The famous inverse scattering method and its less famous discrete version

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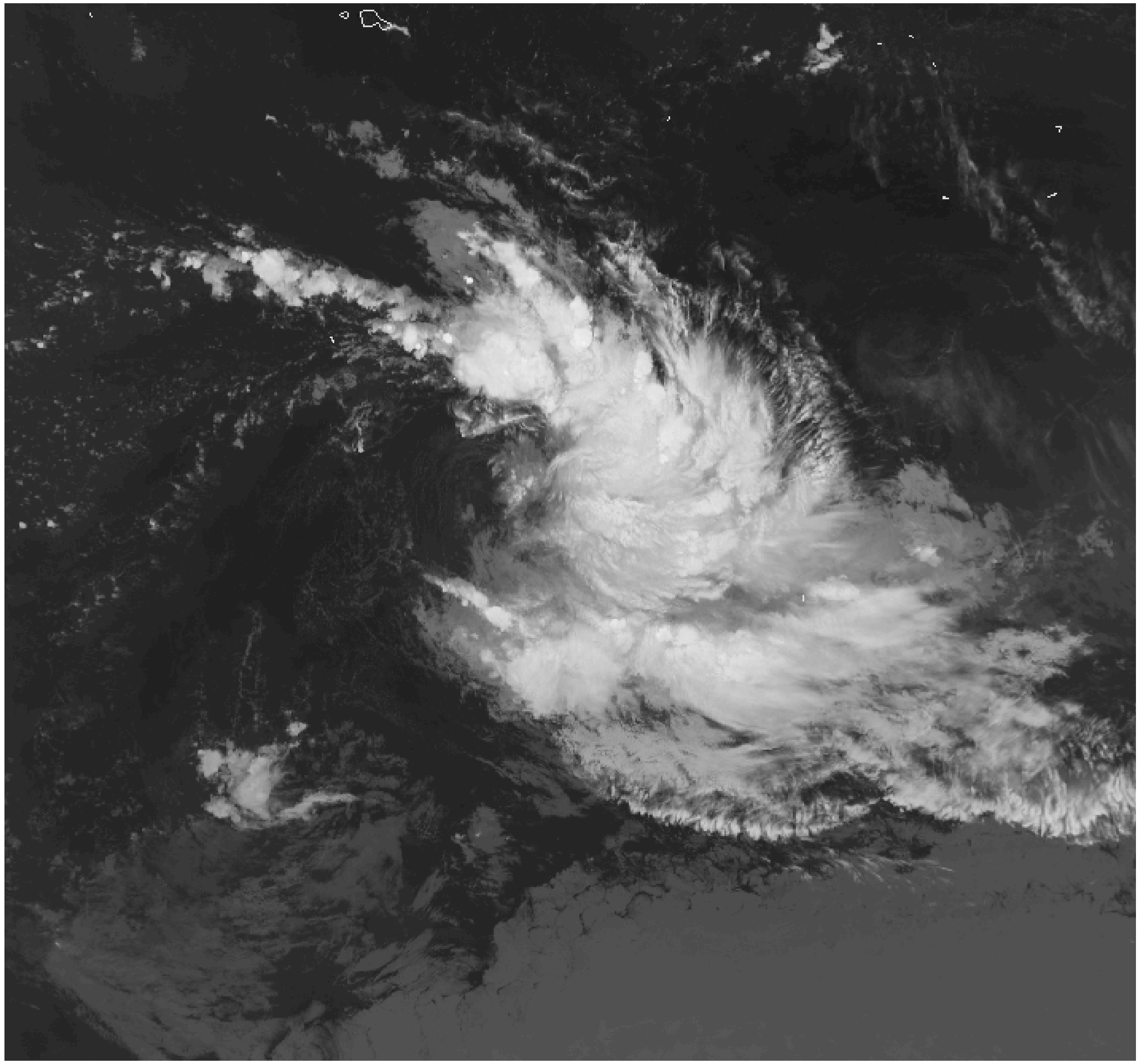
“Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with non-linear partial differential equations and, in fact, with virtually all types of non-linear problems in pure mathematics.”

– John von Neumann, 1946

Chaos

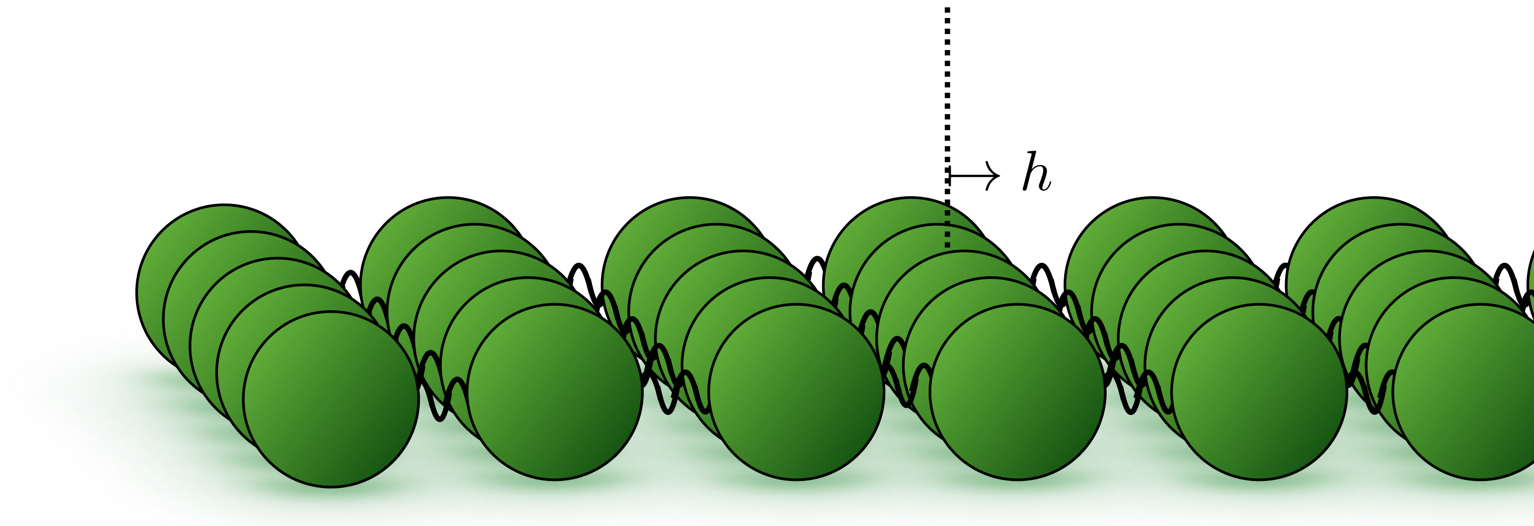
"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"



TC Gavin 1997 SW Pacific

Order

At about the same time as chaos, astonishingly well-ordered & predictable behaviour was found in models used to describe thermal properties of metals.



FPU lattice

Particle-like Waves

Leading to the discovery of *solitons:*

“solitary waves” preserving speed, height, shape, … as they travel and interact in space and time.

Zabusky & Kruskal 1965

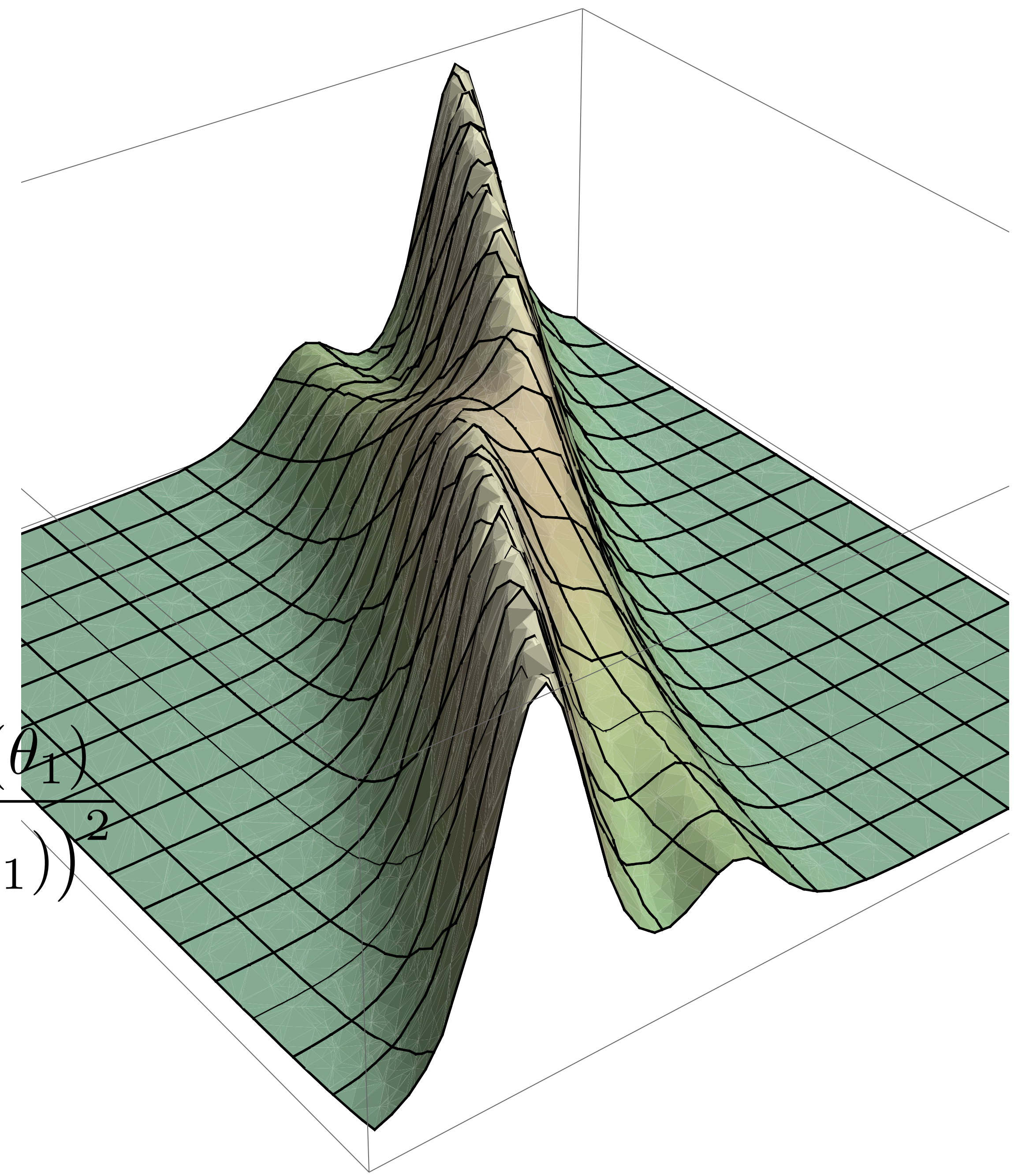
Solitons

The Korteweg-de Vries equation

$$u_t - 6uu_x - u_{xxx} = 0$$

has *N*-soliton solutions. For constant η_i, κ_i

$$u(x,t) = 2(\eta_2^2 - \eta_1^2)\cdot$$
$$\cdot \frac{\eta_2^2 \operatorname{csch}^2(\theta_2) + \eta_1^2 \operatorname{sech}^2(\theta_1)}{\left(\eta_2 \coth(\theta_2) - \eta_1 \tanh(\theta_1)\right)^2}$$



is a 2-soliton solution, where

$$\theta_i = \eta_i\, x - 4\eta_i^3 t + \kappa_i$$

Solution Method

- The Korteweg-de Vries equation

  $$u_t = u_{xxx} + 6uu_x$$

  is integrable because it has an underlying linear structure.

