Gelfand–Tsetlin bases for classical Lie algebras*

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Contents

1 Introduction

The theory of semisimple Lie algebras and their representations is in the heart of modern mathematics. It has numerous connections with other areas of mathematics and physics. The simple Lie algebras over the field of complex numbers were classified in the works of Cartan and Killing in the 1930's. There are four infinite series A_n, B_n, C_n, D_n which are called the *classical Lie algebras*, and five exceptional Lie algebras E_6, E_7, E_8, F_4, G_2. The structure of these Lie algebras is uniformly described in terms of certain finite sets of vectors in a Euclidean space called the *root systems*. Due to Weyl's complete reducibility theorem, the theory of finite-dimensional representations of the semisimple Lie algebras is largely reduced to the study of irreducible representations. The irreducibles are parametrized by their *highest weights*. The characters and dimensions are explicitly known by the *Weyl formula*. The reader is refered to, e.g., the books of Bourbaki [11], Dixmier [19] or Humphreys [54] for a detailed exposition of the theory.

However, the Weyl formula for the dimension does not use any explicit construction of the representations. Such constructions remained unknown until 1950 when Gelfand and Tsetlin[1] published two short papers [36] and [37] (in Russian) where they solved the problem for the general linear Lie algebras (type A_n) and the orthogonal Lie algebras (types B_n and D_n), respectively. Later, Baird and Biedenharn [4] (1963) commented on [36] as follows:

> "This paper is extremely brief (three pages) and does not appear to have been translated in either the usual Journal translations or the translations on group-theoretical subjects of the American Mathematical Society, or even referred to in the review articles on group theory by Gelfand himself. Moreover, the results are presented without the slightest hint as to the methods employed and contain not a single reference or citation of other work. In an effort to understand the meaning of this very impressive work, we were led to develop the proofs . . . ."

Baird and Biedenharn employed the calculus of the *Young patterns* to derive the Gelfand–Tsetlin formulas.[2] Their interest to the formulas was also motivated by the connection with the fundamental *Wigner coefficients*; see Section 2.4 below.

A year earlier (1962) Zhelobenko published an independent work [144] where he derived the *branching rules* for all classical Lie algebras. In his approach the representations are realized in a space of polynomials satisfying the "indicator system" of differential equations. He outlined a method to construct the *lowering operators* and

[1]Some authors and translators write this name in English as *Zetlin, Tzetlin, Cetlin,* or *Tseitlin.*

[2]An indication of the proof of the formulas of [36] is contained in a footnote in the paper Gelfand and Graev [38] (1965). It says that for the proof one has "to verify the commutation relations . . . ; this is done by direct calculation".

to derive the matrix element formulas for the case of the general linear Lie algebra $\mathfrak{gl}_n$. An explicit "infinitesimal" form for the lowering operators as elements of the enveloping algebra was found by Nagel and Moshinsky [95] (1964) and independently by Hou Pei-yu [52] (1966). The latter work relies on Zhelobenko's results [144] and also contains a derivation of the Gelfand–Tsetlin formulas alternative to that of Baird and Biedenharn. This approach was further developed in the book by Zhelobenko [145] which contains its detailed account.

The work of Nagel and Moshinsky was extended to the orthogonal Lie algebras $\mathfrak{o}_N$ by Pang and Hecht [111] and Wong [141] who produced explicit infinitesimal expressions for the lowering operators and gave a derivation of the formulas of Gelfand and Tsetlin [37].

During the half a century passed since the work of Gelfand and Tsetlin, many different approaches were developed to construct bases of the representations of the classical Lie algebras. New interpretations of the lowering operators and new proofs of the Gelfand–Tsetlin formulas were discovered by several authors. In particular, Gould [42, 43, 44, 45] employed the *characteristic identities* of Bracken and Green [12, 48] to calculate the Wigner coefficients and matrix elements of generators of $\mathfrak{gl}_n$ and $\mathfrak{o}_N$. The *extremal projector* discovered by Asherova, Smirnov and Tolstoy [1, 2, 3] turned out to be a powerful instrument in the representation theory of the simple Lie algebras. It plays an essential role in the theory of *Mickelsson algebras* developed by Zhelobenko which has a wide spectrum of applications from the branching rules and reduction problems to the classification of Harish-Chandra modules; see Zhelobenko's expository paper [150] and his book [151]. Two different *quantum minor* interpretations of the lowering and raising operators were given by Nazarov and Tarasov [98] and the author [85]. These techniques are based on the theory of quantum algebras called the *Yangians* and allow an independent derivation of the matrix element formulas. We shall discuss the above approaches in more detail in Sections 2.3, 2.4 and 2.5 below.

A quite different method to construct modules over the classical Lie algebras is developed in the papers by King and El-Sharkaway [62], Berele [6], King and Welsh [63], Koike and Terada [66], Proctor [116], Nazarov [96]. It is based on the Weyl realization of the representations of the classical groups in tensor spaces; see Weyl [136]. In particular, bases in the representations of the orthogonal and symplectic Lie algebras parametrized by $\mathfrak{o}_N$-standard or $\mathfrak{sp}_{2n}$-standard Young tableaux are constructed. This method provides an algorithm for calculation of the representation matrices.

Bases with special properties in the universal enveloping algebra for a simple Lie algebra $\mathfrak{g}$ and in some modules over $\mathfrak{g}$ were constructed by Lakshmibai, Musili and Seshadri [68], Littelmann [70, 71], Chari and Xi [15] (*monomial* bases); De Concini and Kazhdan [18], Xi [143] (*special* bases and their q-analogs); Gelfand and Zelevinsky [41], Retakh and Zelevinsky [118], Mathieu [74] (*good* bases); Lusztig [72], Kashiwara [59], Du [32, 33] (*canonical* or *crystal* bases); see also Mathieu [75] for a review and more references. In general, no explicit formulas are known, however, for the matrix elements of the generators in such bases other than those of Gelfand and Tsetlin type. It is known, although, that for the canonical bases the matrix elements of the standard generators are nonnegative integers. Some classes of representations of the symplectic, odd orthogonal and the Lie algebras of type G_2 were explicitly constructed by Donnelly [20, 21, 23] and Donnelly, Lewis and Pervine [24]. The constructions were applied to establish combinatorial properties of the supporting graphs of the representations and were inspired by the earlier results of Proctor [112, 113, 115]. Another graph-theoretic approach is developed by Wildberger [137, 138, 139, 140] to construct simple Lie algebras and their minuscule representations; see also Stembridge [124].

We now discuss the main idea which leads to the construction of the Gelfand–Tsetlin bases. The first point is to regard a given classical Lie algebra not as a single object but a part of a chain of subalgebras with natural embeddings. We illustrate this idea using representations of the symmetric groups $\mathfrak{S}_n$ as an example. Consider the chain of subgroups

$$\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots \subset \mathfrak{S}_n, \tag{1.1}$$

where the subgroup $\mathfrak{S}_k$ of $\mathfrak{S}_{k+1}$ consists of the permutations which fix the index $k+1$ of the set $\{1, 2, \ldots, k+1\}$. The irreducible representations of the group $\mathfrak{S}_n$ are indexed by partitions λ of n. A partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ is depicted graphically as a *Young diagram* which consists of l left-justified rows of boxes so that the top row contains λ_1 boxes, the second row λ_2 boxes, etc. Denote by $V(\lambda)$ the irreducible representation of $\mathfrak{S}_n$ corresponding to the partition λ. One of the central results of the representation theory of the symmetric groups is the following *branching rule* which describes the restriction of $V(\lambda)$ to the subgroup $\mathfrak{S}_{n-1}$:

$$V(\lambda)|_{\mathfrak{S}_{n-1}} = \bigoplus_{\mu} V'(\mu),$$

summed over all partitions μ whose Young diagram is obtained from that of λ by removing one box. Here $V'(\mu)$ denotes the irreducible representation of $\mathfrak{S}_{n-1}$ corresponding to a partition μ. Thus, the restriction of $V(\lambda)$ to $\mathfrak{S}_{n-1}$ is *multiplicity-free*, i.e., is contains each irreducible representation of $\mathfrak{S}_{n-1}$ at most once. This makes it possible to obtain a natural parameterization of the basis vectors in $V(\lambda)$ by taking its further restrictions to the subsequent subgroups of the chain (1.1). Namely, the basis vectors will be parametrized by sequences of partitions

$$\lambda^{(1)} \to \lambda^{(2)} \to \cdots \to \lambda^{(n)} = \lambda,$$

where $\lambda^{(k)}$ is obtained from $\lambda^{(k+1)}$ by removing one box. Equivalently, each sequence of this type can be regarded as a *standard tableau of shape* λ which is obtained by writing the numbers $1, \ldots, n$ into the boxes of λ in such a way that the numbers increase along the rows and down the columns. In particular, the dimension of $V(\lambda)$ equals the number of standard tableaux of shape λ. There is only one irreducible representation of the trivial group $\mathfrak{S}_1$ therefore the procedure defines basis vectors up to a scalar factor. The corresponding basis is called the *Young basis*. The symmetric group $\mathfrak{S}_n$ is generated by the adjacent transpositions $s_i = (i, i+1)$. The construction of the representation $V(\lambda)$ can be completed by deriving explicit formulas for the action of the elements s_i in the basis which are also due to A. Young. The details can be found, e.g., in James and Kerber [56] and Sagan [119]; see also Okounkov and Vershik [101] where an alternative construction of the Young basis is developed. Branching rules and corresponding Bratteli diagrams were employed by Halverson and Ram [49], [117] to compute irreducible representations of the Iwahori–Hecke algebras and the characters of some families of "centralizer" algebras.

Quite a similar method can be applied to representations of the classical Lie algebras. Consider the general linear Lie algebra $\mathfrak{gl}_n$ which consists of complex $n \times n$-matrices with the usual matrix commutator. The chain (1.1) is now replaced by

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n,$$

with natural embeddings $\mathfrak{gl}_k \subset \mathfrak{gl}_{k+1}$. The orthogonal Lie algebra $\mathfrak{o}_N$ can be regarded as a subalgebra of $\mathfrak{gl}_N$ which consists of the skew-symmetric matrices. Again, we have a natural chain

$$\mathfrak{o}_2 \subset \mathfrak{o}_3 \subset \cdots \subset \mathfrak{o}_N. \tag{1.2}$$

Both restrictions $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$ and $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ are multiplicity-free so that the application of the argument which we used for the chain (1.1) produces basis vectors in an irreducible representation of $\mathfrak{gl}_n$ or $\mathfrak{o}_N$. With an appropriate normalization, these bases are precisely those of Gelfand and Tsetlin given in [36] and [37]. Instead of the standard tableaux, the basis vectors here are parametrized by combinatorial objects called the *Gelfand–Tsetlin patterns*.

However, this approach does not work for the symplectic Lie algebras $\mathfrak{sp}_{2n}$ since the restriction $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ is not multiplicity-free. The multiplicities are given by Zhelobenko's branching rule [144] which was re-discovered later by Hegerfeldt [50][3]. Various attempts to fix this problem were made by several authors. A natural idea is to introduce an intermediate Lie algebra "$\mathfrak{sp}_{2n-1}$" and try to restrict an irreducible representation of $\mathfrak{sp}_{2n}$ first to this subalgebra and then to $\mathfrak{sp}_{2n-2}$ in the hope to get simple spectra in the two restrictions. Such intermediate subalgebras and their representations were studied by Gelfand–Zelevinsky [40], Proctor [114], Shtepin [120]. The drawback of this approach is the fact that the Lie algebra $\mathfrak{sp}_{2n-1}$ is not reductive so that the restriction of an irreducible representation of $\mathfrak{sp}_{2n}$ to $\mathfrak{sp}_{2n-1}$ is not completely reducible. In some sense, the separation of multiplicities can be achieved by constructing a filtration of $\mathfrak{sp}_{2n-1}$-modules; cf. Shtepin [120].

[3]Some western authors refered to Hegerfeldt's result as the original derivation of the rule.

Another idea is to use the restriction $\mathfrak{gl}_{2n} \downarrow \mathfrak{sp}_{2n}$. Gould and Kalnins [46, 47] constructed a basis for the representations of the symplectic Lie algebras parametrized by a subset of the Gelfand–Tsetlin $\mathfrak{gl}_{2n}$-patterns. Some matrix element formulas are also derived by using the $\mathfrak{gl}_{2n}$-action. A similar observation is made independently by Kirillov [64] and Proctor [114]. A description of the Gelfand–Tsetlin patterns for $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_N$ can be obtained by regarding them as fixed points of involutions of the Gelfand–Tsetlin patterns for the corresponding Lie algebra $\mathfrak{gl}_N$.

The lowering operators in the symplectic case were given by Mickelsson [80]; see also Bincer [9, 10]. The application of ordered monomials in the lowering operators to the highest vector yields a basis of the representation. However, the action of the Lie algebra generators in such a basis does not seem to be computable. The reason is the fact that, unlike the cases of $\mathfrak{gl}_n$ and $\mathfrak{o}_N$, the lowering operators do not commute so that the basis depends on the chosen ordering. A "hidden symmetry" has been needed (cf. Cherednik [17]) to make a natural choice of an appropriate combination of the lowering operators. New ideas which led to a construction of a Gelfand–Tsetlin type basis for any irreducible finite-dimensional representation of $\mathfrak{sp}_{2n}$ came from the theory of *quantized enveloping algebras*. This is a part of the theory of *quantum groups* originated from the works of Drinfeld [25, 27] and Jimbo [57]. A particular class of quantized enveloping algebras called *twisted Yangians* introduced by Olshanski [107] plays the role of the hidden symmetries for the construction of the basis. We refer the reader to the book by Chari and Pressley [14] and the review papers [91] and [93] for detailed expositions of the properties of these algebras and their origins. For each classical Lie algebra we attach the *Yangian* $\mathrm{Y}(N) = \mathrm{Y}(\mathfrak{gl}_N)$, or the *twisted Yangian* $\mathrm{Y}^{\pm}(N)$ as follows

type A_n	type B_n	type C_n	type D_n
$\mathrm{Y}(n+1)$	$\mathrm{Y}^{+}(2n+1)$	$\mathrm{Y}^{-}(2n)$	$\mathrm{Y}^{+}(2n)$.

The algebra $\mathrm{Y}(N)$ was first introduced in the work of Faddeev and the St.-Petersburg school in relation with the *inverse scattering method*; see for instance Takhtajan–Faddeev [125], Kulish–Sklyanin [67]. Olshanski [107] introduced the twisted Yangians in relation with his *centralizer construction*; see also [94]. In particular, he established the following key fact which plays an important role in the basis construction. Given irreducible representations $V(\lambda)$ and $V'(\mu)$ of $\mathfrak{sp}_{2n}$ and $\mathfrak{sp}_{2n-2}$, respectively, there exists a natural irreducible action of the algebra $\mathrm{Y}^{-}(2)$ on the space $\mathrm{Hom}_{\mathfrak{sp}_{2n-2}}(V'(\mu), V(\lambda))$. The homomorphism space is isomorphic to the subspace $V(\lambda)^{+}_{\mu}$ of $V(\lambda)$ which is spanned by the highest vectors of weight μ for the subalgebra $\mathfrak{sp}_{2n-2}$. Finite-dimensional irreducible representations of the twisted Yangians were classified later in [86]. In particular, it turned out that the representation $V(\lambda)^{+}_{\mu}$ of $\mathrm{Y}^{-}(2)$ can be extended to the Yangian $\mathrm{Y}(2)$. The algebra $\mathrm{Y}(2)$ and its representations are well-studied; see Tarasov [127], Chari–Pressley [13]. A large class of representation of Y(2) admits Gelfand–Tsetlin-type bases associated with the inclusion $\mathrm{Y}(1) \subset \mathrm{Y}(2)$; see [85]. This allows one to get a natural basis in the space $V(\lambda)^+_\mu$ and then by induction to get a basis in the entire space $V(\lambda)$. Moreover, it turns out to be possible to write down explicit formulas for the action of the generators of the symplectic Lie algebra in this basis; see the author's paper [87] for more details. This construction together with the work of Gelfand and Tsetlin thus provides explicit realizations of all finite-dimensional irreducible representations of the classical Lie algebras.

The same method can be applied to the pairs of the orthogonal Lie algebras $\mathfrak{o}_{N-2} \subset \mathfrak{o}_N$. Here the corresponding space $V(\lambda)^+_\mu$ is a natural $\mathrm{Y}^+(2)$-module which can also be extended to a Y(2)-module. This leads to a construction of a natural basis in the representation $V(\lambda)$ and allows one to explicitly calculate the representation matrices; see [88, 89]. This realization of $V(\lambda)$ is alternative to that of Gelfand and Tsetlin [37]. To compare the two constructions, note that the basis of [37] in the orthogonal case lacks the *weight* property, i.e., the basis vectors are not eigenvectors for the Cartan subalgebra. The reason for that is the fact that the chain (1.2) involves Lie algebras of different types (B and D) and the embeddings are not compatible with the root systems. In the new approach we use instead the chains

$$\mathfrak{o}_2 \subset \mathfrak{o}_4 \subset \dots \subset \mathfrak{o}_{2n} \qquad \text{and} \qquad \mathfrak{o}_3 \subset \mathfrak{o}_5 \subset \dots \subset \mathfrak{o}_{2n+1}.$$

The embeddings here "respect" the root systems so that the basis of $V(\lambda)$ possesses the weight property in both the symplectic and orthogonal cases. However, the new weight bases, in their turn, lack the *orthogonality* property of the Gelfand–Tsetlin bases: the latter are orthogonal with respect to the standard inner product in the representation space $V(\lambda)$. It is an open problem to construct a natural basis of $V(\lambda)$ in the B, C and D cases which would simultaneously accommodate the two properties.

This chapter is structured as follows. In Section 2 we review the construction of the Gelfand–Tsetlin basis for the general linear Lie algebra and discuss its various versions. We start by applying the most elementary approach which consists of using explicit formulas for the lowering operators in a way similar to the pioneering works of the sixties. Remarkably, these operators admit several other presentations which reflect different approaches to the problem developed in the literature. First, we outline the general theory of extremal projectors and Mickelsson algebras as a natural way to work with the lowering operators. Next, we describe the $\mathfrak{gl}_n$-type *Mickelsson–Zhelobenko algebra* which is then used to prove the branching rule and derive the matrix element formulas. Further, we outline the Gould construction based upon the characteristic identities. Finally, we produce quantum minor formulas for the lowering operators inspired by the Yangian approach and describe the action of the Drinfeld generators in the Gelfand–Tsetlin basis.

In Section 3 we produce weight bases for representations of the orthogonal and symplectic Lie algebras. Here we describe the relevant Mickelsson–Zhelobenko algebra, formulate the branching rules and construct the basis vectors. Then we outline the properties of the (twisted) Yangians and their representations and explain their relationship with the lowering and raising operators. Finally, we sketch the main ideas in the calculation of the matrix element formulas.

Section 4 is devoted to the Gelfand–Tsetlin bases for the orthogonal Lie algebras. We outline the basis construction along the lines of the general method of Mickelsson algebras.

At the end of each section we give brief bibliographical comments pointing towards the original articles and to the references where the proofs or further details can be found.

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2 Gelfand–Tsetlin basis for representations of $\mathfrak{gl}_n$

Let $E_{ij},\ i,j = 1,\dots,n$ denote the standard basis of the general linear Lie algebra $\mathfrak{gl}_n$ over the field of complex numbers. The subalgebra $\mathfrak{gl}_{n-1}$ is spanned by the basis elements E_{ij} with $i,j = 1,\dots,n-1$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in $\mathfrak{gl}_n$. The elements $E_{11},\dots,E_{nn}$ form a basis of $\mathfrak{h}$.

