INTERMEDIATE WAKIMOTO MODULES FOR AFFINE $\mathfrak{sl}(n+1)$

BEN L. COX AND VYACHESLAV FUTORNY

Abstract. We construct certain boson type realizations of affine $\mathfrak{sl}(n+1)$ that depend on a parameter $0 \leq r \leq n$ such that when $r = 0$ we get a Fock space realization appearing in Cox[4] and when $r = n$ they are the Wakimoto modules described in the work of Feigin and Frenkel[5].

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1. Introduction

Wakimoto modules for affine Lie algebras were introduced by B. Feigin and E. Frenkel in 1988 by a homological characterization[5]. These modules admit a remarkable boson realization on the Fock space due to Wakimoto[14] for $\hat{\mathfrak{sl}}(2)$, and B. Feigin and E. Frenkel[6] for $\hat{\mathfrak{sl}}(n)$ which plays an important role in the conformal field theory providing a new bosonization rule for the Wess-Zumino-Witten models. Wakimoto modules have a geometric interpretation as certain sheaves on a semi-infinite flag manifold described in B. Feigin and E. Frenkel[6]. They belong to the category $\mathcal{O}$ and generically are isomorphic to corresponding Verma modules. There are numerous other authors who have explicitly constructed Wakimoto modules for affine Lie algebras other than $\mathfrak{sl}(n+1)$.

Affine Lie algebras admit Verma type modules associated with non-standard Borel subalgebras which is described in the work of B. Cox[3], S. Futorny and H. Saifi[9] and H. Jakobsen and V. Kac[11]. In particular modules associated with the *natural Borel subalgebra* were first introduced by H. Jakobsen and V. Kac in 1985[11]. They were studied by V. Futorny[8] under the name of *imaginary Verma modules*.

A Fock space realization of the imaginary Verma modules for $\hat{\mathfrak{sl}}(2)$ were constructed by Bernard and Felder[1] and then extended by the first author to the case of $\hat{\mathfrak{sl}}(n)$[4]. These realizations are given generically by certain Wakimoto type modules.

The main motivation for our work was a problem of finding suitable boson type realizations for all Verma type modules over $\hat{\mathfrak{sl}}(n+1)$. In Theorem 3.1 we construct such realizations, *intermediate Wakimoto modules*, for a series of generic Verma type modules depending on the parameter $0 \leq r \leq n$. If $r = n$ this construction coincides with the boson realization of Wakimoto modules in B. Feigin and E. Frenkel[5]. On the other hand when $r = 0$ the obtained representation gives a Fock space realization described in the work of the first author[4]. Using this realization we plan to discuss the detailed structure of intermediate Wakimoto modules in a subsequent paper.

2. Preliminaries

Fix a positive integer n, $0 \leq r \leq n$, $\gamma \in \mathbb{C}^*$. Set $k = \gamma^2 - (r+1)$. Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ and let E_{ij}, $i,j = 1,\dots,n+1$ be the standard basis for $\mathfrak{gl}(n+1,\mathbb{C})$. Set $H_i := E_{ii} - E_{i+1,i+1}$, $E_i := E_{i,i+1}$, $F_i := E_{i+1,i}$ which is a basis for $\mathfrak{sl}(n+1,\mathbb{C})$. Furthermore we denote the Killing form by $(X|Y) = \operatorname{tr}(XY)$ and $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1,\dots,\alpha_n\}$ be a base for Δ^+, the positive set of roots for $\mathfrak{g}$, such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with base $\{\alpha_1,\dots,\alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{g}_r = \mathfrak{sl}(r+1,\mathbb{C})$. A Cartan subalgebra $\mathfrak{H}$ (respectively $\mathfrak{H}_r$) of $\mathfrak{g}$ (respectively $\mathfrak{g}_r$) is spanned by H_i, $i = 1,\dots,n$ (respectively $i = 1,\dots,r$) and set $\mathfrak{H}_0 = 0$.

For any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathbb{C}[t,t^{-1}] \bigotimes \mathfrak{a}$ be the loop algebra of $\mathfrak{a}$. Then $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}(n+1,\mathbb{C}) = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{g}}_r = L(\mathfrak{g}_r) \oplus \mathbb{C}c \oplus \mathbb{C}d$ are the associated affine Kac-Moody algebras with $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{H}}_r = \mathfrak{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$ respectively.

The algebra $\hat{\mathfrak{g}}$ has generators E_{im}, F_{im}, H_{im}, $i = 1,\dots,n$, $m \in \mathbb{Z}$, and central element c with the product

$$[X_m, Y_n] = t^{m+n}[X,Y] + \delta_{m+n,0}\, m(X|Y)c.$$

2.1. **Oscillator algebras.** Let $\hat{\mathfrak{a}}$ be the infinite dimensional Heisenberg algebra with generators $a_{ij,m}$, $a^*_{ij,m}$, and $\mathbf{1}$, $1 \le i \le j \le n$ and $m \in \mathbb{Z}$, subject to the relations

$$\begin{aligned}[][a_{ij,m}, a_{kl,n}] &= [a^*_{ij,m}, a^*_{kl,n}] = 0,\\ [a_{ij,m}, a^*_{kl,n}] &= \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1},\\ [a_{ij,m}, \mathbf{1}] &= [a^*_{ij,m}, \mathbf{1}] = 0.\end{aligned}$$

Such an algebra has a representation $\tilde{\rho} : \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[\mathbf{x}])$ where

$$\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_{ij,m} | i, j, m \in \mathbb{Z},\, 1 \le i \le j \le n]$$

denotes the algebra over $\mathbb{C}$ generated by the indeterminates $x_{ij,m}$ and $\tilde{\rho}$ is defined by

$$\tilde{\rho}(a_{ij,m}) := \begin{cases} \partial/\partial x_{ij,m} & \text{if} \quad m \ge 0, \text{ and } j \le r \\ x_{ij,m} & \text{otherwise,} \end{cases}$$

$$\tilde{\rho}(a^*_{ij,m}) := \begin{cases} x_{ij,-m} & \text{if} \quad m \le 0, \text{ and } j \le r \\ -\partial/\partial x_{ij,-m} & \text{otherwise.} \end{cases}$$

and $\tilde{\rho}(\mathbf{1}) = 1$. In this case $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_{ij,m}|0\rangle = 0, \quad m \ge 0 \text{ and } j \le r, \quad a^*_{ij,m}|0\rangle = 0, \quad m > 0 \text{ or } j > r.$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a^*_{ij,m}$ and $\mathbf{1}$, where $1 \le i \le j \le r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $A_n = ((\alpha_i|\alpha_j))$ be the Cartan matrix for $\mathfrak{sl}(n+1, \mathbb{C})$ and let $\mathfrak{B}$ be the matrix whose entries are

$$\mathfrak{B}_{ij} := (\alpha_i|\alpha_j)(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1})$$

where

$$\delta_{i>r} = \begin{cases} 1 & \text{if} \quad i > r, \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$\mathfrak{B} := \gamma^2 A_n - (r+1)\begin{pmatrix} 0 & 0 \\ 0 & A_{n-r} \end{pmatrix} + rE_{r+1,r+1}.$$

We also have the Heisenberg Lie algebra $\hat{\mathfrak{b}}$ with generators b_{im}, $1 \le i \le n$, $m \in \mathbb{Z}$, $\mathbf{1}$, and relations $[b_{im}, b_{jp}] = m\,\mathfrak{B}_{ij}\delta_{m+p,0}\mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \le i \le n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ has a representation $\rho_\lambda : \hat{\mathfrak{b}} \to \mathrm{End}(\mathbb{C}[\mathbf{y}]_\lambda)$ where

$$\mathbb{C}[\mathbf{y}] := \mathbb{C}[y_{i,m} | i, m \in \mathbb{N}^*,\, 1 \le i \le n]$$

and ρ_λ is defined on $\mathbb{C}[\mathbf{y}]$ defined by

$$\rho_\lambda(b_{i0}) = \lambda_i, \quad \rho_\lambda(b_{i,-m}) = \mathbf{e}_i \cdot \mathbf{y}_m, \quad \rho_\lambda(b_{im}) = m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \quad \text{for} \quad m > 0$$

and $\rho_\lambda(\mathbf{1}) = 1$. Here

$$\mathbf{y}_m = (y_{1m}, \cdots, y_{nm}), \quad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \cdots, \frac{\partial}{\partial y_{nm}}\right)$$

and $\mathbf{e}_i$ are vectors in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$ where $\cdot$ means the usual dot product.

Note that since $\mathfrak{B}_{ij}$ is symmetric, it is orthogonally diagonalizable, (i.e. there exists an orthogonal matrix P such that $P^t\mathfrak{B}P$ is a diagonal matrix) and hence we can find vectors $\mathbf{e}_i$ in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$. In fact for $m > 0$ and $n < 0$ we get

$$\begin{aligned}[b_{im}, b_{jn}] &= [m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m}, \mathbf{e}_j \cdot \mathbf{y}_{-n}] \\ &= m\sum_{k,l}[e_{ik}\frac{\partial}{\partial y_{km}}, e_{jl}y_{l,-n}] \\ &= m\delta_{m+n,0}\sum_k e_{ik}e_{jk} = m\delta_{m+n,0}\mathfrak{B}_{ij}.\end{aligned}$$

(See also the work of B. Feigin and E. Frenkel[7].)