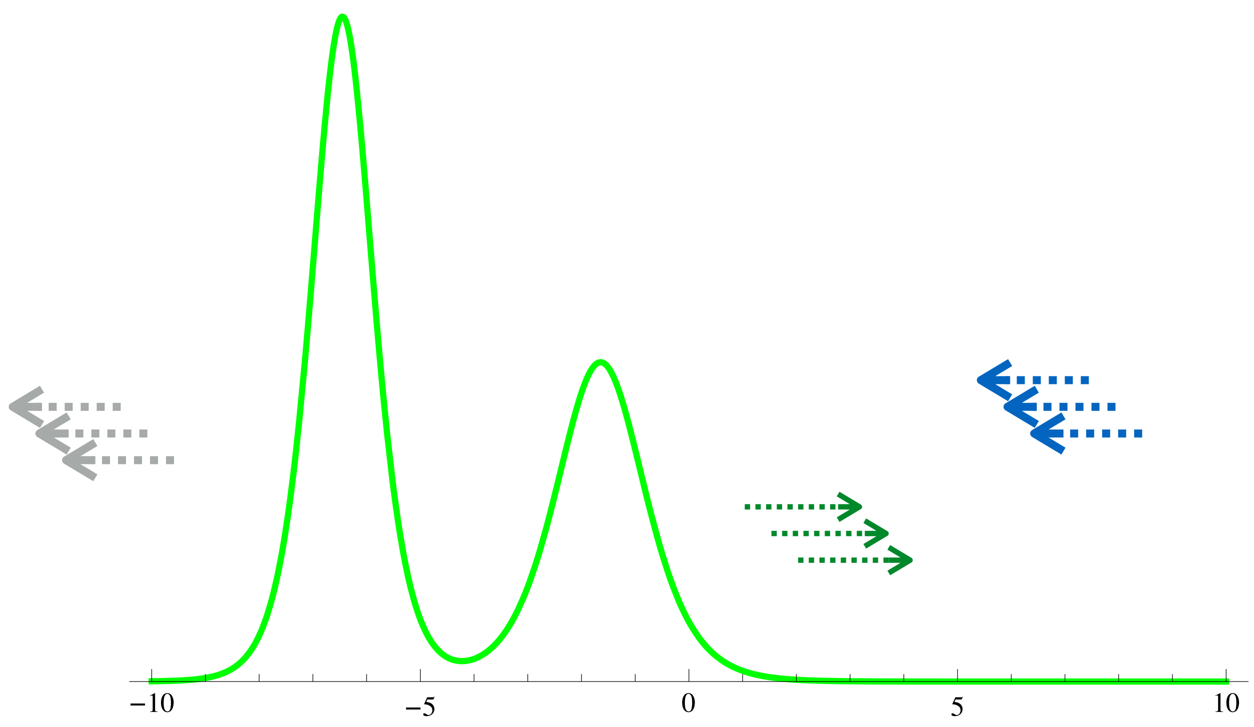
- It is the compatibility condition of the spectral problem

  $$\begin{cases} \psi_{xx} + u(x,t)\,\psi = \lambda\,\psi \\ \psi_t = 4\psi_{xxx} + 6u\psi_x + 3u_x\,\psi \end{cases}$$

  called the Lax pair, used to solve its initial value problem.

How the KdV equation is solved

- Given *u(x, 0)*, solve the Schrödinger equation with this as potential.
- Find reflection, transmission coefficients and bound states
- Evolve these in time.
- Reconstruct the solution of the KdV: *u(x, t).*



GGKM, 1967

Inverse Scattering Transform

Continuous

- Gardner, Greene, Kruskal & Miura 1967
- Zakharov & Shabat 1971
- Wadati 1972
- Ablowitz, Kaup, Newell & Segur 1973
- Calogero & Degasperis 1976
- Deift & Trubowitz 1979
- Fokas 1997
- Fokas & Pelloni 1998
- Degasperis, Manakov & Santini 2001

Differential-Discrete

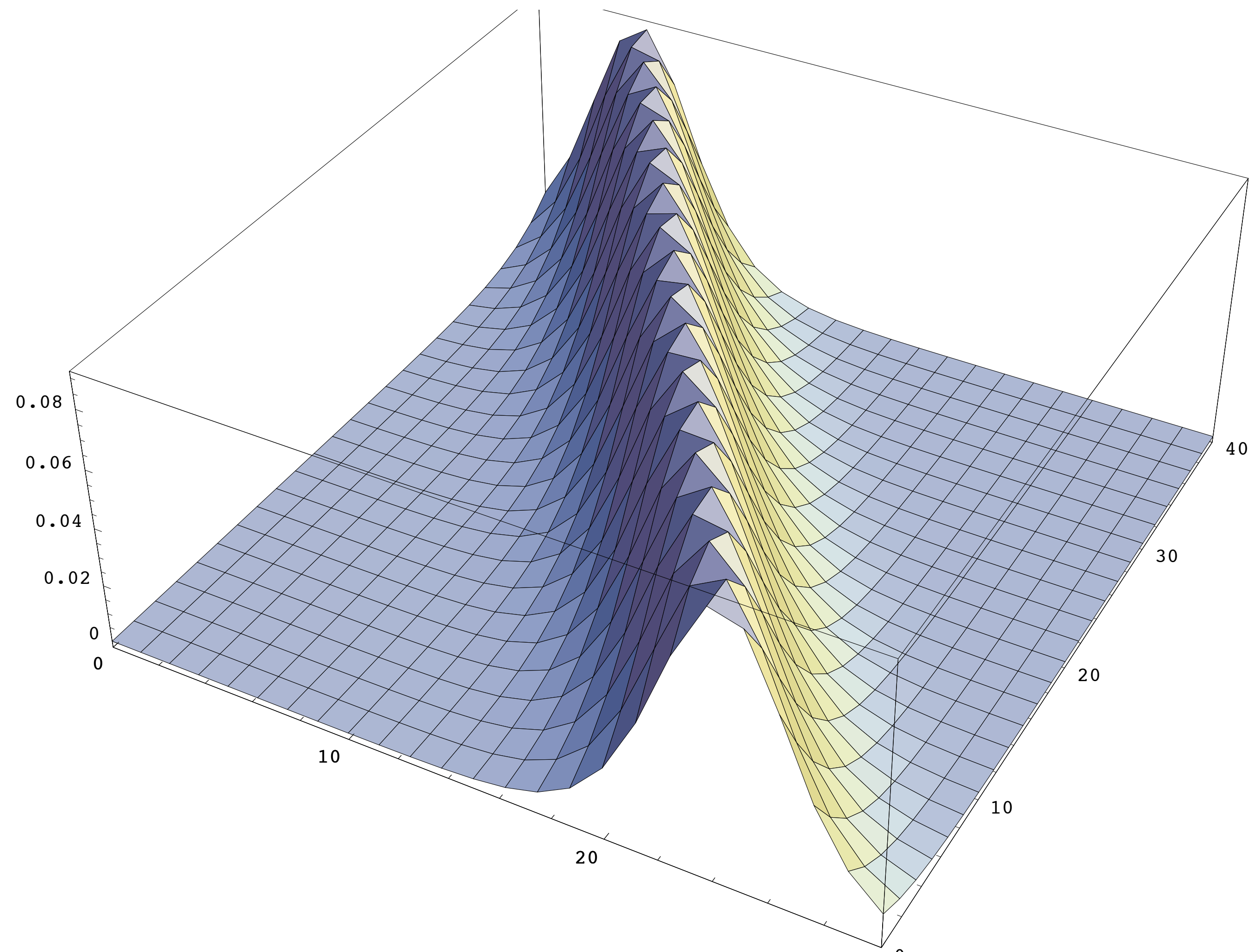
- Case & Kac 1973
- Case 1973
- Flaschka 1974
- Ablowitz & Ladik 1975
- Kac & van Moerbeke 1975
- Levi & Ragnisco 1978
- Pilloni & Levi 1982
- Ragnisco, Santini et al 1987
- Bruckestein & Kailath 1987
- Ruijsenaars 2002

Partial Difference Equations

The discrete potential KdV equation

$$\big(w_{n+1,m+1} - w_{n,m}\big)\big(w_{n,m+1} - w_{n+1,m}\big) = 4(\mu - \lambda)$$

- What is the corresponding inverse scattering transform method?



Part 1

- Continuous-discrete
- Partial difference equations
- Discrete inverse scattering
- Discrete iso-monodromy problems

Duality

- The Weber equation:

$$w'' + \left(\alpha + \frac{1}{2} - \frac{1}{4}x^2\right) w = 0$$

has recurrence relations: $w(x) = D_\alpha(x)$

$$D'_\alpha(x) = -\frac{x}{2} D_\alpha(x) + \alpha\, D_{\alpha-1}(x)$$

$$D'_{\alpha-1}(x) = \frac{x}{2} D_{\alpha-1}(x) - D_\alpha(x)$$

which imply

$$D_{\alpha+1}(x) - x\, D_\alpha(x) + \alpha D_{\alpha-1}(x) = 0$$

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$$D_{\alpha+1}(x) - x\, D_\alpha(x) + \alpha D_{\alpha-1}(x) = 0$$

Discrete

Transformations

- The potential Korteweg-de Vries equation is

$$w_t = w_{xxx} + 3\,w_x^2\,, \qquad u = w_x$$

- Given a parameter λ, the Bäcklund transformation

$$\big(\widetilde{w} + w\big)_x = 2\,\lambda - \tfrac{1}{2}\big(\widetilde{w} - w\big)^2$$

  relates two solutions $\widetilde{w}, w$ of the potential KdV equation.

Wahlquist & Estabrook, 1976

Composition

- Take two such transformations

$$BT_\lambda : w \overset{\lambda}{\mapsto} \widetilde{w}, \quad \left(\widetilde{w} + w\right)_x = 2\lambda - \frac{1}{2}\left(\widetilde{w} - w\right)^2$$

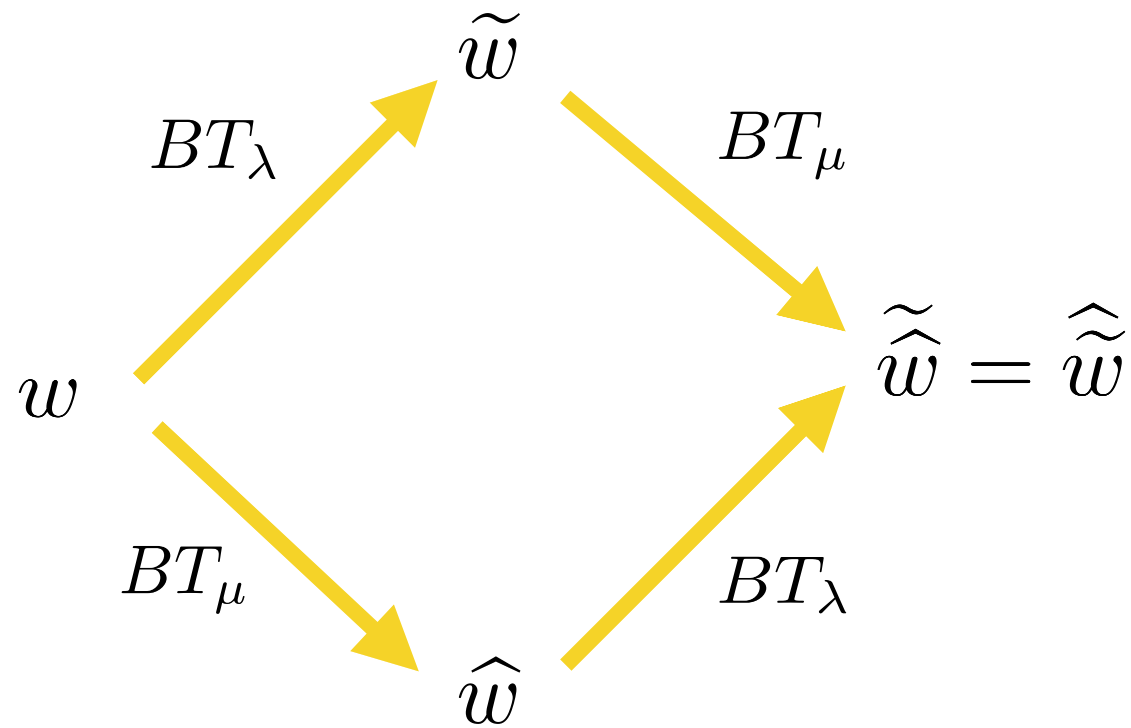
$$BT_\mu : w \overset{\mu}{\mapsto} \widehat{w}, \quad \left(\widehat{w} + w\right)_x = 2\mu - \frac{1}{2}\left(\widehat{w} - w\right)^2$$

- Compose the transformations in two different ways

$$\widehat{\widetilde{w}} = BT_\mu \circ BT_\lambda \, w, \quad \widetilde{\widehat{w}} = BT_\lambda \circ BT_\mu \, w$$

- Are they the same solution?