Finite-dimensional irreducible representations of $\mathfrak{gl}_n$ are in a one-to-one correspondence with n-tuples of complex numbers $\lambda = (\lambda_1,\dots,\lambda_n)$ such that

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1,\dots,n-1. \tag{2.1}$$

Such an n-tuple λ is called the *highest weight* of the corresponding representation which we shall denote by $L(\lambda)$. It contains a unique, up to a multiple, nonzero vector ξ (the *highest vector*) such that $E_{ii}\,\xi = \lambda_i\,\xi$ for $i = 1,\dots,n$ and $E_{ij}\,\xi = 0$ for $1 \le i < j \le n$.

The following theorem is the *branching rule* for the reduction $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$.

Theorem 2.1 *The restriction of $L(\lambda)$ to the subalgebra $\mathfrak{gl}_{n-1}$ is isomorphic to the direct sum of pairwise inequivalent irreducible representations*

$$L(\lambda)|_{\mathfrak{gl}_{n-1}} \simeq \bigoplus_{\mu} L'(\mu),$$

summed over the highest weights μ satisfying the betweenness conditions

$$\lambda_i - \mu_i \in \mathbb{Z}_+ \quad \textit{and} \quad \mu_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \textit{for} \quad i = 1,\dots,n-1. \tag{2.2}$$

The rule could presumably be attributed to I. Schur who was the first to discover the representation-theoretic significance of a particular class of symmetric polynomials which now bear his name. Without loss of generality we may regard λ as a partition: we can take the composition of $L(\lambda)$ with an appropriate automorphism of $\mathrm{U}(\mathfrak{gl}_n)$ which sends E_{ij} to $E_{ij} + \delta_{ij}\, a$ for some $a \in \mathbb{C}$. The *character* of $L(\lambda)$ regarded as a GL_n-module is the *Schur polynomial* $s_\lambda(x)$, $x = (x_1, \dots, x_n)$ defined by

$$s_\lambda(x) = \mathrm{tr}\,\big(g,\, L(\lambda)\big),$$

where $x_1, \dots, x_n$ are the eigenvalues of $g \in GL_n$. The Schur polynomial is symmetric in the x_i and can be given by the explicit combinatorial formula

$$s_\lambda(x) = \sum_T x^T, \tag{2.3}$$

summed over the *semistandard tableaux* T of shape λ (cf. Remark 2.2 below), where x^T is the monomial containing x_i with the power equal to the number of occurrences of i in T; see, e.g., Macdonald [73, Chapter 1] or Sagan [119, Chapter 4] for more details. To find out what happens when $L(\lambda)$ is restricted to GL_{n-1} we just need to put $x_n = 1$ into the formula (2.3). The right hand side will then be written as the sum of the Schur polynomials $s_\mu(x_1, \dots, x_{n-1})$ with μ satisfying (2.2).

On the other hand, the multiplicity-freeness of the reduction $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$ can be explained by the fact that the vector space $\mathrm{Hom}_{\mathfrak{gl}_{n-1}}(L'(\mu), L(\lambda))$ bears a natural irreducible representation of the centralizer $\mathrm{U}(\mathfrak{gl}_n)^{\mathfrak{gl}_{n-1}}$; see, e.g., Dixmier [19, Section 9.1]. However, the centralizer is a commutative algebra and therefore if the homomorphism space is nonzero then it must be one-dimensional.

The branching rule is implicit in the formulas of Gelfand and Tsetlin [36]. Its proof based upon an explicit realization of the representations of GL_n was given by Zhelobenko [144]. We outline a proof of Theorem 2.1 below in Section 2.3 which employs the modern theory of Mickelsson algebras following Zhelobenko [151].

The subsequent applications of the branching rule to the subalgebras of the chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$$

yield a parameterization of basis vectors in $L(\lambda)$ by the combinatorial objects called the *Gelfand–Tsetlin patterns*. Such a pattern Λ (associated with λ) is an array of row vectors

$$\begin{array}{ccccccc}
\lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
& \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & \\
& & \cdots & \cdots & \cdots & & \\
& & \lambda_{21} & & \lambda_{22} & & \\
& & & \lambda_{11} & & &
\end{array}$$

where the upper row coincides with λ and the following conditions hold

$$\lambda_{ki} - \lambda_{k-1,i} \in \mathbb{Z}_+, \qquad \lambda_{k-1,i} - \lambda_{k,i+1} \in \mathbb{Z}_+, \qquad i = 1, \dots, k-1 \tag{2.4}$$

for each $k = 2, \dots, n$.

Remark 2.2 If the highest weight λ is a partition then there is a natural bijection between the patterns associated with λ and the *semistandard* λ-tableaux with entries in $\{1, \dots, n\}$. Namely, the pattern Λ can be viewed as the sequence of partitions

$$\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \dots \subseteq \lambda^{(n)} = \lambda,$$

with $\lambda^{(k)} = (\lambda_{k1}, \dots, \lambda_{kk})$. Conditions (2.4) mean that the skew diagram $\lambda^{(k)}/\lambda^{(k-1)}$ is a *horizontal strip*; see, e.g., Macdonald [73, Chapter 1].

The *Gelfand–Tsetlin basis* of $L(\lambda)$ is provided by the following theorem. Let us set $l_{ki} = \lambda_{ki} - i + 1$.

Theorem 2.3 *There exists a basis $\{\xi_\Lambda\}$ in $L(\lambda)$ parametrized by all patterns Λ such that the action of generators of $\mathfrak{gl}_n$ is given by the formulas*

$$E_{kk}\, \xi_\Lambda = \left(\sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i} \right) \xi_\Lambda, \tag{2.5}$$

$$E_{k,k+1}\, \xi_\Lambda = - \sum_{i=1}^{k} \frac{(l_{ki} - l_{k+1,1}) \cdots (l_{ki} - l_{k+1,k+1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})}\, \xi_{\Lambda + \delta_{ki}}, \tag{2.6}$$

$$E_{k+1,k}\, \xi_\Lambda = \sum_{i=1}^{k} \frac{(l_{ki} - l_{k-1,1}) \cdots (l_{ki} - l_{k-1,k-1})}{(l_{ki} - l_{k1}) \cdots \wedge \cdots (l_{ki} - l_{kk})}\, \xi_{\Lambda - \delta_{ki}}. \tag{2.7}$$

The arrays $\Lambda \pm \delta_{ki}$ are obtained from Λ by replacing λ_{ki} by $\lambda_{ki} \pm 1$. It is supposed that $\xi_\Lambda = 0$ if the array Λ is not a pattern; the symbol $\wedge$ indicates that the zero factor in the denominator is skipped.

A construction of the basis vectors is given in Theorem 2.7 below. A derivation of the matrix element formulas (2.5)–(2.7) is outlined in Section 2.3.

The vector space $L(\lambda)$ is equipped with a contravariant inner product $\langle\, ,\, \rangle$. It is uniquely determined by the conditions

$$\langle\, \xi, \xi \,\rangle = 1 \qquad \text{and} \qquad \langle\, E_{ij}\, \eta, \zeta \,\rangle = \langle\, \eta, E_{ji}\, \zeta \,\rangle$$

for any vectors $\eta, \zeta \in L(\lambda)$ and any indices i, j. In other words, for the adjoint operator for E_{ij} with respect to the inner product we have $(E_{ij})^* = E_{ji}$.

Proposition 2.4 *The basis $\{\xi_\Lambda\}$ is orthogonal with respect to the inner product $\langle\ ,\ \rangle$. Moreover, we have*

$$\langle\, \xi_\Lambda, \xi_\Lambda \,\rangle = \prod_{k=2}^{n} \prod_{1\le i\le j<k} \frac{(l_{ki}-l_{k-1,j})!}{(l_{k-1,i}-l_{k-1,j})!} \prod_{1\le i<j\le k} \frac{(l_{ki}-l_{kj}-1)!}{(l_{k-1,i}-l_{kj}-1)!}.$$

The formulas of Theorem 2.3 can therefore be rewritten in the orthonormal basis

$$\zeta_\Lambda = \xi_\Lambda / \|\,\xi_\Lambda\|, \qquad \|\,\xi_\Lambda\|^2 = \langle\, \xi_\Lambda, \xi_\Lambda \,\rangle. \tag{2.8}$$

They were presented in this form in the original work by Gelfand and Tsetlin [36]. A proof of Proposition 2.4 will be outlined in Section 2.3.

2.1 Construction of the basis: lowering and raising operators

For each $i = 1, \dots, n-1$ introduce the following elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$

$$z_{in} = \sum_{i>i_1>\cdots>i_s\ge 1} E_{ii_1} E_{i_1 i_2} \cdots E_{i_{s-1} i_s} E_{i_s n} (h_i - h_{j_1}) \cdots (h_i - h_{j_r}), \tag{2.9}$$

$$z_{ni} = \sum_{i<i_1<\cdots<i_s<n} E_{i_1 i} E_{i_2 i_1} \cdots E_{i_s i_{s-1}} E_{n i_s} (h_i - h_{j_1}) \cdots (h_i - h_{j_r}), \tag{2.10}$$

where s runs over nonnegative integers, $h_i = E_{ii} - i + 1$ and $\{j_1, \dots, j_r\}$ is the complementary subset to $\{i_1, \dots, i_s\}$ in the set $\{1, \dots, i-1\}$ or $\{i+1, \dots, n-1\}$, respectively. For instance,

$$z_{13} = E_{13}, \qquad z_{23} = E_{23}\,(h_2 - h_1) + E_{21}E_{13},$$
$$z_{32} = E_{32}, \qquad z_{31} = E_{31}\,(h_1 - h_2) + E_{21}E_{32}.$$

Consider now the irreducible finite-dimensional representation $L(\lambda)$ of $\mathfrak{gl}_n$ with the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ and the highest vector ξ. Denote by $L(\lambda)^+$ the subspace of $\mathfrak{gl}_{n-1}$-highest vectors in $L(\lambda)$:

$$L(\lambda)^+ = \{\eta \in L(\lambda) \mid E_{ij}\,\eta = 0, \qquad 1 \le i < j < n\}.$$

Given a $\mathfrak{gl}_{n-1}$-weight $\mu = (\mu_1, \dots, \mu_{n-1})$ we denote by $L(\lambda)^+_\mu$ the corresponding weight subspace in $L(\lambda)^+$:

$$L(\lambda)^+_\mu = \{\eta \in L(\lambda)^+ \mid E_{ii}\,\eta = \mu_i\,\eta, \qquad i = 1, \dots, n-1\}.$$

The main property of the elements z_{ni} and z_{in} is described by the following lemma.

Lemma 2.5 *Let $\eta \in L(\lambda)^+_\mu$. Then for any $i = 1, \dots, n-1$ we have*

$$z_{in}\,\eta \in L(\lambda)^+_{\mu+\delta_i} \qquad \text{and} \qquad z_{ni}\,\eta \in L(\lambda)^+_{\mu-\delta_i},$$

where the weight $\mu \pm \delta_i$ is obtained from μ by replacing μ_i with $\mu_i \pm 1$.

This result allows us to regard the elements z_{in} and z_{ni} as operators in the space $L(\lambda)^+$. They are called the *raising* and *lowering operators*, respectively. By the branching rule (Theorem 2.1) the space $L(\lambda)^+_\mu$ is one-dimensional if the conditions (2.2) hold and it is zero otherwise. The following lemma will be proved in Section 2.3.

Lemma 2.6 *Suppose that μ satisfies the betweenness conditions* (2.2). *Then the vector*

$$\xi_\mu = z_{n1}^{\lambda_1-\mu_1} \cdots z_{n,n-1}^{\lambda_{n-1}-\mu_{n-1}}\, \xi$$

is nonzero. Moreover, the space $L(\lambda)^+_\mu$ is spanned by ξ_μ.

The $\mathrm{U}(\mathfrak{gl}_{n-1})$-span of each nonzero vector ξ_μ is a $\mathfrak{gl}_{n-1}$-module isomorphic to $L'(\mu)$. Iterating the construction of the vectors ξ_μ for each pair of Lie algebras $\mathfrak{gl}_{k-1} \subset \mathfrak{gl}_k$ we shall be able to get a basis in the entire space $L(\lambda)$.

Theorem 2.7 *The basis vectors ξ_Λ of Theorem 2.3 can be given by the formula*

$$\xi_\Lambda = \prod_{k=2,\dots,n}^{\rightarrow} \left(z_{k1}^{\lambda_{k1}-\lambda_{k-1,1}} \cdots z_{k,k-1}^{\lambda_{k,k-1}-\lambda_{k-1,k-1}} \right) \xi, \tag{2.11}$$

where the factors in the product are ordered in accordance with increase of the indices.

2.2 The Mickelsson algebra theory

The lowering and raising operators z_{ni} and z_{in} in the space $L(\lambda)^+$ (see Lemma 2.5) satisfy some quadratic relations with rational coefficients in the parameters of the highest weights. These relations can be regarded in a representation independent form with a suitable interpretation of the coefficients as rational functions in the elements of the Cartan subalgebra $\mathfrak{h}$. In such an abstract form the algebras of lowering and raising operators were introduced by Mickelsson [81] who, however, did not use any rational extensions of the algebra $\mathrm{U}(\mathfrak{h})$. The importance of this extension was realized by Zhelobenko [146, 147] who developed a general structure theory of these algebras which he called the *Mickelsson algebras*. Another important ingredient is the theory of *extremal projectors* originated from the works of Asherova, Smirnov and Tolstoy [1, 2, 3] and further developed by Zhelobenko [150, 151].

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{k}$ be its subalgebra reductive in $\mathfrak{g}$. This means that the adjoint $\mathfrak{k}$-module $\mathfrak{g}$ is completely reducible. In particular, $\mathfrak{k}$ is a reductive Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ and a triangular decomposition

$$\mathfrak{k} = \mathfrak{k}^- \oplus \mathfrak{h} \oplus \mathfrak{k}^+.$$

The subalgebras $\mathfrak{k}^-$ and $\mathfrak{k}^+$ are respectively spanned by the negative and positive root vectors $e_{-\alpha}$ and e_α with α running over the set of positive roots Δ^+ of $\mathfrak{k}$ with respect to $\mathfrak{h}$. The root vectors will be assumed to be normalized in such a way that

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \qquad \alpha(h_\alpha) = 2 \tag{2.12}$$

for all $\alpha \in \Delta^+$.

Let $\mathrm{J} = \mathrm{U}(\mathfrak{g})\,\mathfrak{k}^+$ be the left ideal of $\mathrm{U}(\mathfrak{g})$ generated by $\mathfrak{k}^+$. Its normalizer $\mathrm{Norm}\,\mathrm{J}$ is a subalgebra of $\mathrm{U}(\mathfrak{g})$ defined by

$$\mathrm{Norm}\,\mathrm{J} = \{u \in \mathrm{U}(\mathfrak{g}) \mid \mathrm{J}\,u \subseteq \mathrm{J}\}.$$

Then J is a two-sided ideal of Norm J and the *Mickelsson algebra* $\mathrm{S}(\mathfrak{g},\mathfrak{k})$ is defined as the quotient

$$\mathrm{S}(\mathfrak{g},\mathfrak{k}) = \mathrm{Norm}\,\mathrm{J}/\mathrm{J}.$$

Let $\mathrm{R}(\mathfrak{h})$ denote the field of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$. In what follows it is convenient to consider the extension $\mathrm{U}'(\mathfrak{g})$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ defined by

$$\mathrm{U}'(\mathfrak{g}) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}).$$

Let $\mathrm{J}' = \mathrm{U}'(\mathfrak{g})\,\mathfrak{k}^+$ be the left ideal of $\mathrm{U}'(\mathfrak{g})$ generated by $\mathfrak{k}^+$. Exactly as with the ideal J above, J' is a two-sided ideal of the normalizer $\mathrm{Norm}\,\mathrm{J}'$ and the *Mickelsson–Zhelobenko algebra*[4] $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ is defined as the quotient

$$\mathrm{Z}(\mathfrak{g},\mathfrak{k}) = \mathrm{Norm}\,\mathrm{J}'/\mathrm{J}'.$$

Clearly, $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ is an extension of the Mickelsson algebra $\mathrm{S}(\mathfrak{g},\mathfrak{k})$,

$$\mathrm{Z}(\mathfrak{g},\mathfrak{k}) = \mathrm{S}(\mathfrak{g},\mathfrak{k}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}).$$

An equivalent definition of the algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ can be given by using the quotient space

$$\mathrm{M}(\mathfrak{g},\mathfrak{k}) = \mathrm{U}'(\mathfrak{g})/\mathrm{J}'.$$

The Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ coincides with the subspace of $\mathfrak{k}$-highest vectors in $\mathrm{M}(\mathfrak{g},\mathfrak{k})$

$$\mathrm{Z}(\mathfrak{g},\mathfrak{k}) = \mathrm{M}(\mathfrak{g},\mathfrak{k})^+,$$

where

$$\mathrm{M}(\mathfrak{g},\mathfrak{k})^+ = \{v \in \mathrm{M}(\mathfrak{g},\mathfrak{k}) \mid \mathfrak{k}^+\, v = 0\}.$$

The algebraic structure of the algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ can be described with the use of the *extremal projector* for the Lie algebra $\mathfrak{k}$. In order to define it, suppose that the positive roots are $\Delta^+ = \{\alpha_1,\dots,\alpha_m\}$. Consider the vector space $\mathrm{F}_\mu(\mathfrak{k})$ of formal series of weight μ monomials

$$e_{-\alpha_1}^{k_1}\cdots e_{-\alpha_m}^{k_m}\, e_{\alpha_m}^{r_m}\cdots e_{\alpha_1}^{r_1}$$

[4]Zhelobenko sometimes used the names *Z-algebra* or *extended Mickelsson algebra*. The author believes the new name is more emphatic and justified from the scientific point of view.

with coefficients in $\mathrm{R}(\mathfrak{h})$, where

$$(r_1 - k_1)\,\alpha_1 + \cdots + (r_m - k_m)\,\alpha_m = \mu.$$

Introduce the space $\mathrm{F}(\mathfrak{k})$ as the direct sum

$$\mathrm{F}(\mathfrak{k}) = \bigoplus_{\mu} \mathrm{F}_\mu(\mathfrak{k}).$$

That is, the elements of $\mathrm{F}(\mathfrak{k})$ are finite sums $\sum x_\mu$ with $x_\mu \in \mathrm{F}_\mu(\mathfrak{k})$. It can be shown that $\mathrm{F}(\mathfrak{k})$ is an algebra with respect to the natural multiplication of formal series. The algebra $\mathrm{F}(\mathfrak{k})$ is equipped with a Hermitian anti-involution (antilinear involutive anti-automorphism) defined by

$$e_\alpha^* = e_{-\alpha}, \qquad \alpha \in \Delta^+.$$

Further, call an ordering of the positive roots *normal* if any composite root lies between its components. For instance, there are precisely two normal orderings for the root system of type B_2,

$$\Delta^+ = \{\alpha, \alpha+\beta, \alpha+2\beta, \beta\} \qquad \text{and} \qquad \Delta^+ = \{\beta, \alpha+2\beta, \alpha+\beta, \alpha\},$$

where α and β are the simple roots. In general, the number of normal orderings coincides with the number of reduced decompositions of the longest element of the corresponding Weyl group.

