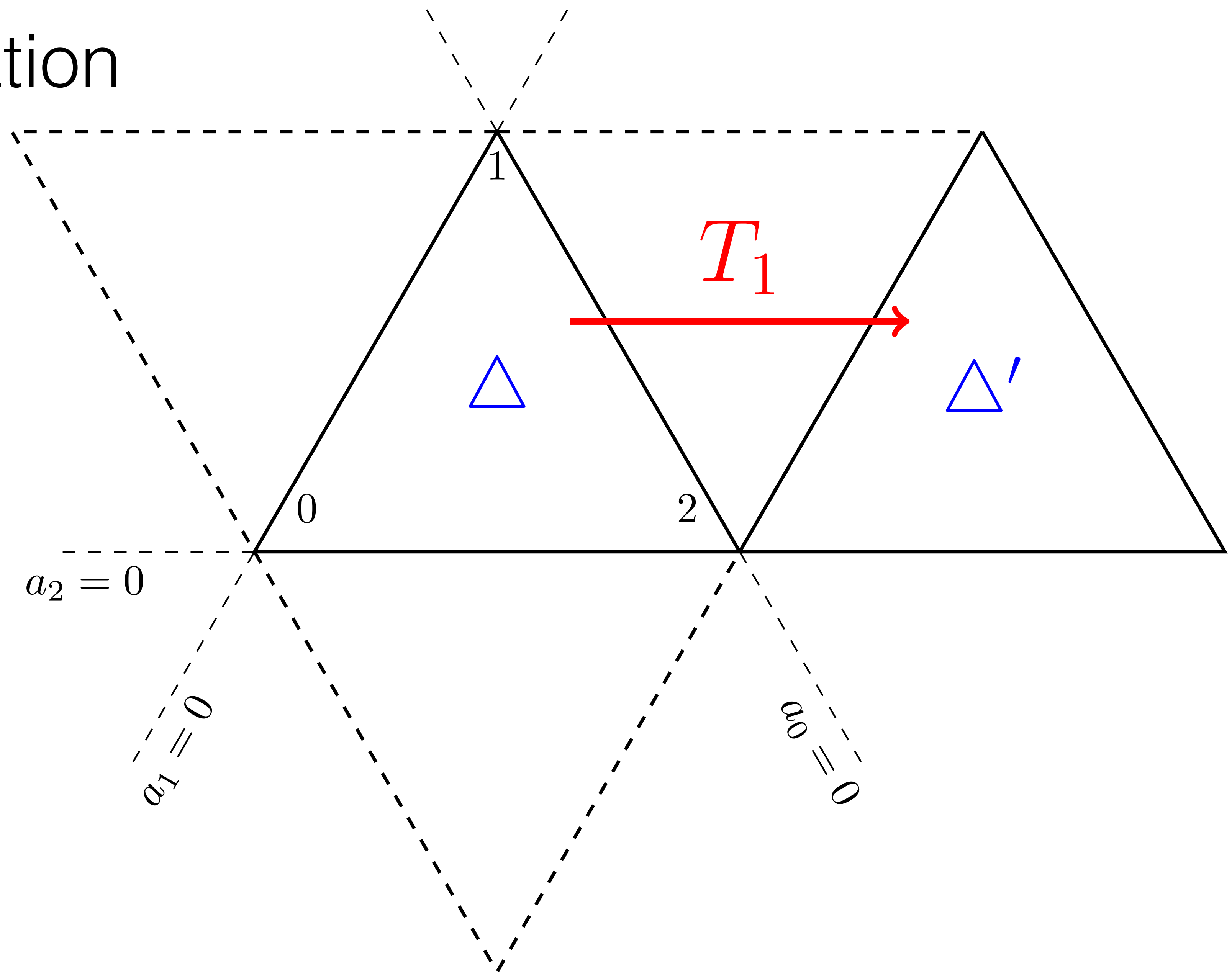
Translation by longest root

$A_2^{(1)}$



Discrete Dynamics I

- Translation



Translations

We have

$$
\begin{aligned}
T_1(a_0) &= \pi\, s_2\, s_1(a_0) \\
&= \pi\, s_2\,(a_0 + a_1) \\
&= \pi\,(a_0 + a_1 + 2a_2) \\
&= a_1 + a_2 + 2\,a_0 = a_0 + k
\end{aligned}
$$

$\Rightarrow$

$$
T_1(a_0) = a_0 + k, \quad T_1(a_1) = a_1 - k, \quad T_1(a_2) = a_2
$$

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

Discrete Dynamics IV

Noting that

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \qquad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

Discrete Dynamics IV

Noting that

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \qquad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

This is dPI again.