Permutability



Two different compositions of BTs give the same solution.

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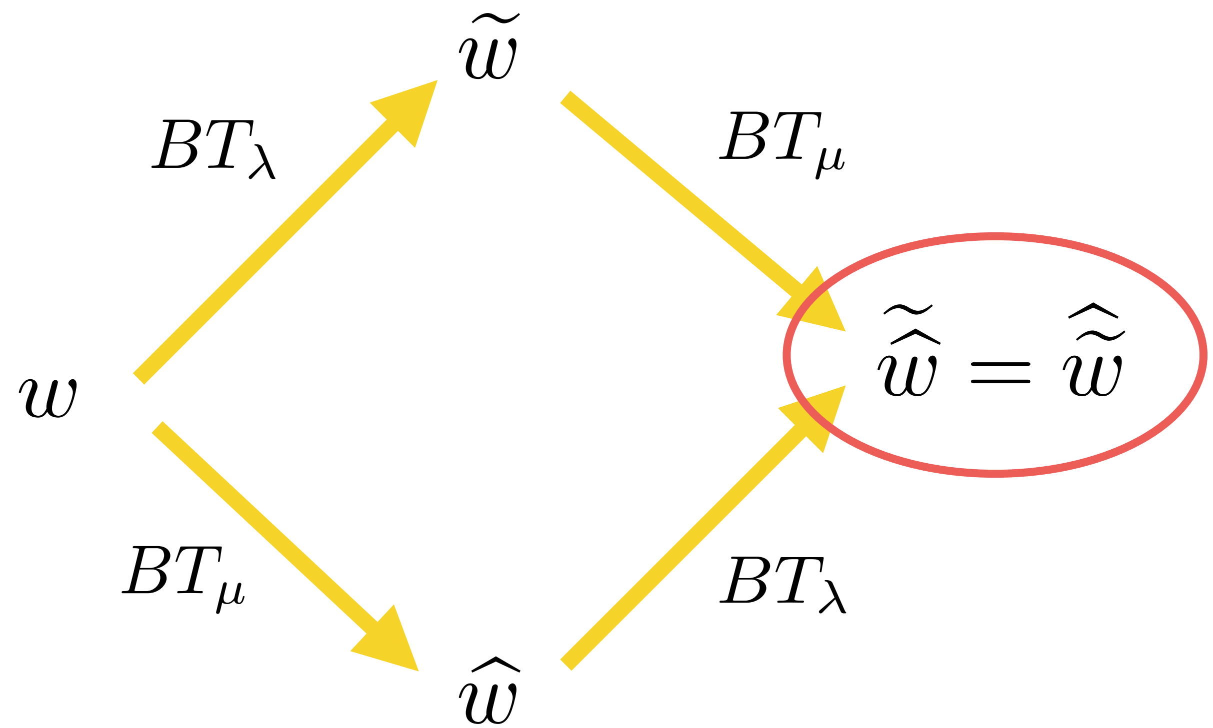


Two different compositions of BTs give the same solution.

Lattice Equations

- Eliminating derivatives between BT_λ, BT_μ and *their* derivatives, we find

$$\left(w_{n+1,m+1} - w_{n,m}\right)\left(w_{n,m+1} - w_{n+1,m}\right) = 4(\mu - \lambda)$$

$$(\widehat{\widetilde{w}} - w)(\widehat{w} - \widetilde{w}) = 4(\mu - \lambda)$$

called the *lattice potential KdV equation, where*

$$w_{n,m} = BT_\lambda^n \circ BT_\mu^m \, w$$

Discrete Solitons

- This has soliton solutions:

$$w = am + bn + k\tanh(kx + \beta m + \gamma n + \xi)$$

where $a^2 - b^2 = 4(\mu - \lambda)$, ,

$$\beta = \frac{1}{2}\log((a+k)/(a-k))$$
$$\gamma = \frac{1}{2}\log((b+k)/(b-k))$$

Nijhoff, Quispel, Capel,1983
Nijhoff, Quispel, van der Linden, Capel,1983

N-dimensional BTs

$$w_{n,m,l,\ldots} = BT_p^n \circ BT_q^m \circ BT_r^l \circ \ldots\, w$$

$$\widetilde{w} = w_{n+1,m,l,\ldots},\, \widehat{w} = w_{n,m+1,l,\ldots},\, \overline{w} = w_{n,m,l+1,\ldots},\, \cdots$$

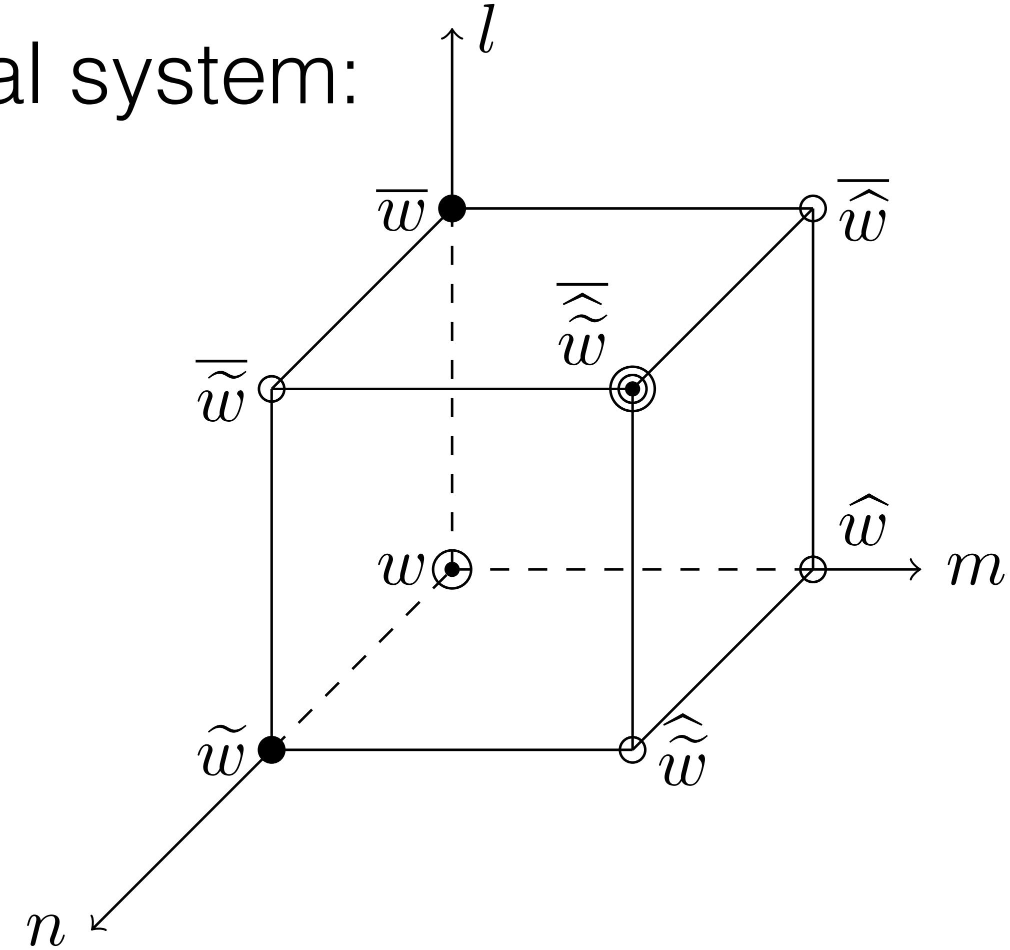
- We get a multidimensional system:

