CUSPIDAL MODULES AS SUMMANDS OF A GELFAND-GRAEV MODULE

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ABSTRACT. Let $G = \mathrm{GL}_n(q)$ be the general linear group over a finite field $\mathbb{F}_q$ with q elements. We call a Gel'fand-Graev module to be the module which affords the Gel'fand-Graev character defined in 1.1. It is known that every cuspidal module of G is isomorphic to a (unique) direct summand of a Gel'fand-Graev module. In this article, we investigate a certain endomorphism so that each irreducible cuspidal module is contained in a certain eigenspace corresponding to the cuspidal character. Furthermore, we determine the eigenvalue of that endomorphism by using character theory of finite general linear group.

1. PRELIMINARIES

Let $G = \mathrm{GL}_n(q)$, the general linear group of degree n over a finite field $\mathbb{F}_q$. Let B be the group of upper triangular matrices (a Borel subgroup) and H the diagonal matrices (torus). For each $i \in \{1, 2, \ldots, n-1\}$, define s_i be the permutation matrix corresponding to the transposition $(i\, i+1)$. Let $S = \{s_i \mid 1 \leqslant i \leqslant n-1\}$. Then $W = \langle S \rangle$ is the Weyl group and isomorphic to the symmetric group on $\{1, 2, \ldots, n\}$. We shall frequently identify elements of W with the corresponding permutations.

Identify the root system of W by

$$\Phi = \{\, \alpha_{i\,j} \;\mid\; i \neq j\,,\, 1 \leqslant i, j \leqslant n \,\}.$$

where $\alpha_{i\,j}$ is the i-th minus the j-th ortonormal basis vectors of an n - dimensional Euclidean space.

The subgroup $U = \{\, (a_{ij}) \in B \;\mid\; a_{ii} = 1 \,\}$ (upper unitriangular matrices) is a maximal unipotent subgroup and Sylow p-subgroup of G. We have $B = U \rtimes H$. For each $\alpha = \alpha_{ij} \in \Phi$, define

$$x_\alpha(t) = x_{ij}(t) = \; I_n \, + \, t\mathbf{e}_{ij}$$

for any $t \in \mathbb{F}_q$, where I_n is the identity matrix. The group $X_\alpha = X_{ij} = \{\, x_{ij}(t) \mid t \in \mathbb{F}_q \,\}$ is the root subgroup corresponding to α. For any $w \in W$,

$$wx_{ij}(t)w^{-1} = x_{w(i),w(j)}(t). \tag{1}$$

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Furthermore, $U = \prod_{1\leqslant i<j\leqslant n} X_{ij}$.

Througout this paper, let $\theta : \mathbb{F}_q^+ \longrightarrow \mathbb{C}^*$ be a fixed nontrivial homomorphism.

Definition 1.1. Define a linear character $\Psi : U \longrightarrow \mathbb{C}^*$ by

$$\Psi((u_{ij})) = \theta\left(\sum_{i=1}^{n-1} u_{i\,i+1}\right)$$

The linear character Ψ is in general position in the sense that the restriction of Ψ to each fundamental root subgroup $X_{i\,i+1}$ is nontrivial. The induced character Ψ^G is called the *Gel'fand-Graev character* of G.

Define an idempotent e in $\mathbb{C}G$ by

$$e := \frac{1}{|U|}\sum_{u\in U}\Psi(u^{-1})u. \tag{2}$$

Then the left ideal $\mathbb{C}Ge$ of $\mathbb{C}G$ affords the Gel'fand-Graev representation.

Let M be an irreducible cuspidal module of G. Then M has a unique (up to scalars) vector b such that

$$u.b = \Psi(u)b$$

for all $u \in U$. Thus $\langle b\rangle$ is the unique $\mathbb{C}U$-submodule of $\mathrm{Res}_U^G M$ affording Ψ (See [7]). Since $eb = b$, the left ideal $\mathbb{C}Ge$ of $\mathbb{C}G$ (which affords the Gel'fand-Graev representation) has a (unique) direct summand isomorphic to M. Identify M with this summand, that is $M = \mathbb{C}Gb$. Then, up to a scalar, b is an idempotent that commutes with e.

2. MAIN RESULT

Hereafter let $\mathbb{F}_{q^n}$, $(n \geqslant 1)$ be an extension of degree n of $\mathbb{F}_q$. Consider a multiplicative character χ of $\mathbb{F}_{q^n}$ satisfying the following condition:

Condition C: For all $d \in \mathbb{Z}^+$ such that n is divisible by d and $d \neq n$, χ^{q^d-1} is not identically equal to unity.

Such a multiplicative character χ defines an irreducible cuspidal character ξ_χ of G (See [8]). If M is an irreducible cuspidal module affording ξ_χ then the elements zI of the center Z of G act on M via the restriction of χ to $\mathbb{F}_q^*$; that is

$$(zI)m = \chi(z)m$$

for all $z \in \mathbb{F}_q^*$ and $m \in M$. From now on fix such a multiplicative character χ of $\mathbb{F}_{q^n}$.

Let

$$\Lambda(\chi) := \frac{(-1)^{n+1}}{q^{n-1}} \sum_{\substack{\lambda\in\mathbb{F}_{q^n}\\ \mathbf{N}(\lambda)=(-1)^{n+1}}} \theta(\mathbf{Tr}(\lambda))\chi(\lambda^{-1}),$$

where $\mathbf{N}$ and $\mathbf{Tr}$ are the usual norm and trace maps of $\mathbb{F}_{q^n}$.

Define a $\mathbb{C}G$-endomorphism f of $\mathbb{C}Ge$ by

$$f(\alpha) = \alpha(es_{n-1}s_{n-2}\dots s_1 e) \tag{3}$$

for all $\alpha \in \mathbb{C}Ge$ (right multiplication by $es_{n-1}s_{n-2}\dots s_1 e$).

The main result of this paper is as follows.

Theorem 2.1. *With above notation, we have*

$$es_{n-1}s_{n-2}\dots s_1 b = \Lambda(\chi)b.$$

Furthermore an irreducible cuspidal module $\mathbb{C}Gb$ corresponding the multiplicative character χ of $\mathbb{F}_{q^n}$ satisfying Condition C given above is contained in the $\Lambda(\chi)$–eigenspace of the endomorphism f defined as in (3) *of $\mathbb{C}Ge$.*

The rest of the paper is devoted to proving this theorem. The proof is done in two stages. The first stage is to prove the following lemma.

Lemma 2.2.

$$es_{n-1}s_{n-2}\dots s_1 b = \frac{\kappa(1)}{q^{n-2}}b$$

where $\kappa(1)$ is a certain constant in complex number field.

The second stage is to prove, by using the character theory of G, that

$$\frac{\kappa(1)}{q^{n-2}} = \Lambda(\chi).$$

3. PROOF OF LEMMA 2.2

For each root $\alpha \in \Phi$, we define

$$\overline{X}_\alpha := \frac{1}{q}\sum_{a\in\mathbb{F}_q} x_\alpha(a),$$

$$\widetilde{X}_\alpha := \frac{1}{q}\sum_{a\in\mathbb{F}_q} \theta(-a)x_\alpha(a).$$

Thus $\overline{X}_\alpha$ and $\widetilde{X}_\alpha$ are respectively the idempotents of the trivial character and θ. Furthermore,

$$e = \prod_{\substack{1\leqslant i\leqslant n-2\\ 3\leqslant j\leqslant n\\ i+1<j}} \overline{X}_{ij} \prod_{1\leqslant k\leqslant n-1} \widetilde{X}_{k\,k+1}. \tag{4}$$

The following results are trivial and so we omit the proofs.

Lemma 3.1. *For each $\alpha \in \Phi$,*

(i) $\widetilde{X}_\alpha\widetilde{X}_\alpha = \widetilde{X}_\alpha$ *and* $\overline{X}_\alpha\overline{X}_\alpha = \overline{X}_\alpha$.

(ii) $\overline{X}_\alpha\widetilde{X}_\alpha = \widetilde{X}_\alpha\overline{X}_\alpha = 0$.