2.2. **Formal Distributions.** We need some more notation that will simplify some of the arguments later. This notation follows the books of A. Matsuo and K. Nagatomo[14] and V. Kac[12]: A *formal distribution* is an expression of the form

$$a(z, w, \dots) = \sum_{m,n,\dots\in\mathbb{Z}} a_{m,n,\dots}z^m w^n$$

where the $a_{m,n,\dots}$ lie in some fixed vector space V. We define $\partial a(z) = \partial_z a(z) = \sum_n n a_n z^{n-1}$. We also have expansion about zero: there are two canonical embeddings of fields $\iota_{z,w} : \mathbb{C}(z-w) \to \mathbb{C}[[z,w]]$ and $\iota_{w,z} : \mathbb{C}(z-w) \to \mathbb{C}[[z,w]]$ where $\iota_{z,w}(a(z,w))$ is formal Laurent series expansion in z^{-1} and $-\iota_{w,z}(a(z,w))$ is formal Laurent series expansion in z. The *formal delta function* $\delta(z-w)$ is the formal distribution

$$\delta(z-w) = z^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^n = \iota_{z,w}\left(\frac{1}{z-w}\right) - \iota_{w,z}\left(\frac{1}{z-w}\right).$$

For any sequence of elements $\{a_{(m)}\}_{m\in\mathbb{Z}}$ in the ring $\mathrm{End}(V)$, V a vector space, the formal distribution

$$a(z) := \sum_{m\in\mathbb{Z}} a_{(m)}z^{-m-1}$$

is called a *field*, if for any $v \in V$, $a_{(m)}v = 0$ for $m \gg 0$. For a field such that $a_{(m)}$ are creation operators for $m \ll 0$, we set

$$a^-(z) := \sum_{m\geq 0} a_{(m)}z^{-m-1}, \quad \text{and} \quad a^+(z) := \sum_{m<0} a_{(m)}z^{-m-1}.$$

Observe that $a_{ij}(z)$ for $j > r$ is not a field whereas $a^*_{ij}(z)$ is always a field. We also define

$$\delta^-(z-w) = \iota_{z,w}\left(\frac{1}{z-w}\right), \quad \delta^+(z-w) = -\iota_{w,z}\left(\frac{1}{z-w}\right).$$

Note that

$$-\partial_z\delta(z-w) = \partial_w\delta(z-w) = \iota_{z,w}\left(\frac{1}{(z-w)^2}\right) - \iota_{w,z}\left(\frac{1}{(z-w)^2}\right).$$

Finally we use the convention

$$a^-_{ij}(z) := 0, \quad \text{and} \quad a^+_{ij}(z) := a_{ij}(z),$$

$$a^{*-}_{ij}(z) := a^*_{ij}(z), \quad \text{and} \quad a^{*+}_{ij}(z) := 0 \quad \text{for} \quad j > r.$$

The *normal ordered product* of two formal distributions $a(z)$ and $b(w)$ is defined by

$$: a(z)b(w) := a^+(z)b(w) + b(w)a^-(z).$$

For any $1 \le i \le j \le n$, we define

$$a^*_{ij}(z) = \sum_{n\in\mathbb{Z}} a^*_{ij,n} z^{-n}, \quad a_{ij}(z) = \sum_{n\in\mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n\in\mathbb{Z}} b_{in} z^{-n-1}.$$

In this case

$$[b_i(z), b_j(w)] = \mathfrak{B}_{ij}\partial_w \delta(z-w),$$
$$[a_{ij}(z), a^*_{kl}(w)] = \delta_{ik}\delta_{jl}\mathbf{1}\delta(z-w).$$

Let

(2.1) $$\lfloor ab \rfloor = a(z)b(w) - : a(z)b(w) := [a^-(z), b(w]$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$. For example if $j, l \le r$, then

(2.2) $$\lfloor a_{ij}a^*_{kl} \rfloor = \sum_{m\ge 0} \delta_{ik}\delta_{jl} z^{-m-1}w^m = \delta_{i,k}\delta_{j,l}\delta^-(z-w) = \delta_{ik}\delta_{jl}\, \iota_{z,w}\left(\frac{1}{z-w}\right)$$

(2.3) $$\lfloor a^*_{kl}a_{ij} \rfloor = -\sum_{n<0} \delta_{ik}\delta_{jl} z^n w^{-n-1} = -\delta_{ik}\delta_{jl}\delta^+(w-z) = \delta_{ik}\delta_{jl}\, \iota_{z,w}\left(\frac{1}{w-z}\right).$$

We need the very useful Wick's Theorem[2,12,14]:

Theorem 2.1. *Let $a_i(z)$ and $b_j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$, *satisfying*

(1) $[\lfloor a_i b_j \rfloor, c_k(z)] = 0$, *for all i, j, k and $c = a$ or $c = b$,*

(2) $[a^\pm_i(z), b^\pm_j(w)] = 0$ *for all i and j.*

Then

$$: a_1(z)a_2(z)\cdots a_k(z) : b(w) = \sum_{i=1}^k : a_1(z)\cdots \lfloor a_i b \rfloor \cdots a_k(z) :$$

and

$$:a_1(z)\cdots a_m(z) :: b_1(w)\cdots b_k(w) :=$$
$$\sum_{s=0}^{\min(m,k)} \sum_{\substack{i_1<\cdots<i_s,\\ j_1\neq\cdots\neq j_s}} \lfloor a_{i_1}b_{j_1} \rfloor \cdots \lfloor a_{i_s}b_{i_s} \rfloor : a_1(z)\cdots a_m(z)b_1(w)\cdots b_k(w) :_{(i_1,\dots,i_s;j_1,\dots,j_s)}$$

where the subscript $(i_1,\dots,i_s; j_1,\dots,j_s)$ means that those factors $a_i(z)$, $b_j(w)$ with indices $i \in \{i_1,\dots,i_s\}$, $j \in \{j_1,\dots,j_s\}$ are to be omitted from the product $: a_1(z)\cdots a_m(z)b_1(w)\cdots b_k(w) :$.

The proof is identical to that in Kac[12] even though it is stated for fields $a_i(z)$ and $b_j(z)$ in that text.

We will also need the following two results.

Theorem 2.2 (Taylor's Theorem[12,14]). *Let $a(z)$ be a formal distribution. Then in the region $|z-w|<|w|$,*

$$a(z)=\sum_{j=0}^{\infty}\partial_w^{(j)}a(w)(z-w)^j. \tag{2.4}$$

Theorem 2.3 (Kac[12], Theorem 2.3.2). *Let $a(z)$ and $b(z)$ be the formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$. *The following are equivalent*

(i) $[a(z),b(w)]=\sum_{j=0}^{N-1}\partial_w^{(j)}\delta(z-w)c^j(w)$, *where $c^j(w)$ is a formal distribution with coefficients in* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$.

(ii) $\lfloor ab\rfloor=\sum_{j=0}^{N-1}\iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c^j(w)$.

In other words the singular part of the *operator product expansion*

$$\lfloor ab\rfloor=\sum_{j=0}^{N-1}\iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. One writes

$$a(z)b(w)\sim\sum_{j=0}^{N-1}\frac{c^j(w)}{(z-w)^{j+1}}.$$

For example

$$b_i(z)b_j(w)\sim\frac{\delta_{ij}}{(z-w)^2}.$$

2.3. **Verma type modules.** For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

Let $\mathfrak{g}_\alpha$ be a root subspace of $\mathfrak{g}$ corresponding to a root α, $\mathfrak{n}^{\pm}=\oplus_{\alpha\in\Delta^+}\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{H}\oplus\mathfrak{n}^+$ a Cartan decomposition of $\mathfrak{g}$. Denote also $\mathfrak{n}_r^{\pm}=\mathfrak{n}^{\pm}\cap\mathfrak{g}_r$, $\mathfrak{n}^+(r)=\mathfrak{n}^+\setminus\mathfrak{n}_r^+$,

$$\bar{B}_r=L(\mathfrak{n}^+(r))\oplus(\mathfrak{n}_r^+\otimes\mathbb{C}[t])\oplus((\mathfrak{n}_r^-)\oplus\mathfrak{H})\otimes\mathbb{C}[t]t).$$

Then $B_r=\bar{B}_r\oplus\hat{\mathfrak{H}}$ is a Borel subalgebra of $\hat{\mathfrak{g}}$ for any $0\le r\le n$.

Fix $\tilde{\lambda}\in\hat{\mathfrak{H}}^*$ and consider a $\hat{\mathfrak{g}}$-module

$$M_r(\tilde{\lambda})=U(\hat{\mathfrak{g}})\otimes_{U(B_r)}\mathbb{C}v_{\tilde{\lambda}}$$

where $\bar{B}_rv_{\tilde{\lambda}}=0$ and $hv_{\tilde{\lambda}}=\tilde{\lambda}(h)v_{\tilde{\lambda}}$ for all $h\in\hat{\mathfrak{H}}$.

Module $M_r(\tilde{\lambda})$ is a particular case of a Verma type module studied in Cox[3], Futorny and Saifi[9]. When $r=n$ it gives a usual Verma module construction. If $r=0$ we get an imaginary Verma module.

Let $\tilde{\lambda}_r=\tilde{\lambda}|_{\hat{\mathfrak{H}}_r}$. Verma type module $M_r(\tilde{\lambda})$ contains a $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r)=U(\hat{\mathfrak{g}}_r)(1\otimes v_{\tilde{\lambda}})$ which is isomorphic to a usual Verma module for $\hat{\mathfrak{g}}_r$.

Note that the proof given in Kac's book cited above works also in the setting that the distributions are not necessarily fields.

Theorem 2.4 (Cox[3], Futorny and Saifi[9])**.** *Let $\tilde{\lambda}(c) \neq 0$. Then the submodule structure of $M_r(\tilde{\lambda})$ is completely determined by the submodule structure of $M(\tilde{\lambda}_r)$. In particular, $M_r(\tilde{\lambda})$ is irreducible if $M(\tilde{\lambda}_r)$ is irreducible.*

3. Intermediate Wakimoto modules

Define for $1 \leq i \leq n$,

$$E_i(z) = \sum_{n\in\mathbb{Z}} E_{in} z^{-n-1}, \quad F_i(z) = \sum_{n\in\mathbb{Z}} F_{in} z^{-n-1}, \quad H_i(z) = \sum_{n\in\mathbb{Z}} H_{in} z^{-n-1}.$$

The defining relations between the generators of $\hat{\mathfrak{g}}$ can be written as follows

(R1) $$[H_i(z), H_j(w)] = (\alpha_i|\alpha_j) c \partial_w \delta(w-z)$$

(R2) $$[H_i(z), E_j(w)] = (\alpha_i|\alpha_j) E_j(z)\delta(w-z)$$

(R3) $$[H_i(z), F_j(w)] = -(\alpha_i|\alpha_j) F_j(z)\delta(w-z)$$

(R4) $$[E_i(z), F_j(w)] = \delta_{i,j}(H_i(z)\delta(w-z) + c\partial_w \delta(w-z))$$

(R5) $$[F_i(z), F_j(w)] = [E_i(z), E_j(w)] = 0 \quad \text{if} \quad (\alpha_i|\alpha_j) \neq -1$$

(R6) $$[F_i(z_1), F_i(z_2), F_j(w)] = [E_i(z_1), E_i(z_2), E_j(w)] = 0 \quad \text{if} \quad (\alpha_i|\alpha_j) = -1$$

where $[X,Y,Z] := [X,[Y,Z]]$ is the Engel bracket for any three operators X, Y, Z.