$$(\widehat{w} - \widetilde{w})(w - \widehat{\widetilde{w}}) = p^2 - q^2$$

$$(\overline{w} - \widetilde{w})(w - \overline{\widetilde{w}}) = p^2 - r^2$$

$$(\widehat{w} - \overline{w})(w - \overline{\widehat{w}}) = r^2 - q^2$$

$$\vdots$$



Nijhoff & Walker, 1998

Multi-dimensional consistency

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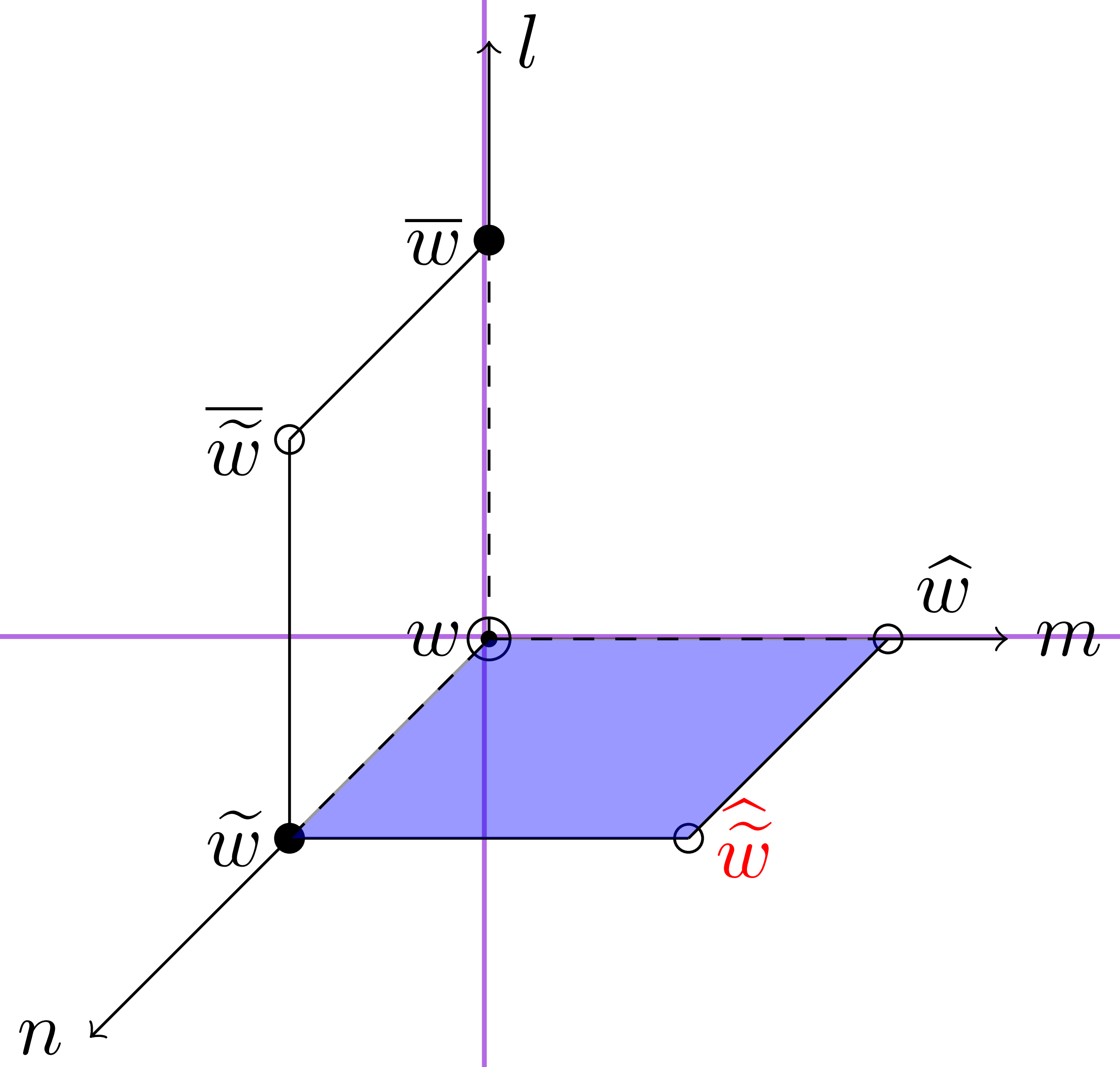
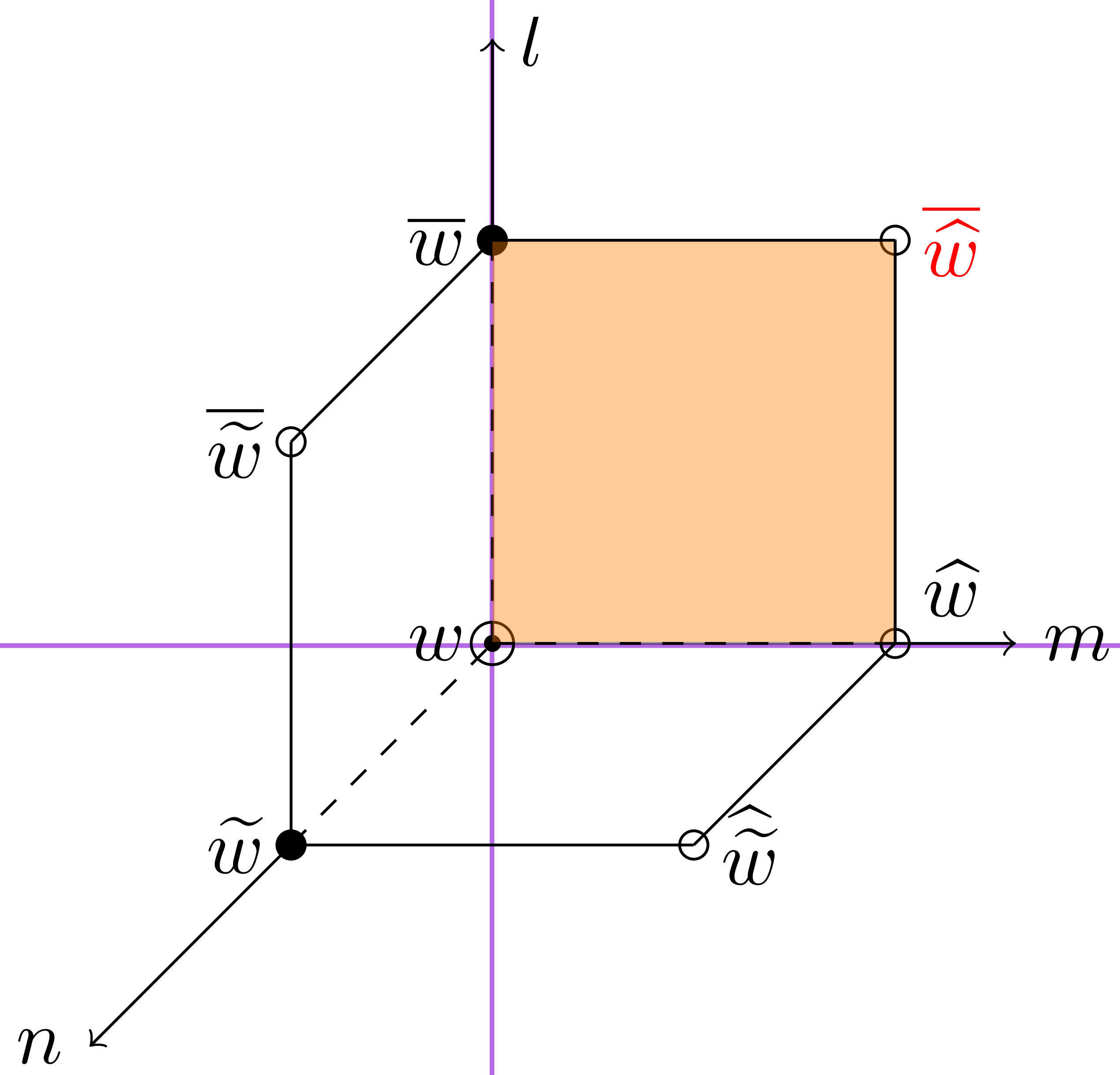


Multi-dimensional consistency



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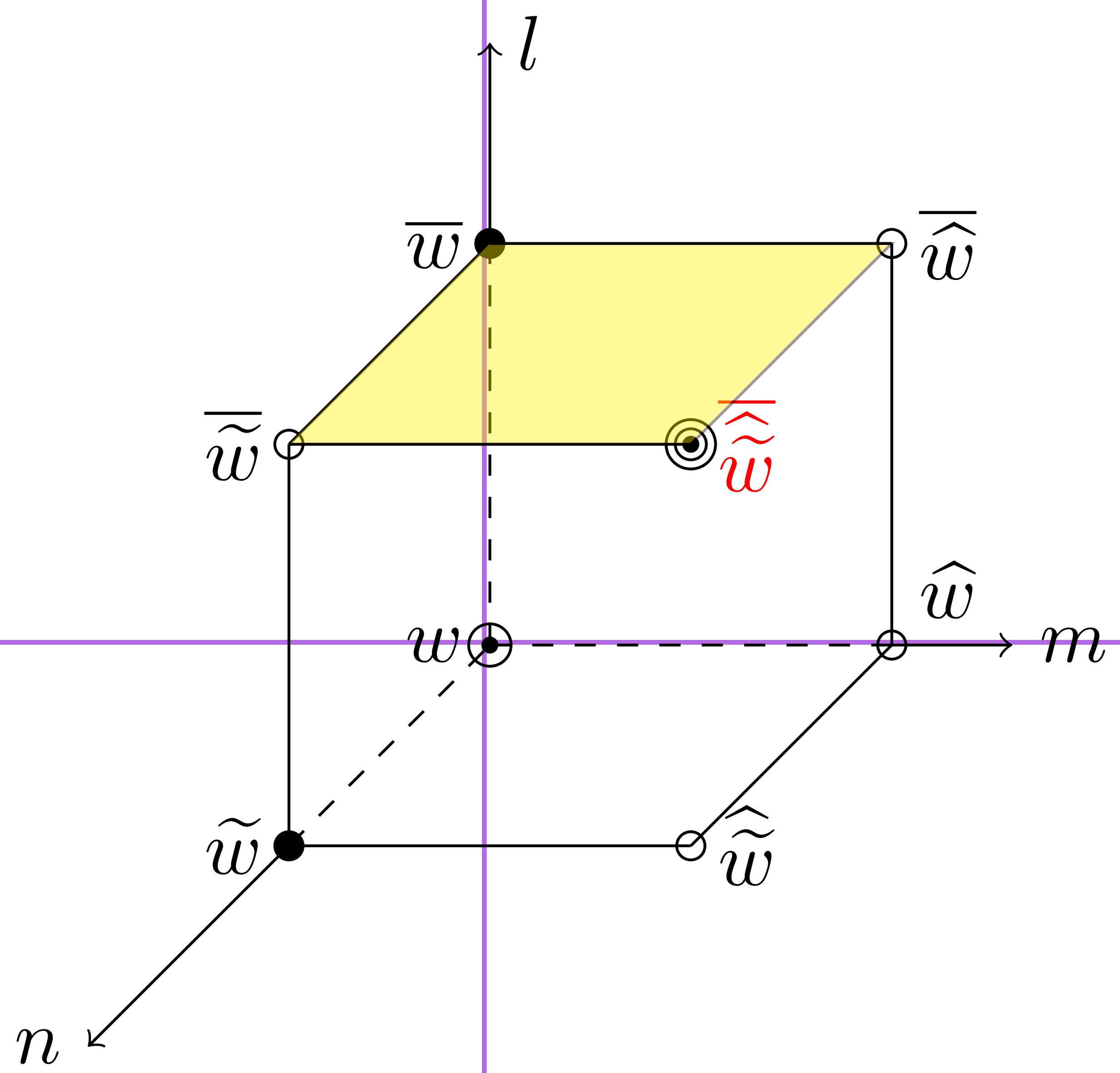

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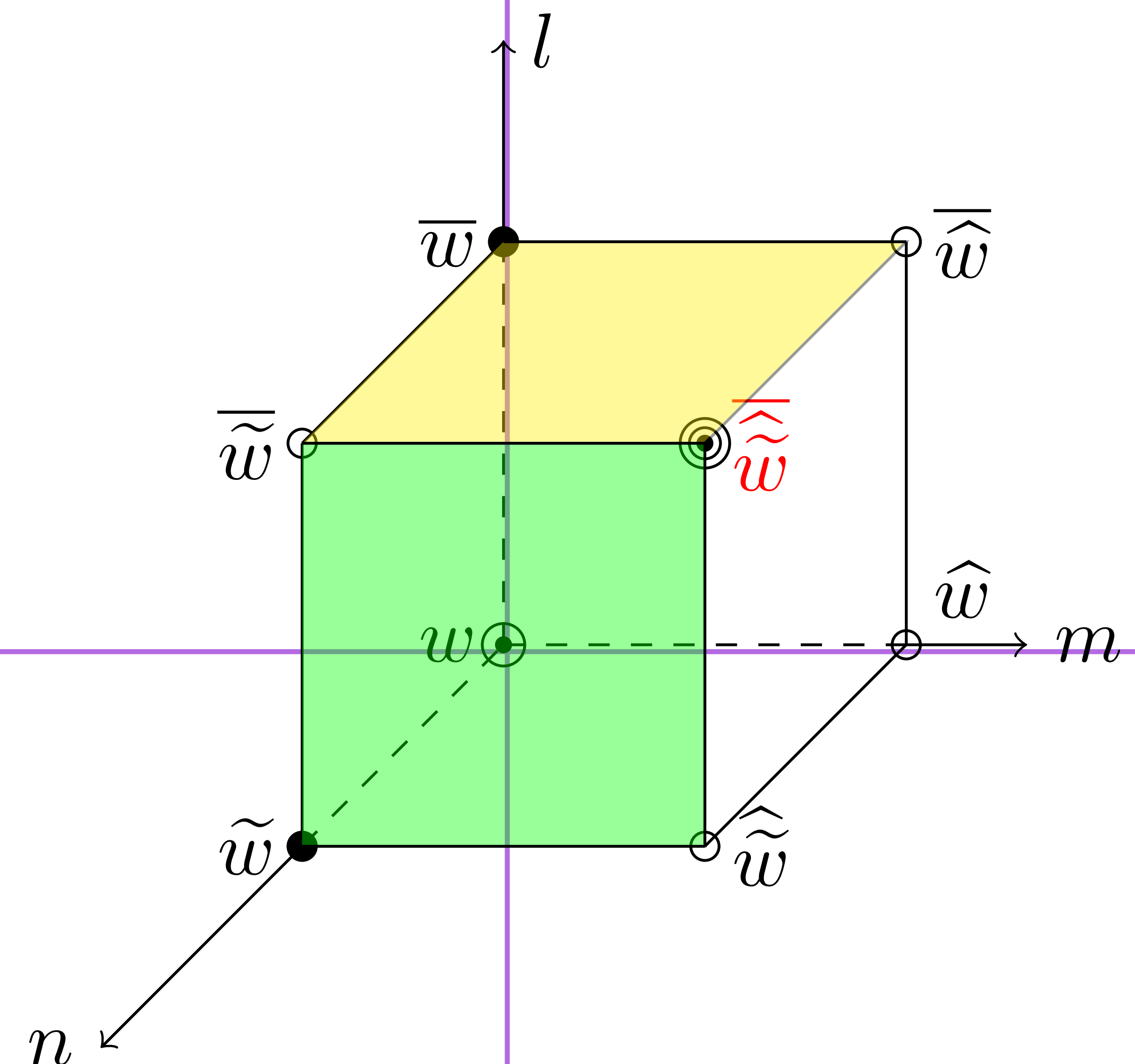
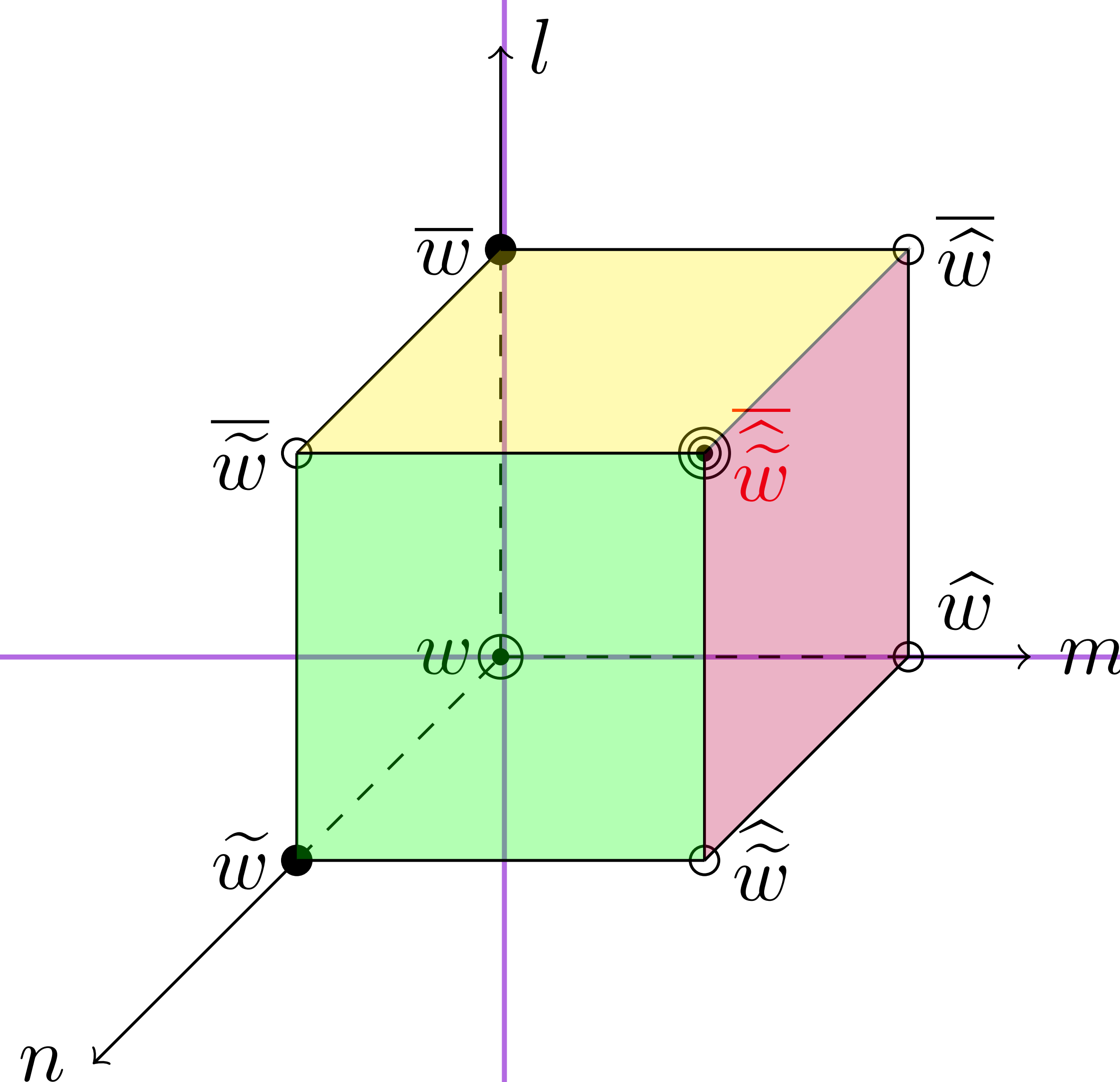