For any $\alpha \in \Delta^+$ introduce the element of $\mathrm{F}(\mathfrak{k})$ by

$$p_\alpha = 1 + \sum_{k=1}^{\infty} e_{-\alpha}^k\, e_\alpha^k \, \frac{(-1)^k}{k!\,(h_\alpha + \rho(h_\alpha) + 1)\cdots(h_\alpha + \rho(h_\alpha) + k)}, \tag{2.13}$$

where h_α is defined in (2.12) and ρ is the half sum of the positive roots. Finally, define the *extremal projector* $p = p_{\mathfrak{k}}$ by

$$p = p_{\alpha_1} \cdots p_{\alpha_m}$$

with the product taken in a normal ordering of the positive roots α_i.

Theorem 2.8 *The element $p \in \mathrm{F}(\mathfrak{k})$ does not depend on the normal ordering on Δ^+ and satisfies the conditions*

$$e_\alpha\, p = p\, e_{-\alpha} = 0 \qquad \textit{for all} \qquad \alpha \in \Delta^+. \tag{2.14}$$

Moreover, $p^* = p$ and $p^2 = p$.

In fact, the relations (2.14) uniquely determine the element p, up to a factor from $\mathrm{R}(\mathfrak{h})$. The extremal projector naturally acts on the vector space $\mathrm{M}(\mathfrak{g}, \mathfrak{k})$. The following corollary states that the Mickelsson–Zhelobenko algebra coincides with its image.

Corollary 2.9 *We have*

$$\mathrm{Z}(\mathfrak{g},\mathfrak{k}) = p\,\mathrm{M}(\mathfrak{g},\mathfrak{k}).$$

To get a more precise description of the algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ consider a $\mathfrak{k}$-module decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Choose a weight basis $e_1, \dots, e_n$ (with respect to the adjoint action of $\mathfrak{h}$) of the complementary module $\mathfrak{p}$.

Theorem 2.10 *The elements*

$$a_i = p\,e_i, \qquad i = 1, \dots, n$$

are generators of the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$*. Moreover, the monomials*

$$a_1^{k_1} \cdots a_n^{k_n}, \qquad k_i \in \mathbb{Z}_+,$$

form a basis of $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$*.*

It can be proved that the generators a_i of $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ satisfy quadratic defining relations; see [150]. For the pairs $(\mathfrak{g},\mathfrak{k})$ relevant to the constructions of bases of the Gelfand–Tsetlin type, the relations can be explicitly written down; cf. Sections 2.3 and 3.1 below.

Regarding $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ as a right $\mathrm{R}(\mathfrak{h})$-module, it is possible to introduce the normalized elements

$$z_i = a_i\,\pi_i, \qquad \pi_i \in \mathrm{U}(\mathfrak{h})$$

by multiplying a_i by its *right denominator* π_i. Therefore the z_i can be viewed as elements of the Mickelsson algebra $\mathrm{S}(\mathfrak{g},\mathfrak{k})$.

To formulate the final theorem of this section, for any $\mathfrak{g}$-module V set

$$V^+ = \{v \in V \mid \mathfrak{k}^+\,v = 0\}.$$

Theorem 2.11 *Let* $V = \mathrm{U}(\mathfrak{g})\,v$ *be a cyclic* $\mathrm{U}(\mathfrak{g})$*-module generated by an element* $v \in V^+$*. Then the subspace* V^+ *is linearly spanned by the elements*

$$z_1^{k_1} \cdots z_n^{k_n}\,v, \qquad k_i \in \mathbb{Z}_+.$$

2.3 Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$

For any positive integer m consider the general linear Lie algebra $\mathfrak{gl}_m$. The positive roots of $\mathfrak{gl}_m$ with respect to the diagonal Cartan subalgebra $\mathfrak{h}$ (with the standard choice of the positive root system) are naturally enumerated by the pairs (i, j) with $1 \leq i < j \leq m$. In accordance with the general theory outlined in the previous section, for each pair introduce the formal series $p_{ij} \in \mathrm{F}(\mathfrak{gl}_m)$ by

$$p_{ij} = 1 + \sum_{k=1}^{\infty} (E_{ji})^k (E_{ij})^k \, \frac{(-1)^k}{k!\,(h_i - h_j + 1) \cdots (h_i - h_j + k)},$$

where, as before, $h_i = E_{ii} - i + 1$. Then define the element $p = p_m$ by

$$p = \prod_{i<j} p_{ij},$$

where the product is taken in a normal ordering on the pairs (i, j). By Theorem 2.8,

$$E_{ij}\, p = pE_{ji} = 0 \qquad \text{for} \quad 1 \leq i < j \leq m. \tag{2.15}$$

Now set $m = n - 1$. By Theorem 2.10, ordered monomials in the elements E_{nn}, pE_{in} and pE_{ni} with $i = 1, \dots, n-1$ form a basis of $\mathrm{Z}(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$ as a left or right $\mathrm{R}(\mathfrak{h})$-module. These elements can explicitly be given by

$$\begin{aligned} pE_{in} &= \sum_{i>i_1>\cdots>i_s\geq 1} E_{ii_1} E_{i_1 i_2} \cdots E_{i_{s-1} i_s} E_{i_s n} \frac{1}{(h_i - h_{i_1}) \cdots (h_i - h_{i_s})}, \\ pE_{ni} &= \sum_{i<i_1<\cdots<i_s<n} E_{i_1 i} E_{i_2 i_1} \cdots E_{i_s i_{s-1}} E_{n i_s} \frac{1}{(h_i - h_{i_1}) \cdots (h_i - h_{i_s})}, \end{aligned} \tag{2.16}$$

where $s = 0, 1, \dots$. Indeed, by choosing appropriate normal orderings on the positive roots, we can write

$$pE_{in} = p_{1i} \cdots p_{i-1,i}\, E_{in} \qquad \text{and} \qquad pE_{ni} = p_{i,i+1} \cdots p_{i,n-1}\, E_{ni}.$$

The lowering and raising operators introduced in Section 2.1 coincide with the normalized generators:

$$\begin{aligned} z_{in} &= pE_{in}\,(h_i - h_{i-1}) \cdots (h_i - h_1), \\ z_{ni} &= pE_{ni}\,(h_i - h_{i+1}) \cdots (h_i - h_{n-1}), \end{aligned} \tag{2.17}$$

which belong to the Mickelsson algebra $\mathrm{S}(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$. Thus, Lemma 2.5 is an immediate corollary of (2.15).

Proposition 2.12 *The lowering and raising operators satisfy the following relations*

$$z_{ni}z_{nj} = z_{nj}z_{ni} \qquad \textit{for all} \quad i, j, \tag{2.18}$$

$$z_{in}z_{nj} = z_{nj}z_{in} \qquad \textit{for} \quad i \neq j, \tag{2.19}$$

and

$$z_{in}z_{ni} = \prod_{j=1,\, j\neq i}^{n} (h_i - h_j - 1) + \sum_{j=1}^{n-1} z_{nj}z_{jn} \prod_{k=1,\, k\neq j}^{n-1} \frac{h_i - h_k - 1}{h_j - h_k}. \tag{2.20}$$

Proof. We use the properties of p. Assume that $i < j$. Then (2.15) and (2.16) imply that in $\mathrm{Z}(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$

$$pE_{ni}\, pE_{nj} = pE_{ni}E_{nj}, \qquad pE_{nj}\, pE_{ni} = pE_{ni}E_{nj}\, \frac{h_i - h_j + 1}{h_i - h_j}.$$

Now (2.18) follows from (2.17). The proof of (2.19) is similar. The "long" relation (2.20) can be verified by analogous but more complicated direct calculations. We give its different proof based upon the properties of the *Capelli determinant* $\mathcal{C}(u)$. Consider the $n \times n$-matrix E whose ij-th entry is E_{ij} and let u be a formal variable. Then $\mathcal{C}(u)$ is a polynomial with coefficients in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ defined by

$$\mathcal{C}(u) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \cdot (u + E)_{\sigma(1),1} \cdots (u + E - n + 1)_{\sigma(n),n}. \tag{2.21}$$

It is well known that all its coefficients belong to the center of $\mathrm{U}(\mathfrak{gl}_n)$ and generate the center; see e.g. Howe–Umeda [53]. This also easily follows from the properties of the *quantum determinant* of the Yangian for the Lie algebra $\mathfrak{gl}_n$; see e.g. [93]. Therefore, these coefficients act in $L(\lambda)$ as scalars which can be easily found by applying $\mathcal{C}(u)$ to the highest vector ξ:

$$\mathcal{C}(u)|_{L(\lambda)} = (u + l_1) \cdots (u + l_n), \qquad l_i = \lambda_i - i + 1. \tag{2.22}$$

On the other hand, the center of $\mathrm{U}(\mathfrak{gl}_n)$ is a subalgebra in the normalizer $\mathrm{Norm}\,\mathrm{J}$. We shall keep the same notation for the image of $\mathcal{C}(u)$ in the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$. To get explicit expressions of the coefficients of $\mathcal{C}(u)$ in terms of the lowering and raising operators we consider $\mathcal{C}(u)$ modulo the ideal J' and apply the projection p. A straightforward calculation yields two alternative formulas

$$\mathcal{C}(u) = (u + E_{nn}) \prod_{i=1}^{n-1} (u + h_i - 1) - \sum_{i=1}^{n-1} z_{in}z_{ni} \prod_{j=1,\, j\neq i}^{n-1} \frac{u + h_j - 1}{h_i - h_j} \tag{2.23}$$

and

$$\mathcal{C}(u) = \prod_{i=1}^{n} (u + h_i) - \sum_{i=1}^{n-1} z_{ni}z_{in} \prod_{j=1,\, j\neq i}^{n-1} \frac{u + h_j}{h_i - h_j}. \tag{2.24}$$

The formulas show that $\mathcal{C}(u)$ can be regarded as an interpolation polynomial for the products $z_{in}z_{ni}$ and $z_{ni}z_{in}$. Namely, for $i = 1, \dots, n-1$ we have

$$\mathcal{C}(-h_i + 1) = (-1)^{n-1} z_{in} z_{ni} \qquad \text{and} \qquad \mathcal{C}(-h_i) = (-1)^{n-1} z_{ni} z_{in} \tag{2.25}$$

with the agreement that when we evaluate u in $\mathrm{U}(\mathfrak{h})$ we write the coefficients of the polynomial to the left from powers of u. Comparing the values of (2.23) and (2.24) at $u = -h_i + 1$ we get (2.20). □

Note that the relation inverse to (2.20) can be obtained by comparing the values of (2.23) and (2.24) at $u = -h_i$.

Next we outline the proofs of the branching rule (Theorem 2.1) and the formulas for the basis elements of $L(\lambda)^+$ (Lemma 2.6). The module $L(\lambda)$ is generated by the highest vector ξ and we have

$$z_{in}\, \xi = 0, \qquad i = 1, \dots, n-1.$$

So, by Theorem 2.11, the vector space $L(\lambda)^+$ is spanned by the elements

$$z_{n1}^{k_1} \cdots z_{n,n-1}^{k_{n-1}}\, \xi, \qquad k_i \in \mathbb{Z}_+. \tag{2.26}$$

Let us set $\mu_i = \lambda_i - k_i$ for $1 \le i \le n-1$ and denote the vector (2.26) by ξ_μ. That is,

$$\xi_\mu = z_{n1}^{\lambda_1 - \mu_1} \cdots z_{n,n-1}^{\lambda_{n-1} - \mu_{n-1}}\, \xi. \tag{2.27}$$

It is now sufficient to show that the vector ξ_μ is nonzero if and only if the betweenness conditions (2.2) hold. The linear independence of the vectors ξ_μ will follow from the fact that their weights are distinct. If $\xi_\mu \neq 0$ then using the relations (2.18) we conclude that each vector $z_{ni}^{\lambda_i - \mu_i}\, \xi$ is nonzero. On the other hand, $z_{ni}^{k_i}\, \xi$ is a $\mathfrak{gl}_{n-1}$-highest vector of the weight obtained from $(\lambda_1, \dots, \lambda_{n-1})$ by replacing λ_i with $\lambda_i - k_i$. Threfore, if $k_i \ge \lambda_i - \lambda_{i+1} + 1$ then the conditions (2.1) are violated for this weight which implies $z_{ni}^{k_i}\, \xi = 0$. Hence, $\lambda_i - \mu_i \le \lambda_i - \lambda_{i+1}$ for each i, and μ satisfies (2.2).

For the proof of the converse statement we shall employ the following key lemma which will also be used for the proof of Theorem 2.3.

Lemma 2.13 *We have for each* $i = 1, \dots, n-1$

$$z_{in}\, \xi_\mu = -(m_i - l_1) \cdots (m_i - l_n)\, \xi_{\mu + \delta_i}, \tag{2.28}$$

where

$$m_i = \mu_i - i + 1, \qquad l_i = \lambda_i - i + 1.$$

It is supposed that $\xi_{\mu+\delta_i} = 0$ *if* $\lambda_i = \mu_i$.

Proof. The relation (2.19) implies that if $\lambda_i = \mu_i$ then $z_{in}\, \xi_\mu = 0$ which agrees with (2.28). Now let $\lambda_i - \mu_i \geq 1$. Using (2.18) and (2.25) we obtain

$$z_{in}\, \xi_\mu = z_{in} z_{ni}\, \xi_{\mu+\delta_i} = (-1)^{n-1} \mathcal{C}(-h_i + 1)\, \xi_{\mu+\delta_i} = (-1)^{n-1} \mathcal{C}(-m_i)\, \xi_{\mu+\delta_i}.$$

The relation (2.28) now follows from (2.22) and the centrality of $\mathcal{C}(u)$. □

If the betweenness conditions (2.2) hold then by Lemma 2.13, applying appropriate operators z_{in} repeatedly to the vector ξ_μ we can obtain the highest vector ξ with a nonzero coefficient. This gives $\xi_\mu \neq 0$.

Thus, we have proved that the vectors ξ_Λ defined in (2.11) form a basis of the representation $L(\lambda)$. The orthogonality of the basis vectors (Proposition 2.4) is implied by the fact that the operators pE_{ni} and pE_{in} are adjoint to each other with respect to the restriction of the inner product $\langle\, ,\, \rangle$ to the subspace $L(\lambda)^+$. Therefore, for the adjoint operator to z_{ni} we have

$$z_{ni}^* = z_{in} \frac{(h_i - h_{i+1} - 1) \cdots (h_i - h_{n-1} - 1)}{(h_i - h_1) \cdots (h_i - h_{i-1})}$$

and Proposition 2.4 is deduced from Lemma 2.13 by induction.

Now we outline a derivation of formulas (2.5)–(2.7). First, since $E_{nn}\, z_{ni} = z_{ni}\, (E_{nn} + 1)$ for any i, we have

$$E_{nn}\, \xi_\mu = \Big(\sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i \Big)\, \xi_\mu$$

which implies (2.5). To prove (2.6) is suffices to calculate $E_{n-1,n}\, \xi_{\mu\nu}$, where

$$\xi_{\mu\nu} = z_{n-1,1}^{\mu_1 - \nu_1} \cdots z_{n-1,n-2}^{\mu_{n-2} - \nu_{n-2}}\, \xi_\mu$$

and the ν_i satisfy the betweenness conditions

$$\mu_i - \nu_i \in \mathbb{Z}_+ \quad \text{and} \quad \nu_i - \mu_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-2.$$

Since $E_{n-1,n}$ commutes with the $z_{n-1,i}$,

$$E_{n-1,n}\, \xi_{\mu\nu} = z_{n-1,1}^{\mu_1 - \nu_1} \cdots z_{n-1,n-2}^{\mu_{n-2} - \nu_{n-2}}\, E_{n-1,n}\, \xi_\mu.$$

The following lemma is implied by the explicit formulas for the lowering and raising operators (2.9) and (2.10).

Lemma 2.14 *We have the relation in* $\mathrm{U}'(\mathfrak{gl}_n)$ *modulo the ideal* J'*,*

$$E_{n-1,n} = \sum_{i=1}^{n-1} z_{n-1,i}\, z_{in} \frac{1}{(h_i - h_1) \cdots \wedge_i \cdots (h_i - h_{n-1})},$$

where $z_{n-1,n-1} = 1$.

By Lemmas 2.13 and 2.14,

$$E_{n-1,n}\,\xi_{\mu\nu} = -\sum_{i=1}^{n-1} \frac{(m_i - l_1)\cdots(m_i - l_n)}{(m_i - m_1)\cdots\wedge_i\cdots(m_i - m_{n-1})}\,\xi_{\mu+\delta_i,\nu} \tag{2.29}$$

which proves (2.6). To prove (2.7) we use Proposition 2.4. Relation (2.29) implies that

$$E_{n,n-1}\xi_{\mu\nu} = \sum_{i=1}^{n-1} c_i(\mu,\nu)\,\xi_{\mu-\delta_i,\nu}$$

for some coefficients $c_i(\mu,\nu)$. Apply the operator $z_{j,n-1}$ to both sides of this relation. Since $z_{j,n-1}$ commutes with $E_{n,n-1}$ we obtain from Lemma 2.13 a recurrence relation for the $c_i(\mu,\nu)$: if $\mu_j - \nu_j \geq 1$ then

$$c_i(\mu,\nu+\delta_j) = c_i(\mu,\nu)\,\frac{m_i - \gamma_j - 1}{m_i - \gamma_j},$$

where $\gamma_j = \nu_j - j + 1$. The proof is completed by induction. The initial values of $c_i(\mu,\nu)$ are found by applying the relation

$$E_{n,n-1}\,z_{n-1,i} = z_{ni}\frac{1}{h_i - h_{n-1}} + z_{n-1,i}E_{n,n-1}\frac{h_i - h_{n-1} - 1}{h_i - h_{n-1}}$$

to the vector ξ_μ and taking into account that $E_{n,n-1} = z_{n,n-1}$. Performing the calculation we get

$$E_{n,n-1}\,\xi_{\mu\nu} = \sum_{i=1}^{n-1} \frac{(m_i - \gamma_1)\cdots(m_i - \gamma_{n-2})}{(m_i - m_1)\cdots\wedge_i\cdots(m_i - m_{n-1})}\,\xi_{\mu-\delta_i,\nu}$$

thus proving (2.7).