(iii)

$$\overline{X}_\alpha b = \begin{cases} b & \textit{if} \quad \alpha \in \Phi^+ \setminus \Pi, \\ 0 & \textit{if} \quad \alpha \in \Pi, \end{cases}$$

$$\widetilde{X}_\alpha b = \begin{cases} 0 & \textit{if} \quad \alpha \in \Phi^+ \setminus \Pi, \\ b & \textit{if} \quad \alpha \in \Pi. \end{cases}$$

Lemma 3.2.

$$es_{n-1}s_{n-2}\dots s_1 b = \widetilde{X}_{n-1\,n} \prod_{1\leqslant i\leqslant n-2} \overline{X}_{in} s_{n-1}s_{n-2}\dots s_1 b.$$

Proof. Observe that for all i, j, $k \in \{1,2,\dots,n\}$ with $1 \leqslant i \leqslant n-3$, $3 \leqslant j \leqslant n-1$, $i+1 < j$, $1 \leqslant k \leqslant n-2$,

$$\begin{aligned} s_1 s_2 \dots s_{n-1}(i\,j) &= (i+1\,j+1) \\ s_1 s_2 \dots s_{n-1}(k\,k+1) &= (k+1\,k+2). \end{aligned}$$

Hence, by (4), (1) and Lemma 3.1(iii),

$$\begin{aligned} es_{n-1}s_{n-2}\dots s_1 b &= \prod_{\substack{1\leqslant i\leqslant n-2 \\ 3\leqslant j\leqslant n \\ i+1<j}} \overline{X}_{ij} \prod_{1\leqslant k\leqslant n-1} \widetilde{X}_{k\,k+1} s_{n-1}s_{n-2}\dots s_1 b \\ &= \left(\widetilde{X}_{n-1\,n} \prod_{1\leqslant i\leqslant n-2} \overline{X}_{in} s_{n-1}s_{n-2}\dots s_1 \right) \prod_{\substack{1\leqslant i\leqslant n-3 \\ 3\leqslant j\leqslant n-1 \\ i+1<j}} \overline{X}_{i+1\,j+1} \prod_{1\leqslant k\leqslant n-2} \widetilde{X}_{k+1\,k+2} b \\ &= \widetilde{X}_{n-1\,n} \prod_{1\leqslant i\leqslant n-2} \overline{X}_{in} s_{n-1}s_{n-2}\dots s_1 b. \end{aligned}$$

□

For each $w \in W$, we define $l(w)$ to be the length of an expression of w as a product of generators $s \in S$. Let w_J be the longest element of $W_J = \langle J \rangle$ for each $J \subseteq S$. In particular, w_S is the longest element of W.

For each $w \in W$, we define the linear character $\Psi^w : U^w \longrightarrow \mathbb{C}$ by $\Psi^w(u) = \Psi(wuw^{-1})$ for all $u \in U^w = w^{-1}Uw$. Let Ψ_w be the restriction of Ψ^w to the subgroup $U_{w_S w} = U \cap U^w$ of U^w. So for all $\alpha \in U_{w_S w} = \prod_{\alpha\in \mathrm{Pos}(w)} X_\alpha$ and $t \in \mathbb{F}_q$,

$$\Psi_w(x_\alpha(t)) = \begin{cases} 1 & \text{if} \quad w(\alpha) \in \Phi^+ \setminus \Pi \\ \theta(t) & \text{if} \quad w(\alpha) \in \Pi, \end{cases} \tag{5}$$

by (1).

Note that If $w = w_S s_\alpha$ where $\alpha \in \Pi$, then $U \cap U^w = X_\alpha$ and Ψ_w is the linear character corresponding to the idempotent $\widetilde{X}_\alpha$.

Definition 3.3. For each $w \in W$, define

$$e_w := \frac{1}{|U_{w_S w}|} \sum_{u \in U_{w_S w}} \Psi_w(u^{-1})u.$$

Lemma 3.4. *For all $w \in W$ and $u \in U_{w_S w}$, we have*

$$ue_w = e_w u = \Psi_w(u)e_w.$$

The proof is obvious.

For each $w \in W$, let

$$\begin{aligned} \mathrm{Neg}(w) &:= \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^- \} \\ \mathrm{Pos}(w) &:= \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+ \}. \end{aligned}$$

Thus $\mathrm{Pos}(w) = \Phi^+ \setminus \mathrm{Neg}(w)$ and $\mathrm{Neg}(w) = \mathrm{Pos}(w_S w)$.

For each $w \in W$, we define

$$S(w) = \{ \alpha \in \Phi^+ \mid w(\alpha) \in \Pi \},$$

the set of positive roots made simple by w, and

$$S'(w) = \mathrm{Pos}(w) \setminus S(w).$$

Clearly, $\mathrm{Pos}(w) = S(w) \cup S'(w)$. It is a well-known fact that $U_{w_S w} = U \cap U^w = \prod_{\alpha \in \mathrm{Pos}(w)} X_\alpha$ (see Proposition 2.5.16, [3]). It follows from (5) that for each $w \in W$, the idempotent e_w can be written as

$$\prod_{\substack{\alpha \in S'(w) \\ \beta \in S(w)}} \overline{X}_\alpha \widetilde{X}_\beta \tag{6}$$

All finite groups of Lie type satisfy the following commutator relation (see [3]). Let α, β be linearly independent roots. Then

$$[\, X_\alpha, X_\beta \,] \subseteq \prod X_{i\alpha + j\beta} \tag{7}$$

where the product is taken (in some order) over the set of all roots which are of the form $i\alpha + j\beta$, i, $j > 0$.

The Commutator Formula (7) shows that

$$[X_\gamma, X_\beta] \subseteq \prod_{\alpha \in S'(w)} X_\alpha$$

for each β, $\gamma \in \mathrm{Pos}(w)$. Consequently, the positions of $\widetilde{X}_\beta$ factors in the above expression can be varied arbitrarily. Hence by Lemma 3.1,

$$e_w \overline{X}_\alpha = \overline{X}_\alpha e_w = e_w \tag{8}$$

$$e_w \widetilde{X}_\beta = \widetilde{X}_\beta e_w = e_w \tag{9}$$

for all $\alpha \in S'(w)$ and $\beta \in S(w)$.

Lemma 3.5. *For each* $w \in W$,

$$e_{w^{-1}}wb = wb.$$

The proof is straightforward and so is omitted.

The following fact is useful in proving Lemma 3.7.

Lemma 3.6. *Let* $w \in W$. *For any sequence* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *of distinct positive roots,*

$$U^w \cap (X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_k}) = X_{\beta_1} X_{\beta_2} \ldots X_{\beta_m}$$

where $\beta_1, \beta_2, \ldots, \beta_m$ *is the subsequence of* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *consisting of those* α_i *such that* $w(\alpha_i)$ *is positive.*

The proof is omitted.

Lemma 3.7. *Let* $w_1, w_2 \in W$. *For any* $w \in W$, $e_{w_1}we_{w_2} \neq 0$ *if and only if*

$$\alpha \in S(w_1) \quad \textit{and} \quad w^{-1}(\alpha) \in S(w_2)$$

for all $\alpha \in \operatorname{Pos}(w_1) \cap \operatorname{Pos}(w^{-1}) \cap \operatorname{Pos}(w_2 w^{-1})$. *Equivalently,* $e_{w_1}we_{w_2} \neq 0$ *if and only if*

$$\beta \in S(w_2) \quad \textit{and} \quad w(\beta) \in S(w_1)$$

for all $\beta \in \operatorname{Pos}(w_2) \cap \operatorname{Pos}(w) \cap \operatorname{Pos}(w_1 w)$

Proof. Observe that

$$e_{w_1}we_{w_2} \neq 0 \quad \text{if and only if} \quad \Psi_{w_1}(u) = \Psi_{w_2}(w^{-1}uw)$$

for all $u \in U_{w_S w_1} \cap wU_{w_S w_2}w^{-1}$. By Lemma 3.6, we have

$$U_{w_S w_1} \cap wU_{w_S w_2}w^{-1} = \prod_{\alpha \in \operatorname{Pos}(w_1) \cap \operatorname{Pos}(w^{-1}) \cap \operatorname{Pos}(w_2 w^{-1})} X_\alpha.$$

Hence $e_{w_1}we_{w_2} \neq 0$ if and only if

$$\Psi_{w_1}(x_\alpha(t)) = \Psi_{w_2}(w^{-1}x_\alpha(t)w) \qquad \text{for all} \quad t \in \mathbb{F}_q \tag{10}$$

for all $\alpha \in \operatorname{Pos}(w_1) \cap \operatorname{Pos}(w^{-1}) \cap \operatorname{Pos}(w_2 w^{-1})$. By (5), if $\alpha \in \operatorname{Pos}(w_1) \cap \operatorname{Pos}(w^{-1}) \cap \operatorname{Pos}(w_2 w^{-1})$ then (10) becomes

$$\Psi(x_{w_1(\alpha)}(t)) = \Psi(x_{w_2 w^{-1}(\alpha)}(t)) \qquad \text{for all} \quad t \in \mathbb{F}_q,$$

and this holds if and only if $w_1(\alpha)$ and $w_2 w^{-1}(\alpha)$ are both in Π or both in $\Phi^+ \setminus \Pi$. Thus $e_{w_1}we_{w_2} \neq 0$ if and only if the conditions $\alpha \in S(w_1)$ and $w^{-1}(\alpha) \in S(w_2)$ are equivalent for $\alpha \in \operatorname{Pos}(w_1) \cap \operatorname{Pos}(w^{-1}) \cap \operatorname{Pos}(w_2 w^{-1})$.