Recall that $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module with respect to the representation $\tilde{\rho}$ and $\mathbb{C}[\mathbf{y}]$ is a $\hat{\mathfrak{b}}$-module with respect to ρ_λ. The main result of the paper is the following theorem where we define a representation

$$\rho : \hat{\mathfrak{g}} \to \mathfrak{gl}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]).$$

We use the notation $\rho(X_m) := \rho(X)_m$, for $X \in \mathfrak{g}$.

Theorem 3.1. *Let $\lambda \in \mathfrak{H}^*$ and set $\lambda_i = \lambda(H_i)$. The generating functions*

$$\rho(c) = \gamma^2 - (r+1),$$

$$\rho(F_i)(z) = a_{ii} + \sum_{j=i+1}^{n} a_{ij} a^*_{i+1,j},$$

$$\rho(H_i)(z) = 2 : a_{ii} a^*_{ii} : + \sum_{j=1}^{i-1} \big(: a_{ji} a^*_{ji} : - : a_{j,i-1} a^*_{j,i-1} : \big)$$
$$+ \sum_{j=i+1}^{n} \big(: a_{ij} a^*_{ij} : - : a_{i+1,j} a^*_{i+1,j} : \big) + b_i,$$

$$\rho(E_i)(z) =: a^*_{ii} \left(\sum_{k=1}^{i-1} a_{k,i-1} a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki} a^*_{ki} \right) : + \sum_{k=i+1}^{n} a_{i+1,k} a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki}$$
$$- a^*_{ii} b_i - \big(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2 \big) \partial a^*_{ii},$$

define an action of the generators E_{im}, F_{im}, H_{im}, $i = 1, \ldots, n$, $m \in \mathbb{Z}$ and c, on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$. In the above a_{ij}, a^*_{ij} and b_i denotes $a_{ij}(z)$, $a^*_{ij}(z)$ and $b_i(z)$ respectively.

Theorem 3.1 defines a boson type realization of $\hat{\mathfrak{sl}}(n+1)$ and a module structure on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ that depends on the parameter r, $0 \leq r \leq n$. We will call such a module, an *intermediate Wakimoto module* and denote it by $W_{n,r}(\lambda, \gamma)$. The intermediate Wakimoto modules $W_{n,r}(\lambda, \gamma)$ have the property that the subalgebra $\bar{B}_r$ annihilates the vector $1 \otimes 1 \in \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$, $h(1 \otimes 1) = \lambda(h)(1 \otimes 1)$ for all $h \in \mathfrak{H}$ and $c(1 \otimes 1) = (\gamma^2 - (r+1))(1 \otimes 1)$. Consider the $\hat{\mathfrak{g}}_r$-submodule $W = U(\hat{\mathfrak{g}}_r)(1 \otimes 1) \simeq W_{r,r}(\lambda, \gamma)$ of $W_{n,r}(\lambda, \gamma)$. Then W is isomorphic to the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ of Feigin and Frenkel[5] where $\lambda(r) = \lambda|_{\mathfrak{H}_r}$, $\tilde{\gamma} = \gamma^2 - (r+1)$.

Consider $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ such that $\tilde{\lambda}|_{\mathfrak{H}} = \lambda$, $\tilde{\lambda}(c) = \gamma^2 - (r+1)$, a Verma type module $M_r(\tilde{\lambda})$ and its $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r)$. Suppose that $M(\tilde{\lambda}_r)$ is irreducible. In this case the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ is isomorphic to $M(\tilde{\lambda}_r)$. Let $\tilde{W} = U(\hat{\mathfrak{g}})W_{\lambda(r),\tilde{\gamma}}$ and assume that $\lambda(c) \neq 0$. Then by Theorem 2.4 the module $M_r(\tilde{\lambda})$ is irreducible and therefore isomorphic to $\tilde{W}$. Hence Theorem 3.1 provides a boson type realization for generic Verma type modules.

We believe that generically Verma type modules and intermediate Wakimoto modules are isomorphic. A similar realization must exist for all Verma type modules over $\hat{\mathfrak{sl}}(n+1)$ and other affine Lie algebras.

4. Formal distribution computations

Set

$$\mathcal{H}_i(z) := 2 : a_{ii}a_{ii}^* : + \sum_{j=1}^{i-1} \left(: a_{ji}a_{ji}^* : - : a_{j,i-1}a_{j,i-1}^* : \right) + \sum_{j=i+1}^{n} \left(: a_{ij}a_{ij}^* : - : a_{i+1,j}a_{i+1,j}^* : \right).$$

In the above a_{ij}, and a_{ij}^* denotes $a_{ij}(z)$, $a_{ij}^*(z)$ respectively.

For any $\alpha \in \Delta^+$ we can find unique $1 \leq k \leq l \leq n$ such that

$$\alpha_{kl} := \alpha = \alpha_k + \cdots + \alpha_l. \tag{4.1}$$

Set $a_\alpha := a_{kl}$ and $a_\alpha^* := a_{kl}^*$. Observe that

$$\begin{aligned}(\alpha_i|\alpha) = \sum_{j=k}^{l} (\alpha_i|\alpha_j) &= (2\delta_{ik}\delta_{il} + \delta_{k<i}(\delta_{il} - \delta_{i-1,l}) + \delta_{l>i}(\delta_{ik} - \delta_{i+1,k})) \\ &= \delta_{ik} - \delta_{i+1,k} - \delta_{i,l+1} + \delta_{i+1,l+1}.\end{aligned}$$

Since this is the case we can rewrite

$$\mathcal{H}_i(z) := \sum_{\alpha \in \Delta^+} (\alpha_i|\alpha) : a_\alpha a_\alpha^* : .$$

Moreover we have

$$\lfloor a_\alpha a_\beta^* \rfloor = \begin{cases} \delta_{\alpha,\beta}\, \iota_{z,w}\left(\frac{1}{z-w}\right) & \text{if} \quad \alpha, \beta \in \Delta_r^+, \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

$$\lfloor a_\alpha^* a_\beta \rfloor = \begin{cases} -\delta_{\alpha,\beta}\, \iota_{z,w}\left(\frac{1}{z-w}\right) & \text{if} \quad \alpha, \beta \in \Delta_r^+, \\ -\delta_{\alpha,\beta}\, \delta(w-z) & \text{otherwise} \end{cases} \tag{4.3}$$

As an example of a computation using formal distributions we have the following

Lemma 4.1. *For* $1 \le i \le j \le n$, $\alpha, \beta \in \Delta^+$,

$$\begin{aligned}
[\mathcal{H}_i(z), a_\alpha(w)] &= -(\alpha_i|\alpha) a_\alpha(z)\delta(z-w)\,, \\
[\mathcal{H}_i(z), a^*_\alpha(w)] &= (\alpha_i|\alpha) a^*_\alpha(z)\delta(z-w), \\
[\mathcal{H}_i(z), \partial_w a^*_\alpha(w)] &= (\alpha_i|\alpha) a^*_\alpha(z)\partial_w(z-w), \\
[\mathcal{H}_i(z), \mathcal{H}_j(w)] &= -(\alpha_i|\alpha_j)\Big((1-\delta_{i>r}\delta_{j>r})(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\Big)\partial_w\delta(z-w),
\end{aligned}$$

$$\begin{aligned}
&[\mathcal{H}_i(z), :a_\alpha(w)a^*_\beta(w)a^*_\gamma(w):] \\
&\qquad = (\alpha_i|\beta+\gamma-\alpha) :a_\alpha(w)a^*_\beta(w)a^*_\gamma(w): \delta(z-w) \\
&\qquad\quad - \delta_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)\left(\delta_{\alpha,\beta}a^*_\gamma(w) + \delta_{\alpha,\gamma}a^*_\beta(w)\right)\partial_w\delta(z-w),
\end{aligned}$$

$$\begin{aligned}
&[\mathcal{H}_{\alpha_i}(z), :a_\alpha(w)a_\beta(w)a^*_\gamma(w):] \\
&\qquad = (\alpha_i|\gamma-\alpha-\beta) :a_\alpha(w)a_\beta(w)a^*_\gamma(w): \delta(z-w) \\
&\qquad\quad - \delta_{\gamma\in\Delta_r^+}(\alpha_i|\gamma)\left(\delta_{\gamma,\beta}a_\gamma(w) + \delta_{\alpha,\gamma}a_\beta(w)\right)\partial_w\delta(z-w).
\end{aligned}$$

Proof. Now by (4.2) and (4.3) and by Wick's Theorem

$$\sum_j :a_{ij}(z)a^*_{ij}(z): a_{kl}(w) \sim \delta_{ik}a_{kl}(z)\lfloor a^*_{ij}a_{kl}\rfloor$$

and if $\alpha = \alpha_k + \cdots + \alpha_l$, then

$$\begin{aligned}
\mathcal{H}_i(z)a_{kl}(w) &= \Big(2:a_{ii}a^*_{ii}: + \sum_{j=1}^{i-1}\left(:a_{ji}a^*_{ji}: - :a_{j,i-1}a^*_{j,i-1}:\right) \\
&\qquad + \sum_{j=i+1}^{n}\left(:a_{ij}a^*_{ij}: - :a_{i+1,j}a^*_{i+1,j}:\right)\Big)a_{kl}(w) \\
&\sim -\delta_{1\le l\le r}\left(\delta_{ik}-\delta_{i+1,k}-\delta_{i,l+1}+\delta_{i+1,l+1}\right)a_{kl}(z)\iota_{z,w}\left(\frac{1}{z-w}\right) \\
&\qquad - \delta_{r<l}(\delta_{ik}-\delta_{i+1,k}-\delta_{i,l+1}+\delta_{i+1,l+1})a_{kl}(z)\delta(z-w) \\
&\sim \delta_{1\le l\le r}(\alpha_i|\alpha)a_{kl}(z)\iota_{z,w}\left(\frac{1}{w-z}\right) - \delta_{r<l}(\alpha_i|\alpha)a_{kl}(z)\delta(z-w).
\end{aligned}$$

On the other hand

$$a_{kl}(w)\mathcal{H}_i(z) \sim \delta_{1\le l\le r}(\alpha_i|\alpha)a_{kl}(w)\iota_{w,z}\left(\frac{1}{z-w}\right).$$

Combining the above operator product expansions we get the first identity. A similar computation yields the second identity. The third identity comes from differentiating the second with respect to w.

