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A multiplicity formula for tensor products of SL_2 modules and an explicit Sp_{2n} to $Sp_{2n-2} \times Sp_2$ branching formula.

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Abstract. In the restriction of an irreducible representation of Sp_{2n} to the standard Sp_{2n-2} the multiplicity spaces are naturally $Sp_2 \cong SL_2$ modules. We show that these multiplicity spaces are each equivalent to a specified tensor product of n irreducible SL_2 modules. The key to these results is a generalization of the Clebsch-Gordan formula and a result of J. Lepowsky that gives the C_n branching to $C_{n-1} \times C_1$ as a difference of two simple partition functions.

1. Introduction

The purpose of this note is to give an elementary decomposition of the restriction of an irreducible representation of C_n to $C_{n-1} \times C_1$. By a decomposition we mean an explicit description of the C_1-module structure of the multiplicity spaces that occur in the restriction of an irreducible representation of C_n to C_{n-1}. By elementary we mean using relatively simple combinatorial methods. In principle the results of this note can be derived from those of [[4], Theorem 5.2] which uses the theory of Yangians and is far from elementary. As a byproduct of our work we derive a formula for the decomposition of arbitrary tensor products of irreducible representations of SL_2, generalizing the Clebsch-Gordan formula. Here the multiplicities are given as a difference of two generalized Kostant partition functions.

2. Tensor products of $SL(2,\mathbb{C})$ representations

Let $H = SL(2,\mathbb{C})$ and let F^k be the irreducible representation of H of dimension $k+1$. The Clebsch-Gordan formula implies that if $r_1 \geq r_2$ then

$$F^{r_1} \otimes F^{r_2} \cong F^{r_1+r_2} \oplus F^{r_1+r_2-2} \oplus \cdots \oplus F^{r_1-r_2}. \tag{2.1}$$

In this section we extend the Clebsch-Gordan formula to an arbitrary tensor product of representations of H.

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We begin by setting up some notation. Let $\{v_1, ..., v_n\}$ be the standard basis for $\mathbb{R}^n$ and set $\Sigma_n = \{v_1 \pm v_n, ..., v_{n-1} \pm v_n\}$. We identify $\mathbb{R}^n$ with $(\mathbb{R}^n)^{**}$; thus if $v \in \mathbb{R}^n$, e^v is a function on $(\mathbb{R}^n)^*$. Denote by $\mathcal{P}_n(v)$ the coefficient of e^v in the formal product

$$\prod_{w \in \Sigma_n} \frac{1}{1-e^w} .$$

This says that $\mathcal{P}_n(v)$ is the number of ways of writing

$$v = \sum_{w \in \Sigma_n} c_w w, \; c_w \in \mathbb{N}.$$

Finally let

$$m_l(r_1, ..., r_n) = \dim Hom_H(F^l, F^{r_1} \otimes \cdots \otimes F^{r_n}).$$

The following is a reinterpretation of formula (2.1).

LEMMA 2.1. *Let $r_1, r_2, l \in \mathbb{N}$. Then*

$$m_l(r_1, r_2) = \mathcal{P}_2(r_1 v_1 + r_2 v_2 - l v_2) - \mathcal{P}_2(r_1 v_1 + r_2 v_2 + (l+2) v_2).$$

PROOF. Note that $\mathcal{P}_2(a v_1 + b v_2) = 1$ if and only if $b \in \{-a, 2-a, ..., a-2, a\}$. The result follows by considering the cases $r_1 \le r_2$ and $r_1 > r_2$ separately. □

The result of this section is a generalization of Lemma 2.1 to a tensor product of an arbitrary number of irreducible H-modules. First we develop some combinatorial properties of $\mathcal{P}_n$.

Let $\Sigma_n^+ = \{v_1 + v_n, ..., v_{n-1} + v_n\}$ and $\Sigma_n^- = \{v_1 - v_n, ..., v_{n-1} - v_n\}$. Denote by $\mathcal{P}_n^{\pm}(v)$ the coefficient of e^v in

$$\prod_{w \in \Sigma_n^{\pm}} \frac{1}{1-e^w} .$$

It is easy to see that

$$\mathcal{P}_n(v) = \sum_{u+w=v} \mathcal{P}_n^+(u) \mathcal{P}_n^-(w).$$

Since Σ_n^+, Σ_n^- are linearly independent the corresponding partition functions take only values 0 or 1. Furthermore, one can easily check that

$$\mathcal{P}_n^+(a_1 v_1 + \cdots + a_n v_n) = 1 \Leftrightarrow a_1, ..., a_{n-1} \in \mathbb{N} \text{ and } \sum_{j=1}^{n-1} a_j = a_n$$

$$\mathcal{P}_n^-(b_1 v_1 + \cdots + b_n v_n) = 1 \Leftrightarrow b_1, ..., b_{n-1} \in \mathbb{N} \text{ and } \sum_{j=1}^{n-1} b_j = -b_n$$