The equivalent statement can be proved in a similar way. $\square$

Corollary 3.8. *Let w_1, $w_2 \in W$. Then for any $w \in W$, $e_{w_1} w e_{w_2} = 0$ if there exists $\alpha \in S(w_1)$ such that*

$$w^{-1}(\alpha) \in S'(w_2)$$

or if there exists $\beta \in S(w_2)$ such that

$$w(\beta) \in S'(w_1).$$

This is straightforward from Lemma 3.7.

Lemma 3.9. *Let w_1, w_2, $w \in W$ and $h \in H$. Then $e_{w_1} h w e_{w_2} \neq 0$ if and only if $e_{w_1} w e_{w_2} \neq 0$ and h centralizes the root subgroup X_β for all $\beta \in w(S(w_2)) \cap S(w_1)$.*

This lemma can be proved by using the technique introduced in the proof of Lemma 3.7.

Observe that

$$\mathrm{Neg}(s_1) = \{\, \alpha_{12} \,\} \tag{11}$$

$$\mathrm{Neg}(s_{n-1}) = \{\, \alpha_{n-1\,n} \,\}. \tag{12}$$

$$S(s_1) = \{\alpha_{13}, \alpha_{34}, \ldots, \alpha_{n-1\,n}\}, \tag{13}$$

$$S(s_{n-1}) = \{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-3\,n-2}\alpha_{n-2\,n}\}, \tag{14}$$

Let $\Omega = \{s_i \mid 2 \leqslant i \leqslant n-2\}$ and $W_\Omega = \langle \Omega \rangle$.

Lemma 3.10, Lemma 3.11, Lemma 3.12 and Lemma 3.13 are essentially proved for more general cases in [12].

Lemma 3.10. *The set*

$$\{\, w \in W_\Omega \;\mid\; e_{s_1} w e_{s_{n-1}} \neq 0 \,\}$$

consists of only $s_2 s_3 \ldots s_{n-2}$.

Proof. Suppose that $w \in W_\Omega$ with $e_{\sigma_k^{-1}} w e_{(\sigma_l')^{-1}} \neq 0$. Note that $w(1) = 1$ and $w(n) = n$, since $w \in W_\Omega$. Clearly,

$$P\left(\begin{pmatrix} 1 & 2 & 3 & \ldots & n-2 & n-1 & n \\ 1 & 3 & 4 & \ldots & n-1 & 2 & n \end{pmatrix}\right) = s_2 s_3 \ldots s_{n-2}.$$

We use induction on j to show that

$$w(j) = 1 + j \qquad \text{for all } j \in \{2, 3, \ldots, n-2\},.$$

By (14), $\alpha_{1\,3} \in S(s_1)$. Then

$$w^{-1}(\alpha_{1\,3}) = \alpha_{1\,w^{-1}(3)} \in \Phi^+,$$

and by (12), we see that $\alpha_{1\,w^{-1}(3)} \notin \mathrm{Neg}(s_{n-1})$. So we must have $\alpha_{1\,w^{-1}(3)} \in \mathrm{Pos}(s_{n-1})$. Lemma 3.7 gives that $\alpha_{1\,w^{-1}(3)} \in S(s_{n-1})$. By (14), $w^{-1}(3) = 2$ and so $w(2) = 3$.

Suppose now that $j \in \{3, 4, \ldots, n-2\}$ and that $w(m) = 1 + m$ for all $m \in \{2, 3, \ldots, j-1\}$ (induction hypothesis). In particular, $j = w(j-1) \leqslant n-1$ since $w(n) = n$. By (13), $\alpha_{j\,j+1} \in S(s_1)$. Now

$$w^{-1}(\alpha_{j\,j+1}) = \alpha_{w^{-1}(j)\,w^{-1}(j+1)} = \alpha_{j-1\,w^{-1}(j+1)},$$

and this must be a positive root since $w^{-1}(j+1) \neq w^{-1}(1) = 1$ and

$$w^{-1}(j+1) \notin \{w^{-1}(3), w^{-1}(4), \ldots, w^{-1}(j-1)\} = \{2, 3, \ldots, j-2\}.$$

By (12), $\alpha_{j-1\,w^{-1}(j+1)} \notin \mathrm{Neg}(s_{n-1})$ as $j-1 < n-3$. So we must have $\alpha_{j-1\,w^{-1}(j+1)} \in \mathrm{Pos}(s_{n-1})$. Lemma 3.7 gives that $\alpha_{j-1\,w^{-1}(j+1)} \in S(s_{n-1})$. Since $j-1 \leqslant n-3$, it follows from (14) that $w^{-1}(j+1) = j$, completing the induction. □

Write $w_1 = s_2 s_3 \ldots s_{n-2}$.

For each $k \in \mathbb{Z}^+$ with $0 \leqslant k \leqslant n$, and each $x \in \mathbb{F}_q^*$, let $d_k(x) = (h_{ij}) \in H$, where

$$h_{ij} = \begin{cases} 1 & \text{if } i = j \neq k \\ x & \text{if } i = j = k \end{cases}$$

Let $D_k = \{\, d_k(x) \in H \mid x \in \mathbb{F}_q^* \,\}$.

Lemma 3.11. *With above notation, we have*

$$\{\, h \in D_2 D_3 \ldots D_n \mid e_{s_1} h w_1 e_{s_{n-1}} \neq 0 \,\} = D_2.$$

Proof. Lemma 3.9 gives that $e_{s_1} h w_1 e_{s_{n-1}} \neq 0$ if and only if h centralizes X_α for all $\alpha \in S(s_1) \cap w_1(S(s_{n-1}))$. By (14) and Lemma 3.10, we have

$$S(s_1) \cap w_1(S(s_{n-1})) = \{\alpha_{13}, \alpha_{34}, \ldots, \alpha_{n-1\,n}\}$$

Let $h = d_2(y_2) d_3(y_3) \ldots d_n(y_n) \in D_2 D_3 \ldots D_n$ where $y_i \in \mathbb{F}_q^*$ for $2 \leqslant i \leqslant n$. For any $t \in \mathbb{F}_q$, $h^{-1} x_{\alpha_{1\,3}}(t) h = x_{\alpha_{1\,3}}(t)$ if and only if $y_3 = 1$. For $3 \leqslant i \leqslant n-1$,

$$h^{-1} x_{\alpha_{i\,i+1}}(t) h = x_{\alpha_{i\,i+1}}(t)$$

if and only if $y_i = y_{i+1}$, as required. □

Lemma 3.12. *For any $h \in D_2$,*

$$e_{s_1} h w_1 s_{n-1} b = \prod_{\substack{\alpha \in S'(s_1) \cap \mathrm{Neg}(s_{n-1} w_1^{-1}) \\ \beta \in S(s_1) \cap \mathrm{Neg}(s_{n-1} w_1^{-1})}} \overline{X}_\alpha \widetilde{X}_\beta h w_1 s_{n-1} b.$$

Proof. By (6), we have

$$e_{s_1} h w_1 s_{n-1} b = \prod_{\substack{\alpha \in S'(s_1) \\ \beta \in S(s_1)}} \overline{X}_\alpha \widetilde{X}_\beta h w_1 s_{n-1} b.$$

As $w^{-1}X_\alpha w = X_{w^{-1}(\alpha)}$ for all $w \in W$ and all $\alpha \in \Phi$,

$$\begin{aligned} e_{s_1} h w_1 s_{n-1} b &= \prod_{\alpha,\,\beta,\,\gamma,\,\mu} \overline{X}_\alpha \widetilde{X}_\beta \overline{X}_\gamma \widetilde{X}_\mu \, h w_1 s_{n-1} b \\ &= \prod_{\alpha,\,\beta,\,\mu} \overline{X}_\alpha \widetilde{X}_\beta \widetilde{X}_\mu \, h w_1 \prod_{\gamma} \overline{X}_{s_{n-1} w_1^{-1}(\gamma)} \, b \end{aligned}$$

where

$$\begin{aligned} &\alpha \in S'(s_1) \cap \mathrm{Neg}(s_{n-1} w_1^{-1}) \\ &\beta \in S(s_1) \cap \mathrm{Neg}(s_{n-1} w_1^{-1}) \\ &\gamma \in S'(s_1) \cap \mathrm{Pos}(s_{n-1} w_1^{-1}) \\ &\mu \in S(s_1) \cap \mathrm{Pos}(s_{n-1} w_1^{-1}). \end{aligned}$$