2.4 Characteristic identities

Denote by L the vector representation of $\mathfrak{gl}_n$ and consider its contragredient L^*. Note that L^* is isomorphic to $L(0,\dots,0,-1)$. Let $\{\varepsilon_1,\dots,\varepsilon_n\}$ denote the basis of L^* dual to the canonical basis $\{e_1,\dots,e_n\}$ of L. Introduce the $n\times n$-matrix E whose ij-th entry is the generator E_{ij}. We shall interpret E as the element

$$E = \sum_{i,j=1}^{n} e_{ij}\otimes E_{ij} \in \operatorname{End} L^*\otimes \mathrm{U}(\mathfrak{gl}_n),$$

where the e_{ij} are the standard matrix units acting on L^* by $e_{ij}\,\varepsilon_k = \delta_{jk}\,\varepsilon_i$. The basis element E_{ij} of $\mathfrak{gl}_n$ acts on L^* as $-e_{ji}$ and hence E may also be thought of as the image of the element

$$e = -\sum_{i,j=1}^{n} E_{ji}\otimes E_{ij} \in \mathrm{U}(\mathfrak{gl}_n)\otimes\mathrm{U}(\mathfrak{gl}_n).$$

On the other hand, using the standard coproduct Δ on $\mathrm{U}(\mathfrak{gl}_n)$ defined by

$$\Delta(E_{ij}) = E_{ij} \otimes 1 + 1 \otimes E_{ij},$$

we can write e in the form

$$e = \frac{1}{2}\Big(z \otimes 1 + 1 \otimes z - \Delta(z)\Big), \qquad (2.30)$$

where z is the second order Casimir element

$$z = \sum_{i,j=1}^{n} E_{ij}E_{ji} \in \mathrm{U}(\mathfrak{gl}_n).$$

We have the tensor product decomposition

$$L^* \otimes L(\lambda) \simeq L(\lambda - \delta_1) \oplus \cdots \oplus L(\lambda - \delta_n), \qquad (2.31)$$

where $L(\lambda-\delta_i)$ is considered to be zero if $\lambda_i = \lambda_{i+1}$. On the level of characters this is a particular case of the *Pieri rule* for the expansion of the product of a Schur polynomial by an elementary symmetric polynomial; see, e.g., Macdonald [73, Chapter 1]. The Casimir element z acts as a scalar operator in any highest weight representation $L(\lambda)$. The corresponding eigenvalue is given by

$$z|_{L(\lambda)} = \sum_{i=1}^{n} \lambda_i(\lambda_i + n - 2i + 1).$$

Regarding now E as an operator on $L^* \otimes L(\lambda)$ and using (2.30) we derive that the restriction of E to the summand $L(\lambda - \delta_r)$ in (2.31) is the scalar operator with the eigenvalue $\lambda_r + n - r$ which we shall denote by α_r. This implies the *characteristic identity* for the matrix E,

$$\prod_{r=1}^{n}(E - \alpha_r) = 0, \qquad (2.32)$$

as an operator in $L^* \otimes L(\lambda)$. Moreover, the projection $P[r]$ of $L^* \otimes L(\lambda)$ to the summand $L(\lambda - \delta_r)$ can be written explicitly as

$$P[r] = \frac{(E - \alpha_1) \cdots \wedge_r \cdots (E - \alpha_n)}{(\alpha_r - \alpha_1) \cdots \wedge_r \cdots (\alpha_r - \alpha_n)}$$

with $\wedge_r$ indicating that the r-th factor is omitted. Together with (2.32) this yields the *spectral decomposition* of E,

$$E = \sum_{r=1}^{n} \alpha_r \, P[r]. \qquad (2.33)$$

Consider the orthonormal Gelfand–Tsetlin bases $\{\zeta_\Lambda\}$ of $L(\lambda)$ and $\{\zeta_{\Lambda^{(r)}}\}$ of $L(\lambda-\delta_r)$ for $r = 1, \dots, n$; see (2.8). Regarding the matrix element $P[r]_{ij}$ as an operator in $L(\lambda)$ we obtain

$$\langle\, \zeta_{\Lambda'}, P[r]_{ij}\, \zeta_\Lambda \,\rangle = \langle\, \varepsilon_i \otimes \zeta_{\Lambda'}, P[r]\, (\varepsilon_j \otimes \zeta_\Lambda) \,\rangle, \tag{2.34}$$

where we have extended the inner products on L^* and $L(\lambda)$ to $L^* \otimes L(\lambda)$ by setting

$$\langle\, \eta \otimes \zeta, \eta' \otimes \zeta' \,\rangle = \langle\, \eta, \eta' \,\rangle \langle\, \zeta, \zeta' \,\rangle$$

with $\eta, \eta' \in L^*$ and $\zeta, \zeta' \in L(\lambda)$. Furthermore, using the expansions

$$\varepsilon_j \otimes \zeta_\Lambda = \sum_{s=1}^{n} \sum_{\Lambda^{(s)}} \langle\, \varepsilon_j \otimes \zeta_\Lambda, \zeta_{\Lambda^{(s)}} \,\rangle\, \zeta_{\Lambda^{(s)}},$$

bring (2.34) to the form

$$\sum_{\Lambda^{(r)}} \langle\, \varepsilon_i \otimes \zeta_{\Lambda'}, \zeta_{\Lambda^{(r)}} \,\rangle \langle\, \varepsilon_j \otimes \zeta_\Lambda, \zeta_{\Lambda^{(r)}} \,\rangle,$$

where we have used the fact that $P[r]$ is the identity map on $L(\lambda - \delta_r)$, and zero on $L(\lambda - \delta_s)$ with $s \neq r$. The numbers $\langle\, \varepsilon_i \otimes \zeta_{\Lambda'}, \zeta_{\Lambda^{(r)}} \,\rangle$ are the *Wigner coefficients* (a particular case of the *Clebsch–Gordan coefficients*). They can be used to express the matrix elements of the generators E_{ij} in the Gelfand–Tsetlin basis as follows. Using the spectral decomposition (2.33) we get

$$E_{ij} = \sum_{r=1}^{n} \alpha_r\, P[r]_{ij}.$$

Therefore, we derive the following result from (2.34).

Theorem 2.15 *We have*

$$\langle\, \zeta_{\Lambda'}, E_{ij}\, \zeta_\Lambda \,\rangle = \sum_{r=1}^{n} \alpha_r \sum_{\Lambda^{(r)}} \langle\, \varepsilon_i \otimes \zeta_{\Lambda'}, \zeta_{\Lambda^{(r)}} \,\rangle \langle\, \varepsilon_j \otimes \zeta_\Lambda, \zeta_{\Lambda^{(r)}} \,\rangle.$$

Employing the characteristic identities for both the Lie algebras $\mathfrak{gl}_{n+1}$ and $\mathfrak{gl}_n$ it is possible to determine the values of the Wigner coefficients and thus to get an independent derivation of the formulas of Theorem 2.3. In fact, explicit formulas for the matrix elements of E_{ij} with $|i - j| > 1$ can also be given; see Gould [44] for details.

The approach based upon the characteristic identities also leads to an alternative presentation of the lowering and raising operators. Taking ζ_Λ to be the highest vector ξ in (2.34) we conclude that $P[r]_{ij}\, \xi = 0$ for $j > r$. Consider now $\mathfrak{gl}_n$ as a subalgebra of $\mathfrak{gl}_{n+1}$. Suppose that ξ is a highest vector of weight λ in a representation $L(\lambda')$ of $\mathfrak{gl}_{n+1}$. The previous observation implies that the vector

$$\sum_{i=r}^{n} E_{n+1,i}\, P[r]_{ir}\, \xi$$

is again a $\mathfrak{gl}_n$-highest vector of weight $\lambda - \delta_r$.

Proposition 2.16 *We have the identity of operators on the space* $L(\lambda')^+_\lambda$*:*

$$p\, E_{n+1,r} = \sum_{i=r}^{n} E_{n+1,i}\, P[r]_{ir}$$

where p *is the extremal projector for* $\mathfrak{gl}_n$*.*

Outline of the proof. Since the both sides represent lowering operators they must be proportional. It is therefore sufficient to apply both sides to a vector $\xi \in L(\lambda')^+_\lambda$ and compare the coefficients at $E_{n+1,r}\,\xi$. For the calculation we use the explicit formula (2.16) for $p\, E_{n+1,r}$ and the relation

$$P[r]_{rr}\, \xi = \prod_{s=r+1}^{n} \frac{h_r - h_s - 1}{h_r - h_s}\, \xi$$

which can be derived from the characteristic identities. □

An analogous argument leads to a similar formula for the raising operators. Here one starts with the dual characteristic identity

$$\prod_{r=1}^{n} (\overline{E} - \bar{\alpha}_r) = 0,$$

where the ij-th matrix element of $\overline{E}$ is $-E_{ij}$, $\bar{\alpha}_r = -\lambda_r + r - 1$ and the powers of $\overline{E}$ are defined recursively by

$$(\overline{E}^p)_{ij} = \sum_{k=1}^{n} (\overline{E}^{p-1})_{kj} \overline{E}_{ik}.$$

For any $r = 1, \dots, n$ the dual projection operator is given by

$$\overline{P}[r] = \frac{(\overline{E} - \bar{\alpha}_1) \cdots \wedge_r \cdots (\overline{E} - \bar{\alpha}_n)}{(\bar{\alpha}_r - \bar{\alpha}_1) \cdots \wedge_r \cdots (\bar{\alpha}_r - \bar{\alpha}_n)}.$$

Proposition 2.17 *We have the identity of operators on the space* $L(\lambda')^+_\lambda$*:*

$$p\, E_{r,n+1} = \sum_{i=1}^{r} E_{i,n+1}\, \overline{P}[r]_{ri}.$$

2.5 Quantum minors

For a complex parameter u introduce the $n \times n$-matrix $E(u) = u\,1 + E$. Given sequences $a_1, \dots, a_s$ and $b_1, \dots, b_s$ of elements of $\{1, \dots, n\}$ the corresponding *quantum minor* of the matrix $E(u)$ is defined by the following equivalent formulas:

$$E(u)^{a_1 \cdots a_s}_{b_1 \cdots b_s} = \sum_{\sigma \in \mathfrak{S}_s} \operatorname{sgn} \sigma \cdot E(u)_{a_{\sigma(1)} b_1} \cdots E(u - s + 1)_{a_{\sigma(s)} b_s} \tag{2.35}$$

$$= \sum_{\sigma \in \mathfrak{S}_s} \operatorname{sgn} \sigma \cdot E(u - s + 1)_{a_1 b_{\sigma(1)}} \cdots E(u)_{a_s b_{\sigma(s)}}. \tag{2.36}$$

This is a polynomial in u with coefficients in $\mathrm{U}(\mathfrak{gl}_n)$. It is skew symmetric under permutations of the indices a_i, or b_i.

For any index $1 \le i < n$ introduce the polynomials

$$\tau_{ni}(u) = E(u)^{\,i+1 \cdots n}_{\,i \cdots n-1} \qquad \text{and} \qquad \tau_{in}(u) = (-1)^{i-1} E(u)^{\,1 \cdots i}_{\,1 \cdots i-1, n}.$$

For instance,

$$\tau_{13}(u) = E_{13}, \qquad \tau_{23}(u) = -E_{23}\,(u + E_{11}) + E_{21} E_{13},$$
$$\tau_{32}(u) = E_{32}, \qquad \tau_{31}(u) = E_{21} E_{32} - E_{31}\,(u + E_{22} - 1).$$

Proposition 2.18 *If $\eta \in L(\lambda)^+_\mu$ then*

$$\tau_{ni}(-\mu_i)\,\eta \in L(\lambda)^+_{\mu - \delta_i} \qquad \textit{and} \qquad \tau_{in}(-\mu_i + i - 1)\,\eta \in L(\lambda)^+_{\mu + \delta_i}.$$

Outline of the proof. The proof is based upon the following relations

$$[E_{ij}, E(u)^{\,a_1 \cdots a_s}_{\,b_1 \cdots b_s}] = \sum_{r=1}^{s} \Big(\delta_{j a_r} E(u)^{\,a_1 \cdots i \cdots a_s}_{\,b_1 \cdots b_s} - \delta_{i b_r} E(u)^{\,a_1 \cdots a_s}_{\,b_1 \cdots j \cdots b_s} \Big), \tag{2.37}$$

where i and j on the right hand side take the r-th positions. □

The relations (2.37) imply the important property of the quantum minors: for any indices i, j we have

$$[E_{a_i b_j}, E(u)^{\,a_1 \cdots a_s}_{\,b_1 \cdots b_s}] = 0.$$

In particular, this implies the centrality of the Capelli determinant $\mathcal{C}(u) = E(u)^{\,1 \cdots n}_{\,1 \cdots n}$; see (2.21).

The lowering and raising operators of Proposition 2.18 can be shown to essentially coincide with those defined in Section 2.1. To write down the formulas we shall need to evaluate the variable u in $\mathrm{U}(\mathfrak{h})$. To make this operation well-defined we accept the agreement used in the evaluation of the Capelli determinant in (2.25).

Proposition 2.19 *We have the identities for any* $i = 1, \ldots, n-1$

$$\tau_{ni}(-h_i - i + 1) = z_{ni} \qquad \text{and} \qquad \tau_{in}(-h_i) = z_{in}. \tag{2.38}$$

Using this interpretation of the lowering operators one can express the Gelfand–Tsetlin basis vector (2.11) in terms of the quantum minors $\tau_{ki}(u)$. The action of certain other quantum minors on these vectors can be explicitly found. This will provide one more independent proof of Theorem 2.3. For $m \geq 1$ introduce the polynomials $A_m(u)$, $B_m(u)$ and $C_m(u)$ by

$$A_m(u) = E(u)\,^{1\cdots m}_{1\cdots m}, \qquad B_m(u) = E(u)\,^{1\cdots m}_{1\cdots m-1,m+1}, \qquad C_m(u) = E(u)\,^{1\cdots m-1,m+1}_{1\cdots m}.$$

We use the notation $l_{mi} = \lambda_{mi} - i + 1$ and $l_i = \lambda_i - i + 1$.

Theorem 2.20 *Let* $\{\xi_\Lambda\}$ *be the Gelfand–Tsetlin basis of* $L(\lambda)$. *Then*

$$A_m(u)\,\xi_\Lambda = (u + l_{m1}) \cdots (u + l_{mm})\,\xi_\Lambda, \tag{2.39}$$

$$B_m(-l_{mj})\,\xi_\Lambda = -\prod_{i=1}^{m+1} (l_{m+1,i} - l_{mj})\,\xi_{\Lambda+\delta_{mj}} \quad \textit{for} \quad j = 1, \ldots, m,$$

$$C_m(-l_{mj})\,\xi_\Lambda = \prod_{i=1}^{m-1} (l_{m-1,i} - l_{mj})\,\xi_{\Lambda-\delta_{mj}} \quad \textit{for} \quad j = 1, \ldots, m, \tag{2.40}$$

where $\Lambda \pm \delta_{mj}$ *is obtained from* Λ *by replacing the entry* λ_{mj} *with* $\lambda_{mj} \pm 1$. □

Applying the Lagrange interpolation formula we can find the action of $B_m(u)$ and $C_m(u)$ for any u. Note that these polynomials have degree $m-1$ with the leading coefficients $E_{m,m+1}$ and $E_{m+1,m}$, respectively. Theorem 2.3 is therefore an immediate corollary of Theorem 2.20.

Formula (2.40) prompts a quite different construction of the basis vectors of $L(\lambda)$ which uses the polynomials $C_m(u)$ instead of the traditional lowering operators z_{ni}. Indeed, for a particular value of u, $C_m(u)$ takes a basis vector into another one, up to a factor. Given a pattern Λ associated with λ, define the vector $\kappa_\Lambda \in L(\lambda)$ by

$$\kappa_\Lambda = \prod_{k=1,\ldots,n-1}^{\rightarrow} \Big\{ C_{n-1}(-l_{n-1,k} - 1) \cdots C_{n-1}(-l_k + 1)\, C_{n-1}(-l_k) \times C_{n-2}(-l_{n-2,k} - 1) \cdots C_{n-2}(-l_k + 1)\, C_{n-2}(-l_k) \times \cdots \times C_k(-l_{kk} - 1) \cdots C_k(-l_k + 1)\, C_k(-l_k) \Big\}\, \xi.$$

Theorem 2.21 *The vectors* κ_Λ *with* Λ *running over all patterns associated with* λ *form a basis of* $L(\lambda)$ *and one has* $\kappa_\Lambda = N_\Lambda\, \xi_\Lambda$, *for a nonzero constant* N_Λ.

The value of the constant N_Λ can be found from (2.40). Using the relations between the elements $A_m(u)$, $B_m(u)$ and $C_m(u)$ one can derive Theorem 2.20 from Theorem 2.21 with the use of Proposition 2.22 below; see Nazarov and Tarasov [98] for details.

Observe that $A_m(u)$ is the Capelli determinant (2.21) for the Lie algebra $\mathfrak{gl}_m$. Therefore, its coefficients a_{mi} defined by

$$A_m(u) = u^m + a_{m1}\, u^{m-1} + \cdots + a_{mm}$$

are generators of the center of the enveloping algebra $\mathrm{U}(\mathfrak{gl}_m)$. All together the elements a_{mi} with $1 \leq i \leq m \leq n$ generate a commutative subalgebra $\mathcal{A}_n$ of $\mathrm{U}(\mathfrak{gl}_n)$ which is called the *Gelfand–Tsetlin subalgebra*. By (2.39), the basis vectors ξ_Λ are simultaneous eigenvectors for the elements of the subalgebra $\mathcal{A}_n$. Introduce the corresponding eigenvalues of the generators a_{mi} by

$$a_{mi}\, \xi_\Lambda = \alpha_{mi}(\Lambda)\, \xi_\Lambda. \tag{2.41}$$

Thus, $\alpha_{mi}(\Lambda)$ is the i-th elementary symmetric polynomial in $l_{m1}, \dots, l_{mm}$.

Proposition 2.22 *For any pattern Λ associated with λ, the one-dimensional subspace of $L(\lambda)$ spanned by the basis vector ξ_Λ is uniquely determined by the set of eigenvalues $\{\alpha_{mi}(\Lambda)\}$.*

Bibliographical notes

The explicit formulas for the lowering and raising operators (2.9) and (2.10) first appeared in Nagel and Moshinsky [95]; see also Hou Pei-yu [52] and Zhelobenko [145]. The derivation of the Gelfand–Tsetlin formulas outlined in Section 2.1 follows Zhelobenko [145] and Asherova, Smirnov and Tolstoy [2]. The extremal projectors were originally introduced by Asherova, Smirnov and Tolstoy [1] (see also [3]). In a subsequent paper [2] the projectors were used to construct the lowering operators and derive the relations between them. A systematic study of the extremal projectors and the corresponding Mickelsson algebras was undertaken by Zhelobenko: a detailed exposition is given in his paper [150] and book [151]. The application to the Gelfand–Tsetlin formulas is contained in his paper [148]. Section 2.2 is a brief outline of the general results which are used in the basis constructions.

The first proof of Theorem 2.11 was given by van den Hombergh [51] as an answer to the question posed by Mickelsson [81]. A derivation of the relations in the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{gl}_n, \mathfrak{gl}_m)$ with the use of the Capelli-type determinants is contained in the author's paper [90]. A proof of the formulas (2.23) and (2.24) is also given there. The results of Section 2.4 are due to Gould [42, 43, 44]. The characteristic identity (2.32) was proved by Green [48]. The significance of the

Wigner coefficients in mathematical physics is discussed in the book by Biedenharn and Louck [8]. The definition (2.35) of the quantum minors is inspired by the theory of "quantum" algebras called the *Yangians*; see [91, 93] for a review of the theory. The polynomials $A_m(u)$, $B_m(u)$ and $C_m(u)$ are essentially the images of the *Drinfeld generators* of the Yangian $\mathrm{Y}(n)$ under the evaluation homomorphism to the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. The quantum minor presentation of the lowering operators (2.38) is due to the author [85]; see also [90]. The construction of the Gelfand–Tsetlin basis vectors κ_Λ with the use of the Drinfeld generators (Theorem 2.21) was devised by Nazarov and Tarasov [98].

Analogs of the extremal projector were given by Tolstoy [128, 129, 130, 131, 132] for a wide class of Lie (super)algebras and their quantized enveloping algebras. The corresponding super and quantum versions of the Mickelsson–Zhelobenko algebras are studied in [130, 131, 132]. An alternative "tensor formula" for the extremal projector was provided by Tolstoy and Draayer [133]. The techniques of extremal projectors were applied by Khoroshkin and Tolstoy [60] for calculation of the universal R-matrices for quantized enveloping algebras. A basis of Gelfand–Tsetlin type for representations of the exceptional Lie algebra G_2 was constructed by Sviridov, Smirnov and Tolstoy [122, 123].