On the other hand by Wick's Theorem

$$\begin{aligned}
&:a_\nu(z)a^*_\mu(z)::a_\alpha(w)a^*_\beta(w): \\
&\qquad =:a_\alpha(w)a^*_\beta(w)a_\nu(z)a^*_\mu(z): + \lfloor a_\alpha a^*_\mu\rfloor :a_\nu(z)a^*_\beta(w): \\
&\qquad\quad + \lfloor a^*_\beta a_\nu\rfloor :a_\alpha(w)a^*_\mu(z): + \lfloor a_\alpha a^*_\mu\rfloor\lfloor a^*_\beta a_\nu\rfloor.
\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{H}_{\alpha_i}(z)\mathcal{H}_{\alpha_j}(w) &= \sum_{\alpha,\beta\in\Delta^+}(\alpha_i|\alpha)(\alpha_j|\beta) : a_\alpha(z)a^*_\alpha(z) :: a_\beta(w)a^*_\beta(w) : \\ &= \sum_{\alpha,\beta\in\Delta^+}(\alpha_i|\alpha)(\alpha_j|\beta) : a_\alpha(z)a_\beta(w)a^*_\alpha(z)a^*_\beta(w) : \\ &\quad + \sum_{\beta\in\Delta^+}(\alpha_i|\beta)(\alpha_j|\beta) : a_\beta(w)a^*_\beta(z) : \lfloor a^*_\beta a_\beta\rfloor \\ &\quad + \sum_{\alpha\in\Delta^+}(\alpha_i|\alpha)(\alpha_j|\alpha) : a_\alpha(z)a^*_\alpha(w) : \lfloor a_\alpha a^*_\alpha\rfloor \\ &\quad + \sum_{\alpha\in\Delta^+_r}(\alpha_i|\alpha)(\alpha_j|\alpha)\lfloor a_\alpha a^*_\alpha\rfloor\lfloor a^*_\alpha a_\alpha\rfloor,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}[\mathcal{H}_{\alpha_i}(z),\mathcal{H}_{\alpha_j}(w)] &= \sum_{\alpha\in\Delta^+_r}(\alpha_i|\alpha)(\alpha_j|\alpha)\left(\iota_{w,z}\frac{1}{(w-z)^2}-\iota_{z,w}\frac{1}{(z-w)^2}\right) \\ &= -(\alpha_i|\alpha_j)\left((1-\delta_{i>r}\delta_{j>r})(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\partial_w\delta(z-w).\end{aligned}$$

This follows from the calculation below for root system of $\mathfrak{sl}(r+1)$: If $j\le r$, then

$$\sum_{\alpha\in\Delta^+_r}(\alpha_j|\alpha)\alpha=(r+1)\alpha_j$$

and

$$\sum_{\alpha\in\Delta^+_r}(\alpha_{r+1}|\alpha)^2=r.$$

Again by (4.2), (4.3) and Wick's Theorem

$$\begin{aligned}&: a_\nu(z)a^*_\nu(z) :: a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) : \\ &\quad =: a_\nu(z)a^*_\nu(z)a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) : \\ &\quad + \lfloor a^*_\nu a_\alpha\rfloor : a_\nu(z)a^*_\beta(w)a^*_\gamma(w) : \\ &\quad + \lfloor a_\nu a^*_\beta\rfloor : a_\alpha(w)a^*_\nu(z)a^*_\gamma(w) : \\ &\quad + \lfloor a_\nu a^*_\gamma\rfloor : a_\alpha(w)a^*_\beta(w)a^*_\nu(z) : \\ &\quad + \left(\lfloor a^*_\nu a_\alpha\rfloor\lfloor a_\nu a^*_\beta\rfloor a^*_\gamma(w)+\lfloor a^*_\nu a_\alpha\rfloor\lfloor a_\nu a^*_\gamma\rfloor a^*_\beta(w)\right).\end{aligned}$$

Hence the last identity follows from

$$
\begin{aligned}
\mathcal{H}_{\alpha_i}(z) : a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) := \sum_{\nu\in\Delta^+}(\alpha_i|\nu) : a_\nu(z)a^*_\nu(z) :: a_\alpha(w)a^*_\beta(w)a^*_\gamma(w) :\\
\sim \sum_{\nu\in\Delta^+}(\alpha_i|\nu)\Bigg(\Big(\lfloor a^*_\nu a_\alpha\rfloor : a_\nu(z)a^*_\beta(w)a^*_\gamma(w) :\\
+ \lfloor a_\nu a^*_\beta\rfloor : a_\alpha(w)a^*_\nu(z)a^*_\gamma(w) : + \lfloor a_\nu a^*_\gamma\rfloor : a_\alpha(w)a^*_\beta(z)a^*_\nu(z)\Big) :\\
+ \big(\lfloor a^*_\nu a_\alpha\rfloor\lfloor a_\nu a^*_\beta\rfloor a^*_\gamma(w) + \lfloor a^*_\nu a_\alpha\rfloor\lfloor a_\nu a^*_\gamma\rfloor a^*_\beta(w)\big)\\
\sim \Big((\alpha_i|\alpha)\lfloor a^*_\alpha a_\alpha\rfloor : a_\alpha(z)a^*_\beta(w)a^*_\gamma(w) : + (\alpha_i|\beta)\lfloor a_\beta a^*_\beta\rfloor : a_\alpha(w)a^*_\beta(z)a^*_\gamma(w) :\\
+ (\alpha_i|\gamma)\lfloor a_\gamma a^*_\gamma\rfloor : a_\alpha(z)a^*_\beta(z)a^*_\mu(w)\Big) :\\
+ (\alpha_i|\alpha)\left(\delta_{\alpha,\beta}a^*_\gamma(w) + \delta_{\alpha,\gamma}a^*_\beta(w)\right)\lfloor a^*_\alpha a_\alpha\rfloor\lfloor a_\alpha a^*_\alpha\rfloor.
\end{aligned}
$$

□

Lemma 4.2.

$$
\begin{aligned}
[a_{ij}(z), a^*_{kl}(w)] &= \delta_{ik}\delta_{jl}\delta(z-w)\\
[a_{ij}(z)a^*_{ij}(z), a_{ij}(w)a^*_{ij}(w)] &= -\delta_{1\le i,j\le r}\partial_w\delta(z-w)\\
[a_{ij}(z), \partial_w a^*_{kl}(w)] &= \delta_{ik}\delta_{jl}\partial_w\delta(z-w)\\
\partial_w a^*_{ij}(w)\delta(z-w) &= a^*_{ij}(z)\partial_w\delta(z-w) - a^*_{ij}(w)\partial_w\delta(z-w)
\end{aligned}
$$

The following result collects some other computations involving the formal distributions that will make future calculations less tedious.

Lemma 4.3.

(a)

$$
\begin{aligned}
\sum_{k=i+1,l=j+1}^{n}&\left[a_{ik}(z)a^*_{i+1,k}(z), a_{jl}(w)a^*_{j+1,l}(w)\right]\\
&= \Big(\delta_{i,j+1}\sum_{k=j+2}^{n} a_{jk}(z)a^*_{j+2,k}(z) - \delta_{j,i+1}\sum_{k=i+2}^{n} a_{ik}(z)a^*_{i+2,k}(z)\Big)\delta(z-w)
\end{aligned}
$$

(b)

$$
\sum_{k=1}^{i-1}\sum_{l=j+1}^{n}\left[a_{k,i-1}(z)a^*_{ki}(z), a_{jl}(w)a^*_{j+1,l}(w)\right] = -\delta_{j,i-1}a_{i-1,i-1}(z)a^*_{ii}(z)\delta(z-w)
$$

(c)

$$
\sum_{l=j+1}^{n}\Bigg[: a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right) :, a_{jl}(w)a^*_{j+1,l}(w)\Bigg] = 0,
$$

(d) $$\Big[: a_{ii}^*(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(z)a_{k,i-1}^*(z) - \sum_{k=1}^{i} a_{ki}(z)a_{ki}^*(z)\right) :, a_{jj}(w)\Big]$$
$$= -\delta_{ij}\left(: \sum_{k=1}^{i-1} a_{k,i-1}(z)a_{k,i-1}^*(z) - \sum_{k=1}^{i} a_{ki}(z)a_{ki}^*(z) :\right)\delta(z-w)$$
$$- \big(\delta_{j,i-1}a_{i-1,i-1}(z)a_{i,i}^*(z) - \delta_{i,j} : a_{ii}(z)a_{ii}^*(z) :\big)\,\delta(z-w)$$

(e) $$\Big[: a_{ii}^*\Big(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\Big) :, : a_{jj}^*\left(\sum_{l=1}^{j-1} a_{l,j-1}a_{l,j-1}^*\right) : \Big]$$
$$= \quad \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(z)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a_{k,i-2}^*(w)\right) : \delta(z-w)$$
$$- \delta_{i,j-1} : a_{jj}^*(w)a_{j-1,j-1}^*(w)\left(\sum_{l=1}^{j-2} a_{l,j-2}(z)a_{l,j-2}^*(z)\right) : \delta(z-w)$$
$$- (i-1)\delta_{1\le i-1\le r}\delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) : \partial_w\delta(z-w)$$

(f) $$\Big[: a_{ii}^*\Big(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\Big) :, : a_{jj}^*\left(\sum_{l=1}^{j} a_{l,j}a_{l,j}^*\right) : \Big]$$
$$= \delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(w)a_{k,i-1}^*(w)\right) : \delta(z-w)$$
$$- \delta_{ij} : a_{ii}^*(w)a_{ii}^*(w)\left(\sum_{l=1}^{i-1} a_{l,i-1}(z)a_{l,i-1}^*(z)\right) : \delta(z-w)$$
$$- i\delta_{1\le i-1\le r}\delta_{j,i-1} : a_{ii}^*(z)a_{i-1,i-1}^*(w) : \partial_w\delta(z-w)$$

(g) $$\Big[: a_{ii}^*\Big(\sum_{k=1}^{i} a_{ki}a_{ki}^*\Big) :, : a_{jj}^*\left(\sum_{l=1}^{j} a_{lj}a_{lj}^*\right) : \Big]$$
$$= -(3+i)\delta_{1\le i\le r}\delta_{ij} : a_{ii}^*(z)a_{ii}^*(w) : \partial_w\delta(z-w)$$

(h) $$\sum_{l=j+1}^{n}\Big[: a_{ii}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^{i} a_{ki}a_{ki}^*\right) :, a_{j+1,l}(w)a_{jl}^*(w)\Big]$$

$$= -\delta_{j,i-1} : a_{i-1,i}^*(w)\left(\sum_{k=1}^{i-1} a_{k,i-1}(z)a_{k,i-1}^*(z) - \sum_{k=1}^{i} a_{ki}(z)a_{ki}^*(z) :\right)\delta(z-w)$$
$$+ \delta_{i\le r}\delta_{j,i-1}a_{i-1,i}^*(z)\partial_w\delta(z-w),$$