By Lemma 3.10, $e_{s_1} w_1 e_{s_{n-1}} \neq 0$, and so, by Lemma 3.7 if $\gamma \in S'(s_1) \cap \mathrm{Pos}(s_{n-1} w_1^{-1})$ then $w_1^{-1}(\gamma) \in S'(s_{n-1})$. So

$$\overline{X}_{s_{n-1} w_1^{-1}(\gamma)} b = b \qquad \text{(by Lemma 3.1 (iii))}.$$

Similarly, by Lemma 3.7, if $\mu \in S(s_1) \cap \mathrm{Pos}(s_{n-1} w_1^{-1})$ then $w_1^{-1}(\mu) \in S(s_{n-1})$, and, by Lemma 3.11, $\widetilde{X}_\mu h = h \widetilde{X}_\mu$. Thus we have

$$e_{s_1} h w_1 s_{n-1} b = \prod_{\alpha,\,\beta} \overline{X}_\alpha \widetilde{X}_\beta h w_1 s_{n-1} \prod_{\mu} \widetilde{X}_{s_{n-1} w_1^{-1}(\mu)} \, b = \prod_{\alpha,\,\beta} \overline{X}_\alpha \widetilde{X}_\beta h w_1 s_{n-1} \, b$$

by Lemma 3.1(iii). This completes the proof. $\square$

Lemma 3.13. *For any $w \in W_\Omega$, $h \in H$ and u_1, $u_2 \in U$, we have*

$$e_{s_1} u_1 h w u_2 e_{s_{n-1}} = \kappa \, e_{s_1} h w e_{s_{n-1}}$$

for some $\kappa \in \mathbb{C}^*$.

Proof. Lemma 3.4 implies that

$$e_{s_1} u_1 h = \kappa_1 \, e_{s_1} h u_1',$$

for some $\kappa_1 \in \mathbb{C}^*$ and some $u_1' \in U_{s_1} = U \cap U^{w_S s_1} = \prod_{\alpha \in \mathrm{Neg}(s_1)} X_\alpha$. If $w \in W_\Omega$ and $\alpha \in \mathrm{Neg}(s_1)$ then by (11) we see that $w^{-1}(\alpha) \in \Phi^+$. Hence $w^{-1} u_1' w \in U$ and so

$$e_{s_1} u_1 h w u_2 e_{s_{n-1}} = \kappa_1 e_{s_1} h w (w^{-1} u_1' w) u_2 e_{s_{n-1}} = \kappa_1 e_{s_1} h w u_3 e_{s_{n-1}} \tag{15}$$

for some $u_3 \in U$. By Lemma 3.4, we have

$$u_3 e_{s_{n-1}} = \kappa_2 u_3' e_{s_{n-1}}$$

for some $\kappa_2 \in \mathbb{C}^*$ and $u_3' \in U_{s_{n-1}} = \prod_{\alpha \in \mathrm{Neg}(s_{n-1})} X_\alpha$. Now since $w \in W_\Omega$ and $\alpha \in \mathrm{Neg}(s_{n-1})$, by (11), $w(\alpha) = \alpha_{i\,n}$ for some $i \geqslant 2$ and so by (11) and (12) it follows that $w(\alpha) \notin \mathrm{Neg}(s_1)$, whence $w(\alpha) \in \mathrm{Pos}(s_1)$. Thus

$$w u_3' w^{-1} \in \prod_{\gamma \in \mathrm{Pos}(s_1)} X_\gamma = U_{w_S s_1}$$

So

$$e_{s_1} h w u_3 e_{s_{n-1}} = \kappa_2 e_{s_1} h (w u_3' w^{-1}) w e_{s_{n-1}} = \kappa' e_{s_1} h w e_{s_{n-1}} \tag{16}$$

for some $\kappa' \in \mathbb{C}^*$, by Lemma 3.4. Combining (15) and (16) gives the result. $\square$

Theorem 3.14.

$$s_1 b = \sum_{h \in D_2} \kappa(h) \prod_{3 \leqslant i \leqslant n} \overline{X}_{\alpha_{2\,i}} h s_2 s_3 \ldots s_{n-2} s_{n-1} b$$

where $\kappa(h)$ is a constant depending on h.

Proof. By [Theorem 2.3, [11]], we have

$$s_1 b \in \mathbb{C} C s_{n-1} b,$$

where

$$C = \left\{ \begin{pmatrix} 1 & * & \ldots & * & * \\ 0 & & & & * \\ \vdots & & \mathrm{GL}_{n-2}(q) & & \vdots \\ 0 & & & & * \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \right\}.$$

Then for some scalar $\kappa(c)$, by Bruhat decomposition,

$$s_1 b = \sum_{c \in C} \kappa(c)\, c s_{n-1} b = \sum_{\substack{u_1,\, u_2 \in U \\ h \in D_2 D_3 \ldots D_{n-1} \\ w \in W_\Omega}} \kappa(u_1 h w u_2)\, u_1 h w u_2 s_{n-1} b$$

Lemma 3.5 implies that (for some scalars $\kappa'(hw)$, $\kappa(h)$)

$$\begin{aligned} s_1 b &= \sum_{\substack{u_1,\, u_2 \in U \\ h \in D_2 D_3 \ldots D_{n-1} \\ w \in W_\Omega}} \kappa(u_1 h w u_2)\, e_{s_1} u_1 h w u_2 e_{s_{n-1}} s_{n-1} b \\ &= \sum_{\substack{h \in D_2 D_3 \ldots D_{n-1} \\ w \in W_\Omega}} \kappa'(hw)\, e_{s_1} h w e_{s_{n-1}} s_{n-1} b \qquad \text{by Lemma 3.13.} \end{aligned}$$

By Lemma 3.10, Lemma 3.11 and Lemma 3.5,

$$s_1 b = \sum_{h \in D_2} \kappa(h) e_{s_1} h w_1 e_{s_{n-1}} s_{n-1} b$$

Lemma 3.12 gives

$$s_1 b = \sum_{h \in D_2} \kappa_1(h) \prod_{\substack{\alpha \in S'(s_1) \cap \mathrm{Neg}(s_{n-1} w_1^{-1}) \\ \beta \in S(s_1) \cap \mathrm{Neg}(s_{n-1} w_1^{-1})}} \overline{X}_\alpha \widetilde{X}_\beta h w_1 s_{n-1} b.$$

Observe that

$$\mathrm{Neg}(s_{n-1} w_1^{-1}) = \{\, \alpha_{23},\, \alpha_{24}, \ldots, \alpha_{2\,n} \,\}.$$

So $S(s_1) \cap \mathrm{Neg}(s_{n-1}w_1^{-1})$ is empty and

$$S'(s_1) \cap \mathrm{Neg}(s_{n-1}w_1^{-1}) = \mathrm{Neg}(s_{n-1}w_1^{-1}) = \{\, \alpha_{2\,i} \mid 3 \leqslant i \leqslant n \,\}.$$

This completes the proof. $\square$

For $2 \leqslant i < j \leqslant n-1$, let

$$\Upsilon_{i\,j} = \sum_{t \in \mathbb{F}_q^*} x_{i\,j}(t^{-1}) d_i(-t^{-1}) d_j(t)$$

$$\widetilde{\Upsilon}_{i\,j} = \sum_{t \in \mathbb{F}_q^*} \theta(t^{-1}) x_{i\,j}(t^{-1}) d_i(-t^{-1}) d_j(t). \tag{17}$$

We write Υ_i for $\Upsilon_{i\,i+1}$ and $\widetilde{\Upsilon}_i$ for $\widetilde{\Upsilon}_{i\,i+1}$ for all $2 \leqslant i \leqslant n-1$.