Scaled dPI

$$\begin{cases} u_n & \mapsto \epsilon^{-1/2}\, u(s) \\ v_n & \mapsto \epsilon^{-1/2}\, v(s), \;\; s = \epsilon\, n \end{cases}$$

- dPI (with c=0) becomes

$$\big(v(s+\epsilon) + u(s) + v(s-\epsilon)\big)u(s) = \alpha\, s + \epsilon\,\beta + \epsilon^{1/2}\,\gamma\, u(s)$$
$$\big(u(s+\epsilon) + v(s) + u(s-\epsilon)\big)v(s) = \alpha\, s + \epsilon\,\beta + \epsilon^{1/2}\,\gamma\, u(s)$$

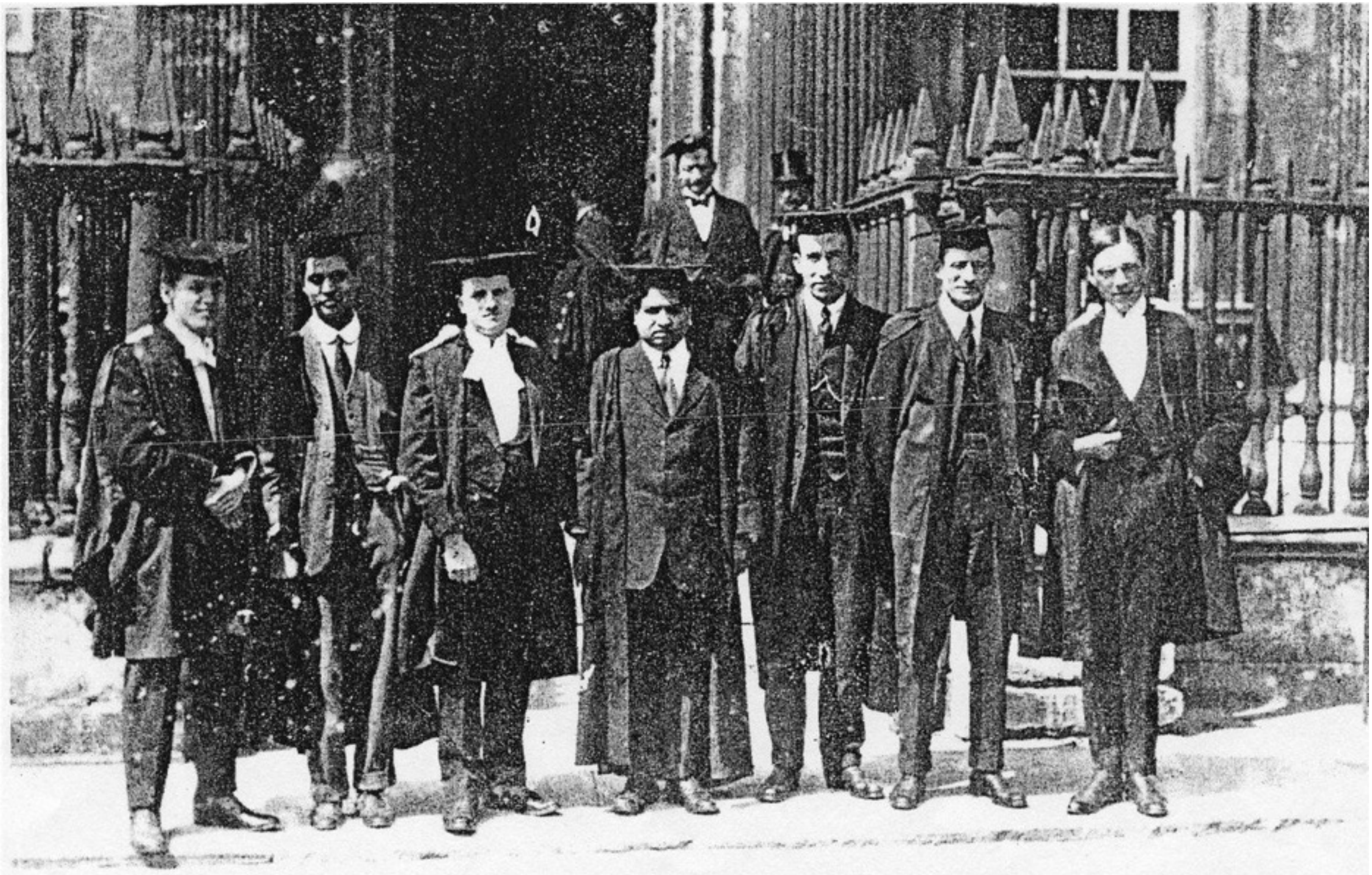
- Asymptotic behaviours as $\epsilon \to 0$
  - General behaviours are close to elliptic functions
  - Special solutions are given by power series

Joshi 1997, Joshi & Lustri 2015

Vereschagin 1995

Summary

- New mathematical models of physics pose new questions for applied mathematics
- Global dynamics of solutions of non-linear equations, whether they are differential or discrete, can be found through geometry.
- Geometry provides the only analytic approach available in $\mathbb{C}$ for discrete equations.
- Tantalising questions about finite properties of solutions remain open.



The mathematician's pattern's, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. *GH Hardy, A Mathematician’s Apology, 1940*