(i) $$\Big[: a^*_{ii} \left(\sum_{k=1}^{i-1} a_{k,i-1} a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki} a^*_{ki} \right) :, \sum_{l=1}^{j-1} a_{l,j-1} a^*_{lj} \Big]$$
$$= \Bigg(: a^*_{ii} \left(\delta_{i-1,j} \sum_{l=1}^{i-2} a_{l,i-2} a^*_{l,i-1} - 2\delta_{ij} \sum_{l=1}^{i-1} a_{l,i-1} a^*_{li} + \delta_{i,j-1} \sum_{l=1}^{i} a_{l,i} a^*_{l,i+1} \right) :$$
$$- \delta_{i,j-1} : a^*_{i,i+1} \left(\sum_{l=1}^{i-1} a_{l,i-1} a^*_{l,i-1} - \sum_{l=1}^{i} a_{li} a^*_{li} \right) : \Bigg) \delta(z-w)$$

(j) $$\Big[\sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki}, \sum_{l=1}^{j-1} a_{l,j-1} a^*_{lj} \Big]$$
$$= \left(\delta_{j,i-1} \sum_{l=1}^{i-2} : a_{l,i-2} a^*_{li} : - \delta_{i,j-1} \sum_{l=1}^{j-2} : a_{l,j-2} a^*_{lj} : \right) \delta(z-w)$$

(k) $$\Big[\sum_{k=i+1}^{n} a_{i+1,k}(z) a^*_{ik}(z), \sum_{l=j+1}^{n} a_{j+1,l}(w) a^*_{jl}(w) \Big]$$
$$= \left(\delta_{i,j-1} \sum_{l=i+2}^{n} : a_{i+2,l} a^*_{il} : - \delta_{j,i-1} \sum_{l=j+2}^{n} : a_{j+2,k} a^*_{jk} : \right) \delta(z-w)$$

(l) $$\Big[\sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki}, \sum_{l=j+1}^{n} a_{j+1,l} a^*_{jl} \Big] = 0.$$

5. PROOF OF THEOREM 3.1

We can now check the defining relations.

Lemma 5.1 (R1)**.**
$$[\rho(H_i)(z), \rho(H_j)(w)] = (\alpha_i|\alpha_j)\rho(c)\partial_w \delta(z-w).$$

Proof. We use Lemma 4.1 in the following calculation:
$$[\rho(H_i)(z), \rho(H_j)(w)] = [\mathcal{H}_i(z) + b_i(z),\ \mathcal{H}_j(z) + b_j(z)]$$
$$= \Bigg(- (\alpha_i|\alpha_j) \left((1 - \delta_{i>r}\delta_{j>r})(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1} \right)$$
$$+ (\alpha_i|\alpha_j) \left(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1} \right) \Bigg) \partial_w \delta(z-w)$$
$$= (\alpha_i|\alpha_j)\rho(c)\partial_w \delta(z-w).$$

□

Lemma 5.2 (R2)**.**
$$[\rho(H_i)(z), \rho(E_j)(w)] = (\alpha_i|,\alpha_j)\rho(E_j)(z)\delta(z/w).$$

Proof. We will use Lemma 4.1 repeatedly and the convention (4.1):

$$\begin{aligned}
&[\mathcal{H}_i(z), \rho(E_j)(w)] \\
&= \sum_{k=1}^{j-1} [\mathcal{H}_i(z),\ :a^*_{jj}a_{k,j-1}a^*_{k,j-1}:] - \sum_{k=1}^{j} [\mathcal{H}_i(z),\ :a^*_{jj}a_{kj}a^*_{kj}:] \\
&\quad + \sum_{k=j+1}^{n} [\mathcal{H}_i(z),\ a_{j+1,k}a^*_{jk}] - \sum_{k=1}^{j-1} [\mathcal{H}_i(z),\ a_{k,j-1}a^*_{kj}] \\
&\quad - [\mathcal{H}_i(z),\ a^*_{jj}]b_j - \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)[\mathcal{H}_i(z),\ \partial_w a^*_{jj}(w)]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{j-1} \left((\alpha_i|\alpha_j) :a^*_{jj}a_{k,j-1}a^*_{k,j-1}: \delta(z-w) - \delta_{j-1\le r}(\alpha_i|\alpha_{k,j-1})a^*_{jj}(w)\partial_w\delta(z-w)\right) \\
&\quad - \sum_{k=1}^{j} \left((\alpha_i|\alpha_j) :a^*_{jj}a_{kj}a^*_{kj}: \delta(z-w) - \delta_{j\le r}(\alpha_i|\alpha_{kj})(\delta_{jk}a^*_{kk}(w) + a^*_{jj}(w))\partial_w\delta(z-w)\right) \\
&\quad + (\alpha_i|\alpha_j) \sum_{k=j+1}^{n} a_{j+1,k}(z)a^*_{jk}(w)\delta(z-w) - (\alpha_i|\alpha_j)\sum_{k=1}^{j-1} a_{k,j-1}(z)a^*_{kj}(w)\delta(z-w) \\
&\quad - (\alpha_i|\alpha_j)a^*_{jj}(z)b_j(w)\delta(z-w) \\
&\quad - (\alpha_i|\alpha_j)\left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)
\end{aligned}$$

$$\begin{aligned}
&= (\alpha_i|\alpha_j)\Bigg(:a^*_{jj}\Big(\sum_{k=1}^{j-1} a_{k,j-1}a^*_{k,j-1} - \sum_{k=1}^{j} a_{kj}a^*_{kj}\Big): \\
&\quad + \sum_{k=j+1}^{n} a_{j+1,k}a^*_{jk} - \sum_{k=1}^{j-1} a_{k,j-1}a^*_{kj} - a^*_{jj}b_j\Bigg)\delta(z-w) \\
&\quad + \sum_{k=1}^{j} \delta_{1\le j\le r}(\alpha_i|\alpha_k + \cdots + \alpha_j)(\delta_{jk}a^*_{kk}(w) + a^*_{jj}(w))\partial_w\delta(z-w) \\
&\quad - \sum_{k=1}^{j-1} \left(\delta_{1\le j-1\le r}(\alpha_i|\alpha_k + \cdots + \alpha_{j-1})a^*_{jj}(w)\partial_w\delta(z-w)\right) \\
&\quad - (\alpha_i|\alpha_j)(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2)a^*_{jj}(z)\partial_w\delta(z-w).
\end{aligned}$$

The last term of $\rho(H_i)(z)$ gives us

$$[b_i(z),\ \rho(E_j)(w)] = -\mathfrak{B}_{ij}a_{jj}(w)\partial_w\delta(z-w).$$

There are three cases to consider:
Case I. $j \le r$: Then

$$\begin{aligned}
&\sum_{k=1}^{j}(\alpha_i|\alpha_k+\cdots+\alpha_j)(\delta_{jk}a^*_{kk}(w)+a^*_{jj}(w))\partial_w\delta(z-w)\\
&-\sum_{k=1}^{j-1}(\alpha_i|\alpha_k+\cdots+\alpha_{j-1})a^*_{jj}(w)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)\left(\delta_{j>r}(r+1)+\delta_{j\le r}(j+1)-\gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)(\gamma^2-\delta_{i>r}\delta_{j>r}(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1})a_{jj}(w)\partial_w\delta(z-w)\\
&=(j+1)(\alpha_i|\alpha_j)a^*_{jj}(w)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)(j+1-\gamma^2)a^*_{jj}(z)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_j)\gamma^2a_{jj}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_j)(j+1-\gamma^2)\partial_w a_{jj}(w)\delta(z-w)
\end{aligned}$$

by Lemma 4.2.

Case II: $j=r+1$:

$$\begin{aligned}
&-\sum_{k=1}^{r}\left((\alpha_i|\alpha_k+\cdots+\alpha_r)a^*_{r+1,r+1}(w)\partial_w\delta(z-w)\right)\\
&-(\alpha_i|\alpha_{r+1})\left((r+1)-\gamma^2\right)a^*_{r+1,r+1}(z)\partial_w\delta(z-w)\\
&-(\alpha_i|\alpha_{r+1})(\gamma^2-\delta_{i>r}(r+1)+\frac{r}{2}\delta_{i,r+1})a_{r+1,r+1}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_{r+1})\left((r+1)-\gamma^2\right)(\partial_w a^*_{r+1,r+1}(w))\delta(z-w)
\end{aligned}$$

which follows from

$$\begin{aligned}
&-\sum_{k=1}^{r}(\alpha_i|\alpha_k+\cdots+\alpha_r)+(\alpha_i|\alpha_{r+1})(\delta_{i>r}(r+1)-\frac{r}{2}\delta_{i,r+1})\\
&=\begin{cases}-2 & \text{if}\quad 1=i=r\\ 0 & \text{if}\quad 1\le i<r\\ -(r+1) & \text{if}\quad 1<i=r\\ 2(r+1) & \text{if}\quad 1\le i=r+1\\ (\alpha_i|\alpha_{r+1})(r+1) & \text{if}\quad i>r+1\end{cases}\\
&=(\alpha_i|\alpha_{r+1})(r+1)
\end{aligned}$$

Cases III: $j>r+1$:

$$\begin{aligned}
&-(\alpha_i|\alpha_j)\left(r+1-\gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)\\
&-\mathfrak{B}_{ij}a_{jj}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_j)\left(r+1-\gamma^2\right)a^*_{jj}(z)\partial_w\delta(z-w)\\
&\quad+(\alpha_i|\alpha_j)(\gamma^2-\delta_{i>r}(r+1))a_{jj}(w)\partial_w\delta(z-w)\\
&=-(\alpha_i|\alpha_j)\left(r+1-\gamma^2\right)(\partial_w a^*_{jj}(w))\delta(z-w)
\end{aligned}$$

by Lemma 4.2 and the fact that $(\alpha_i|\alpha_j)=0$ for $i\le r<r+1<j$.