Lemma 3.15. *For all* $2 \leqslant i \leqslant n-2$,

$$\overline{X}_{i+1\,j} \Upsilon_i s_i s_{i+1} \dots s_{n-1} b = \begin{cases} \widetilde{X}_{i\,i+2} \Upsilon_i s_i s_{i+1} \dots s_{n-1} b & \text{if } j = i+2 \\ \overline{X}_{i\,j} \Upsilon_i s_i s_{i+1} \dots s_{n-1} b & \text{if } i+3 \leqslant j \leqslant n. \end{cases}$$

Proof. By the definition of $\overline{X}_{i+1\,i+2}$ and Υ_i,

$$\begin{aligned} &\overline{X}_{i+1\,i+2} \Upsilon_i s_i s_{i+1} \dots s_{n-1} b \\ &= \frac{1}{q} \sum_{\substack{t \in \mathbb{F}_q^* \\ l \in \mathbb{F}_q}} x_{i+1\,i+2}(l) x_{i\,i+1}(t^{-1}) d_i(-t^{-1}) d_{i+1}(t) s_i s_{i+1} \dots s_{n-1} b \\ &= \frac{1}{q} \sum_{\substack{t \in \mathbb{F}_q^* \\ l \in \mathbb{F}_q}} x_{i\,i+1}(t^{-1}) x_{i\,i+2}(-lt^{-1}) x_{i+1\,i+2}(l) d_i(-t^{-1}) d_{i+1}(t) s_i s_{i+1} \dots s_{n-1} b. \end{aligned}$$

Note that $x_{i+1\,i+2}(l)$ commutes with $d_i(-t^{-1})$. Since

$$\begin{aligned} x_{i+1\,i+2}(l) d_{i+1}(t) s_i s_{i+1} \dots s_{n-1} b &= d_{i+1}(t) x_{i+1\,i+2}(lt^{-1}) s_i s_{i+1} \dots s_{n-1} b \\ &= d_{i+1}(t) s_i s_{i+1} \dots s_{n-1} x_{i\,i+1}(lt^{-1}) b \\ &= \theta(lt^{-1}) d_{i+1}(t) s_i s_{i+1} \dots s_{n-1} b, \end{aligned}$$

we have

$$\begin{aligned} &\overline{X}_{i+1\,i+2} \Upsilon_i s_i s_{i+1} \dots s_{n-1} b \\ &= \sum_{t \in \mathbb{F}_q^*} \left(\frac{1}{q} \sum_{l \in \mathbb{F}_q} \theta(lt^{-1}) x_{i\,i+2}(-lt^{-1}) \right) x_{i\,i+1}(t^{-1}) d_i(-t^{-1}) d_{i+1}(t) s_i s_{i+1} \dots s_{n-1} b \\ &= \widetilde{X}_{i\,i+2} \Upsilon_i s_i s_{i+1} \dots s_{n-1} b, \end{aligned}$$

since lt^{-1} runs through all elements of $\mathbb{F}_q$ as l does.

Now suppose that $i+3 \leqslant j \leqslant n$. Then

$$\begin{aligned}
&\overline{X}_{i+1\,j}\Upsilon_i s_i s_{i+1}\dots s_{n-1}b \\
&= \frac{1}{q}\sum_{\substack{t\in\mathbb{F}_q^* \\ l\in\mathbb{F}_q}} x_{i+1\,j}(l)x_{i\,i+1}(t^{-1})d_i(-t^{-1})d_{i+1}(t)s_i s_{i+1}\dots s_{n-1}b \\
&= \frac{1}{q}\sum_{\substack{t\in\mathbb{F}_q^* \\ l\in\mathbb{F}_q}} x_{i\,i+1}(t^{-1})x_{i\,j}(-lt^{-1})x_{i+1\,j}(l)d_i(-t^{-1})d_{i+1}(t)s_i s_{i+1}\dots s_{n-1}b.
\end{aligned}$$

Note that $x_{i+1\,j}(l)$ commutes with $d_i(-t^{-1})$, and

$$x_{i+1\,j}(l)d_{i+1}(t) = d_{i+1}(t)x_{i+1\,j}(lt^{-1}).$$

Since $x_{i+1\,j}(lt^{-1})s_i s_{i+1}\dots s_{n-1}b = s_i s_{i+1}\dots s_{n-1}b$,

$$\begin{aligned}
&\overline{X}_{i+1\,j}\Upsilon_i s_i s_{i+1}\dots s_{n-1}b \\
&= \sum_{t\in\mathbb{F}_q^*}\left(\frac{1}{q}\sum_{l\in\mathbb{F}_q} x_{i\,j}(-lt^{-1})\right) x_{i\,i+1}(t^{-1})d_i(-t^{-1})d_{i+1}(t)s_i s_{i+1}\dots s_{n-1}b \\
&= \overline{X}_{i\,j}\Upsilon_i s_i s_{i+1}\dots s_{n-1}b,
\end{aligned}$$

as required. □

The following corollary is an immediate consequence of the above lemma.

Corollary 3.16. *For all* $2 \leqslant i \leqslant n-2$,

$$\left(\prod_{j=i+2}^{n}\overline{X}_{i+1\,j}\right)\Upsilon_i s_i s_{i+1}\dots s_{n-1}b = \widetilde{X}_{i\,i+2}\left(\prod_{j=i+3}^{n}\overline{X}_{i\,j}\right)\Upsilon_i s_i s_{i+1}\dots s_{n-1}b.$$

Lemma 3.17. *For any* $t\in\mathbb{F}_q^*$ *and any* i *with* $1\leqslant i\leqslant n-1$,

$$s_i x_{i\,i+1}(t)s_i = x_{i\,i+1}(t^{-1})d_i(-t^{-1})d_{i+1}(t)s_i x_{i\,i+1}(t^{-1}).$$

Proof. Observe that for any $t\in\mathbb{F}_q^*$,

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&t^{-1}\\0&1\end{pmatrix}\begin{pmatrix}-t^{-1}&0\\0&t\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&t^{-1}\\0&1\end{pmatrix}$$

from which the result follows. □

Lemma 3.18. *For* $1\leqslant i\leqslant n-2$,

$$s_i\overline{X}_{i\,i+1}s_i s_{i+1}\dots s_{n-1}b = \frac{1}{q}s_{i+1}s_{i+2}\dots s_{n-1}b + \frac{1}{q}\Upsilon_i s_i s_{i+1}\dots s_{n-1}b,$$

and $s_{n-1}\overline{X}_{n-1\,n}s_{n-1}b = \frac{1}{q}b + \frac{1}{q}\widetilde{\Upsilon}_{n-1}\,s_{n-1}b$.

Proof. Lemma 3.17 implies that

$$\begin{aligned} s_i\overline{X}_{i\,i+1}s_i &= \frac{1}{q}\sum_{t\in\mathbb{F}_q} s_ix_{i\,i+1}(t)s_i \\ &= \frac{1}{q}+\frac{1}{q}\sum_{t\in\mathbb{F}_q^*} x_{i\,i+1}(t^{-1})d_i(-t^{-1})d_{i+1}(t)s_ix_{i\,i+1}(t^{-1}) \end{aligned} \tag{18}$$

for all $1\leqslant i\leqslant n-1$. Assume that $1\leqslant i\leqslant n-2$. Since for all $t\in\mathbb{F}_q^*$

$$x_{i\,i+1}(t^{-1})s_{i+1}s_{i+2}\dots s_{n-1}b = s_{i+1}s_{i+2}\dots s_{n-1}x_{i\,n}(t^{-1})b = s_{i+1}s_{i+2}\dots s_{n-1}b,$$

we have

$$\begin{aligned} s_i\overline{X}_{i\,i+1}s_is_{i+1}\dots s_{n-1}b &= \frac{1}{q}s_{i+1}s_{i+2}\dots s_{n-1}b \\ &\quad + \frac{1}{q}\sum_{t\in\mathbb{F}_q^*} x_{i\,i+1}(t^{-1})d_i(-t^{-1})d_{i+1}(t)s_ix_{i\,i+1}(t^{-1})s_{i+1}s_{i+2}\dots s_{n-1}b \\ &= \frac{1}{q}s_{i+1}s_{i+2}\dots s_{n-1}b + \frac{1}{q}\sum_{t\in\mathbb{F}_q^*} x_{i\,i+1}d_i(-t^{-1})d_{i+1}(t)s_is_{i+1}\dots s_{n-1}b \\ &= \frac{1}{q}s_{i+1}s_{i+2}\dots s_{n-1}b + \frac{1}{q}\Upsilon_is_is_{i+1}\dots s_{n-1}b. \end{aligned}$$

As $x_{n-1\,n}(t^{-1})b=\theta(t^{-1})b$,

$$s_{n-1}\overline{X}_{n-1\,n}s_{n-1}b = \frac{1}{q}b+\frac{1}{q}\widetilde{\Upsilon}_{n-1}\,s_{n-1}b,$$

by (18). □

It is known that

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \tag{19}$$

$$s_is_j = s_js_i \qquad \text{if} \quad |i-j|>1 \tag{20}$$

for all i, j such that $1\leqslant i\leqslant n-2$ and $1\leqslant j\leqslant n-1$.

For each $k\in\mathbb{Z}$ with $1\leqslant k\leqslant n-2$, we define an element τ_k of W by

$$\tau_k := s_{n-k}s_{n-k+1}\dots s_{n-2}s_{n-1}s_{n-2}\dots s_{n-k+1}s_{n-k}.$$

Note that $\tau_k^{-1}=\tau_k$.