Bases of Gelfand–Tsetlin type have been constructed for representations of various types of algebras. For the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ such bases were constructed by Jimbo [58], Ueno, Takebayashi and Shibukawa [135], Nazarov and Tarasov [98], Tolstoy [131]. The results of [98] include q-analogs of Theorems 2.20 and 2.21, while [131] contains matrix element formulas for the generators corresponding to arbitrary roots. Gelfand–Tsetlin bases for 'generic' representations of the Yangian $\mathrm{Y}(n)$ were constructed in [85]. Theorem 2.20 was proved there in a more general context of representations of the *Yangian of level p* for $\mathfrak{gl}_n$ which was previously introduced by Cherednik [17]. In particular, the enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ coincides with the Yangian of level 1. A more general class of the *tame* Yangian modules was introduced and explicitly constructed by Nazarov and Tarasov [99] via the *trapezium* or *skew* analogs of the Gelfand–Tsetlin patterns. Their approach was motivated by the so-called *centralizer construction* devised by Olshanski [103, 105, 106] and also employed by Cherednik [16, 17]. Basis vectors in the tame Yangian modules are characterized in a way similar to Proposition 2.22. The skew Yangian modules were also studied in [90] with the use of the quantum Sylvester theorem and the Mickelsson algebras.

The center of $\mathrm{U}(\mathfrak{gl}_n)$ possesses several natural families of generators and so does the Gelfand–Tsetlin subalgebra $\mathcal{A}_n$. The corresponding eigenvalues in $L(\lambda)$ are known explicitly; see, e.g., [91] for a review. An alternative description of $\mathcal{A}_n$ was given by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [39, Section 7.3] as an application of their theory of noncommutative symmetric functions and quasi-determinants.

The combinatorics of the skew Gelfand–Tsetlin patterns is employed by Berenstein and Zelevinsky [7] to obtain multiplicity formulas for the skew representations of $\mathfrak{gl}_n$. Applications to continuous piecewise linear actions of the symmetric group were found by Kirillov and Berenstein [65].

The explicit realization of irreducible finite-dimensional representations of $\mathfrak{gl}_n$ via the Gelfand–Tsetlin bases has important applications in the representation theory of the quantum affine algebras and Yangians. In particular, Theorem 2.20 and its Yangian analog [85] are crucial in the proof of the irreducibility criterion of the tensor products of the Yangian evaluation modules (a generalization to $\mathfrak{gl}_n$ of Theorem 3.8 below); see [92].

Analogs of the Gelfand–Tsetlin bases for representations of some Lie superalgebras were given by Ottoson [108, 109], Palev and Tolstoy [110], Tolstoy, Istomina and Smirnov [134].

The explicit formulas of Theorem 2.3 make it possible to define a class of infinite-dimensional representations of $\mathfrak{gl}_n$ by altering the inequalities (2.4). Families of such representations were introduced by Gelfand and Graev [38]. However, as was later observed by Lemire and Patera [69], some necessary conditions were missing in [38] so that only a part of those families actually provides representations. More general theory of the so-called *Gelfand–Tsetlin modules* is developed by Drozd, Futorny and Ovsienko [28, 29, 30, 31] and Mazorchuk [76, 77]. The starting point of the theory is to axiomatize the property of the basis vectors (2.41) and to consider the module generated by an eigenvector for the Gelfand–Tsetlin subalgebra with a given arbitrary sets of eigenvalues $\{\alpha_{mi}\}$. Some q-analogs of such modules were constructed by Mazorchuk and Turowska [79].

The formulas of Theorem 2.3 were applied by Olshanski [102, 104] to study unitary representations of the pseudo-unitary groups $U(p,q)$. In particular, he classified all irreducible unitarizable highest weight representations of the Lie algebra $\mathfrak{u}(p,q)$ [102]. This work was extended by the author to a family of the Enright–Varadarajan modules over $\mathfrak{u}(p,q)$ [84]. Analogs of the Gelfand–Tsetlin bases for the unitary highest weight modules were constructed in [83].

Applications of the Gelfand–Tsetlin bases in mathematical physics are reviewed in the books by Barut and Rączka [5] and Biedenharn and Louck [8].

3 Weight bases for representations of $\mathfrak{o}_N$ and $\mathfrak{sp}_{2n}$

Let $\mathfrak{g}_n$ denote the rank n simple complex Lie algebra of type B, C, or D. That is,

$$\mathfrak{g}_n = \mathfrak{o}_{2n+1}, \quad \mathfrak{sp}_{2n}, \quad \text{or} \quad \mathfrak{o}_{2n}, \tag{3.1}$$

respectively. Let $V(\lambda)$ denote the finite-dimensional irreducible representation of $\mathfrak{g}_n$ with the highest weight λ. The restriction of $V(\lambda)$ to the subalgebra $\mathfrak{g}_{n-1}$ is not multiplicity-free in general. This means that if $V'(\mu)$ is the finite-dimensional irreducible representation of $\mathfrak{g}_{n-1}$ with the highest weight μ, then the space

$$\mathrm{Hom}_{\mathfrak{g}_{n-1}}(V'(\mu), V(\lambda)) \tag{3.2}$$

need not be one-dimensional. In order to construct a basis of $V(\lambda)$ associated with the chain of subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n$$

we need to construct a basis of the space (3.2) which is isomorphic to the subspace $V(\lambda)^+_\mu$ of $\mathfrak{gl}_{n-1}$-highest vectors of weight μ in $V(\lambda)$. The subspace $V(\lambda)^+_\mu$ possesses a natural structure of a representation of the centralizer $\mathrm{C}_n = \mathrm{U}(\mathfrak{g}_n)^{\mathfrak{g}_{n-1}}$ of $\mathfrak{g}_{n-1}$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$. It was shown by Olshanski [107] that there exist natural homomorphisms

$$\mathrm{C}_1 \leftarrow \mathrm{C}_2 \leftarrow \cdots \leftarrow \mathrm{C}_n \leftarrow \mathrm{C}_{n+1} \leftarrow \cdots .$$

The projective limit of this chain turns out to be an extension of the *twisted Yangian* $\mathrm{Y}^+(2)$ or $\mathrm{Y}^-(2)$, in the orthogonal and symplectic case, respectively; see [107], [93] and [94] for the definition and properties of the twisted Yangians. In particular, there is an algebra homomorphism $\mathrm{Y}^\pm(2) \to \mathrm{C}_n$ which allows one to equip the space $V(\lambda)^+_\mu$ with a $\mathrm{Y}^\pm(2)$-module structure. By the results of [86], the representation $V(\lambda)^+_\mu$ can be extended to a larger algebra, the *Yangian* $\mathrm{Y}(2)$. This is a key fact which allows us to construct a natural basis in each space $V(\lambda)^+_\mu$. In the C and D cases the $\mathrm{Y}(2)$-module $V(\lambda)^+_\mu$ is irreducible while in the B case it is a direct sum of two irreducible submodules. This does not lead, however, to major differences in the constructions, and the final formulas are similar in all the three cases.

The calculations of the matrix elements of the generators of $\mathfrak{g}_n$ are based on the relationship between the twisted Yangian $\mathrm{Y}^\pm(2)$ and the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$; see Section 2.2. We construct an algebra homomorphism $\mathrm{Y}^\pm(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ which allows us to express the generators of the twisted Yangian, as operators in $V(\lambda)^+_\mu$, in terms of the lowering and raising operators.

3.1 Raising and lowering operators

Whenever possible we consider the three cases (3.1) simultaneously, unless otherwise stated. The rows and columns of $2n \times 2n$-matrices will be enumerated by the indices $-n, \dots, -1, 1, \dots, n$, while the rows and columns of $(2n+1) \times (2n+1)$-matrices will be enumerated by the indices $-n, \dots, -1, 0, 1, \dots, n$. Accordingly, the index 0 will usually be skipped in the former case. For $-n \le i, j \le n$ set

$$F_{ij} = E_{ij} - \theta_{ij}\, E_{-j,-i} \tag{3.3}$$

where the E_{ij} are the standard matrix units, and

$$\theta_{ij} = \begin{cases} 1 & \text{in the orthogonal case,} \\ \operatorname{sgn} i \cdot \operatorname{sgn} j & \text{in the symplectic case.} \end{cases} \tag{3.4}$$

The matrices F_{ij} span the Lie algebra $\mathfrak{g}_n$. The subalgebra $\mathfrak{g}_{n-1}$ is spanned by the elements (3.3) with the indices i, j running over the set $\{-n+1, \dots, n-1\}$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in $\mathfrak{g}_n$. The elements $F_{11}, \dots, F_{nn}$ form a basis of $\mathfrak{h}$.

The finite-dimensional irreducible representations of $\mathfrak{g}_n$ are in a one-to-one correspondence with n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ where the numbers λ_i satisfy the conditions

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-1, \tag{3.5}$$

and

$$\begin{aligned} -2\,\lambda_1 &\in \mathbb{Z}_+ && \text{for} \quad \mathfrak{g}_n = \mathfrak{o}_{2n+1}, \\ -\lambda_1 &\in \mathbb{Z}_+ && \text{for} \quad \mathfrak{g}_n = \mathfrak{sp}_{2n}, \\ -\lambda_1 - \lambda_2 &\in \mathbb{Z}_+ && \text{for} \quad \mathfrak{g}_n = \mathfrak{o}_{2n}. \end{aligned} \tag{3.6}$$

Such an n-tuple λ is called the *highest weight* of the corresponding representation which we shall denote by $V(\lambda)$. It contains a unique, up to a constant factor, nonzero vector ξ (the *highest vector*) such that

$$\begin{aligned} F_{ii}\,\xi &= \lambda_i\,\xi && \text{for} \quad i = 1, \dots, n, \\ F_{ij}\,\xi &= 0 && \text{for} \quad -n \leq i < j \leq n. \end{aligned}$$

Denote by $V(\lambda)^+$ the subspace of $\mathfrak{g}_{n-1}$-highest vectors in $V(\lambda)$:

$$V(\lambda)^+ = \{\eta \in V(\lambda) \mid F_{ij}\,\eta = 0, \qquad -n < i < j < n\}.$$

Given a $\mathfrak{g}_{n-1}$-weight $\mu = (\mu_1, \dots, \mu_{n-1})$ we denote by $V(\lambda)^+_\mu$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)^+_\mu = \{\eta \in V(\lambda)^+ \mid F_{ii}\,\eta = \mu_i\,\eta, \qquad i = 1, \dots, n-1\}.$$

Consider the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ introduced in Section 2.2. Let $p = p_{n-1}$ be the extremal projector for the Lie algebra $\mathfrak{g}_{n-1}$. It satisfies the conditions

$$F_{ij}\,p = p\,F_{ji} = 0 \qquad \text{for} \quad -n < i < j < n.$$

By Theorem 2.10, the elements

$$F_{nn}, \qquad pF_{ia}, \qquad a = -n, n, \quad i = -n+1, \dots, n-1 \tag{3.7}$$

are generators of $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ in the orthogonal case. In the symplectic case, the algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ is generated by the elements (3.7) together with $F_{n,-n}$ and $F_{-n,n}$. To write down explicit formulas for the generators, introduce the numbers ρ_i, where $i = 1, \dots, n$, by

$$\rho_i = \begin{cases} -i + 1/2 & \text{for} \quad \mathfrak{g}_n = \mathfrak{o}_{2n+1}, \\ -i & \text{for} \quad \mathfrak{g}_n = \mathfrak{sp}_{2n}, \\ -i + 1 & \text{for} \quad \mathfrak{g}_n = \mathfrak{o}_{2n}. \end{cases}$$

The numbers $-\rho_i$ are coordinates of the half-sum of positive roots with respect to the upper triangular Borel subalgebra. Now set

$$f_i = F_{ii} + \rho_i, \qquad f_{-i} = -f_i$$

for $i = 1, \dots, n$. Moreover, in the case of $\mathfrak{o}_{2n+1}$ also set $f_0 = -1/2$. The generators pF_{ia} can be given by a uniform expression in all the three cases. Let $a \in \{-n, n\}$ and $i \in \{-n+1, \dots, n-1\}$. Then we have modulo the ideal J',

$$pF_{ia} = F_{ia} + \sum_{i > i_1 > \cdots > i_s > -n} F_{ii_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s a} \frac{1}{(f_i - f_{i_1}) \cdots (f_i - f_{i_s})}, \tag{3.8}$$

summed over $s \geq 1$. It will be convenient to work with normalized generators of $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$. Set

$$z_{ia} = pF_{ia}\, (f_i - f_{i-1}) \cdots (f_i - f_{-n+1})$$

in the B, C cases, and

$$z_{ia} = pF_{ia} (f_i - f_{i-1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{-n+1})$$

in the D case, where the hat indicates the factor to be omitted if it occurs. We shall also use the elements z_{ai} defined by

$$z_{ai} = (-1)^{n-i}\, z_{-i,-a} \qquad \text{and} \qquad z_{ai} = (-1)^{n-i} \operatorname{sgn} a \cdot z_{-i,-a},$$

in the orthogonal and symplectic case, respectively. The elements z_{ia} satisfy some quadratic relations which can be shown to be the defining relations of the algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$. In particular, we have for all $a, b \in \{-n, n\}$ and $i + j \neq 0$,

$$z_{ia} z_{jb} + z_{ja} z_{ib} (f_i - f_j - 1) = z_{ib} z_{ja} (f_i - f_j). \tag{3.9}$$

Thus, z_{ia} and z_{ja} commute for $i + j \neq 0$. Also, z_{ia} and z_{ib} commute for $i \neq 0$ and all values of a and b. Analogs of the relation (2.20) in the algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ can be explicitly written down as well. However, we shall avoid using them in a way similar to the proof of Lemma 2.13.

The elements z_{ia} naturally act in the space $V(\lambda)^+$ by *raising* or *lowering* the weights. We have for $i = 1, \dots, n-1$:

$$z_{ia} : V(\lambda)^+_\mu \to V(\lambda)^+_{\mu+\delta_i}, \qquad z_{ai} : V(\lambda)^+_\mu \to V(\lambda)^+_{\mu-\delta_i},$$

where $\mu \pm \delta_i$ is obtained from μ by replacing μ_i with $\mu_i \pm 1$. In the B case the operators z_{0a} preserve each subspace $V(\lambda)^+_\mu$.

We shall need the following element which can be checked to belong to the normalizer $\mathrm{Norm}\, \mathrm{J}'$, and so it can be regarded as an element of the algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$:

$$z_{n,-n} = \sum_{n>i_1>\cdots>i_s>-n} F_{ni_1} F_{i_1 i_2} \cdots F_{i_s,-n} \, (f_n - f_{j_1}) \cdots (f_n - f_{j_k})$$

in the B, C cases, and

$$z_{n,-n} = \sum_{n>i_1>\cdots>i_s>-n} F_{ni_1} F_{i_1 i_2} \cdots F_{i_s,-n} \, \frac{(f_n - f_{j_1}) \cdots (f_n - f_{j_k})}{2 f_n}$$

in the D case, where $s = 0, 1, \dots$ and $\{j_1, \dots, j_k\}$ is the complement to the subset $\{i_1, \dots, i_s\}$ in $\{-n+1, \dots, n-1\}$. The following is a counterpart of Lemma 2.14 and is crucial in the calculation of the matrix elements of the generators in the bases.

Lemma 3.1 *For $a \in \{-n, n\}$ we have*

$$F_{n-1,a} = \sum_{i=-n+1}^{n-1} z_{n-1,i} \, z_{ia} \frac{1}{(f_i - f_{-n+1}) \cdots \wedge_i \cdots (f_i - f_{n-1})}$$

in the B, C cases, and

$$F_{n-1,a} = \sum_{i=-n+1}^{n-1} z_{n-1,i} \, z_{ia} \frac{1}{(f_i - f_{-n+1}) \cdots \wedge_{-i,i} \cdots (f_i - f_{n-1})}$$

in the D case, where $z_{n-1,n-1} = 1$ and the equalities are considered in $\mathrm{U}'(\mathfrak{g}_n)$ modulo the ideal J'. The wedge indicates the indices to be skipped.

In order to write down the basis vectors, introduce the interpolation polynomials $Z_{n,-n}(u)$ with coefficients in the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ by

$$Z_{n,-n}(u) = \sum_{i=1}^{n} z_{ni} z_{i,-n} \prod_{j=1,\, j \neq i}^{n} \frac{u^2 - g_j^2}{g_i^2 - g_j^2} \tag{3.10}$$

in the B, C cases, and

$$Z_{n,-n}(u) = \sum_{i=1}^{n-1} z_{ni} z_{i,-n} \prod_{j=1,\, j \neq i}^{n-1} \frac{u^2 - g_j^2}{g_i^2 - g_j^2} \tag{3.11}$$

in the D case, where $g_i = f_i + 1/2$. Accordingly, we have

$$Z_{n,-n}(g_i) = z_{ni} z_{i,-n} \tag{3.12}$$

with the agreement that when u is evaluated in $\mathrm{U}(\mathfrak{h})$, the coefficients of the polynomial $Z_{n,-n}(u)$ are written to the left of the powers of u, as appears in the formulas (3.10) and (3.11).

3.2 Branching rules, patterns and basis vectors

The restriction of $V(\lambda)$ to the subalgebra $\mathfrak{g}_{n-1}$ is given by

$$V(\lambda)|_{\mathfrak{g}_{n-1}} \simeq \bigoplus_{\mu} c(\mu)\, V'(\mu),$$

where $V'(\mu)$ is the irreducible finite-dimensional representation of $\mathfrak{g}_{n-1}$ with the highest weight μ. The multiplicity $c(\mu)$ coincides with the dimension of the space $V(\lambda)^+_\mu$, and its exact value is found from the Zhelobenko branching rules [144]. We formulate them separately for each case recalling the conditions (3.5) and (3.6) on the highest weight λ. In the formulas below we use the notation

$$l_i = \lambda_i + \rho_i + 1/2, \qquad \gamma_i = \nu_i + \rho_i + 1/2,$$

where the ν_i are the parameters defined in the branching rules.

A parameterization of basis vectors in $V(\lambda)$ is obtained by applying the branching rules to its subsequent restrictions to the subalgebras of the chain

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n.$$

This leads to the definition of the Gelfand–Tsetlin patterns for the B, C and D types. Then we give formulas for the basis vectors of the representation $V(\lambda)$. We use the notation

$$l_{ki} = \lambda_{ki} + \rho_i + 1/2, \qquad l'_{ki} = \lambda'_{ki} + \rho_i + 1/2,$$

where the λ_{ki} and λ'_{ki} are the entries of the patterns defined below.

B type case. The multiplicity $c(\mu)$ equals the number of n-tuples $(\nu'_1, \nu_2, \dots, \nu_n)$ satisfying the inequalities

$$\begin{aligned} &-\lambda_1 \geq \nu'_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_{n-1} \geq \nu_n \geq \lambda_n, \\ &-\mu_1 \geq \nu'_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_{n-1} \geq \mu_{n-1} \geq \nu_n \end{aligned}$$

with ν'_1 and all the ν_i being simultaneously integers or half-integers together with the λ_i. Equivalently, $c(\mu)$ equals the number of $(n+1)$-tuples $\nu = (\sigma, \nu_1, \dots, \nu_n)$, with the entries given by

$$(\sigma, \nu_1) = \begin{cases} (0, \nu'_1) & \text{if } \nu'_1 \leq 0, \\ (1, -\nu'_1) & \text{if } \nu'_1 > 0. \end{cases}$$

Lemma 3.2 *The vectors*

$$\xi_\nu = z_{n0}^{\sigma} \prod_{i=1}^{n-1} z_{ni}^{\nu_i-\mu_i} z_{i,-n}^{\nu_i-\lambda_i} \cdot \prod_{k=l_n}^{\gamma_n-1} Z_{n,-n}(k)\,\xi$$

form a basis of the space $V(\lambda)^+_\mu$.