Multi-dimensional consistency



Part 2

- Continuous-discrete
- Partial difference equations
- Discrete inverse scattering
- Discrete iso-monodromy problems

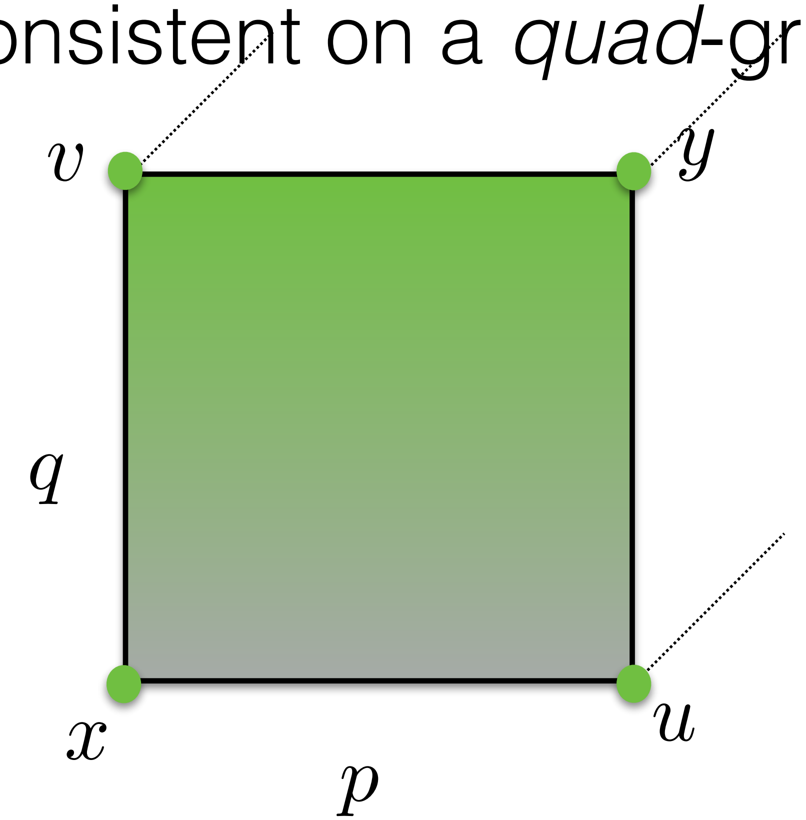
ABS Classification

- Adler, Bobenko & Suris (2003) classified all affine linear equations

$$Q(w, \widetilde{w}, \widehat{w}, \widehat{\widetilde{w}}; p, q) = 0$$

which are multi-dimensionally consistent on a *quad*-graph

$$Q(x, u, v, y; p, q) = 0$$



Some ABS Equations

- H1:

$$(x-y)(u-v)+p^2-q^2=0$$

- H3:

$$\mathcal{Q}(xu+vy)-\mathcal{P}(uv+uy)+\frac{p^2-q^2}{\mathcal{P}\mathcal{Q}}=0$$

where $\mathcal{P}^2=a^2-p^2, \mathcal{Q}^2=a^2-q^2$

- Q3:

$$\mathcal{P}(uv+uy)-\mathcal{Q}(xu+vy)-(p^2-q^2)\left(uv+xy+\frac{\delta^2}{4\mathcal{P}\mathcal{Q}}\right)=0$$

where

$$\mathcal{P}^2=(p^2-a^2)(p^2-b^2)$$
$$\mathcal{Q}^2=(q^2-a^2)(q^2-b^2)$$

At the top

$$
\text{(Q4)} \quad a_0 xuvy + a_1(xuv + uvy + vyx + yxu) + a_2(xy + uv) + \bar{a}_2(xu + vy) \\
+\tilde{a}_2(xv + uy) + a_3(x + u + v + y) + a_4 = 0,
$$

where the coefficients a_i are expressed through (α, a) and (β, b) with $a^2 = r(\alpha)$, $b^2 = r(\beta)$, $r(x) = 4x^3 - g_2 x - g_3$, by the following formulae:

$$a_0 = a + b, \quad a_1 = -\beta a - \alpha b, \quad a_2 = \beta^2 a + \alpha^2 b,$$

$$\bar{a}_2 = \frac{ab(a+b)}{2(\alpha - \beta)} + \beta^2 a - (2\alpha^2 - \frac{g_2}{4})b,$$

$$\tilde{a}_2 = \frac{ab(a+b)}{2(\beta - \alpha)} + \alpha^2 b - (2\beta^2 - \frac{g_2}{4})a,$$

$$a_3 = \frac{g_3}{2} a_0 - \frac{g_2}{4} a_1, \quad a_4 = \frac{g_2^2}{16} a_0 - g_3 a_1.$$

ABS, 2003

At the top

$$\text{(Q4)}\quad a_0xuvy + a_1(xuv + uvy + vyx + yxu) + a_2(xy + uv) + \bar{a}_2(xu + vy) + \tilde{a}_2(xv + uy) + a_3(x + u + v + y) + a_4 = 0,$$

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$$a_3 = \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \quad a_4 = \frac{g_2^2}{16}a_0 - g_3 a_1.$$

ABS, 2003

In what sense are these partial difference equations integrable?

Linear Problems

Consider a 2D lattice equation on a quadrilateral face.

The third direction provides a “spectral” problem.

Linear Problems

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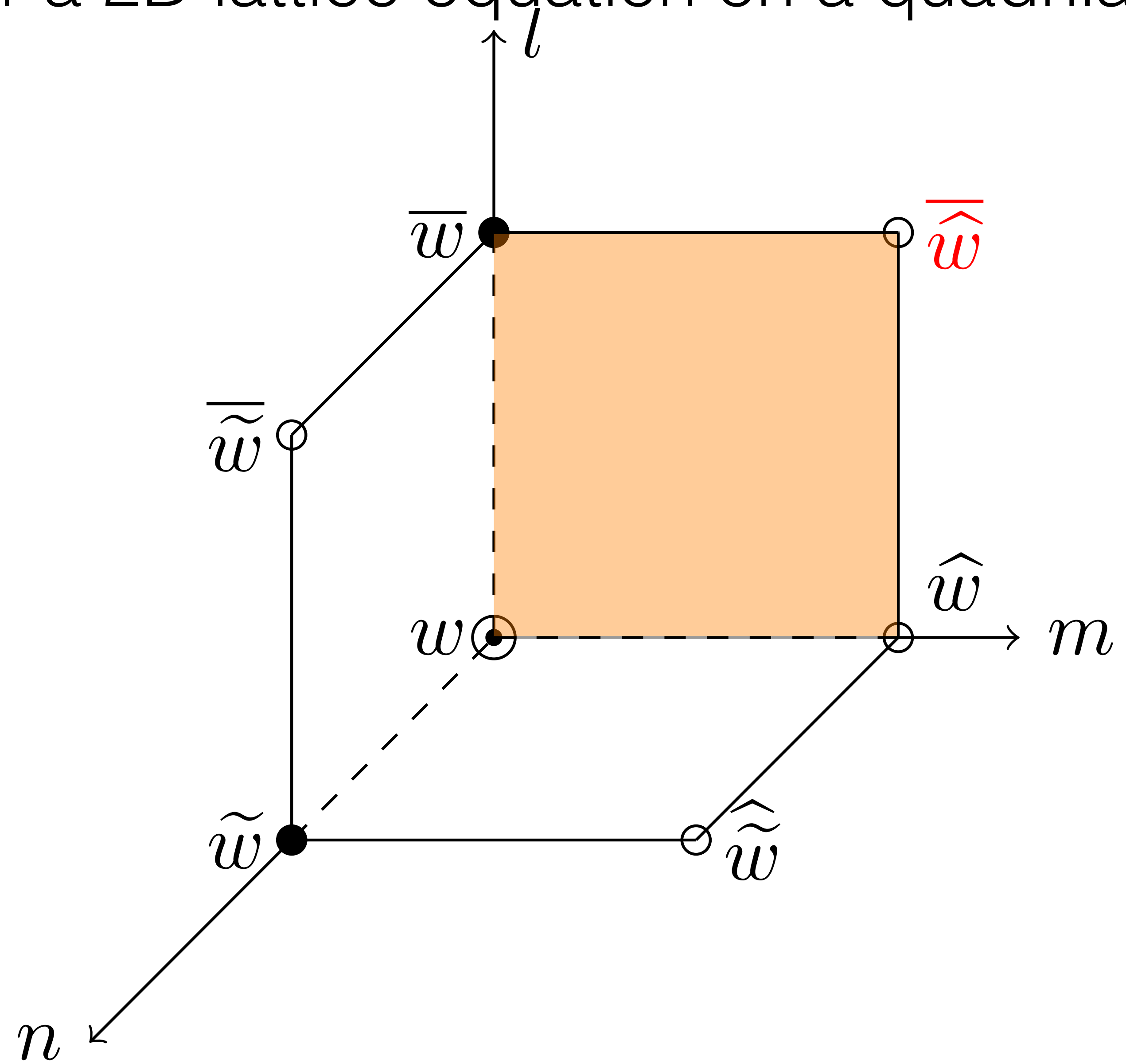

The third direction provides a “spectral” problem.