Observe that by (19) and (20)

$$\begin{aligned} \tau_k &= s_{n-k}s_{n-k+1}\dots s_{n-2}s_{n-1}s_{n-2}\dots s_{n-k+1}s_{n-k} \\ &= s_{n-k}s_{n-k+1}\dots s_{n-3}(s_{n-1}s_{n-2}s_{n-1})s_{n-3}\dots s_{n-k+1}s_{n-k} \\ &= s_{n-1}s_{n-k}s_{n-k+1}\dots s_{n-3}s_{n-2}s_{n-3}\dots s_{n-k+1}s_{n-k}s_{n-1} \\ &\dots \\ &= s_{n-1}s_{n-2}\dots s_{n-k+1}s_{n-k}s_{n-k+1}\dots s_{n-2}s_{n-1} \end{aligned} \tag{21}$$

for all k with $1\leqslant k\leqslant n-2$.

For all $y \in \mathbb{F}_q^*$, write $\kappa(y) = \kappa(d_2(y))$ (the constant appearing in Theorem 3.14). Let

$$\gamma_i = \sum_{y \in \mathbb{F}_q^*} \kappa(y) d_i(y). \tag{22}$$

Theorem 3.19. *We have*

$$s_{n-1}s_{n-2}\dots s_1 b = \frac{1}{q^{n-2}}\gamma_n b + \gamma_n \sum_{k=2}^{n-1} \mathcal{E}_k \tau_{n-k} b$$

where

$$\mathcal{E}_k = \begin{cases} \frac{1}{q^{k-1}} \widetilde{X}_{k\,k+1} \left(\prod_{j=k+2}^{n-1} \overline{X}_{k\,j}\right) \Upsilon_{k\,n} & \text{if } 2 \leqslant k \leqslant n-2 \\ \frac{1}{n-2} \widetilde{\Upsilon}_{n-1\,n} & \text{if } k = n-1 \end{cases}$$

and $\widetilde{\Upsilon}_{i\,j}$ *as in* (17).

Proof. Theorem 3.14 implies that

$$s_{n-1}s_{n-2}\dots s_1 b = (s_{n-1}s_{n-2}\dots s_2)\left(\gamma_2 \left(\prod_{j=3}^{n} \overline{X}_{2\,j}\right) s_2 s_3 \dots s_{n-1} b\right) \tag{23}$$

by (22). Now

$$\begin{aligned} s_2\gamma_2 \left(\prod_{j=3}^{n} \overline{X}_{2\,j}\right) s_2 s_3 \dots s_{n-1} b &= \gamma_3 \left(\prod_{j=4}^{n} \overline{X}_{3\,j}\right) (s_2 \overline{X}_{23} s_2) s_3 s_4 \dots s_{n-1} b \\ &= \gamma_3 \frac{1}{q} \left(\prod_{j=4}^{n} \overline{X}_{3\,j}\right) s_3 s_4 \dots s_{n-1} b + \gamma_3 \frac{1}{q} \left(\prod_{j=4}^{n} \overline{X}_{3\,j}\right) \Upsilon_{2\,3} s_2 s_3 \dots s_{n-1} b \end{aligned}$$

by Lemma 3.18, provided $n > 3$. If $n = 3$ then (by Lemma 3.18) the same expression is obtained with $\Upsilon_{2\,3}$ replaced by $\widetilde{\Upsilon}_{2\,3}$, and so

$$s_{n-1}s_{n-2}\dots s_1 b = s_2 s_1 b = \frac{\gamma_3}{q} + \frac{\gamma_3}{q} \widetilde{\Upsilon}_{2\,3} s_2 b,$$

as desired. For $n > 3$ the second term above can be written as

$$\frac{\gamma_3}{q} \widetilde{X}_{24} \left(\prod_{j=5}^{n} \overline{X}_{2\,j}\right) \Upsilon_{2\,3} s_2 s_3 \dots s_{n-1} b$$

by Corollary 3.16, and so multiplying $s_{n-1}s_{n-2}\dots s_3$ we obtain

$$\begin{aligned} s_{n-1}s_{n-2}\dots s_1 b = \frac{1}{q}(s_{n-1}s_{n-2}\dots s_3)\gamma_3 \left(\prod_{j=4}^{n} \overline{X}_{3\,j}\right) s_3 s_4 \dots s_{n-1} b \\ + \frac{\gamma_n}{q} \widetilde{X}_{23} \left(\prod_{j=4}^{n-1} \overline{X}_{2\,j}\right) \Upsilon_{2\,n} s_{n-1} \dots s_3 s_2 s_3 \dots s_{n-1} b \\ = \frac{1}{q}(s_{n-1}s_{n-2}\dots s_3)\gamma_3 \left(\prod_{j=4}^{n} \overline{X}_{3\,j}\right) s_3 s_4 \dots s_{n-1} b + \gamma_n \mathcal{E}_2 \tau_{n-2} b. \end{aligned}$$

The first term here is similar to the right hand side of (23), and so by calculations parallel to those above it can be expressed as

$$\frac{1}{q^2}(s_{n-1}s_{n-2}\dots s_4)\gamma_4\left(\prod_{j=5}^{n}\overline{X}_{4\,j}\right)+\gamma_n\mathcal{E}_3\tau_{n-3}b$$

(where the first term is just $\frac{1}{q^2}\gamma_4 b$ in the case $n=4$). Repeating the process a further $n-4$ times yields the desired formula. □

Lemma 3.20. *For all k with $1\leqslant k\leqslant n-2$*

$$\left(\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}\right)\tau_k b=0.$$

Proof. Since τ_k interchanges $n-k$ and n,

$$\overline{X}_{n-k-1\,n}\tau_k b=\tau_k\overline{X}_{n-k-1\,n-k}b=0$$

by Lemma 3.1(iii). □

Corollary 3.21. *We have*

$$\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}s_{n-1}s_{n-2}\dots s_1 b=\frac{1}{q^{n-2}}\gamma_n b$$

Proof. Multiply the equation given in Theorem 3.19 by $\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}$. Since $\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}$ is a subgroup normalized by U and H,

$$\left(\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}\right)\gamma_n\mathcal{E}_k\tau_{n-k}b=\gamma_n\mathcal{E}_k\left(\prod_{i=1}^{n-2}\overline{X}_{i\,n}\right)\tau_{n-k}b=0$$

for all $k\in\{2,3,\dots,n-1\}$, by Lemma 3.20. Hence

$$\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}s_{n-1}s_{n-2}\dots s_1 b=\frac{1}{q^{n-2}}\gamma_n b,$$

as required. □

Proof of Lemma 2.2.
By Lemma 3.2,

$$es_{n-1}s_{n-2}\dots s_1 b=\widetilde{X}_{n-1,\,n}\prod_{1\leqslant i\leqslant n-2}\overline{X}_{i\,n}s_{n-1}s_{n-2}\dots s_1 b.$$

Multiplying the equation in Corollary 3.21 by $\widetilde{X}_{n-1\,n}$ gives

$$es_{n-1}s_{n-2}\dots s_1 b=\frac{1}{q^{n-2}}\widetilde{X}_{n-1,\,n}\gamma_n b.$$

If $t \in \mathbb{F}_q^*$ then

$$\begin{aligned}\widetilde{X}_{n-1\,n} d_n(t) b &= \frac{1}{q} \sum_{l \in \mathbb{F}_q} \theta(-l) x_{n-1\,n}(l) d_n(t) b \\ &= \frac{1}{q} \sum_{l \in \mathbb{F}_q} \theta(-l) d_n(t) x_{n-1\,n}(tl) b \\ &= \frac{1}{q} \sum_{l \in \mathbb{F}_q} \theta(-l) \theta(tl) d_n(t) b \\ &= \begin{cases} b & \text{if } t = 1 \\ 0 & \text{if } t \neq 1, \end{cases}\end{aligned}$$

since the character θ^t of $\mathbb{F}_q$ defined by $\theta^t(l) = \theta(lt)$ is not equivalent to θ when $t \neq 1$. Hence

$$\widetilde{X}_{n-1\,n} \gamma_n b = \kappa(1) b,$$

from which the result follows.

Now the aim is to determine the constant $k(1)$, (where $1 \in \mathbb{F}_q^*$) by using the character theory of finite general linear groups.

4. CONJUGACY CLASSES AND CUSPIDAL CHARACTERS

This section is devoted to give some well-known facts about conjugacy classes and cuspidal characters of G.