Let $v = c_1 v_1 + \cdots + c_n v_n$ and suppose $v = u + w$ with $u = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$. Then $a_j + b_j = c_j$ for $j = 1, ..., n$. If $\mathcal{P}_n^+(u)\mathcal{P}_n^-(w) = 1$ then

$$c_n = \sum_{j=1}^{n-1} a_j - b_j. \tag{2.2}$$

Define a **bisection** of a natural number m to be a two-part partition of m. Then $\mathcal{P}_n(v)$ counts the number of bisections of $c_1, ..., c_{n-1}$ that satisfy (2.2). This description provides a useful recursive formula.

LEMMA 2.2.

$$\mathcal{P}_n(c_1v_1 + \cdots + c_nv_n) = \sum_{i=0}^{c_{n-1}} \mathcal{P}_{n-1}(c_1v_1 + \cdots + c_{n-2}v_{n-2} + (c_{n-1} + c_n - 2i)v_{n-1})$$

PROOF. The i^{th} summand on the right hand side counts the number of bisections of $c_1, ..., c_{n-2}$ that satisfy $c_{n-1} + c_n - 2i = \sum_{j=1}^{n-2} a_j - b_j$. (Here $c_j = a_j + b_j$ for $j = 1, ..., n-2$.) These bisections correspond to the bisections of $c_1, ..., c_{n-1}$ that satisfy $c_n = \sum_{j=1}^{n-1} a_j - b_j$ with $a_{n-1} = i$ and $b_{n-1} = c_{n-1} - i$. □

THEOREM 2.3. *Let $r_1, ..., r_n, l \in \mathbb{N}$. Then*

$$\boldsymbol{m}_l(r_1, ..., r_n) = \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n - lv_n) - \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n + (l+2)v_n).$$

PROOF. We proceed by induction on $n \geq 2$. If $n = 2$ use Lemma 2.1. Now suppose $n > 2$ and the claim holds for $n-1$. Let $r_1, ..., r_n, l \in \mathbb{N}$ and to simplify matters write $S_k = \sum_{j=1}^{k} r_jv_j$ and $Q(t) = \mathcal{P}_{n-1}(S_{n-2} + tv_{n-1})$. By Lemma 2.2 we obtain

$$\mathcal{P}_n(S_n - lv_n) - \mathcal{P}_n(S_n + (l+2)v_n) = \sum_{i=0}^{r_{n-1}} Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2).$$

If $r_{n-1} \leq r_n$ then $r_{n-1} + r_n - 2i \geq 0$ so by the inductive hypothesis

$$Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2) = \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1} + r_n - 2i).$$

By the Clebsch-Gordan formula

$$\sum_{i=0}^{r_{n-1}} \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1} + r_n - 2i) = \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1}, r_n).$$

If $r_{n-1} > r_n$ the situation is not as straightforward. As above we have

$$\mathcal{P}_n(S_n - lv_n) - \mathcal{P}_n(S_n + (l+2)v_n) = \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1}, r_n) + E$$

where

$$E = \sum_{i=r_n+1}^{r_{n-1}} Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2).$$

Rewrite E as

$$\sum_{i=1}^{r_{n-1}-r_n} Q(r_{n-1} - r_n - 2i - l) - Q(r_{n-1} - r_n - 2i + l + 2)$$

and notice that

$$r_{n-1} - r_n - 2i - l = -(r_{n-1} - r_n - 2(r_{n-1} - r_n + 1 - i) + l + 2).$$