Putting these computations together we get

$$[\rho(H_i)(z),\rho(E_j)(w)] = (\alpha_i|\alpha_j)\rho(E_j)(w)\delta(z-w).$$

□

Lemma 5.3 (R3)**.**

$$[\rho(H_i)(z),\rho(F_j)(w)] = -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w).$$

Proof. The proof follows from Lemma 4.1 :

$$\begin{aligned}
[\rho(H_i)(z),\rho(F_j)(w)] &= [\mathcal{H}_i(z), a_{jj}(w) + \sum_{k=j+1}^{n} a_{jk}(w)a^*_{j+1,k}(w)] \\
&= -(\alpha_i|\alpha_j)a_{jj}(w)\delta(z-w) \\
&\quad + \sum_{k=j+1}^{n}[\mathcal{H}_i(z), a_{jk}(w)]a^*_{j+1,k}(w) + \sum_{k=j+1}^{n} a_{jk}(w)[\mathcal{H}_i(z), a^*_{j+1,k}(w)] \\
&= \Big(-(\alpha_i|\alpha_j)a_{jj}(w) - \sum_{k=j+1}^{n}(\alpha_i|\alpha_j+\dots+\alpha_k)a_{jk}(z)a^*_{j+1,k}(w) \\
&\qquad + \sum_{k=j+1}^{n}(\alpha_i|\alpha_{j+1}+\dots+\alpha_k)a_{jk}(w)a^*_{j+1,k}(z)\Big)\delta(z-w) \\
&= -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w)
\end{aligned}$$

□

Lemma 5.4 (R4)**.**

$$[\rho(E_i)(z),\rho(F_j)(w)] = \delta_{i,j}(\rho(H_i)(z))\delta(z-w) + \rho(c)\partial_w\delta(z-w))$$

Proof. First we take $i=j$. Now for the convenience of the reader we recall that $\rho(E_i)(z)$ is equal to

$$\begin{aligned}
&: a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right) : + \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki} \\
&\quad - a^*_{ii}b_i - \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\partial a^*_{ii}
\end{aligned}$$

and thus the first summand of $\rho(F_i)(w) = a_{ii} + \sum_{l=i+1}^{n} a_{il}a^*_{i+1,l}$ brackets with $\rho(E_i)(z)$ to give us (by Lemma 4.3 (d) and Lemma 4.2)

$$\begin{aligned}
&\Big(2 : a_{ii}(z)a^*_{ii}(z) : - : \sum_{k=1}^{i-1}\left(a_{k,i-1}a^*_{k,i-1} - a_{ki}a^*_{ki}\right) : + b_i(z)\Big)\delta(z-w) \\
&+ \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\partial_z\delta(z-w).
\end{aligned}$$

The second summation in $\rho(F_i)(w)$ contributes

$$\begin{aligned}\sum_{l=i+1}^{n}\Big[\rho(E_i)(z),\ a_{il}(w)a^*_{i+1,l}(w)\big] &= \sum_{l=i+1}^{n}\Big[\sum_{k=i+1}^{n} a_{i+1,k}(z)a^*_{ik}(z), a_{il}(w)a^*_{i+1,l}(w)\Big] \\ &= \sum_{l=i+1}^{n}\Big(a_{il}(z)a^*_{il}(z) - a_{i+1,l}(z)a^*_{i+1,l}(z)\Big)\delta(z-w) \\ &\quad - \delta_{i+1\le r}(r-i)\partial_w\delta(z-w).\end{aligned}$$

Adding these two summations up, we arrive at the desired result.

Now consider the case $|i-j|\ge 1$. Then $\rho(F_j)(w)$ is $a_{jj}+\sum_{l=j+1}^{n} a_{jl}a^*_{j+1,l}$. First we have

$$\begin{aligned}[E_i(z), a_{jj}(w)] &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right):, a_{jj}(w)\Big] \\ &= -\delta_{j,i-1}a_{i-1,i-1}(z)a^*_{i,i}(z)\delta(z-w)\end{aligned}$$

by Lemma 4.3 (d). Next we have

$$\begin{aligned}&[E_i(z), \sum_{l=j+1}^{n} a_{jl}(w)a^*_{j+1,l}(w)] \\ &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right): + \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki} \\ &\quad - a^*_{ii}b_i + \left(\gamma^2 - \delta_{i+1\le r}(i+1)\right)\partial a^*_{ii},\ \sum_{l=j+1}^{n} a_{jl}(w)a^*_{j+1,l}(w)\Big] \\ &= \Big[:a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right): - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki},\ \sum_{l=j+1}^{n} a_{jl}(w)a^*_{j+1,l}(w)\Big] \\ &\qquad\qquad \text{as } i\ne j, \\ &= \delta_{j,i-1}a_{i-1,i-1}(z)a^*_{ii}(z)\delta(z-w) \\ &\qquad\qquad \text{by Lemma 4.3 (b) and (c).}\end{aligned}$$

Adding up the last two calculations finishes the proof of this lemma.

□

We are now left with the Serre relations:

Lemma 5.5 (R5/R6)**.**

$$\begin{aligned}[\rho(F_i)(z), \rho(F_j)(w)] = [\rho(E_i)(z), \rho(E_j)(w)] = 0 \quad \textit{if} \quad (\alpha_i|\alpha_j)\ne -1 \\ [\rho(F_i)(z_1), \rho(F_i)(z_2), \rho(F_j)(w)] = [\rho(E_i)(z_1), \rho(E_i)(z_2), \rho(E_j)(w)] = 0, \\ \textit{if} \quad (\alpha_i|\alpha_j) = -1.\end{aligned}$$

Proof. Let us check the relations for $\rho(F_i)$. (The Serre relations were already known to hold true for the F_i, see B. Feigin and E. Frenkel[7], but we provide a proof as some of the calculations will be used in future work.) By Lemma 4.3 (a)

$$[\rho(F_i)(z),\rho(F_j)(w)] = (\delta_{i,j+1}a_{i-1,i} - \delta_{j,i+1}a_{i,i+1})\,\delta(z-w) + \left(\delta_{i,j+1}\sum_{k=i+1}^{n} a_{i-1,k}a^*_{i+1,k} - \delta_{j,i+1}\sum_{k=i+2}^{n} a_{ik}a^*_{i+2,k}\right)\delta(z-w). \tag{5.1}$$

Note the above is zero if $|i-j|\neq 1$ which is precisely when $(\alpha_i|\alpha_j)\neq -1$. As the first index in a_{kl} (resp. a^*_{kl}) in $\rho(F_i)(z)$ is i (resp. $i+1$) we also get

$$[\rho(F_i)(z_1),\rho(F_i)(z_1),\rho(F_j)(w)] = 0.$$

This completes the proof of the relations R5 and R6 for $\rho(F_i)(z)$.

Now we break up $\rho(E_i)(z)$ into three summands

$$\begin{aligned}
\rho(E_i^1)(z) &:=: a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right): \\
\rho(E_i^2)(z) &:= \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki} \\
\rho(E_i^3)(z) &:= -a^*_{ii}b_i - \left(\delta_{i>r}(r+1) + \delta_{i\leq r}(i+1) - \gamma^2\right)\partial a^*_{ii}.
\end{aligned}$$

By Lemma 4.3 (e), (f) and (g) we have

$$\begin{aligned}
&\left[\rho(E_i^1)(z),\rho(E_j^1)(w)\right] \\
&= \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a^*_{k,i-2}(w)\right) : \delta(z-w) \\
&- \delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(w)\left(\sum_{l=1}^{j-2} a_{l,j-2}(z)a^*_{l,j-2}(z)\right)\delta(z-w) \\
&- \delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-1} a_{k,i-1}(w)a^*_{k,i-1}(w)\right) : \delta(z-w) \\
&+ \delta_{i,j-1} : a^*_{jj}(z)a^*_{j-1,j-1}(z)\left(\sum_{k=1}^{j-1} a_{k,j-1}(w)a^*_{k,j-1}(w)\right) : \delta(z-w) \\
&- (i-1)\delta_{1\leq i-1\leq r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w) \\
&+ i\delta_{1\leq i-1\leq r}\delta_{j,i-1} : a^*_{ii}(z)a^*_{i-1,i-1}(w) : \partial_w\delta(z-w) \\
&- j\delta_{1\leq j-1\leq r}\delta_{i,j-1} : a^*_{jj}(w)a^*_{j-1,j-1}(z) : \partial_z\delta(z-w) \\
&- (3+i)\delta_{1\leq i\leq r}\delta_{ij} : a^*_{ii}(z)a^*_{ii}(w) : \partial_w\delta(z-w).
\end{aligned}$$

By Lemma 4.3 (h) and (i),

$$
\begin{aligned}
&\Big[\rho(E_i^1)(z), \rho(E_j^2)(w)\Big] + \Big[\rho(E_i^2)(z), \rho(E_j^1)(w)\Big] \\
&= -\delta_{j,i-1} : a_{i-1,i}^* \left(\sum_{k=1}^{i-2} a_{k,i-2} a_{k,i-2}^* - \sum_{k=1}^{i} a_{ki} a_{ki}^* : \right) \delta(z-w) \\
&\quad + \delta_{i,j-1} : a_{i,i+1}^* \left(\sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^{i+1} a_{k,i+1} a_{k,i+1}^* : \right) \delta(z-w) \\
&\quad - : a_{ii}^* \left(\delta_{i-1,j} \sum_{l=1}^{i-2} a_{l,i-2} a_{l,i-1}^* + \delta_{i,j-1} \sum_{l=1}^{i} a_{l,i} a_{l,i+1}^* \right) \delta(z-w) \\
&\quad + : a_{jj}^* \left(\delta_{j-1,i} \sum_{l=1}^{j-2} a_{l,j-2} a_{l,j-1}^* + \delta_{j,i-1} \sum_{l=1}^{j} a_{l,j} a_{l,j+1}^* \right) \delta(z-w) \\
&\quad + \delta_{i\le r} \delta_{j,i-1} a_{i-1,i}^*(z) \partial_w \delta(z-w) \\
&\quad - \delta_{j\le r} \delta_{i,j-1} a_{j-1,j}^*(w) \partial_z \delta(z-w).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\Big[\rho(E_i^1)(z)\rho(E_j^3)(w)\Big] + \Big[\rho(E_i^3)(z), \rho(E_j^1)(w)\Big] \\
&= \left(-\delta_{i-1,j} a_{ii}^* a_{i-1,i-1}^* b_j + \delta_{j-1,i} a_{jj}^* a_{j-1,j-1}^* b_i\right) \delta(z-w) \\
&+ \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right) \left(-\delta_{i-1,j} a_{ii}^*(z) a_{i-1,i-1}^*(z) + \delta_{ij} a_{ii}^*(z) a_{ii}^*(z)\right) \partial_w \delta(z-w) \\
&+ \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right) \left(\delta_{j-1,i} a_{jj}^*(w) a_{j-1,j-1}^*(w) - \delta_{ij} a_{jj}^*(w) a_{jj}^*(w)\right) \partial_z \delta(z-w).
\end{aligned}
$$