Spectral problem for H1

$$\overline{w} =: W$$

$$(W - \widetilde{w})(\widetilde{W} - w) = k^2 - p^2 \Rightarrow \ \widetilde{W} = \frac{wW + (k^2 - p^2 - w\widetilde{w})}{W - \widetilde{w}}$$

$$(W - \widehat{w})(\widehat{W} - w) = k^2 - q^2 \Rightarrow \ \widehat{W} = \frac{wW + (k^2 - q^2 - w\widehat{w})}{W - \widehat{w}}$$

Linearize by using $W = F/G$, then separate variables:

$$\widetilde{\varphi} = L\varphi \qquad \varphi = \begin{pmatrix} F \\ G \end{pmatrix}$$
$$\widehat{\varphi} = M\varphi$$

$$L = \gamma \begin{pmatrix} w & k^2 - p^2 - w\widetilde{w} \\ 1 & -\widetilde{w} \end{pmatrix}, \quad M = \gamma' \begin{pmatrix} w & k^2 - q^2 - w\widehat{w} \\ 1 & -\widehat{w} \end{pmatrix}$$

where k is the spectral parameter.

Spectral problem for H1

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$$\widetilde{\varphi} = L\varphi \qquad \widehat{\varphi} = M\varphi \qquad \varphi = \begin{pmatrix} F \\ G \end{pmatrix}$$

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where k is the spectral parameter.

Compatibility, again

$$\widehat{\widetilde{\varphi}} = \widehat{L\varphi} = \widehat{L}\widehat{\varphi} = \widehat{L}M\varphi$$

$$\widetilde{\widehat{\varphi}} = \widetilde{M\varphi} = \widetilde{M}\widetilde{\varphi} = \widetilde{M}L\varphi$$

$$\Rightarrow \ \widehat{L}M = \widetilde{M}L$$

$$\Leftrightarrow \ H1$$

$$(\widehat{w} - \widetilde{w})(w - \widehat{\widetilde{w}}) = p^2 - q^2$$

Part 3

- Continuous-discrete duality
- Partial difference equations
- Discrete inverse scattering
- Discrete iso-monodromy problems

Initial-value problem

Given an initial value

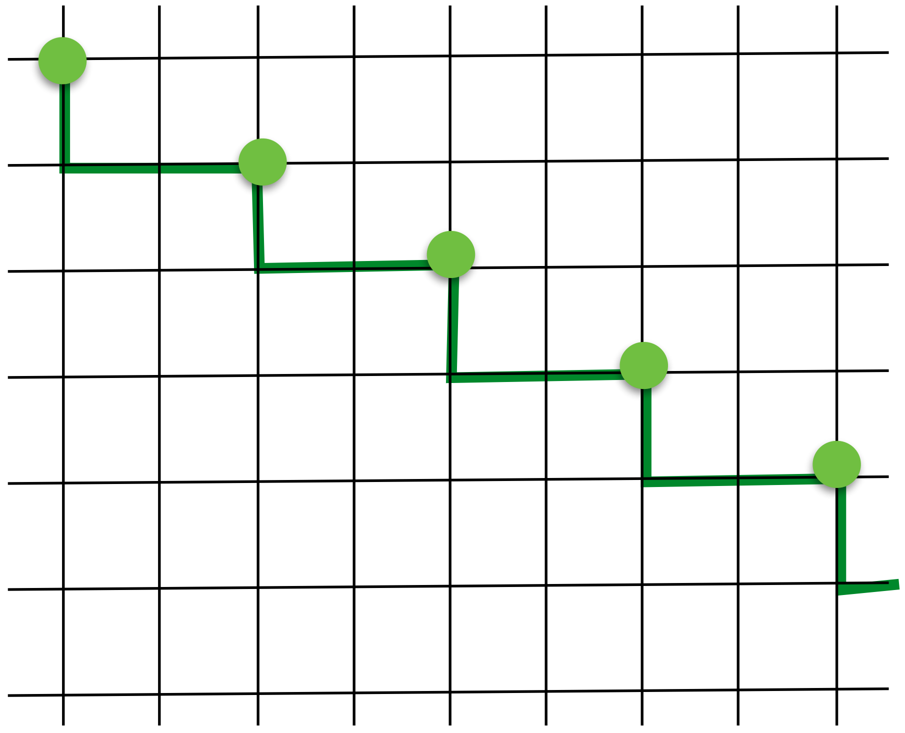
$$u(x,0) = u_0(x) \;\in\; L^1(\mathbb{R})$$

Gardiner, Greene, Kruskal and Miura (1967) showed:

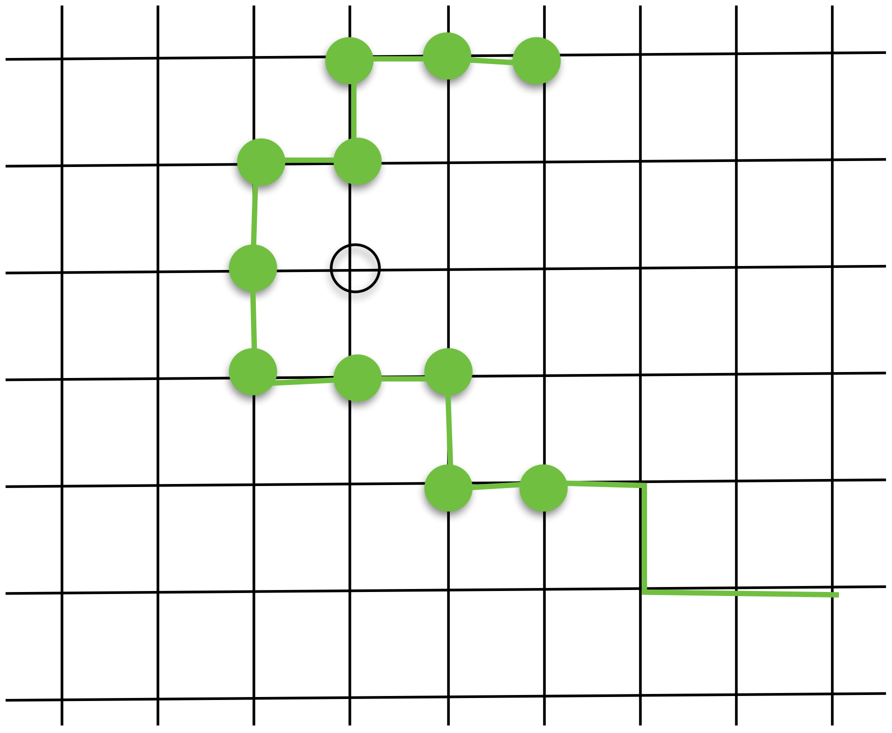
$$\begin{array}{ccc}
u(x,0) & \xrightarrow[\text{Direct Scattering}]{} & \left\{\begin{array}{c} \lambda\{0\}, \psi(x;\lambda\{0\};0) \\ \text{spectral information} \\ S\{0\} \end{array}\right\} \\
\Big\downarrow & & \Big\downarrow \\
u(x,t) & \xleftarrow[\text{Inverse Scattering}]{} & \left\{\begin{array}{c} \lambda\{t\}, \psi(x;\lambda\{t\};t) \\ \text{spectral information} \\ S\{t\} \end{array}\right\}
\end{array}$$

Discrete Initial-value problem

First define an initial value on a discrete oriented “staircase”



Acceptable



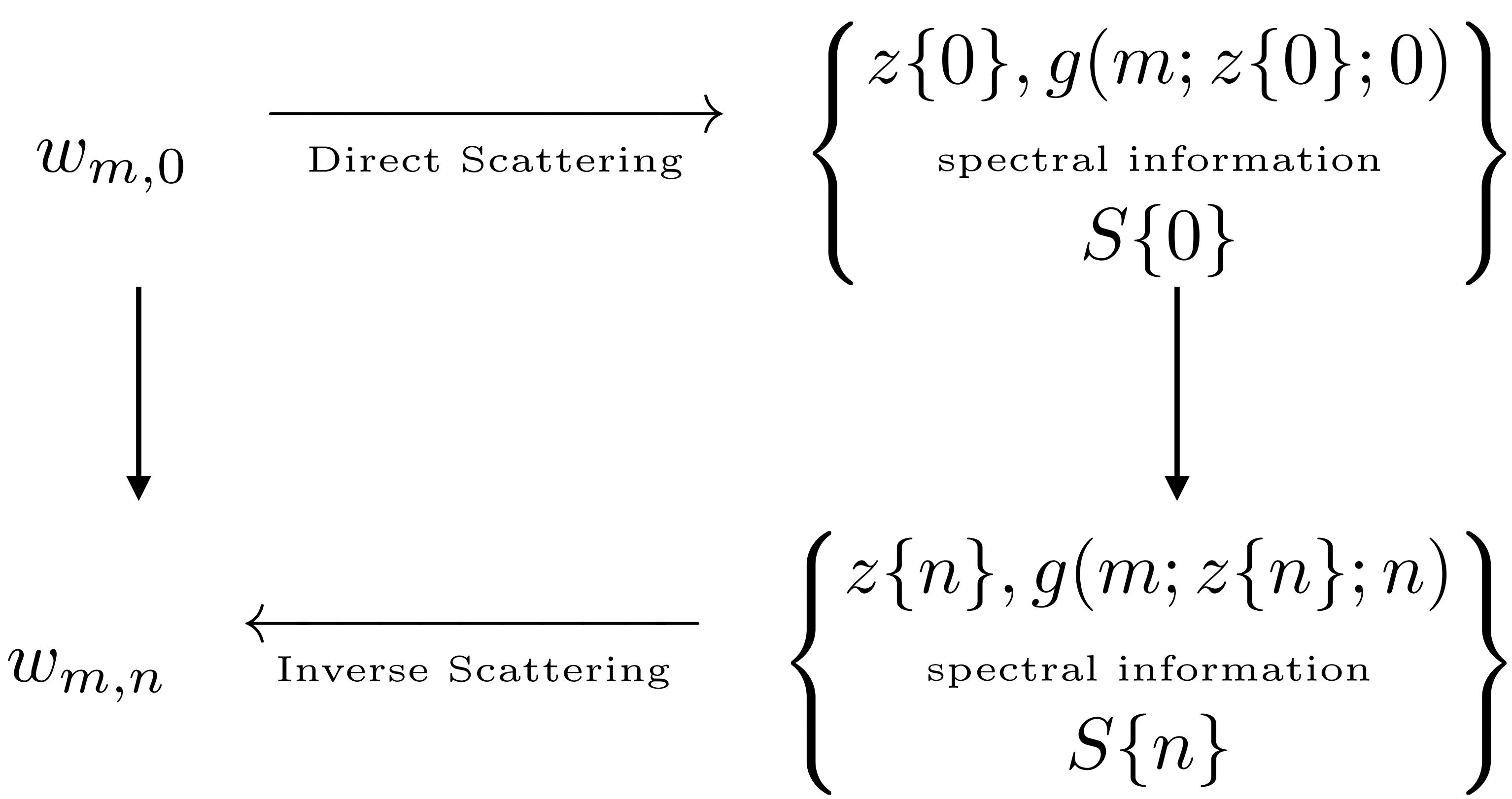
Problematic

Discrete Initial-value problem

Given an initial value for H1 on a line in the lattice, s.t.

$$\sum_{m=-\infty}^{+\infty} \left| w_{m+2,0} - w_{m,0} - 2p \right| (1 + |m|) < \infty$$

$$w_{m+2,0} - w_{m,0} > 0$$



Direct Scattering

- Define a basis set of solutions, the “Jost solutions”
- Obtain the scattering data
- Deduce their analyticity properties in the spectral plane.