Therefore if we set $C_i = r_{n-1} - r_n - 2i - l$ then by rearranging terms

$$E = \sum_{i=1}^{r_{n-1}-r_n} Q(C_i) - Q(-C_i).$$

But clearly $Q(t) = Q(-t)$ so $E = 0$. □

3. An application to Sp_{2n} branching

Label a basis for $\mathbb{C}^{2l}$ as $e_{\pm 1}, ..., e_{\pm l}$ where $e_{-i} = e_{2l+1-i}$. Here we view $\mathbb{C}^{2l}$ as column vectors. Denote by s_l the $l \times l$ matrix with ones on the anti-diagonal and zeros everywhere else. Set

$$J_l = \begin{bmatrix} 0 & s_l \\ -s_l & 0 \end{bmatrix}$$

and define the skew-symmetric bilinear form $\Omega_l(x, y) = x^t J_l y$ on $\mathbb{C}^{2l}$. Let $G = Sp(\mathbb{C}^{2n}, \Omega_n)$ and define subgroups

$$K = \{k \in G : ke_n = e_n \text{ and } ke_{-n} = e_{-n}\}$$
$$H = \{h \in G : he_j = e_j \text{ for } j = \pm 1, ..., \pm n-1\}$$

Then $K \cong Sp(\mathbb{C}^{2(n-1)}, \Omega_{n-1})$ and $H \cong Sp(\mathbb{C}^2, \Omega_1) \cong SL(2, \mathbb{C})$. Let $\Lambda = (\Lambda_1 \geq \cdots \geq \Lambda_n \geq 0)$ be a decreasing sequence of natural numbers. We identify the set of such Λ with the dominant integral weights of G as in [[**1**], Proposition 2.5.11]. Let V^Λ be the finite dimensional irreducible regular representation of G of high weight Λ. Similarly a decreasing sequence of $n-1$ natural numbers $\mu = (\mu_1 \geq \cdots \geq \mu_{n-1} \geq 0)$ is identified with the corresponding dominant integral weights of K. Let V^μ be the finite dimensional irreducible regular representation of K of high weight μ.

We say μ **doubly interlaces** Λ if $\Lambda_i \geq \mu_i \geq \Lambda_{i+2}$ for $i = 1, ..., n-1$ (with $\Lambda_{n+1} = 0$). Given μ, Λ set $r_i(\Lambda, \mu) = x_i - y_i$, where $\{x_1 \geq y_1 \geq \cdots \geq x_n \geq y_n\}$ is the decreasing rearrangement of $\{\Lambda_1, .., \Lambda_n, \mu_1, ..., \mu_{n-1}, 0\}$.

THEOREM 3.1 ([**1**], Proposition 8.1.5). *Let* $n \geq 2$. *Then* $\dim Hom_K(V^\mu, V^\Lambda) > 0$ *if and only if* μ *doubly interlaces* Λ. *If* μ *doubly interlaces* Λ *then* $\dim Hom_K(V^\mu, V^\Lambda) = \prod_{j=1}^{n} (r_i(\Lambda, \mu) + 1)$.

This theorem in particular provides the decomposition of K modules

$$V^\Lambda \cong \bigoplus_\mu V^\mu \otimes Hom_K(V^\mu, V^\Lambda)$$

where the sum is over all μ that doubly interlaces Λ. Here K acts on left factor. Since H is a subgroup of the centralizer of K in G, H acts on the multiplicity spaces $Hom_K(V^\mu, V^\Lambda)$. One is thus led to the natural question: what is the H-module structure of $H_K(\mu, \Lambda) = Hom_K(V^\mu, V^\Lambda)$?

The following theorem, due to J. Lepowsky ([**3**]), provides a partial answer.

THEOREM 3.2 ([**2**], Proposition 9.5.9). *Let* Λ, μ *be as above and set* $r_i = r_i(\Lambda, \mu)$. *Then*

$$\dim Hom_H(F^l, H_K(\mu, \Lambda)) = \mathcal{P}_n(r_1 v_1 + \cdots + r_n v_n - l v_n) - \mathcal{P}_n(r_1 v_1 + \cdots + r_n v_n + (l+2) v_n).$$

We combine this result with Theorem 2.3 to obtain an explicit decomposition of V^Λ as a $K \times H$ module.

THEOREM 3.3. *Let* Λ, μ *be as above and set* $r_i = r_i(\Lambda, \mu)$. *Then as a* $K \times H$-*module*

$$V^\Lambda \cong \bigoplus_\mu V^\mu \otimes (F^{r_1} \otimes \cdots \otimes F^{r_n}).$$

The direct sum is over all μ *that doubly interlace* Λ.

References

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