By Lemma 4.3 (j), (k) and (l) we have

$$
\begin{aligned}
&\Big[\rho(E_i^2)(z), \rho(E_j^2)(w)\Big] \\
&\qquad = \left(\delta_{j,i-1} \sum_{l=1}^{i-2} : a_{l,i-2} a_{li}^* : - \delta_{i,j-1} \sum_{l=1}^{j-2} : a_{l,j-2} a_{lj}^* : \right) \delta(z-w) \\
&\qquad + \left(\delta_{i,j-1} \sum_{l=i+2}^{n} : a_{i+2,l} a_{il}^* : - \delta_{j,i-1} \sum_{l=j+2}^{n} : a_{j+2,k} a_{jk}^* : \right) \delta(z-w)
\end{aligned}
$$

Next we have

$$
\begin{aligned}
&\Big[\rho(E_i^2)(z), \rho(E_j^3)(w)\Big] + \Big[\rho(E_i^3)(z), \rho(E_j^2)(w)\Big] \\
&= \left(-\delta_{j,i+1} a_{i,i+1}^* b_{i+1} + \delta_{j,i-1} a_{i-1,i}^* b_{i-1}\right) \delta(z-w) \\
&\quad - \left(\delta_{j>r}(r+1) + \delta_{j\le r}(j+1) - \gamma^2\right) \left(\delta_{j,i+1} a_{i,i+1}^*(z) - \delta_{j,i-1} a_{i-1,i}^*(z)\right) \partial_w \delta(z-w) \\
&- \left(-\delta_{i,j+1} a_{j,j+1}^* b_{j+1} + \delta_{i,j-1} a_{j-1,j}^* b_{j-1}\right) \delta(z-w) \\
&\quad + \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right) \left(\delta_{i,j+1} a_{j,j+1}^*(w) - \delta_{i,j-1} a_{j-1,j}^*(w)\right) \partial_z \delta(z-w).
\end{aligned}
$$

while

$$\begin{aligned}
&\Big[\rho(E^3_i)(z),\ \ \rho(E^3_j)(w)\Big]\\
&=\Big[-a^*_{ii}b_i-\big(\delta_{i>r}(r+1)+\delta_{i\leq r}(i+1)-\gamma^2\big)\,\partial a^*_{ii},\\
&\qquad\qquad -a^*_{jj}b_j-\big(\delta_{j>r}(r+1)+\delta_{j\leq r}(j+1)-\gamma^2\big)\,\partial a^*_{jj}\Big]\\
&=a^*_{ii}(z)a^*_{jj}(w)\mathfrak{B}_{ij}\partial_w\delta(z-w).
\end{aligned}$$

Now we observe that every reduction above is zero if $|i-j|\notin\{0,1\}$ i.e. if $(\alpha_i|\alpha_j)=0$. Thus

$$\Big[\rho(E_i)(z),\ \ \rho(E_j)(w)\Big]=0,\quad \text{if}\quad (\alpha_i|\alpha_j)=0.$$

When $i=j$, $\Big[\rho(E_i)(z),\ \ \rho(E_j)(w)\Big]$ reduces, by Lemma 4.2, to

$$\begin{aligned}
&\Big[\rho(E_i)(z),\ \ \rho(E_i)(w)\Big]\\
&=-(i-1)\delta_{1\leq i-1\leq r}:a^*_{ii}(z)a^*_{ii}(w):\partial_w\delta(z-w)\\
&-(3+i)\delta_{1\leq i\leq r}:a^*_{ii}(z)a^*_{ii}(w):\partial_w\delta(z-w)\\
&+\big(\delta_{i>r}(r+1)+\delta_{i\leq r}(i+1)-\gamma^2\big)\,a^*_{ii}(z)a^*_{ii}(z)\partial_w\delta(z-w)\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\leq r}(i+1)-\gamma^2\big)\,a^*_{ii}(w)a^*_{ii}(w)\partial_z\delta(z-w)\\
&+2a^*_{ii}(z)a^*_{ii}(w)\left(\gamma^2+(\delta_{1\leq i\leq r}-1)(r+1)+\delta_{i,r+1}\frac{r}{2}\right)\partial_w\delta(z-w)\\
&=-\left((i-1)\delta_{1\leq i-1\leq r}+(3+i)\delta_{1\leq i\leq r}-\delta_{i,r+1}r\right):a^*_{ii}(z)a^*_{ii}(w):\partial_w\delta(z-w)\\
&+\delta_{i\leq r}(i+1)\left(a^*_{ii}(z)a^*_{ii}(z)+a^*_{ii}(w)a^*_{ii}(w)\right)\partial_z\delta(z-w)\\
&+(r+1)\delta_{i>r}\left(a^*_{ii}(z)a^*_{ii}(z)+a^*_{ii}(w)a^*_{ii}(w)-2a^*_{ii}(z)a^*_{jj}(w)\right)\partial_w\delta(z-w)\\
&+\gamma^2\left(2a^*_{ii}(z)a^*_{ii}(w)-a^*_{ii}(z)a^*_{ii}(z)-a^*_{ii}(w)a^*_{ii}(w)\right)\partial_w\delta(z-w)=0.
\end{aligned}$$

This proves the result for $i\neq j\pm 1$.

If $i=j+1$ then we get

$$\begin{aligned}
&\Big[\rho(E_i)(z),\ \ \rho(E_{i-1})(w)\Big]\\
&=: a^*_{ii}(z)a^*_{i-1,i-1}(z)\left(\sum_{k=1}^{i-2}a_{k,i-2}(w)a^*_{k,i-2}(w)-\sum_{k=1}^{i-1}a_{k,i-1}(w)a^*_{k,i-1}(w)\right):\delta(z-w)\\
&-:a^*_{i-1,i}\left(\sum_{k=1}^{i-2}a_{k,i-2}a^*_{k,i-2}-\sum_{k=1}^{i}a_{ki}a^*_{ki}:\right)\delta(z-w)\\
&-:a^*_{ii}\left(\sum_{l=1}^{i-2}a_{l,i-2}a^*_{l,i-1}\right)\delta(z-w)+:a^*_{i-1,i-1}\left(\sum_{l=1}^{i-1}a_{l,i-1}a^*_{li}\right)\delta(z-w)\\
&+\left(\sum_{l=1}^{i-2}:a_{l,i-2}a^*_{li}:\right)\delta(z-w)-\left(\sum_{l=i+1}^{n}:a_{i+1,k}a^*_{i-1,k}:\right)\delta(z-w)\\
&-a^*_{ii}a^*_{i-1,i-1}b_{i-1}\delta(z-w)+a^*_{i-1,i}(b_{i-1}+b_i)\delta(z-w)\\
&+\left(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\right)\partial_w a^*_{i-1,i}(w)\delta(z-w)\\
&+a^*_{ii}(z)\partial_w a^*_{i-1,i-1}(w)\left(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\right)\delta(z-w).
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big[\rho(E_i^1)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&=: a^*_{ii}a_{i-1,i}\left(\sum_{k=1}^{i-2}a_{k,i-2}a^*_{k,i-2}:\right)\delta(z_1-w)\delta(z_2-w)\\
&+:a^*_{ii}a^*_{i-1,i}\left(\sum_{k=1}^{i}a_{ki}a^*_{ki}:-\sum_{k=1}^{i-1}a_{k,i-1}a^*_{k,i-1}\right)\delta(z_1-w)\delta(z_2-w)\\
&-:a^*_{ii}a^*_{i-1,i}\left(\sum_{k=1}^{i}a_{ki}a^*_{ki}:\right)\delta(z_1-w)\delta(z_2-w)\\
&+i\delta_{1\le i-1\le r}:a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2):\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+(i+2)\delta_{1\le i\le r}a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+\delta_{1\le i\le r}a^*_{ii}(z_2)a^*_{i-1,i}(z_1)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-\delta_{1\le i-1\le r}a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-:a^*_{ii}\sum_{l=1}^{i-2}a_{l,i-2}a^*_{li}:\delta(z_1-w)\delta(z_2-w)\\
&-:a^*_{ii}a^*_{i-1,i}:(b_{i-1}+b_i)\delta(z_1-z_2)\delta(z_2-w)\\
&-\left(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\right):a^*_{ii}(z_1)a^*_{i-1,i}(z_1):\delta(z_2-w)\partial_w\delta(z_1-w)\\
&-\left(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\right):a^*_{ii}(z_1)a^*_{ii}(z_1)\partial_w a^*_{i-1,i-1}(w):\delta(z_1-z_2)\delta(z_2-w)\\
&+\left(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\right):a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_1):\partial_w\delta(z_1-w)\delta(z_2-w).
\end{aligned}$$

Next we have

$$
\begin{aligned}
&\Big[\rho(E_i^2)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= \Bigg(- : a_{ii}^*(z_2)a_{i-1,i}^*(z_2)\left(\sum_{k=1}^{i-2} a_{k,i-2}(w)a_{k,i-2}^*(w)\right) :\\
&\qquad + : a_{ii}^*a_{i-1,i}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^*\right) : + : a_{i-1,i}^*\sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*\\
&\qquad + : a_{ii}^*\left(\sum_{l=1}^{i-2} a_{l,i-2}a_{l,i}^*\right) : - : a_{i-1,i}^*\left(\sum_{l=1}^{i-1} a_{l,i-1}a_{li}^*\right)\\
&\qquad\quad + a_{ii}^*a_{i-1,i}^*b_{i-1}\Bigg)\delta(z_1-z_2)\delta(z_2-w)\\
&- \big(\gamma^2 - \delta_{i-1\le r}i - \delta_{i-1>r}(r+1)\big) : a_{ii}^*(z_2)a_{i-1,i}^*(z_1) : \partial_w\delta(z_1-w)\delta(z_2-w).
\end{aligned}
$$