Define the B *type pattern* Λ associated with λ as an array of the form

$$\begin{array}{ccccccccccc}
\sigma_n & & \lambda_{n1} & & \lambda_{n2} & & & \cdots & & & \lambda_{nn} \\
& \lambda'_{n1} & & \lambda'_{n2} & & & \cdots & & & \lambda'_{nn} & \\
\sigma_{n-1} & & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & & & \\
& \lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & & & \\
\cdots & & \cdots & & \cdots & & & & & & \\
\sigma_1 & & \lambda_{11} & & & & & & & & \\
& \lambda'_{11} & & & & & & & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \dots, \lambda_{nn})$, each σ_k is 0 or 1, the remaining entries are all non-positive integers or non-positive half-integers together with the λ_i, and the following inequalities hold

$$\lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \lambda_{k2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk}$$

for $k = 1, \dots, n$, and

$$\lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq \lambda_{k-1,2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk}$$

for $k = 2, \dots, n$. In addition, in the case of the integer λ_i the condition

$$\lambda'_{k1} \leq -1 \qquad \text{if} \quad \sigma_k = 1$$

should hold for all $k = 1, \dots, n$.

Theorem 3.3 *The vectors*

$$\xi_\Lambda = \prod_{k=1,\dots,n}^{\rightarrow} \left(z_{k0}^{\sigma_k} \cdot \prod_{i=1}^{k-1} z_{ki}^{\lambda'_{ki}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{ki}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{kk}-1} Z_{k,-k}(j) \right) \xi$$

parametrized by the patterns Λ *form a basis of the representation* $V(\lambda)$.

C type case. The multiplicity $c(\mu)$ equals the number of n-tuples of integers $\nu = (\nu_1, \dots, \nu_n)$ satisfying the inequalities

$$\begin{aligned}
&0 \geq \nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_{n-1} \geq \nu_n \geq \lambda_n, \\
&0 \geq \nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_{n-1} \geq \mu_{n-1} \geq \nu_n.
\end{aligned} \tag{3.13}$$

Lemma 3.4 *The vectors*

$$\xi_\nu = \prod_{i=1}^{n-1} z_{ni}^{\nu_i-\mu_i} z_{i,-n}^{\nu_i-\lambda_i} \cdot \prod_{k=l_n}^{\gamma_n-1} Z_{n,-n}(k)\,\xi$$

form a basis of the space $V(\lambda)^+_\mu$.

Define the C *type pattern* Λ associated with λ as an array of the form

$$\begin{array}{cccccccc}
 & \lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
\lambda'_{n1} & & \lambda'_{n2} & & \cdots & & \lambda'_{nn} & \\
 & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & \\
\lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & \\
 & \cdots & & \cdots & & & & \\
 & \lambda_{11} & & & & & & \\
\lambda'_{11} & & & & & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \dots, \lambda_{nn})$, the remaining entries are all non-positive integers and the following inequalities hold

$$0 \geq \lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \lambda_{k2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk}$$

for $k = 1, \dots, n$, and

$$0 \geq \lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq \lambda_{k-1,2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk}$$

for $k = 2, \dots, n$.

Theorem 3.5 *The vectors*

$$\xi_\Lambda = \overrightarrow{\prod_{k=1,\dots,n}} \left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{ki}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{ki}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{kk}-1} Z_{k,-k}(j) \right) \xi$$

parametrized by the patterns Λ *form a basis of the representation* $V(\lambda)$.

D type case. The multiplicity $c(\mu)$ equals the number of $(n-1)$-tuples $\nu = (\nu_1, \dots, \nu_{n-1})$ satisfying the inequalities

$$\begin{aligned}
-|\lambda_1| &\geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \nu_{n-1} \geq \lambda_n, \\
-|\mu_1| &\geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots \geq \mu_{n-1} \geq \nu_{n-1}
\end{aligned}$$

with all the ν_i being simultaneously integers or half-integers together with the λ_i. Set $\nu_0 = \max\{\lambda_1, \mu_1\}$.

Lemma 3.6 *The vectors*

$$\xi_\nu = \prod_{i=1}^{n-1} z_{ni}^{\nu_{i-1}-\mu_i} z_{i,-n}^{\nu_{i-1}-\lambda_i} \cdot \prod_{k=l_n}^{\gamma_{n-1}-2} Z_{n,-n}(k)\, \xi$$

form a basis of the space $V(\lambda)^+_\mu$.

Define the *D type pattern* Λ associated with λ as an array of the form

$$\begin{array}{cccccccccc}
\lambda_{n1} & & \lambda_{n2} & & & \cdots & & & & \lambda_{nn} \\
& \lambda'_{n-1,1} & & & \cdots & & & \lambda'_{n-1,n-1} & & \\
\lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & & & & \\
& & \cdots & \cdots & & & & & & \\
\lambda_{21} & & \lambda_{22} & & & & & & & \\
& \lambda'_{11} & & & & & & & & \\
\lambda_{11} & & & & & & & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \dots, \lambda_{nn})$, the remaining entries are all non-positive integers or non-positive half-integers together with the λ_i, and the following inequalities hold

$$\begin{aligned}
-|\lambda_{k1}| \geq \lambda'_{k-1,1} \geq \lambda_{k2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k,k-1} \geq \lambda'_{k-1,k-1} \geq \lambda_{kk}, \\
-|\lambda_{k-1,1}| \geq \lambda'_{k-1,1} \geq \lambda_{k-1,2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k-1,k-1} \geq \lambda'_{k-1,k-1}
\end{aligned}$$

for $k = 2, \dots, n$. Set $\lambda'_{k-1,0} = \max\{\lambda_{k1}, \lambda_{k-1,1}\}$.

Theorem 3.7 *The vectors*

$$\xi_\Lambda = \prod_{k=2,\dots,n}^{\rightarrow} \left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{k-1,i-1}-\lambda_{k-1,i}} z_{i,-k}^{\lambda'_{k-1,i-1}-\lambda_{ki}} \cdot \prod_{j=l_{kk}}^{l'_{k-1,k-1}-2} Z_{k,-k}(j) \right) \xi$$

parametrized by the patterns Λ *form a basis of the representation* $V(\lambda)$.

Proofs of Theorems 3.3, 3.5 and 3.7 will be outlined in the next two sections. These are based on the application of the representation theory of the twisted Yangians. Clearly, due to the branching rules, it is sufficient to construct a basis in the multiplicity space $V(\lambda)^+_\mu$.

3.3 Yangians and their representations

We start by introducing the *Yangian* $\mathrm{Y}(2)$ for the Lie algebra $\mathfrak{gl}_2$. In what follows it will be convenient to use the indices $-n, n$ to enumerate the rows and columns of 2×2-matrices. The Yangian $\mathrm{Y}(2)$ is the complex associative algebra with the generators $t_{ab}^{(1)}, t_{ab}^{(2)}, \dots$ where $a,b\in\{-n,n\}$, and the defining relations

$$(u-v)\,[t_{ab}(u),t_{cd}(v)] = t_{cb}(u)t_{ad}(v) - t_{cb}(v)t_{ad}(u), \tag{3.14}$$

where

$$t_{ab}(u) = \delta_{ab} + t_{ab}^{(1)}u^{-1} + t_{ab}^{(2)}u^{-2} + \dots \in \mathrm{Y}(2)[[u^{-1}]].$$

Introduce the series $s_{ab}(u)$, $a,b\in\{-n,n\}$ by

$$s_{ab}(u) = \theta_{nb}\, t_{an}(u)t_{-b,-n}(-u) + \theta_{-n,b}\, t_{a,-n}(u)t_{-b,n}(-u) \tag{3.15}$$

with θ_{ij} defined in (3.4). Write

$$s_{ab}(u) = \delta_{ab} + s_{ab}^{(1)}u^{-1} + s_{ab}^{(2)}u^{-2} + \cdots .$$

The *twisted Yangian* $\mathrm{Y}^{\pm}(2)$ is defined as the subalgebra of $\mathrm{Y}(2)$ generated by the elements $s_{ab}^{(1)}, s_{ab}^{(2)}, \dots$ where $a,b\in\{-n,n\}$. Also, $\mathrm{Y}^{\pm}(2)$ can be viewed as an abstract algebra with generators $s_{ab}^{(r)}$ and quadratic and linear defining relations which have the following form

$$\begin{aligned}(u^2-v^2)\,[s_{ab}(u),s_{cd}(v)] = (u+v)\big(&s_{cb}(u)s_{ad}(v) - s_{cb}(v)s_{ad}(u)\big)\\ &- (u-v)\big(\theta_{c,-b}s_{a,-c}(u)s_{-b,d}(v) - \theta_{a,-d}s_{c,-a}(v)s_{-d,b}(u)\big)\\ &+ \theta_{a,-b}\big(s_{c,-a}(u)s_{-b,d}(v) - s_{c,-a}(v)s_{-b,d}(u)\big)\end{aligned}$$

and

$$\theta_{ab}\, s_{-b,-a}(-u) = s_{ab}(u) \pm \frac{s_{ab}(u)-s_{ab}(-u)}{2u}.$$

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the orthogonal case and the lower sign to the symplectic case. In particular, we have the relation

$$[s_{n,-n}(u), s_{n,-n}(v)] = 0.$$

The Yangian $\mathrm{Y}(2)$ is a Hopf algebra with the coproduct

$$\Delta(t_{ab}(u)) = t_{an}(u)\otimes t_{nb}(u) + t_{a,-n}(u)\otimes t_{-n,b}(u). \tag{3.16}$$

The twisted Yangian $\mathrm{Y}^{\pm}(2)$ is a left coideal in $\mathrm{Y}(2)$ with

$$\Delta(s_{ab}(u)) = \sum_{c,d\in\{-n,n\}} \theta_{bd}\, t_{ac}(u)t_{-b,-d}(-u)\otimes s_{cd}(u). \tag{3.17}$$

Given a pair of complex numbers (α,β) such that $\alpha-\beta\in\mathbb{Z}_+$ we denote by $L(\alpha,\beta)$ the irreducible representation of the Lie algebra $\mathfrak{gl}_2$ with the highest weight (α,β) with respect to the upper triangular Borel subalgebra. Then $\dim L(\alpha,\beta) = \alpha-\beta+1$.

We equip $L(\alpha, \beta)$ with a $\mathrm{Y}(2)$-module structure by using the algebra homomorphism $\mathrm{Y}(2) \to \mathrm{U}(\mathfrak{gl}_2)$ given by

$$t_{ab}(u) \mapsto \delta_{ab} + E_{ab}u^{-1}, \qquad a, b \in \{-n, n\}.$$

The coproduct (3.16) allows us to construct representations of $\mathrm{Y}(2)$ of the form

$$L = L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k). \tag{3.18}$$

Any finite-dimensional irreducible $\mathrm{Y}(2)$-module is isomorphic to a representation of this type twisted by an automorphism of $\mathrm{Y}(2)$ of the form

$$t_{ab}(u) \mapsto (1 + \varphi_1 u^{-1} + \varphi_2 u^{-2} + \cdots)\, t_{ab}(u), \qquad \varphi_i \in \mathbb{C}.$$

There is an explicit irreducibility criterion for the $\mathrm{Y}(2)$-module L. To formulate the result, with each $L(\alpha, \beta)$ associate the *string*

$$S(\alpha, \beta) = \{\beta, \beta + 1, \ldots, \alpha - 1\} \subset \mathbb{C}.$$

We say that two strings S_1 and S_2 are *in general position* if

$$\text{either} \quad S_1 \cup S_2 \quad \text{is not a string, or} \quad S_1 \subseteq S_2, \quad \text{or} \quad S_2 \subseteq S_1.$$

Theorem 3.8 *The representation* (3.18) *of* $\mathrm{Y}(2)$ *is irreducible if and only if the strings* $S(\alpha_i, \beta_i)$, $i = 1, \ldots, k$, *are pairwise in general position.*

Note that the generators $t_{ab}^{(r)}$ with $r > k$ act as zero operators in L. Therefore, the operators $T_{ab}(u) = u^k\, t_{ab}(u)$ are polynomials in u:

$$T_{ab}(u) = \delta_{ab}\, u^k + t_{ab}^{(1)} u^{k-1} + \cdots + t_{ab}^{(k)}. \tag{3.19}$$

Let ξ_i denote the highest vector of the $\mathfrak{gl}_2$-module $L(\alpha_i, \beta_i)$. Suppose that the $\mathrm{Y}(2)$-module L given by (3.18) is irreducible and the strings $S(\alpha_i, \beta_i)$ are pairwise disjoint. Set

$$\eta = \xi_1 \otimes \cdots \otimes \xi_k. \tag{3.20}$$

Then using (3.16) we easily check that η is the highest vector of the $\mathrm{Y}(2)$-module L. That is, η is annihilated by $T_{-n,n}(u)$, and it is an eigenvector for the operators $T_{nn}(u)$ and $T_{-n,-n}(u)$. Explicitly,

$$\begin{aligned} T_{-n,-n}(u)\, \eta &= (u + \alpha_1) \cdots (u + \alpha_k)\, \eta, \\ T_{nn}(u)\, \eta &= (u + \beta_1) \cdots (u + \beta_k)\, \eta. \end{aligned} \tag{3.21}$$

Let a k-tuple $\gamma = (\gamma_1, \ldots, \gamma_k)$ satisfy the conditions: for each i

$$\alpha_i - \gamma_i \in \mathbb{Z}_+, \qquad \gamma_i - \beta_i \in \mathbb{Z}_+. \tag{3.22}$$

Set

$$\eta_\gamma = \prod_{i=1}^{k} T_{n,-n}(-\gamma_i+1)\cdots T_{n,-n}(-\beta_i-1)T_{n,-n}(-\beta_i)\,\eta.$$

The following theorem provides a Gelfand–Tsetlin type basis for representations of the Yangian Y(2) associated with the embedding $\mathrm{Y}(1)\subset \mathrm{Y}(2)$. Here Y(1) is the (commutative) subalgebra of Y(2) generated by the elements $t_{nn}^{(r)}$, $r\geq 1$.

Theorem 3.9 *Let the* Y(2)*-module* L *given by* (3.18) *be irreducible and the strings* $S(\alpha_i,\beta_i)$ *be pairwise disjoint. Then the vectors* η_γ *with* γ *satisfying* (3.22) *form a basis of* L*. Moreover, the generators of* Y(2) *act in this basis by the rule*

$$\begin{aligned} T_{nn}(u)\,\eta_\gamma &= (u+\gamma_1)\cdots(u+\gamma_k)\,\eta_\gamma,\\ T_{n,-n}(-\gamma_i)\,\eta_\gamma &= \eta_{\gamma+\delta_i},\\ T_{-n,n}(-\gamma_i)\,\eta_\gamma &= -\prod_{m=1}^{k}(\alpha_m-\gamma_i+1)(\beta_m-\gamma_i)\,\eta_{\gamma-\delta_i},\\ T_{-n,-n}(u)\,\eta_\gamma &= \prod_{i=1}^{k}\frac{(u+\alpha_i+1)(u+\beta_i)}{u+\gamma_i+1}\,\eta_\gamma\\ &\quad+\prod_{i=1}^{k}\frac{1}{u+\gamma_i+1}\,T_{-n,n}(u)T_{n,-n}(u+1)\,\eta_\gamma. \end{aligned} \tag{3.23}$$

These formulas are derived from the defining relations for the Yangian (3.14) with the use of the *quantum determinant*

$$d(u) = T_{-n,-n}(u+1)\,T_{nn}(u) - T_{n,-n}(u+1)\,T_{-n,n}(u) \tag{3.24}$$

$$= T_{-n,-n}(u)\,T_{nn}(u+1) - T_{-n,n}(u)\,T_{n,-n}(u+1). \tag{3.25}$$

The coefficients of the quantum determinant belong to the center of Y(2) and so, $d(u)$ acts in L as a scalar which can be found by the application of (3.24) to the highest vector η. Indeed, by (3.21)

$$d(u)\,\eta = (u+\alpha_1+1)\cdots(u+\alpha_k+1)(u+\beta_1)\cdots(u+\beta_k)\,\eta.$$

This allows us to derive the last formula in (3.23) from (3.25). The operators $T_{-n,n}(u)$ and $T_{n,-n}(u)$ are polynomials in u of degree $\leq k-1$; see (3.19). Therefore, their action can be found from (3.23) by using the Lagrange interpolation formula.

We can regard (3.18) as a module over the twisted Yangian $\mathrm{Y}^-(2)$ obtained by restriction. Irreducibility criterion of such a module is provided by the following theorem.

Theorem 3.10 *The representation* (3.18) *of* $\mathrm{Y}^-(2)$ *is irreducible if and only if all the strings* $S(\alpha_i,\beta_i)$*,* $S(-\beta_i,-\alpha_i)$*,* $i=1,\dots,k$*, are pairwise in general position.*

The defining relations (3.14) allow us to rewrite the formula (3.15) for $s_{n,-n}(u)$ in the form

$$s_{n,-n}(u) = \frac{u+1/2}{u}\Big(t_{n,-n}(u)t_{nn}(-u) - t_{n,-n}(-u)t_{nn}(u)\Big).$$

Therefore the operator in L defined by

$$S_{n,-n}(u) = \frac{u^{2k}}{u+1/2} s_{n,-n}(u) = \\ \frac{(-1)^k}{u}\Big(T_{n,-n}(u)T_{nn}(-u) - T_{n,-n}(-u)T_{nn}(u)\Big) \tag{3.26}$$

is an even polynomial in u of degree $\leq 2k-2$. Its action in the basis of L provided in Theorem 3.9 is given by

$$S_{n,-n}(\gamma_i)\,\eta_\gamma = 2\prod_{a=1,\,a\neq i}^{k}(-\gamma_i-\gamma_a)\,\eta_{\gamma+\delta_i}, \qquad i=1,\dots,k.$$

We have thus proved the following corollary.

Corollary 3.11 *Suppose that the* $\mathrm{Y}^-(2)$*-module* L *is irreducible and all the strings* $S(\alpha_i,\beta_i)$ *and* $S(-\beta_i,-\alpha_i)$ *are pairwise disjoint.*[5] *Then the vectors*

$$\xi_\gamma = \prod_{i=1}^{k} S_{n,-n}(\gamma_i-1)\cdots S_{n,-n}(\beta_i+1)S_{n,-n}(\beta_i)\,\eta$$

with γ *satisfying* (3.22) *form a basis of* L*.*

Let us now turn to the orthogonal twisted Yangian $\mathrm{Y}^+(2)$. For any $\delta\in\mathbb{C}$ denote by $W(\delta)$ the one-dimensional representation of $\mathrm{Y}^+(2)$ spanned by a vector w such that

$$s_{nn}(u)\,w = \frac{u+\delta}{u+1/2}\,w, \qquad s_{-n,-n}(u)\,w = \frac{u-\delta+1}{u+1/2}\,w,$$

and $s_{a,-a}(u)\,w = 0$ for $a=-n,n$. By (3.17) we can regard the tensor product $L\otimes W(\delta)$ as a representation of $\mathrm{Y}^+(2)$. The representations of $\mathrm{Y}^+(2)$ of this type, and the representations of $\mathrm{Y}^-(2)$ of type (3.18) essentially exhaust all finite-dimensional irreducible representations of $\mathrm{Y}^\pm(2)$ [86].