Recall continuous Jost solutions

$$\phi_{xx} + (u(x,0) + \zeta^2)\,\phi = 0$$

$$\phi_{xx} + \zeta^2\phi = 0 \;\Rightarrow\; \phi \sim Ae^{i\zeta x} + Be^{-i\zeta x}, \quad |x| \;\to\; \infty,$$

Jost solutions defined by

$$\left.\begin{aligned} \varphi(x,0;\zeta) &\sim e^{-i\zeta x} \\ \mathring{\varphi}(x,0;\zeta) &\sim e^{i\zeta x} \end{aligned}\right\} \text{ as } x \to -\infty$$

$$\left.\begin{aligned} \psi(x,0;\zeta) &\sim e^{i\zeta x} \\ \mathring{\psi}(x,0;\zeta) &\sim e^{-i\zeta x} \end{aligned}\right\} \text{ as } x \to +\infty.$$

Discrete Jost solutions

$$\widetilde{\widetilde{g}} - (2p + \widetilde{u})\widetilde{g} + (p^2 + z^2)\, g = 0$$

where $$2p + \widetilde{u} := \widetilde{\widetilde{w}} - w,\ \ k = i\, z,$$

Note that $$u \to 0 \text{ as } |m| \to \infty$$

Jost solutions are defined by

$$\begin{cases} \varphi & \sim \ (p - iz)^m \\ \overline{\varphi} & \sim \ (p + iz)^m \end{cases} \text{ as } m \to \ -\infty$$

$$\begin{cases} \psi & \sim \ (p + iz)^m \\ \overline{\psi} & \sim \ (p - iz)^m \end{cases} \text{ as } m \to \ +\infty$$

We have $\overline{\varphi}(m; z) = \varphi(m; -z)$ and $\overline{\psi}(m; z) = \psi(m; -z)$

Scaled Jost Solutions

$$\chi(m;z) := \frac{\varphi(m;z)}{(p-iz)^m}, \;\; \overline{\chi}(m;z) := \chi(m;-z)$$

$$\Upsilon(m;z) := \frac{\psi(m;z)}{(p+iz)^m}, \;\; \overline{\Upsilon}(m;z) = \Upsilon(m;-z)$$

Theorem:

- $\chi(m;z), \Upsilon(m;z)$ exist and are analytic in $\Im(z) > 0$
- $\overline{\chi}(m;z), \overline{\Upsilon}(m;z)$ exist and are analytic in $\Im(z) < 0$
- All are continuous on the real line $\Im(z) = 0$

Scattering Data

$$\psi = a\overline{\varphi} + b\varphi$$

$$R = \frac{b}{a},\ T = \frac{1}{a}$$

- The coefficients $a(z)$ and $b(z)$ satisfy the following properties
  - $a(z)$ is analytic in $\Im(z) > 0$ and continuous on $\Im(z) = 0$ except possibly at $z = 0$
  - $b(z)$ is continuous on $\Im(z) = 0$, except possibly at $z = 0$
  - $a(z)$ has a finite number of zeroes z_k in $\Im(z) > 0$. They are simple, lie on $\Re(z) = 0$, and satisfy $|z_k| < p$

‘Time’ Evolution

When *n* evolves, we get the evolution of $a(z)$ and $b(z)$

$$a(n; z) = a(0; z) \equiv a(z)$$

$$b(n; z) = b(0; z) \left(\frac{q - iz}{a + iz} \right)^n = b(z) \left(\frac{q - iz}{a + iz} \right)^n$$

Inverse Scattering

$$\frac{\Upsilon(m;z)}{a(z)} \sim \left(1 + o\left(\frac{1}{z}\right)\right) \quad \text{as} \quad |z| \to +\infty, \quad \Im z \geqslant 0$$

$$\frac{\Upsilon(m;z)}{a(z)} - \overline{\chi}(m;z) = R(z)\,\chi(m;z)\left(\frac{p-iz}{p+iz}\right)^m$$

$$\overline{\chi}(m;z) \sim \left(1 + o\left(\frac{1}{z}\right)\right) \quad \text{as} \quad |z| \to +\infty, \quad \Im z \leqslant 0$$

$z \in \mathbb{C}$

Solution

$$\chi(m;z) = 1 + \frac{2p}{p - iz} \sum_{j=-\infty}^{m} K(m,j)\lambda^{j-m}$$

is given by

$$K(m,L) + B(L) + \sum_{r=-\infty}^{m} K(m,r)(B(r-m+L) + B(r-m+L-1)) = 0$$

where

$$B(T) := \sum_{k=1}^{N} \frac{-\mathrm{i}\epsilon_k}{(p - \mathrm{i}z_k)} \left(\frac{p - \mathrm{i}z_k}{p + \mathrm{i}z_k}\right)^T + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{R(\zeta)}{(p - \mathrm{i}\zeta)} \left(\frac{p - \mathrm{i}\zeta}{p + \mathrm{i}\zeta}\right)^T \mathrm{d}\zeta.$$

leading to

$$u_{m,n} = 2p\left[\frac{1 + K(m+1, m+1)}{1 + K(m,m)} - 1\right]$$

Multi-dimensions

- The discrete inverse scattering transform method can be extended to other ABS equations, up to Q3.
- The initial-value problem can be given on a well-defined multi-dimensional staircase.
- Soliton solutions correspond to reflectionless potentials, described through Cauchy matrices.

Butler 2012

Part 4

- Continuous-discrete
- Partial difference equations
- Discrete inverse scattering
- Discrete iso-monodromy problems

Reductions

Reductions of soliton equations are Painlevé equations

$$w_\tau + 6\,w\,w_\xi + w_{\xi\xi\xi} = 0$$

$$\begin{cases} w = -2\,y(x) - 2\,\tau \\ x = \xi + 6\,\tau^2 \end{cases}$$

$$\Rightarrow \begin{cases} w_\tau & = -24\,\tau\,y_x - 2 \\ w_\xi & = -2\,y_x \\ w_{\xi\xi\xi} & = -2\,y_{xxx} \end{cases}$$

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$$\longrightarrow \qquad y'' = 6\,y^2 - x$$

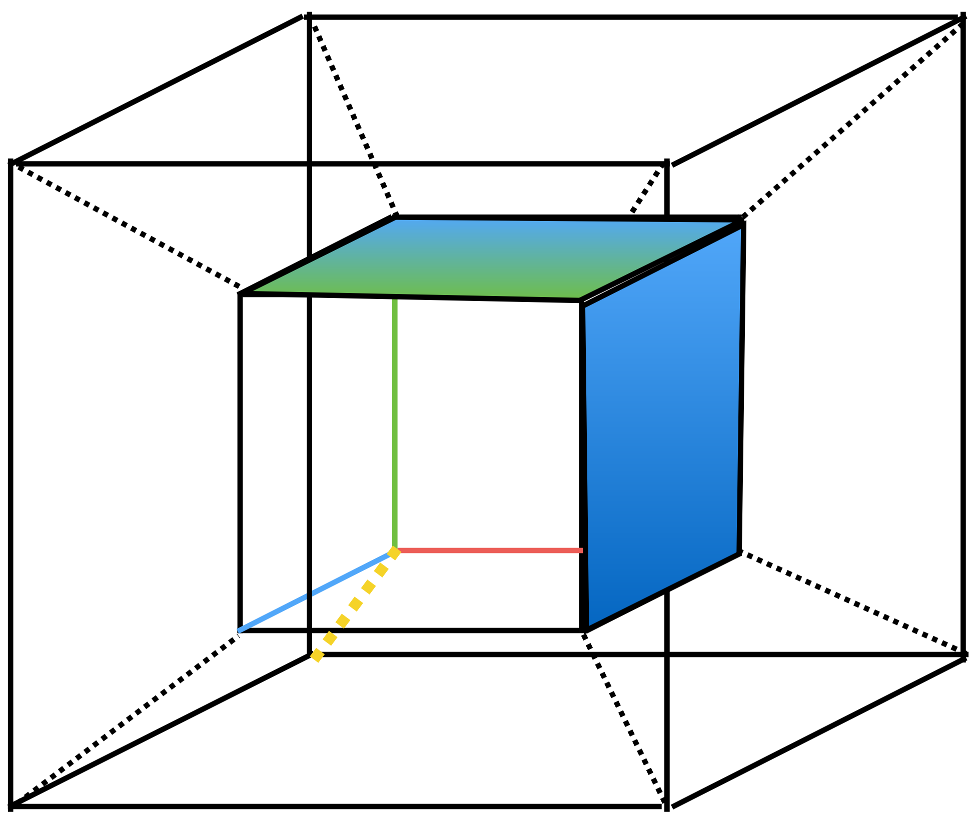
Discrete Reductions?

- What are reductions of discrete soliton equations?
- What are their corresponding linear problems?
- How do we solve them?

H3 & H6 on 4-cube

$$H3_{\delta=0}: \quad \alpha(xu+vy)-\beta(xv+uy)=0$$

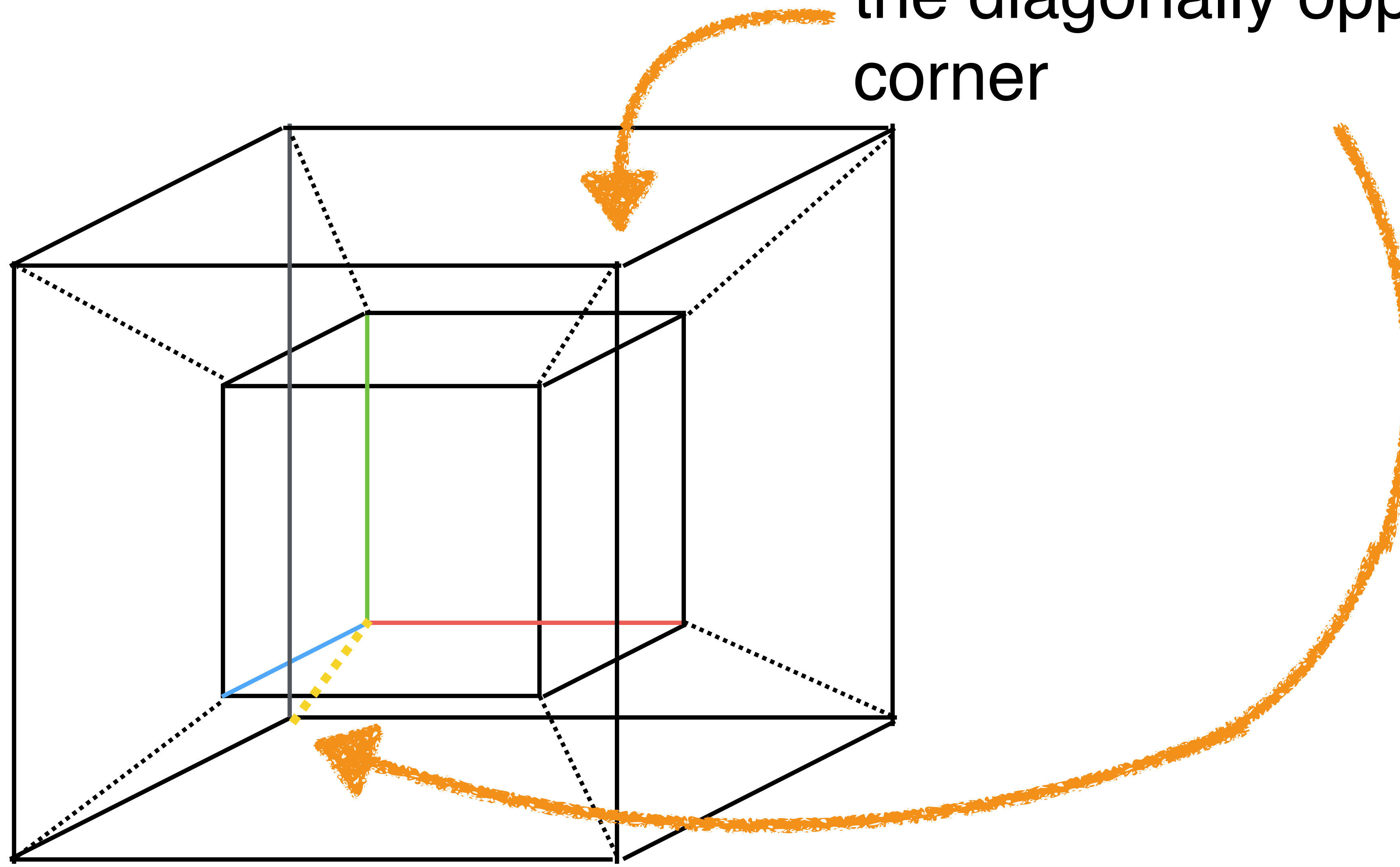
$$H6: \quad xy+uv+\delta_1 xu+\delta_2 vy=0$$



Each sub 3-cube in this 4-cube has 2 copies of H3 and 4 copies of H6 associated to its faces.