Let f be an irreducible monic polynomial over $\mathbb{F}_q$ of degree $d \geqslant 1$ such that $f \neq x$. For each such f fix a matrix $P_f \in G_k$ whose characteristic polynomial is equal to f (such a matrix always exists). Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ be a partition with r parts. Define $|\mu| = \mu_1 + \mu_2 + \ldots + \mu_r$. For each such f and μ define

$$P_\mu(f) = \begin{pmatrix} P_{f,\,\mu_1} & 0 & \ldots & 0 \\ 0 & P_{f,\,\mu_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & P_{f,\,\mu_r} \end{pmatrix},$$

where for each μ_i

$$P_{f,\,\mu_i} = \begin{pmatrix} P_f & I_d & \ldots & 0 & 0 \\ 0 & P_f & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & P_f & I_d \\ 0 & 0 & \ldots & 0 & P_f \end{pmatrix} \qquad \text{(an analog of a Jordan Block)}$$

and I_d is the identity matrix of order d. Clearly $P_{f,\,\mu_i}$ is of degree $\mu_i d$ and $P_\mu(f)$ is of degree $|\mu| d$. Let $\{f_1, f_2, \ldots, f_N\}$ be a set of distinct irreducible monic polynomials and $\{\rho_1, \rho_2, \ldots, \rho_N\}$ be a set of partitions satisfying the following conditions:

(i) $\sum_{1 \leqslant i \leqslant N} |\rho_i| \deg(f_i) = n$,

(ii) $f_i \neq x$ for all i, $1 \leqslant i \leqslant N$.

Consider the following matrix

$$P = \begin{pmatrix} P_{\rho_1}(f_1) & 0 & \dots & 0 \\ 0 & P_{\rho_2}(f_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{\rho_N}(f_N) \end{pmatrix} \in G. \tag{24}$$

Note that the characteristic polynomial of P is equal to $f_1^{|\rho_1|} f_2^{|\rho_2|} \dots f_N^{|\rho_N|}$.

Theorem 4.1. *The set of matrices of the form* (24) *for all possible choices of the f_i and the ρ_i satisfying conditions (i) and (ii) above is a complete system of representatives of the conjugacy classes of G.*

Denote the class of G containing the element of P in (24) by

$$c = [f_1^{\rho_1} f_2^{\rho_2} \dots f_N^{\rho_N}].$$

Let $\mathbf{C}$ be the set of all conjugacy classes of G. Define the following subsets of $\mathbf{C}$:

$$\begin{aligned} \mathbf{C}_{>1} &:= \{\, [f_1^{\rho_1} f_2^{\rho_2} \dots f_N^{\rho_N}] \mid N > 1 \,\} \\ \mathbf{C}_1 &:= \{\, [f^{\rho}] \mid |\rho| \deg f = n \,\} \\ \mathbf{C}^s &:= \{\, [f_1^{(e_1)} f_2^{(e_2)} \dots f_N^{(e_N)}] \mid N \geqslant 1 \,\} \\ \mathbf{C}_1^{(e)} &:= \{\, [f^{(e)}] \in \mathbf{C}_1 \mid e|n,\, \deg(f) = \frac{n}{e} \,\} = \mathbf{C}^s \cap \mathbf{C}_1 \end{aligned}$$

for each $1 \leqslant e \leqslant n$ such that e divides n. Thus $g \in \mathbf{C}_1$ if and only if the characteristic polynomial of g is a power of an irreducible, and $g \in \mathbf{C}_{>1}$ if and only if the characteristic polynomial of g is divisible by more than one irreducible. Furthermore, $\mathbf{C}^s$ is the subset of $\mathbf{C}$ for which the partitions $(\rho_1, \rho_2, \dots, \rho_N)$ have only one part and $\mathbf{C}_1^{(e)}$ is the subset of $\mathbf{C}_1$ for which the partition ρ has just one part. Note that $g \in G$ lies in a class in the set $\mathbf{C}^s$ (the set of regular classes) if and only if the minimal polynomial of g equals its characteristic polynomial.

Let

$$C_f = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

be the companion matrix of a monic polynomial $f = x^n + a_{n-1}x^{n-1} + \dots + a_0$. Note that if λ is one of the roots of f over $\mathbb{F}_q$ then

$$\begin{aligned} \mathbf{Tr}(C_f) &= -a_{n-1} = \mathbf{Tr}(\lambda) \\ \mathbf{Tr}(C_{f^e}) &= e\,\mathbf{Tr}(C_f). \end{aligned} \tag{25}$$

The following proposition can be found in [10].

Proposition 4.2. *For a monic polynomial f of degree n, the minimal polynomial and the characteristic polynomial of the companion matrix C_f are both equal to f. Clearly* $\det C_f = (-1)^n a_0$.

Proposition 4.2 implies that C_f is in $\mathbf{C}^s$.

Let $\phi_r(x) = (x-1)(x^2-1)\ldots(x^{r-1}-1)$. A formula for an irreducible cuspidal character ξ_χ of G can be given by

$$\xi_\chi(c) = \begin{cases} (-1)^{r+n}\phi_r(q^d)\sum_{0\leqslant i\leqslant d-1}\chi(\lambda^{q^i}) & \text{if} \quad c = [f^\rho] \in \mathbf{C}_1 \\ 0 & \text{if} \quad c \in \mathbf{C}_{>1}, \end{cases}$$

where r is the number of the parts of the partition ρ, $\deg(f) = d$ and λ is one of the roots of f over $\mathbb{F}_{q^n}$. (See [7] or [9]). Note also that λ, λ^q, $\ldots$, $\lambda^{q^{d-1}}$ give all roots of f.

In particular,

$$\xi_\chi(C_f) = \begin{cases} (-1)^{n+1}\left(\chi(\lambda_1) + \chi(\lambda_2) + \ldots + \chi(\lambda_d)\right) & \text{if} \quad C_f \in \mathbf{C}_1^{(n/d)}, \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where λ_1, λ_2, $\ldots$, λ_d are all roots of f.

5. THE BESSEL FUNCTION

The following definition was given by Gel'fand in his paper [7]. Also see G.James's paper [9].

Definition 5.1. Define $\mathcal{J} : G \longrightarrow \mathbb{C}$ by

$$\mathcal{J}(g) = \frac{1}{|U|}\sum_{u\in U}\Psi(u^{-1})\xi_\chi(gu).$$

Then the map $\mathcal{J}$ is called the *Bessel function* of the cuspidal representation with respect to Ψ.

Consider the subgroup

$$A := \{\, (a_{ij}) \in |\ a_{nn} = 1,\ a_{ni} = 0,\ \text{if } 1 \leqslant i \leqslant k-1 \,\}$$

of G. Let $l = [A : U]$ and $A/U = \{1 = a_1, a_2, \ldots, a_l\,\}$ be a set of left coset representatives of U in A. It is known that

$$\{\, a_i b \mid 1 \leqslant i \leqslant l \,\}$$

forms a basis for an irreducible cuspidal module M (see for example [7], [9], [2]).

The following lemma is essentially proved in [7], [9] and [1].

Lemma 5.2. *For arbitrary $g \in G$, let*

$$ga_jb = \sum_{1 \leqslant i \leqslant l} \mathcal{J}(a_i^{-1} g a_j) a_i b$$

for all j, $1 \leqslant j \leqslant l$. Then this action defines a representation of G in M and this representation has character ξ_χ.

Let $J_{n-1} = \{s_1, s_2, \dots, s_{n-2}\}$ and $W_{J_{n-1}} = \langle J_{n-1} \rangle$. So $W \cap A = W_{J_{n-1}}$. By using Bruhat decomposition, it can be seen that

$$A = \dot{\bigcup_{w \in W_{J_{n-1}}}} (U_{n-1})_w H_{n-1} w U,$$

where $(U_{n-1})_w = U_{n-1} \cap (U_{n-1})^{w_{J_{n-1}} w}$. So the set

$$\{\, uhw \mid u \in (U_{n-1})_w,\ h \in H_{n-1},\ w \in W_{J_{n-1}} \,\}$$

forms a set of left coset representatives of U in A. In particular, $s_{n-2}s_{n-3}\dots s_1$ is in the set of left coset representatives of U in A. Applying Lemma 5.2 with $a_j = s_{n-2}s_{n-3}\dots s_1$, and s_{n-1} in place of g, we find that the coefficient of b in $s_{n-1}s_{n-2}\dots s_1 b$ is

$$\mathcal{J}(s_{n-1}s_{n-2}\dots s_1).$$

However, by Theorem 3.19, the coefficient of b in $s_{n-1}s_{n-2}\dots s_1 b$ is

$$\frac{1}{q^{n-2}}\kappa(1).$$

Hence

$$\mathcal{J}(s_{n-1}s_{n-2}\dots s_1) = \frac{1}{q^{n-2}}\kappa(1).$$

So the task is now to calculate $\Lambda(\chi) = \mathcal{J}(s_{n-1}s_{n-2}\dots s_1)$.

The following property of $\mathcal{J}$ is known (see [7]):

$$\overline{\mathcal{J}(g)} = \mathcal{J}(g^{-1}) \tag{27}$$

for all $g \in G$.

Let $\sigma = s_1 s_2 \dots s_{n-1}$. Then σ is the companion matrix of $f = x^n - 1$.