The following is an analog of Theorem 3.10.

Theorem 3.12 *The representation* $L\otimes W(\delta)$ *of* $\mathrm{Y}^+(2)$ *is irreducible if and only if all the strings* $S(\alpha_i,\beta_i)$, $S(-\beta_i,-\alpha_i)$, $i=1,\dots,k$, *are pairwise in general position and none of them contains* $-\delta$*.*

[5] A part of this condition was erroneously omitted in the formulation of [87, Proposition 4.2] although it is implicit in the proof.

Using the vector space isomorphism

$$L \otimes W(\delta) \to L, \qquad v \otimes w \mapsto v, \qquad v \in L \tag{3.27}$$

we can regard L as a $\mathrm{Y}^+(2)$-module. Accordingly, using the defining relations (3.14) and the coproduct formula (3.17) we can write $s_{n,-n}(u)$, as an operator in L, in the form

$$s_{n,-n}(u) = \frac{u-\delta}{u}\, t_{n,-n}(u) t_{nn}(-u) + \frac{u+\delta}{u}\, t_{n,-n}(-u) t_{nn}(u).$$

Therefore the operator in L defined by

$$S_{n,-n}(u) = u^{2k}\, s_{n,-n}(u) = \frac{(-1)^k}{u}\Big((u-\delta)\, T_{n,-n}(u) T_{nn}(-u) + (u+\delta)\, T_{n,-n}(-u) T_{nn}(u)\Big) \tag{3.28}$$

is an even polynomial in u of degree $\leq 2k-2$. Its action in the basis of L provided in Theorem 3.9 is given by

$$S_{n,-n}(\gamma_i)\, \eta_\gamma = 2(-\delta-\gamma_i) \prod_{a=1,\, a\neq i}^{k} (-\gamma_i - \gamma_a)\, \eta_{\gamma+\delta_i}, \qquad i = 1, \dots, k.$$

We have thus proved the following corollary.

Corollary 3.13 *Suppose that the* $\mathrm{Y}^+(2)$*-module* $L \otimes W(\delta)$ *is irreducible and all the strings* $S(\alpha_i, \beta_i)$ *and* $S(-\beta_i, -\alpha_i)$ *are pairwise disjoint. Then the vectors*

$$\xi_\gamma = \prod_{i=1}^{k} S_{n,-n}(\gamma_i - 1) \cdots S_{n,-n}(\beta_i + 1) S_{n,-n}(\beta_i)\, \eta$$

with γ *satisfying* (3.22) *form a basis of* L.

3.4 Yangian action on the multiplicity space

Now we construct an algebra homomorphism $\mathrm{Y}^\pm(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ and then use it to define an action of $\mathrm{Y}^\pm(2)$ on the multiplicity space $V(\lambda)^+_\mu$.

For $a, b \in \{-n, n\}$ and a complex parameter u introduce the elements $Z_{ab}(u)$ of the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ by

$$Z_{ab}(u) = -\Big(\delta_{ab}(u + \rho_n + \frac{1}{2}) + F_{ab}\Big) \prod_{i=-n+1}^{n-1} (u + g_i) + \sum_{i=-n+1}^{n-1} z_{ai} z_{ib} \prod_{j=-n+1,\, j\neq i}^{n-1} \frac{u + g_j}{g_i - g_j} \tag{3.29}$$

in the B case,

$$Z_{ab}(u) = \Big(\delta_{ab}(u+\rho_n+\frac{1}{2}) + F_{ab}\Big) \prod_{i=-n+1}^{n-1} (u+g_i) - \sum_{i=-n+1}^{n-1} z_{ai} z_{ib} \prod_{j=-n+1,\, j\neq i}^{n-1} \frac{u+g_j}{g_i - g_j} \tag{3.30}$$

in the C case, and

$$Z_{ab}(u) = -\Bigg(\Big(\delta_{ab}(u+\rho_n+\frac{1}{2})+F_{ab}\Big)\prod_{i=-n+1}^{n-1}(u+g_i) \\ -\sum_{i=-n+1}^{n-1} z_{ai}z_{ib}\,(u+g_{-i})\prod_{j=-n+1,\, j\neq\pm i}^{n-1}\frac{u+g_j}{g_i-g_j}\Bigg)\frac{1}{2u+1} \qquad (3.31)$$

in the D case, where $g_i = f_i + 1/2$ for all i. In particular, it can be verified that each $Z_{n,-n}(u)$ coincides with the corresponding interpolation polynomial given in (3.10) or (3.11).

Consider now the three cases separately. We shall assume $\mu_n = -\infty$ in the notation below.

B type case.

Theorem 3.14 (i) *The mapping*

$$s_{ab}(u) \mapsto -u^{-2n}\, Z_{ab}(u), \qquad a, b \in \{-n, n\} \qquad (3.32)$$

defines an algebra homomorphism $\mathrm{Y}^+(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$.

(ii) *The* $\mathrm{Y}^+(2)$*-module* $V(\lambda)^+_\mu$ *defined via the homomorphism* (3.32) *is isomorphic to the direct sum of two irreducible submodules,* $V(\lambda)^+_\mu \simeq U \oplus U'$*, where*

$$U = L(0,\beta_1)\otimes L(\alpha_2,\beta_2)\otimes\cdots\otimes L(\alpha_n,\beta_n)\otimes W(1/2),$$
$$U' = L(-1,\beta_1)\otimes L(\alpha_2,\beta_2)\otimes\cdots\otimes L(\alpha_n,\beta_n)\otimes W(1/2),$$

if the λ_i *are integers* (*it is supposed that* $U' = \{0\}$ *if* $\beta_1 = 0$)*; or*

$$U = L(-1/2,\beta_1)\otimes L(\alpha_2,\beta_2)\otimes\cdots\otimes L(\alpha_n,\beta_n)\otimes W(0),$$
$$U' = L(-1/2,\beta_1)\otimes L(\alpha_2,\beta_2)\otimes\cdots\otimes L(\alpha_n,\beta_n)\otimes W(1),$$

if the λ_i *are half-integers, and the following notation is used*

$$\alpha_i = \min\{\lambda_{i-1},\mu_{i-1}\} - i + 1, \qquad i = 2,\dots,n,$$
$$\beta_i = \max\{\lambda_i,\mu_i\} - i + 1, \qquad i = 1,\dots,n.$$

C type case.

Theorem 3.15 (i) *The mapping*

$$s_{ab}(u) \mapsto (u+1/2)\, u^{-2n}\, Z_{ab}(u), \qquad a, b \in \{-n, n\} \qquad (3.33)$$

defines an algebra homomorphism $\mathrm{Y}^-(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$.

(ii) *The* $\mathrm{Y}^-(2)$*-module* $V(\lambda)^+_\mu$ *defined via the homomorphism* (3.33) *is irreducible and isomorphic to the tensor product*

$$L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_n, \beta_n),$$

where $\alpha_1 = -1/2$ *and*

$$\begin{aligned} \alpha_i &= \min\{\lambda_{i-1}, \mu_{i-1}\} - i + 1/2, \qquad && i = 2, \dots, n, \\ \beta_i &= \max\{\lambda_i, \mu_i\} - i + 1/2, \qquad && i = 1, \dots, n. \end{aligned}$$

D type case.

Theorem 3.16 (i) *The mapping*

$$s_{ab}(u) \mapsto -2\, u^{-2n+2}\, Z_{ab}(u), \qquad a, b \in \{-n, n\} \tag{3.34}$$

defines an algebra homomorphism $\mathrm{Y}^+(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$.

(ii) *The* $\mathrm{Y}^+(2)$*-module* $V(\lambda)^+_\mu$ *defined via the homomorphism* (3.34) *is irreducible and isomorphic to the tensor product*

$$L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_{n-1}, \beta_{n-1}) \otimes W(-\alpha_0),$$

where $\alpha_1 = \min\{-|\lambda_1|, -|\mu_1|\} - 1/2, \quad \alpha_0 = \alpha_1 + |\lambda_1 + \mu_1|,$

$$\begin{aligned} \alpha_i &= \min\{\lambda_i, \mu_i\} - i + 1/2, \qquad && i = 2, \dots, n-1, \\ \beta_i &= \max\{\lambda_{i+1}, \mu_{i+1}\} - i + 1/2, \qquad && i = 1, \dots, n-1. \end{aligned}$$

Outline of the proof. Part (i) of Theorems 3.14–3.16 is verified by using the composition of homomorphisms

$$\mathrm{Y}^\pm(2) \to \mathrm{C}_n \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1}),$$

where C_n is the centralizer $\mathrm{U}(\mathfrak{g}_n)^{\mathfrak{g}_{n-1}}$. The first arrow is the homomorphism provided by the centralizer construction (see [94], [107]) while the second is the natural projection.

By the results of [86], every irreducible finite-dimensional representation of the twisted Yangian is a highest weight representation. It contains a unique, up to a constant factor, vector which is annihilated by $s_{-n,n}(u)$ and which is an eigenvector of $s_{nn}(u)$. The corresponding eigenvalue (the highest weight) uniquely determines the representation. The vectors in $V(\lambda)^+_\mu$ annihilated by $s_{-n,n}(u)$ can be explicitly constructed by using the lowering operators. One of such vectors is given by

$$\xi_\mu = \prod_{i=1}^{n-1} \left(z_{ni}^{\max\{\lambda_i, \mu_i\} - \mu_i}\, z_{i,-n}^{\max\{\lambda_i, \mu_i\} - \lambda_i} \right) \xi,$$

where ξ is the highest vector of $V(\lambda)$. This is the only vector in the C, D cases, while in the B case there is another one defined by

$$\xi'_\mu = z_{n0}\, \xi_\mu.$$

Calculating the eigenvalues of these vectors we conclude that they respectively coincide with the eigenvalues of the tensor product of the highest vectors of the modules $L(\alpha_i, \beta_i)$; see (3.20). $\square$

While keeping λ and μ fixed we let ν run over the values determined by the branching rules; see Section 3.2. Using the homomorphisms of Theorems 3.14–3.16 we conclude from (3.26) and (3.28) that the element $S_{n,-n}(u)$ acts in the representation $V(\lambda)^+_\mu$ precisely as the operator $-Z_{n,-n}(u)$, $Z_{n,-n}(u)$, or $-2Z_{n,-n}(u)$ in the B, C or D cases, respectively. Thus, by Corollaries 3.11 and 3.13, the following vectors ξ_ν form a basis of the space $V(\lambda)^+_\mu$, where

$$\xi_\nu = z_{n0}^\sigma \prod_{i=1}^{n} Z_{n,-n}(\gamma_i - 1) \cdots Z_{n,-n}(\beta_i + 1) Z_{n,-n}(\beta_i)\, \xi_\mu$$

in the B case,

$$\xi_\nu = \prod_{i=1}^{n} Z_{n,-n}(\gamma_i - 1) \cdots Z_{n,-n}(\beta_i + 1) Z_{n,-n}(\beta_i)\, \xi_\mu \qquad (3.35)$$

in the C case, and

$$\xi_\nu = \prod_{i=1}^{n-1} Z_{n,-n}(\gamma_i - 1) \cdots Z_{n,-n}(\beta_i + 1) Z_{n,-n}(\beta_i)\, \xi_\mu$$

in the D case. Applying the interpolation properties of the polynomials $Z_{n,-n}(u)$ we bring the above formulas to the form given in Lemmas 3.2, 3.4 and 3.6, respectively. Clearly, Theorems 3.3, 3.5 and 3.7 follow.

3.5 Calculation of the matrix elements

Without writing down all explicit formulas we shall demonstrate how the matrix elements of the generators of $\mathfrak{g}_n$ in the basis ξ_Λ provided by Theorems 3.3, 3.5 and 3.7 can be calculated. The interested reader is refered to the papers [87, 88, 89] for details. We choose the following generators

$$F_{k-1,-k}, \quad F_{k-1,k}, \qquad k = 1, \dots, n$$

in the B case,

$$F_{k-1,-k}, \qquad k = 2, \dots, n, \qquad \text{and} \qquad F_{-k,k}, \quad F_{k,-k}, \qquad k = 1, \dots, n$$

in the C case, and

$$F_{k-1,-k}, \quad F_{k-1,k}, \qquad k=2,\dots,n, \qquad \text{and} \qquad F_{21}, \quad F_{-2,1}$$

in the D case.

In the symplectic case the elements F_{kk}, $F_{k,-k}$, $F_{-k,k}$ commute with the subalgebra $\mathfrak{g}_{k-1}$ in $\mathrm{U}(\mathfrak{g}_k)$. Therefore, these operators preserve the subspace of $\mathfrak{g}_{k-1}$-highest vectors in $V(\lambda)$. So, it suffices to compute the action of these operators with $k=n$ in the basis $\{\xi_\nu\}$ of the space $V(\lambda)^+_\mu$; see Lemma 3.4. For F_{nn} we immediately get

$$F_{nn}\,\xi_\nu = \left(2\sum_{i=1}^{n}\nu_i - \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n-1}\mu_i\right)\xi_\nu.$$

Further, by (3.35)

$$Z_{n,-n}(\gamma_i)\,\xi_\nu = \xi_{\nu+\delta_i}, \qquad i=1,\dots,n.$$

However, $Z_{n,-n}(u)$ is a polynomial in u^2 of degree $n-1$ with the highest coefficient $F_{n,-n}$. Applying the Lagrange interpolation formula with the interpolation points γ_i, $i=1,\dots,n$ we obtain

$$Z_{n,-n}(u)\,\xi_\nu = \sum_{i=1}^{n}\prod_{a=1,\,a\neq i}^{n}\frac{u^2-\gamma_a^2}{\gamma_i^2-\gamma_a^2}\,\xi_{\nu+\delta_i}.$$

Taking here the coefficient at u^{2n-2} we get

$$F_{n,-n}\,\xi_\nu = \sum_{i=1}^{n}\prod_{a=1,\,a\neq i}^{n}\frac{1}{\gamma_i^2-\gamma_a^2}\,\xi_{\nu+\delta_i}. \tag{3.36}$$

The action of $F_{-n,n}$ is found in a similar way with the use of Theorem 3.9.

In the orthogonal case the action of F_{nn} is found in the same way. However, the elements $F_{n,-n}$ and $F_{-n,n}$ are zero. We shall use second order elements of the enveloping algebra instead. These are given by

$$\Phi_{-a,a} = \frac{1}{2}\sum_{i=-n+1}^{n-1}F_{-a,i}F_{ia}$$

with $a\in\{-n,n\}$. The elements $\Phi_{-a,a}$ commute with the subalgebra $\mathfrak{g}_{n-1}$ so that, like in the symplectic case, they preserve the space $V(\lambda)^+_\mu$ and their action in the basis $\{\xi_\nu\}$ is given by formulas similar to those for $F_{-a,a}$.

The calculation of the matrix elements of the generators $F_{k-1,-k}$ is similar in all the three cases. We may assume $k=n$. The operator $F_{n-1,-n}$ preserves the subspace of $\mathfrak{g}_{n-2}$ highest vectors in $V(\lambda)$. Consider the symplectic case as an example. Suppose that μ' is a fixed $\mathfrak{g}_{n-2}$ highest weight, ν' is an $(n-1)$-tuple of integers such that the inequalities (3.13) are satisfied with λ, ν, μ respectively replaced by μ, ν', μ', and set $\gamma'_i = \nu'_i + \rho_i + 1/2$. It suffices to calculate the action of $F_{n-1,-n}$ on the basis vectors of the form

$$\xi_{\nu\mu\nu'} = X_{\mu\nu'}\, \xi_{\nu\mu},$$

where $\xi_{\nu\mu} = \xi_\nu$ and $X_{\mu\nu'}$ denotes the operator

$$X_{\mu\nu'} = \prod_{i=1}^{n-2} z_{n-1,i}^{\nu'_i-\mu'_i}\, z_{n-1,-i}^{\nu'_i-\mu_i} \cdot \prod_{a=m_{n-1}}^{\gamma'_{n-1}-1} Z_{n-1,-n+1}(a),$$

where we have used the notation $m_i = \mu_i + \rho_i + 1/2$. The operator $F_{n-1,-n}$ is permutable with the elements $z_{n-1,i}$ and $Z_{n-1,-n+1}(u)$. Hence, we can write

$$F_{n-1,-n}\, \xi_{\nu\mu\nu'} = X_{\mu\nu'}\, F_{n-1,-n}\, \xi_{\nu\mu}.$$

Now we apply Lemma 3.1. It remains to calculate $z_{ni}\, \xi_{\nu\mu}$ and $X_{\mu\nu'}\, z_{n-1,-i}$. Using the relations between the elements of the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ given in (3.9), we find that

$$z_{ni}\, \xi_{\nu\mu} = \xi_{\nu,\mu-\delta_i}$$

if $i > 0$. Otherwise, if $i = -j$ with positive j, write

$$z_{n,-j}\, \xi_{\nu\mu} = z_{n,-j} z_{nj}\, \xi_{\nu,\mu+\delta_j} = Z_{n,-n}(m_j)\, \xi_{\nu,\mu+\delta_j}, \tag{3.37}$$

where we have used the interpolation properties (3.12) of the polynomials $Z_{n,-n}(u)$. Finally, we use the expression (3.35) of the basis vectors and Theorem 3.9 to present (3.37) as a linear combination of basis vectors. The same argument applies to calculate $X_{\mu\nu'}\, z_{n-1,-i}$.

The final formulas for the matrix elements of the generators $F_{n-1,-n}$ in all the three cases are given by multiplicative expressions in the entries of the patterns which exhibit some similarity to the formulas of Theorem 2.3.

In the orthogonal case we also need to find the action of the generators $F_{n-1,n}$. Unlike the case of the generators $F_{n-1,-n}$, the corresponding matrix elements will be given by certain combinations of multiplicative expressions which do not seem to be possible to bring to a product form. There are two alternative ways to calculate these combinations which we briefly outline below. First, as in the previous calculation, we can write

$$F_{n-1,n}\, \xi_{\nu\mu\nu'} = X_{\mu\nu'}\, F_{n-1,n}\, \xi_{\nu\mu}.$$

Applying again Lemma 3.1, we come to the calculation of $z_{in}\, \xi_{\nu\mu}$. This time the interpolation property of $Z_{-n,-n}(u)$ (see (3.29) and (3.31)) allows us to write, e.g., for $i > 0$

$$z_{in}\, \xi_{\nu\mu} = z_{in} z_{ni}\, \xi_{\nu,\mu+\delta_i} = z_{-n,-i} z_{-i,-n}\, \xi_{\nu,\mu+\delta_i} = Z_{-n,-n}(m_i)\, \xi_{\nu,\mu+\delta_i}.$$

Now, as $Z_{-n,-n}(u)$ is, up to a multiple, the image of $S_{-n,-n}(u)$ under the homomorphism $\mathrm{Y}^+(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$, we can express this operator in terms of the Yangian operators $T_{ab}(u)$ and then apply Theorem 3.9 to calculate its action.