Reduction

Push one corner to the diagonally opposite corner



$$\overline{\widehat{\widetilde{w}}} = -i\,\lambda\,w$$

$$\mathring{\lambda} = q\,\lambda$$

Joshi, Nakazono & Shi, 2014

Reductions to *q*-discrete Painlevé equations

$$q\text{-P}_{\text{IV}}:\begin{cases} f(qt) = ab\,g(t)\,\dfrac{1 + c\,h(t)\,(a\,f(t) + 1)}{1 + a\,f(t)\,(b\,g(t) + 1)}, \\ g(qt) = bc\,h(t)\,\dfrac{1 + a\,f(t)\,(b\,g(t) + 1)}{1 + b\,g(t)\,(c\,h(t) + 1)}, \\ h(qt) = ca\,f(t)\,\dfrac{1 + b\,g(t)\,(c\,h(t) + 1)}{1 + c\,h(t)\,(a\,f(t) + 1)}, \end{cases}$$

$$q\text{-P}_{\text{III}}:\begin{cases} g(qt) = \dfrac{a}{g(t)f(t)}\,\dfrac{1 + tf(t)}{t + f(t)}, \\ f(qt) = \dfrac{a}{f(t)g(qt)}\,\dfrac{1 + btg(qt)}{bt + g(qt)}, \end{cases}$$

$$q\text{-P}_{\text{II}}:\quad f(pt) = \frac{a}{f(p^{-1}t)f(t)}\,\frac{1 + tf(t)}{t + f(t)},$$

Discrete Monodromy Problems

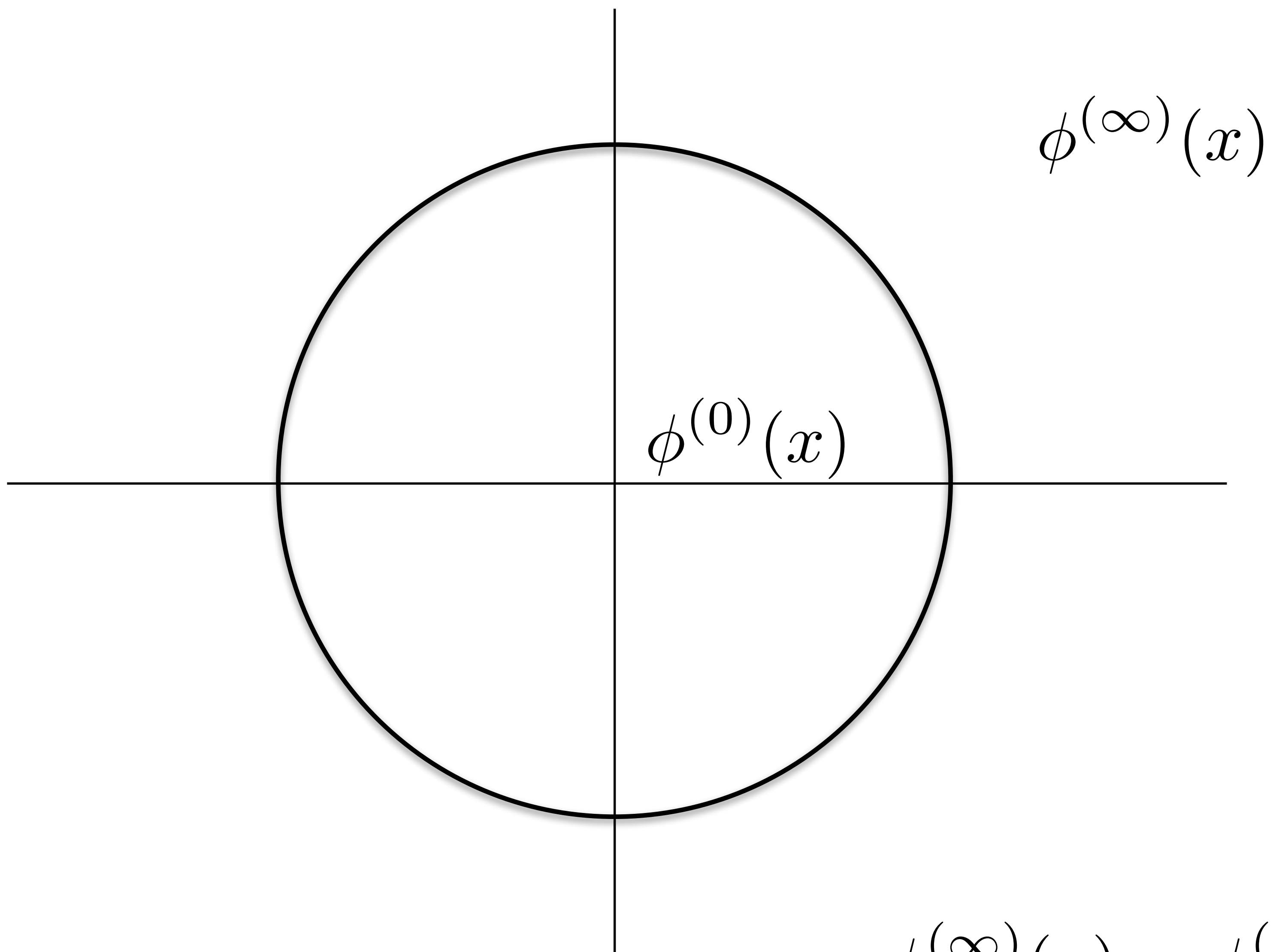
Reductions also provide linear problems, e.g.

$$\phi(qx,t)=\begin{pmatrix}\frac{qt}{h(t)}x & 1\\ -1 & \frac{qh(t)}{t}x\end{pmatrix}.\begin{pmatrix}\frac{act}{f(t)}x & 1\\ -1 & \frac{acf(t)}{t}x\end{pmatrix}.\begin{pmatrix}\frac{at}{g(t)}x & 1\\ -1 & \frac{ag(t)}{t}x\end{pmatrix}.\phi(x,t),$$

$$\phi(x,qt)=\begin{pmatrix}-\frac{(qt^2-1)h(t)}{(1+b+bch(t))tg(t)}x & -1\\ 1 & 0\end{pmatrix}.\phi(x,t).$$

whose compatibility condition is qP_{IV}

Joshi, Nakazono & Shi, 2015

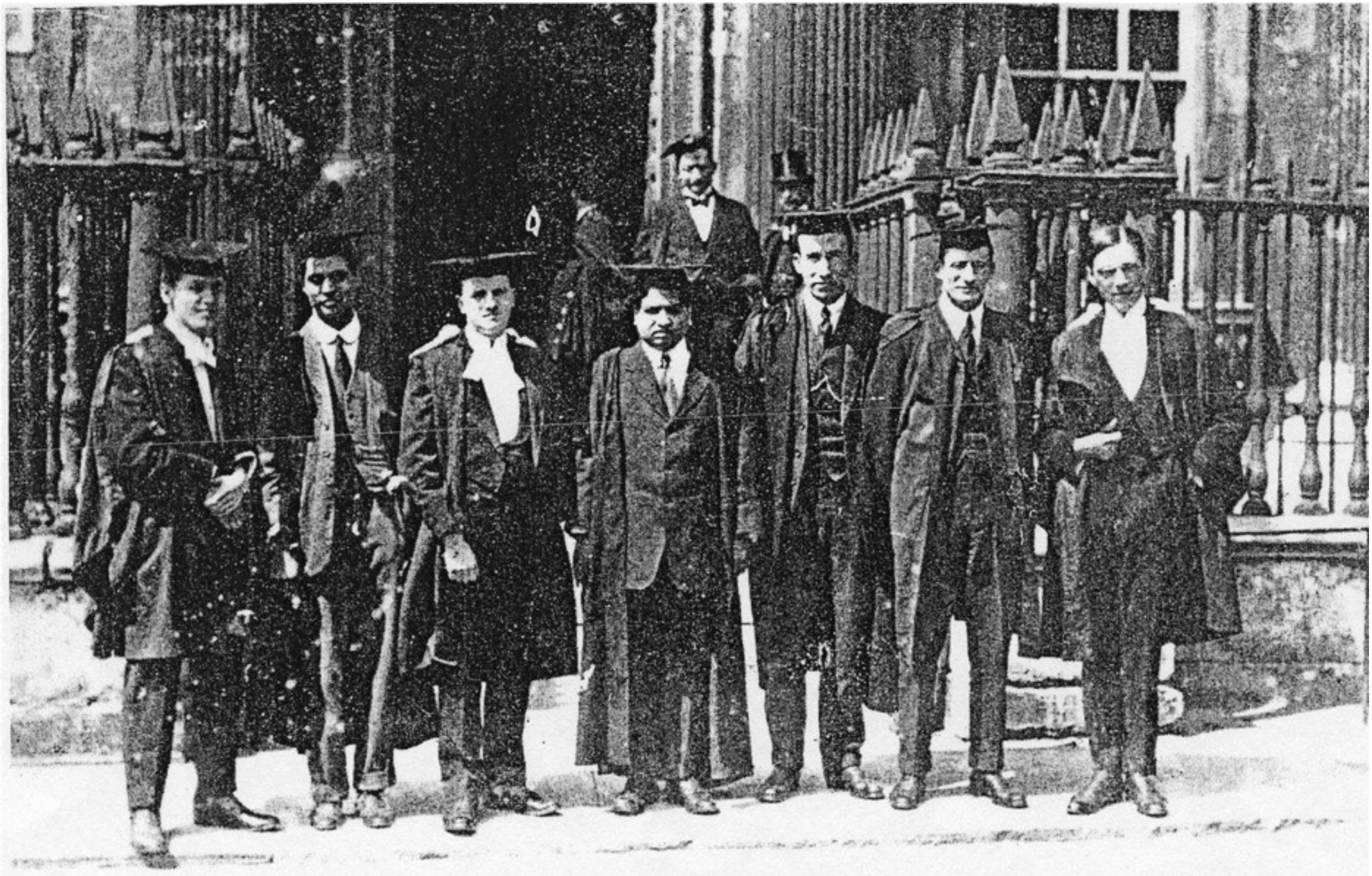


$$\phi^{(\infty)}(x) = \phi^{(0)}(x)\, P(x)$$

Carmichael 1912, Birkhoff & Guenther 1941

Summary and Open Problems

- Discrete versions of integrable PDEs and ODEs have associated linear spectral problems.
- The discrete inverse scattering method can be solved for ABS equations up to Q3 from any well-posed N-dimensional staircase.
- How the discrete connection problems for q-Painlevé and elliptic-Painlevé equations provide information about solutions remains an open question.



The mathematician's patterns, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. *GH Hardy, A Mathematician's Apology, 1940*