Lemma 5.3. *For any monic polynomial f of degree n, the centralizer $C_U(C_f)$ of C_f in U is trivial.*

Proof. Suppose that $u = (u_{ij})$ is upper unitriangular and commutes with C_f. Comparing the (i, j) entries of $C_f u$ and uC_f as i ranges from 1 to $n-1$, and j from i to $n-1$ we find successively

$$u_{12} = 0,\ u_{23} = 0, \dots, u_{1\,n} = 0,\ u_{23} = u_{12} = 0,\ u_{24} = u_{13} = 0,$$

and so on. Finally we obtain that u is the identity. □

The following results are essentially proved for the general case $f = x^n - z$ for all $z \in \mathbb{F}_q^*$ in [12].

Lemma 5.4. $|U \cap \sigma^{-1} U \sigma| = q^{\frac{1}{2}(n-1)(n-2)}$.

The proof is straightforward and so is omitted.

Lemma 5.5. *If f is any monic polynomial of degree n with $f(0) = -1$ then there are exactly $q^{\frac{1}{2}(n-1)(n-2)}$ elements conjugate to C_f in the coset $U\sigma$. Furthermore, every element of $U\sigma$ is conjugate to C_f for some polynomial f. Similarly the double coset $U\sigma U$ contains $q^{\frac{1}{2}n(n-1)}$ elements of each of these classes and no other elements.*

Proof. It is clear (by elementary column operations) that for any such f there is a $u \in U$ with $C_f = \sigma u \in U\sigma U$. By Lemma 5.3, in the conjugation action of U on $U\sigma U$ the orbit containing C_f has length $|U|$, and since the number of polynomials f satisfying the given conditions is q^{n-1}, this accounts for $q^{n-1}|U|$ elements of $U\sigma U$. But by Lemma 5.4 we see that

$$|U\sigma U| = |U||U : U \cap \sigma^{-1} U \sigma| = |U| q^{n-1}$$

and so $U\sigma U$ contains no further elements. Thus $U\sigma U$ contains exactly $|U|$ elements conjugate to C_f. If $\mathcal{C}$ is any conjugacy class then, for each $u \in U$,

$$\mathcal{C} \cap U\sigma u = u^{-1}(\mathcal{C} \cap U\sigma)u$$

and so all cosets Ug contained in $U\sigma U$ contain the same number of elements of $\mathcal{C}$. So the number of elements of $U\sigma$ conjugate to C_f is

$$|U| \frac{|U\sigma|}{|U\sigma U|} = q^{\frac{1}{2}(n-1)(n-2)},$$

as claimed. □

Proposition 5.6.

$$\mathcal{J}(\sigma) = \frac{(-1)^{n+1}}{q^{n-1}} \sum_{\substack{\lambda \in \mathbb{F}_{q^n} \\ \mathbf{N}(\lambda) = (-1)^{n+1}}} \theta(-\mathbf{Tr}(\lambda))\chi(\lambda).$$

Proof. By Definition 5.1, we have

$$\begin{aligned} \mathcal{J}(\sigma) &= \frac{1}{|U|} \sum_{u \in U} \Psi(u^{-1}) \xi_\chi(\sigma u) \\ &= \frac{1}{|U|} \sum_{u \in U} \Psi(u^{-1}) \xi_\chi(u\sigma) \\ &= \frac{1}{|U|} \sum_{u \in U} \Psi(u) \xi_\chi(u^{-1}\sigma). \end{aligned}$$

For any $u \in U$,

$$u^{-1}\sigma = \begin{pmatrix} -u_{12} & * & * & \dots & * & z \\ 1 & -u_{23} & * & \dots & * & 0 \\ 0 & 1 & -u_{34} & \dots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -u_{n-1\,n} & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Then clearly $\mathbf{Tr}(u^{-1}\sigma) = -u_{12} - u_{23} - \ldots - u_{n-1\,n}$. So we have

$$\begin{aligned} \mathcal{J}(\sigma) &= \frac{1}{|U|} \sum_{u^{-1} \in U} \theta(-\mathbf{Tr}(u^{-1}\sigma))\xi_\chi(u^{-1}\sigma) \\ &= \frac{q^{\frac{1}{2}(n-1)(n-2)}}{|U|} \sum \theta(-\mathbf{Tr}(C_f))\xi_\chi(C_f) \end{aligned}$$

by Lemma 5.5, where f ranges over all monic polynomials of degree n with $f(0) = -1$. By (26), $\xi_\chi(C_f) = 0$ unless $C_f \in \mathbf{C}_1^{(e)}$; that is, unless f is a power of an irreducible. For each divisor d of n, let $\mathcal{P}(d)$ be the set of monic irreducible polynomials f_1 of degree d such that $(f_1(0))^e = -1$ where $de = n$. Then we have

$$\begin{aligned} \mathcal{J}(\sigma) &= \frac{1}{q^{n-1}} \sum_{de=n} \sum_{f_1 \in \mathcal{P}(d)} \theta(-\mathbf{Tr}(C_{f_1^e}))\xi_\chi(C_{f_1^e}) \\ &= \frac{1}{q^{n-1}} \sum_{de=n} \sum_{f_1 \in \mathcal{P}(d)} \theta(-e\,\mathbf{Tr}(C_{f_1})\xi_\chi(C_{f_1^e}) \quad \text{by (25)}, \\ &= \frac{(-1)^{n+1}}{q^{n-1}} \sum_{de=n} \sum_{f_1 \in \mathcal{P}(d)} \theta(-e\,\mathbf{Tr}(C_{f_1}))\,(\chi(\lambda_1) + \chi(\lambda_2) + \ldots \chi(\lambda_{d-1})) \end{aligned}$$

by (26), the λ_i being the roots of f_1. Note that each of these roots is an element $\lambda \in \mathbb{F}_{q^d}$ such that $\mathbb{F}_q(\lambda) = \mathbb{F}_{q^d}$ and $\mathbf{N}(\lambda) = \mathbf{N}_{\mathbb{F}_{q^d}:\mathbb{F}_q}(\lambda) = (-1)^{n+1}$; moreover each such λ is a root of some $f_1 \in \mathcal{P}(d)$; and $\mathbf{Tr}_{\mathbb{F}_{q^d}:\mathbb{F}_q}(\lambda) = \mathbf{Tr}(C_{f_1})$. Thus

$$\begin{aligned} \mathcal{J}(\sigma) &= \frac{(-1)^{n+1}}{q^{n-1}} \sum_{de=n} \sum_{\substack{\lambda \in \mathbb{F}_{q^d} \\ \mathbb{F}_q(\lambda) = \mathbb{F}_{q^d} \\ \mathbf{N}(\lambda) = (-1)^{n+1}}} \theta(-e\,\mathbf{Tr}_{\mathbb{F}_{q^d}:\mathbb{F}_q}(\lambda))\chi(\lambda) \\ &= \frac{(-1)^{n+1}}{q^{n-1}} \sum_{de=n} \sum_{\substack{\lambda \in \mathbb{F}_{q^d} \\ \mathbb{F}_q(\lambda) = \mathbb{F}_{q^d} \\ \mathbf{N}(\lambda) = (-1)^{n+1}}} \theta(-\mathbf{Tr}(\lambda))\chi(\lambda) \\ &= \frac{(-1)^{n+1}}{q^{n-1}} \sum_{\substack{\lambda \in \mathbb{F}_{q^n} \\ \mathbf{N}(\lambda) = (-1)^{n+1}}} \theta(-\mathbf{Tr}(\lambda))\chi(\lambda), \end{aligned}$$

by using the fact that if $\mathbb{F}_{q^d} = \mathbb{F}_q(\lambda)$ is an extension of degree d over $\mathbb{F}_q$ (where $1 < d < n$ and d divides n) then $\mathbb{F}_q \subseteq \mathbb{F}_{q^d} \subseteq \mathbb{F}_{q^n}$ and

$$\mathbf{Tr}_{\mathbb{F}_{q^n}:\mathbb{F}_q}(\lambda) = \frac{n}{d}\,\mathbf{Tr}_{\mathbb{F}_{q^d}:\mathbb{F}_q}(\lambda).$$

□

Corollary 5.7.

$$\Lambda(\chi) = \frac{(-1)^{n+1}}{q^{n-1}} \sum_{\substack{\lambda\in\mathbb{F}_{q^n} \\ \mathbf{N}(\lambda)=(-1)^{n+1}z}} \theta(\mathbf{Tr}(\lambda))\chi(\lambda^{-1})$$

Proof. It follows from

$$\Lambda(\chi) = \mathcal{J}(\sigma^{-1}) = \overline{\mathcal{J}(\sigma)},$$

by (27). □

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