The third summation contributes

$$
\begin{aligned}
&\Big[\rho(E_i^3)(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&\ =: a_{i-1,i}^*a_{ii}^*b_i\delta(z_1-z_2)\delta(z_2-w)\\
&\ + \big(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\big)\, a_{ii}^*(z_2)a_{i-1,i}^*(z_2)\partial_{z_1}\delta(z_1-z_2)\delta(z_2-w)\\
&\ + \Bigg(- a_{ii}^*(z_1)a_{i-1,i}^*(z_2)\left(\gamma^2 - \delta_{i>r}(r+1) - \delta_{i,r+1}\right)\\
&\qquad - a_{ii}^*(z_1)a_{ii}^*(z_2)a_{i-1,i-1}^*(z_2)\left(\gamma^2 - \delta_{i-1>r}(r+1)\right)\Bigg)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\Big[\rho(E_i)(z_1),\rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&=\Big(i\delta_{1\le i-1\le r} : a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2) :\\
&-\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a^*_{ii}(z_1)a^*_{ii}(z_1)\partial_w a^*_{i-1,i-1}(w) :\\
&+\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_1) :\\
&-\big(\gamma^2-\delta_{i-1>r}(r+1)\big)\, a^*_{ii}(z_1)a^*_{ii}(z_2)a^*_{i-1,i-1}(z_2)\Big)\delta(z_2-w)\partial_w\delta(z_1-w)\\
&+(i+2)\delta_{1\le i\le r}a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-\delta_{1\le i-1\le r}a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+\delta_{1\le i\le r}a^*_{ii}(z_2)a^*_{i-1,i}(z_1)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i-1)-\gamma^2\big) : a^*_{ii}(z_1)a^*_{i-1,i}(z_1) : \delta(z_2-w)\partial_w\delta(z_1-w)\\
&-\big(\gamma^2-\delta_{i-1\le r}i-\delta_{i-1>r}(r+1)\big) : a^*_{ii}(z_2)a^*_{i-1,i}(z_1) : \partial_w\delta(z_1-w)\delta(z_2-w)\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big)\, a^*_{ii}(z_2)a^*_{i-1,i}(z_2)\delta(z_2-w)\partial_w\delta(z_1-w)\\
&-a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\big(\gamma^2-\delta_{i>r}(r+1)-\delta_{i,r+1}\big)\,\delta(z_2-w)\partial_w\delta(z_1-w)\\
&=\Bigg(\Big((i+2)\delta_{1\le i\le r}-\delta_{1\le i-1\le r}-\gamma^2+\delta_{i>r}(r+1)+\delta_{i,r+1}\Big)a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\\
&+\Big(\delta_{1\le i\le r}-\gamma^2+\delta_{i-1\le r}i+\delta_{i-1>r}(r+1)\Big) : a^*_{ii}(z_2)a^*_{i-1,i}(z_1) :\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i-1)-\gamma^2\big) : a^*_{ii}(z_1)a^*_{i-1,i}(z_1) :\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big)\, a^*_{ii}(z_2)a^*_{i-1,i}(z_2)\Bigg)\delta(z_2-w)\partial_w\delta(z_1-w)\\
&=\Big(-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big) : a^*_{ii}(z_1)a^*_{i-1,i}(z_1) :\\
&+\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big)\, a^*_{ii}(z_1)a^*_{i-1,i}(z_2)\\
&-\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big)\, a^*_{ii}(z_2)a^*_{i-1,i}(z_2)\\
&+\big(\delta_{i>r}(r+1)+\delta_{i\le r}(i+1)-\gamma^2\big) : a^*_{ii}(z_2)a^*_{i-1,i}(z_1) : \Big)\delta(z_2-w)\partial_w\delta(z_1-w)=0.
\end{aligned}
$$

Now we turn to the last series of computations:

$$\begin{aligned}
&\Big[\rho(E^1_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= -(i-2)\delta_{i-2\leq r} : a^*_{ii}(z_2)a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_2)\delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&- (i+2)\delta_{i-1\leq r} : a_{ii}(z_2)a_{i-1,i-1}(z_1)a_{i-1,i-1}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ (i-2)\delta_{i-2\leq r} : a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ 2\delta_{i-1\leq r} : a^*_{i-1,i-1}(z_1)a^*_{i-1,i}(z_2) : \delta(z_2-w)\partial_{z_2}\delta(z_1-z_2)\\
&+ 2 : a^*_{i-1,i-1}a^*_{ii}\left(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{l,i-1}\right)\delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{i-1,i-1}a^*_{i-1,i}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&+ a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&- : a^*_{i-1,i-1}\Big(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{li} : \Big)\delta(z_1-z_2)\delta(z_2-w)\\
&+ : a^*_{i-1,i-1}a^*_{i-1,i-1}a^*_{ii}b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&- \big(\gamma^2-\delta_{i-1\leq r}i-\delta_{i-1>r}(r+1)\big) : a^*_{i-1,i-1}(z_1)a^*_{i-1,i-1}(z_1)a^*_{ii}(z_2) : \delta(z_2-w)\partial_w\delta(z_1-w).
\end{aligned}$$

Next we have

$$\begin{aligned}
&\Big[\rho(E^2_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&=: a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-2}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&- 2 : a^*_{ii}a^*_{i-1,i-1}\Big(\sum_{k=1}^{i-2} a_{k,i-2}a^*_{k,i-1} : \Big)\delta(z_1-z_2)\delta(z_2-w)\\
&- a^*_{i-1,i}a^*_{i-1,i-1}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1}\right) : \delta(z_1-z_2)\delta(z_2-w)\\
&+ : a^*_{i-1,i-1}\left(\sum_{l=1}^{i-2} a_{l,i-2}a^*_{li}\right)\delta(z_1-z_2)\delta(z_2-w)\\
&- a^*_{i-1,i}a^*_{i-1,i-1}b_{i-1}\delta(z_1-z_2)\delta(z_2-w)\\
&+ \big(\gamma^2-\delta_{i-1\leq r}i-\delta_{i-1>r}(r+1)\big)\, a^*_{i-1,i}(z_1)\partial_w a^*_{i-1,i-1}(w)\delta(z_1-z_2)\delta(z_2-w).
\end{aligned}$$

The third summation contributes

$$
\begin{aligned}
&\Big[\rho(E^3_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= - : a^*_{ii} a^*_{i-1,i-1} a^*_{i-1,i-1} : b_{i-1} \delta(z_1 - z_2)\delta(z_2 - w)\\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\le r} i - \gamma^2\big) : a^*_{ii}(z_2) a^*_{i-1,i-1}(z_2) a^*_{i-1,i-1}(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2)\\
&+ : a^*_{i-1,i-1} a^*_{i-1,i} : b_{i-1}\delta(z_1 - z_2)\delta(z_2 - w)\\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\le r} i - \gamma^2\big) : a^*_{i-1,i-1}(z_2) a^*_{i-1,i}(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2)\\
&- (\gamma^2 - (r+1)\delta_{i>r+2} - \delta_{i,r+2}) a^*_{i-1,i-1}(z_1) a^*_{i-1,i}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} + \frac{r}{2}\delta_{i,r+2}) a^*_{i-1,i-1}(z_1) a^*_{i-1,i-1}(z_2) a^*_{i,i}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2).
\end{aligned}
$$

Adding these all up we get

$$
\begin{aligned}
&\Big[\rho(E_{i-1})(z_1),\ \ \rho(E_i)(z_2),\ \ \rho(E_{i-1})(w)\Big]\\
&= -(i-2)\delta_{i-2\le r} : a^*_{ii}(z_2) a^*_{i-1,i-1}(z_1) a^*_{i-1,i-1}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&- (i+2)\delta_{i-1\le r} : a_{ii}(z_2) a_{i-1,i-1}(z_1) a_{i-1,i-1}(z_2) : \delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&+ (i-2)\delta_{i-2\le r} : a^*_{i-1,i-1}(z_1) a^*_{i-1,i}(z_2) : \delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&+ 2\delta_{i-1\le r} : a^*_{i-1,i-1}(z_1) a^*_{i-1,i}(z_2) : \delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&- \big(\gamma^2 - \delta_{i-1\le r} i - \delta_{i-1>r}(r+1)\big) : a^*_{i-1,i-1}(z_1) a^*_{i-1,i-1}(z_1) a^*_{ii}(z_2) : \delta(z_2 - w)\partial_w\delta(z_1 - w)\\
&+ \big(\gamma^2 - \delta_{i-1\le r} i - \delta_{i-1>r}(r+1)\big)\, a^*_{i-1,i}(z_1)\partial_w a^*_{i-1,i-1}(w)\delta(z_1 - z_2)\delta(z_2 - w)\\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\le r} i - \gamma^2\big) : a^*_{ii}(z_2) a^*_{i-1,i-1}(z_2) a^*_{i-1,i-1}(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2)\\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\le r} i - \gamma^2\big) : a^*_{i-1,i-1}(z_2) a^*_{i-1,i}(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2)\\
&- (\gamma^2 - (r+1)\delta_{i>r+2} - \delta_{i,r+2}) a^*_{i-1,i-1}(z_1) a^*_{i-1,i}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} + \frac{r}{2}\delta_{i,r+2}) a^*_{i-1,i-1}(z_1) a^*_{i-1,i-1}(z_2) a^*_{i,i}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)
\end{aligned}
$$

$$
\begin{aligned}
&= - \big(\gamma^2 - \delta_{i-1\le r} i - \delta_{i-1>r}(r+1)\big) : a^*_{i-1,i-1}(z_1) a^*_{i-1,i-1}(z_1) a^*_{ii}(z_2) : \delta(z_2 - w)\partial_w\delta(z_1 - w)\\
&+ 2(\gamma^2 - (r+1)\delta_{i>r+1} - i\delta_{i\le r+1}) a^*_{i-1,i-1}(z_1) a^*_{i-1,i-1}(z_2) a^*_{i,i}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\le r} i - \gamma^2\big) : a^*_{ii}(z_2) a^*_{i-1,i-1}(z_2) a^*_{i-1,i-1}(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2)\\
&+ \big(\delta_{i-1>r}(r+1) + \delta_{i-1\le r} i - \gamma^2\big) : a^*_{i-1,i-1}(z_2) a^*_{i-1,i}(z_2) : \delta(z_2 - w)\partial_{z_1}\delta(z_1 - z_2)\\
&- \big(\gamma^2 - \delta_{i-1\le r} i - \delta_{i-1>r}(r+1)\big)\, a^*_{i-1,i-1}(z_1) a^*_{i-1,i}(z_2)\delta(z_2 - w)\partial_{z_2}\delta(z_1 - z_2)\\
&+ \big(\gamma^2 - \delta_{i-1\le r} i - \delta_{i-1>r}(r+1)\big)\, a^*_{i-1,i}(z_1)\partial_w a^*_{i-1,i-1}(w)\delta(z_1 - z_2)\delta(z_2 - w).\\
&= 0
\end{aligned}
$$

□

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Department of Mathematics, College of Charleston, 66 George Street, Charleston SC 29424, USA
E-mail address: coxbl@cofc.edu

School of Mathematics and Statistics, University of Sydney, Sydney 2006, Australia, On leave from Institute of Mathematics, University of São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970, Brazil
E-mail address: futorny@ime.usp.br