Alternatively, the generator $F_{n-1,n}$ can be written modulo the left ideal J' of $\mathrm{U}'(\mathfrak{g}_n)$ as

$$F_{n-1,n} = \Phi_{n-1,-n}(2)\, \Phi_{-n,n} - \Phi_{-n,n}\Phi_{n-1,-n}(0), \tag{3.38}$$

where

$$\Phi_{n-1,-n}(u) = \sum_{i=-n+1}^{n-1} z_{n-1,i}\, z_{i,-n} \prod_{a=-n+1,\, a\neq i}^{n-1} \frac{1}{f_i - f_a} \cdot \frac{1}{u + f_i + F_{nn}} \tag{3.39}$$

in the B case, and

$$\Phi_{n-1,-n}(u) = \sum_{i=-n+1}^{n-1} z_{n-1,i}\, z_{i,-n} \prod_{a=-n+1,\, a\neq \pm i}^{n-1} \frac{1}{f_i - f_a} \cdot \frac{1}{u + f_i + F_{nn}} \tag{3.40}$$

in the D case. The action of $\Phi_{n-1,-n}(u)$ is found exactly as that of $F_{n-1,-n}$ and the matrix elements have a similar multiplicative form. Note, however, that the formula (3.38), regarded as the equality of operators acting on $V(\lambda)^+$, is only valid provided the denominators in (3.39) or (3.40) do not vanish. Therefore, in order to use (3.38), we first consider $V(\lambda)$ with 'generic' entries of λ and calculate the matrix elements of $F_{n-1,n}$ as functions in the entries of the patterns Λ. The final explicit formulas can be written in a singularity-free form and they are valid in the general case.

Bibliographical notes

The exposition here is based upon the author's papers [87, 88, 89]. Slight changes in the notation were made in order to present the results in a uniform manner for all the three cases. The branching rules for all classical reductions $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ and $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ are due to Zhelobenko [144]; see also Hegerfeldt [50], King [61], Proctor [116], Okounkov [100]. The lowering operators for the symplectic Lie algebras were first constructed by Mickelsson [80]; see also Bincer [9]. The explicit relations in the algebra $\mathrm{Z}(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2})$ were calculated by Zhelobenko [147].

The algebra $\mathrm{Y}(n)$ was first studied in the work of Faddeev and the St.-Petersburg school in relation with the inverse scattering method; see for instance Takhtajan–Faddeev [125], Kulish–Sklyanin [67]. The term "Yangian" was introduced by Drinfeld in [25]. In that paper he defined the Yangian $\mathrm{Y}(\mathfrak{a})$ for each simple finite-dimensional Lie algebra $\mathfrak{a}$. Finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{a})$ were classified by Drinfeld [26] with the use of a previous work by Tarasov [126, 127]. Theorem 3.9 goes back to this work of Tarasov; see also [85], [99]. The criterion of Theorem 3.8 is due to Chari and Pressley [13]. It can also be deduced from the results of [126, 127];

see [86]. The twisted Yangians were introduced by Olshanski [107]; see also [93]. Their finite-dimensional irreducible representations were classified in the author's paper [86] which, in particular, contains the criteria of Theorems 3.10 and 3.12. For more details on the (twisted) Yangians and their applications in the classical representation theory see the expository papers [93], [91] and the recent work of Nazarov [96, 97] where, in particular, the skew representations of the twisted Yangians were studied.

In some particular cases, bases in $V(\lambda)$ were constructed, e.g., by Wong and Yeh [142], Smirnov and Tolstoy [121].

Weight bases for the fundamental representations of $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$ were independently constructed by Donnelly [21, 22, 23] in a different way. He also demonstrated that the bases of his coincide with those of Theorems 3.3 and 3.5, up to a diagonal equivalence.

4 Gelfand–Tsetlin basis for representations of $\mathfrak{o}_N$

In this section we sketch the construction of the bases proposed originally by Gelfand and Tsetlin in [37]. It is based upon the fact that the restriction $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ is multiplicity-free. This makes the construction similar to the $\mathfrak{gl}_n$ case. We shall be applying the general method of Mickelsson algebras outlined in Section 2.2. In particular, the corresponding branching rules can be derived from Theorem 2.11; cf. Section 2.3.

It will be convenient to change the notation for the elements of the orthogonal Lie algebra $\mathfrak{o}_N$ used in Section 3. We shall now use the standard enumeration of the rows and columns of $N \times N$-matrices by the numbers $\{1, \dots, N\}$. Define the involution of this set of indices by setting $i' = N - i + 1$. The Lie algebra $\mathfrak{o}_N$ is spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'}, \qquad i, j = 1, \dots, N. \tag{4.1}$$

We shall keep the notation $\mathfrak{g}_n$ for $\mathfrak{o}_N$ with $N = 2n + 1$ or $N = 2n$.

The finite-dimensional irreducible representations of $\mathfrak{g}_n$ are now parametrized by n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ where the numbers λ_i satisfy the conditions

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-1, \tag{4.2}$$

and

$$\begin{aligned} 2\,\lambda_n \in \mathbb{Z}_+ \qquad &\text{for} \quad \mathfrak{g}_n = \mathfrak{o}_{2n+1}, \\ \lambda_{n-1} + \lambda_n \in \mathbb{Z}_+ \qquad &\text{for} \quad \mathfrak{g}_n = \mathfrak{o}_{2n}. \end{aligned} \tag{4.3}$$

Such an n-tuple λ is called the *highest weight* of the corresponding representation which we shall denote by $V(\lambda)$. It contains a unique, up to a constant factor, nonzero vector ξ (the *highest vector*) such that

$$F_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad i = 1, \dots, n,$$
$$F_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \le i < j \le N.$$

4.1 Lowering operators for the reduction $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n}$

Taking $N = 2n + 1$ in the definition (4.1), we shall consider $\mathfrak{o}_{2n}$ as the subalgebra of $\mathfrak{o}_{2n+1}$ spanned by the elements (4.1) with $i, j \neq n + 1$. In accordance with the branching rule, the restriction of $V(\lambda)$ to the subalgebra $\mathfrak{o}_{2n}$ is given by

$$V(\lambda)|_{\mathfrak{o}_{2n}} \simeq \bigoplus_{\mu} V'(\mu),$$

where $V'(\mu)$ is the irreducible finite-dimensional representation of $\mathfrak{o}_{2n}$ with the highest weight μ and the sum is taken over the weights μ satisfying the inequalities

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n \ge |\mu_n|, \tag{4.4}$$

with all the μ_i being simultaneously integers or half-integers together with the λ_i.

The elements $F_{n+1,i}$ span the $\mathfrak{o}_{2n}$-invariant complement to $\mathfrak{o}_{2n}$ in $\mathfrak{o}_{2n+1}$. Therefore, by the general theory of Section 2.2, the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n})$ is generated by the elements

$$pF_{n+1,i}, \qquad i = 1, \dots, n, n', \dots, 1', \tag{4.5}$$

where p is the extremal projector for the Lie algebra $\mathfrak{o}_{2n}$. Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the basis of $\mathfrak{h}^*$ dual to the basis $\{F_{11}, \dots, F_{nn}\}$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{o}_{2n}$. Set $\varepsilon_{i'} = -\varepsilon_i$ for $i = 1, \dots, n$. Denote by p_{ij} the element p_α given by (2.13) for the positive root $\alpha = \varepsilon_i - \varepsilon_j$. Choosing an appropriate normal ordering on the positive roots, for any $i = 1, \dots, n$ we can write the elements (4.5) in the form

$$pF_{n+1,i} = p_{i,i+1} \cdots p_{in}\, p_{in'} \cdots p_{i1'}\, F_{n+1,i}, \tag{4.6}$$

where the factor $p_{ii'}$ is skipped in the product. Therefore the right denominator of this fraction is

$$\pi_i = f_{i,i+1} \cdots f_{in}\, f_{in'} \cdots f_{i1'},$$

where

$$f_{ij} = \begin{cases} F_{ii} - F_{jj} + j - i & \text{if} \quad j = 1, \dots, n \\ F_{ii} - F_{jj} + j - i - 2 & \text{if} \quad j = 1', \dots, n'. \end{cases}$$

Hence, the elements $s'_{ni} = pF_{n+1,i}\,\pi_i$ with $i = 1, \dots, n$ belong to the Mickelsson algebra $\mathrm{S}(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n})$. One can verify that they are pairwise commuting.

Denote by $V(\lambda)^+$ the subspace of $\mathfrak{o}_{2n}$-highest vectors in $V(\lambda)$. Given a $\mathfrak{o}_{2n}$-highest weight $\mu = (\mu_1, \dots, \mu_n)$ we denote by $V(\lambda)^+_\mu$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)^+_\mu = \{\eta \in V(\lambda)^+ \mid F_{ii}\,\eta = \mu_i\,\eta, \qquad i = 1, \dots, n\}.$$

By the branching rule, the space $V(\lambda)^+_\mu$ is one-dimensional if the condition (4.4) is satisfied. Otherwise, it is zero.

Theorem 4.1 *Suppose that the inequalities* (4.4) *hold. Then the space* $V(\lambda)^+_\mu$ *is spanned by the vector*

$$s'^{\lambda_1-\mu_1}_{n1} \cdots s'^{\lambda_n-\mu_n}_{nn}\,\xi.$$

4.2 Lowering operators for the reduction $\mathfrak{o}_{2n} \downarrow \mathfrak{o}_{2n-1}$

Taking $N = 2n$ in the definition (4.1), we shall consider $\mathfrak{o}_{2n-1}$ as the subalgebra of $\mathfrak{o}_{2n}$ spanned by the elements (4.1) with $i, j \neq n, n'$ together with

$$\frac{1}{\sqrt{2}}\,(F_{ni} - F_{n'i}), \qquad i = 1, \dots, n-1, (n-1)', \dots, 1'.$$

In accordance with the branching rule, the restriction of $V(\lambda)$ to the subalgebra $\mathfrak{o}_{2n-1}$ is given by

$$V(\lambda)|_{\mathfrak{o}_{2n-1}} \simeq \bigoplus_{\mu} V'(\mu),$$

where $V'(\mu)$ is the irreducible finite-dimensional representation of $\mathfrak{o}_{2n-1}$ with the highest weight μ and the sum is taken over the weights μ satisfying the inequalities

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq |\lambda_n|, \tag{4.7}$$

with all the μ_i being simultaneously integers or half-integers together with the λ_i.

The elements

$$F_{nn}, \qquad F'_{ni} = \frac{1}{\sqrt{2}}\,(F_{ni} + F_{n'i}), \qquad i = 1, \dots, n-1, (n-1)', \dots, 1' \tag{4.8}$$

span the $\mathfrak{o}_{2n-1}$-invariant complement to $\mathfrak{o}_{2n-1}$ in $\mathfrak{o}_{2n}$. Therefore, by the general theory of Section 2.2, the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{o}_{2n}, \mathfrak{o}_{2n-1})$ is generated by the elements

$$pF_{nn}, \qquad pF'_{ni}, \qquad i = 1, \dots, n-1, (n-1)', \dots, 1', \tag{4.9}$$

where p is the extremal projector for the Lie algebra $\mathfrak{o}_{2n-1}$. Let $\{\varepsilon_1, \dots, \varepsilon_{n-1}\}$ be the basis of $\mathfrak{h}^*$ dual to the basis $\{F_{11}, \dots, F_{n-1,n-1}\}$ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{o}_{2n-1}$. Set $\varepsilon_{i'} = -\varepsilon_i$ for $i = 1, \dots, n-1$. Denote by p_{ij} and p_i the elements p_α given by (2.13) for the positive roots $\alpha = \varepsilon_i - \varepsilon_j$ and $\alpha = \varepsilon_i$, respectively. Choosing an appropriate normal ordering on the positive roots, for any $i = 1, \dots, n-1$ we can write the elements (4.9) in the form

$$pF'_{ni} = p_{i,i+1} \cdots p_{i,n-1}\, p_i\, p_{i,(n-1)'} \cdots p_{i1'}\, F'_{ni}, \tag{4.10}$$

where the factor $p_{ii'}$ is skipped in the product. Therefore the right denominator of this fraction is

$$\pi_i = f_{i,i+1} \cdots f_{i,n-1}\, f_i\, f'_i\, f_{i,(n-1)'} \cdots f_{i1'},$$

where

$$f_{ij} = \begin{cases} F_{ii} - F_{jj} + j - i & \text{if} \quad j = 1, \dots, n-1 \\ F_{ii} - F_{jj} + j - i - 2 & \text{if} \quad j = 1', \dots, (n-1)' \end{cases}$$

and $f_i = f'_i - 1 = 2(F_{ii} + n - i)$. Hence, the elements $s_{ni} = pF'_{ni}\, \pi_i$ with $i = 1, \dots, n-1$ belong to the Mickelsson algebra $\mathrm{S}(\mathfrak{o}_{2n}, \mathfrak{o}_{2n-1})$. One can verify that they are pairwise commuting.

Denote by $V(\lambda)^+$ the subspace of $\mathfrak{o}_{2n-1}$-highest vectors in $V(\lambda)$. Given a $\mathfrak{o}_{2n-1}$-highest weight $\mu = (\mu_1, \dots, \mu_{n-1})$ we denote by $V(\lambda)^+_\mu$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)^+_\mu = \{\eta \in V(\lambda)^+ \mid F_{ii}\, \eta = \mu_i\, \eta, \qquad i = 1, \dots, n-1\}.$$

By the branching rule, the space $V(\lambda)^+_\mu$ is one-dimensional if the condition (4.7) is satisfied. Otherwise, it is zero.

Theorem 4.2 *Suppose that the inequalities* (4.7) *hold. Then the space* $V(\lambda)^+_\mu$ *is spanned by the vector*

$$s_{n1}^{\lambda_1 - \mu_1} \cdots s_{n,n-1}^{\lambda_{n-1} - \mu_{n-1}}\, \xi.$$

Note that the generator pF_{nn} of the algebra $\mathrm{Z}(\mathfrak{o}_{2n}, \mathfrak{o}_{2n-1})$ does not occur in the formula for the basis vector as it has the zero weight with respect to $\mathfrak{h}$.

4.3 Basis vectors

The representation $V(\lambda)$ of the Lie algebra $\mathfrak{g}_n = \mathfrak{o}_{2n+1}$ or $\mathfrak{o}_{2n}$ is equipped with a contravariant inner product which is uniquely determined by the conditions

$$\langle\, \xi, \xi \,\rangle = 1 \qquad \text{and} \qquad \langle\, F_{ij}\, u, v \,\rangle = \langle\, u, F_{ji}\, v \,\rangle$$

for all $u, v \in V(\lambda)$ and any indices i, j.

Combining Theorems 4.1 and 4.2 we can construct another basis for each representation $V(\lambda)$ of $\mathfrak{g}_n$; cf. Section 3.2.

B type case. We need to modify the definition of the B type pattern Λ introduced in Section 3.2. Here Λ is an array of the form

$$\begin{array}{ccccccccccc}
\lambda_{n1} & & \lambda_{n2} & & & \cdots & & & \lambda_{nn} & & \\
& \lambda'_{n1} & & \lambda'_{n2} & & & \cdots & & & \lambda'_{nn} & \\
& & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & & & \\
& & & \lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & \\
& & & & \cdots & & \cdots & & & & \\
& & & & & \lambda_{11} & & & & & \\
& & & & & & \lambda'_{11} & & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \dots, \lambda_{nn})$, the remaining entries are all integers or half-integers together with the λ_i, and the following inequalities hold

$$\lambda_{k1} \geq \lambda'_{k1} \geq \lambda_{k2} \geq \lambda'_{k2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{kk} \geq |\lambda'_{kk}|$$

for $k = 1, \dots, n$, and

$$\lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq \lambda_{k-1,2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq |\lambda'_{kk}|$$

for $k = 2, \dots, n$.

Theorem 4.3 *The vectors*

$$\eta_\Lambda = s'^{\,\lambda_{11}-\lambda'_{11}}_{11} \prod_{k=2,\dots,n}^{\rightarrow} \left(s'^{\,\lambda_{k1}-\lambda'_{k1}}_{k1} \cdots s'^{\,\lambda_{kk}-\lambda'_{kk}}_{kk} s^{\lambda'_{k1}-\lambda_{k-1,1}}_{k1} \cdots s^{\lambda'_{k,k-1}-\lambda_{k-1,k-1}}_{k,k-1} \right) \xi$$

parametrized by the patterns Λ form an orthogonal basis of the representation $V(\lambda)$.

D type case. Here we define the D type patterns Λ as arrays of the form

$$\begin{array}{ccccccccc}
\lambda_{n1} & & \lambda_{n2} & & & \cdots & & & \lambda_{nn} \\
& \lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & \\
& & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & \\
& & & \cdots & & \cdots & & & \\
& & & & \lambda'_{11} & & & & \\
& & & & & \lambda_{11} & & &
\end{array}$$

such that $\lambda = (\lambda_{n1}, \dots, \lambda_{nn})$, the remaining entries are all integers or half-integers together with the λ_i, and the following inequalities hold

$$\lambda_{k1} \geq \lambda'_{k-1,1} \geq \lambda_{k2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k,k-1} \geq \lambda'_{k-1,k-1} \geq |\lambda_{kk}|$$

for $k=2,\dots,n$, and

$$\lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \lambda_{k2} \geq \cdots \geq \lambda_{k,k-1} \geq \lambda'_{kk} \geq |\lambda_{kk}|$$

for $k=1,\dots,n-1$.

Theorem 4.4 *The vectors*

$$\eta_\Lambda = \prod_{k=1,\dots,n-1}^{\rightarrow} \Big(s_{k+1,1}^{\lambda_{k+1,1}-\lambda'_{k1}} \cdots s_{k+1,k}^{\lambda_{k+1,k}-\lambda'_{kk}} s_{k1}'^{\lambda'_{k1}-\lambda_{k1}} \cdots s_{kk}'^{\lambda'_{kk}-\lambda_{kk}} \Big) \xi$$

parametrized by the patterns Λ *form an orthogonal basis of the representation* $V(\lambda)$.

The norms of the basis vectors η_Λ can be found in an explicit form. The formulas for the matrix elements of the generators of the Lie algebra $\mathfrak{o}_N$ in the original paper by Gelfand and Tsetlin [37] are given in the orthonormal basis

$$\zeta_\Lambda = \eta_\Lambda / \|\eta_\Lambda\|, \qquad \|\eta_\Lambda\|^2 = \langle\, \eta_\Lambda, \eta_\Lambda \,\rangle.$$

Bibliographical notes

The exposition of this section follows Zhelobenko [148]. The branching rules were previously derived by him in [144]. The lowering operators for the reduction $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ were constructed by Pang and Hecht [111] and Wong [141]; see also Mickelsson [82]. They are presented in a form similar to (2.9) and (2.10) although more complicated. A derivation of the matrix element formulas of Gelfand and Tsetlin [37] was also given in [111] and [141] which basically follows the approach outlined in Section 2.1. The defining relations for the algebra $\mathrm{Z}(\mathfrak{o}_N, \mathfrak{o}_{N-1})$ were given in an explicit form by Zhelobenko [147]. Gould's approach based upon the characteristic identities of Bracken and Green [12, 48] for the orthogonal Lie algebras is also applicable; see Gould [42, 43, 45]. It produces an independent derivation of the matrix element formulas. Although the quantum minor approach has not been developed so far for the Gelfand–Tsetlin basis for the orthogonal Lie algebras, it seems to be plausible that the corresponding analogs of the results outlined in Section 2.5 can be obtained.

Analogs of the Gelfand–Tsetlin bases [37] for representations of a nonstandard deformation $\mathrm{U}'_q(\mathfrak{o}_N)$ of $\mathrm{U}(\mathfrak{o}_N)$ were given by Gavrilik and Klimyk [35], Gavrilik and Iorgov [34] and Iorgov and Klimyk [55].

The Gelfand–Tsetlin modules over the orthogonal Lie algebras were studied by Mazorchuk [78] with the use of the matrix element formulas from [37].

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