Surgery on $\widetilde{\mathbb{SL}} \times \mathbb{E}^n$-manifolds

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ABSTRACT

We show that although closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^n$-manifolds do not admit metrics of nonpositive sectional curvature, the arguments of Farrell and Jones can be extended to show that such manifolds are topologically rigid, if $n \geq 2$. [1]

Smooth manifolds with Riemannian metrics of nonpositive curvature are topologically rigid, by the work of Farrell and Jones [3]. In [7] this work was used to establish topological rigidity for $M \times D^k$ for all orientable closed irreducible 3-manifolds M with $\beta_1(M) > 0$ and all $k \geq 3$. We shall adapt the approach of [7] to show that all closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^n$-manifolds with $n \geq 2$ are topologically rigid, although such manifolds do not admit metrics of nonpositive curvature (see [1]). As a corollary we show that $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are rigid up to s-cobordism, settling the one case left open by Theorem 9.11 of [4].

If G is a group let ζG and $\sqrt{G}$ denote the centre and the Hirsch-Plotkin radical of G, respectively. Let $E(n) = Isom(\mathbb{E}^n) = \mathbb{R}^n \rtimes O(n)$. The following lemma is based on Lemma 9.5 of [4].

Lemma 1. *Let π be a finitely generated group with normal subgroups $A \leq N$ such that A is free abelian of rank r, $[\pi : N] < \infty$ and $N \cong A \times N/A$. Then there is a homomorphism $f : \pi \to E(r)$ with image a discrete cocompact subgroup and such that $f|_A$ is injective.*

Proof. We may assume that the index $[\pi : N]$ is minimal among all such normal subgroups containing A as a direct factor. Let $G = \pi/N$. Then G is finite. Let $M = N^{ab} \cong A \oplus (N/AN')$. Then M

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is a finitely generated $\mathbb{Z}[G]$-module, and A is a submodule. The ring $\mathbb{Q}[G]$ is semisimple and so $\mathbb{Q}M \cong \mathbb{Q}A \oplus P$, where the complementary summand P is also a $\mathbb{Q}[G]$-submodule. Let $K < M$ be the kernel of the homomorphism from M to $\mathbb{Q}A$ induced by projection to the first factor, and let $\tilde{K}$ be the preimage of K in π. Then $M/K \cong Z^r$, since it is finitely generated and torsion free of rank r, while K is a $\mathbb{Z}[G]$-submodule and so $\tilde{K}$ is normal in π. Moreover $A \cap \tilde{K} = 1$ and so A projects isomorphically to a subgroup of finite index in $H = \pi/\tilde{K}$. Let T be a finite normal subgroup of H. Then $A \cap T = 1$ and hence $T = 1$, by minimality of the index $[\pi : N]$. Therefore G acts effectively on M/K and so H is isomorphic to a discrete cocompact subgroup of $E(r)$. □

Corollary. *Let M be a 3-manifold which is Seifert fibred over a complete open $\mathbb{H}^2$-orbifold B of finite area. Then M is homeomorphic to a complete open $\mathbb{H}^2 \times \mathbb{E}^1$-manifold.*

Proof. Let $\pi = \pi_1(M)$ and let $A \cong Z$ be the image in π of the fundamental group of the general fibre. Let $p : \pi \to \pi/A \cong \pi_1^{orb}(B)$ be the epimorphism given by the Seifert fibration, and let $\psi : \pi_1^{orb}(B) \to Isom(\mathbb{H}^2)$ be a monomorphism onto a discrete subgroup of finite coarea which determines the hyperbolic structure of B.

Since B is complete and has finite area $\pi_1^{orb}(B)$ is finitely generated and since B is open $\pi_1^{orb}(B)$ has a free normal subgroup F of finite index. Then π is finitely generated. Let $N = p^{-1}(F) \cap C_\pi(A)$. Then $A < N$ and $N \cong A \times (N/A)$, since A is central in N and N/A is free. Hence there is a homomorphism $f : \pi \to E(1)$ which is injective on A, by the lemma. Let $\theta = (\psi p, f) : \pi \to Isom(\mathbb{H}^2 \times \mathbb{E}^1)$. Then θ is injective, and $\theta(\pi)$ is a discrete subgroup of finite covolume. Since $\theta(\pi)$ is torsion free it acts freely and so $N = H^2 \times R/\theta(\pi)$ is a complete open $\mathbb{H}^2 \times \mathbb{E}^1$-manifold of finite volume. Projection from $H^2 \times R$ onto the first factor induces a Seifert fibration of N over B, and since $\pi_1(N) \cong \pi = \pi_1(M)$ it follows that M and N are homeomorphic. □

In particular, if M is a compact 3-manifold with a nontrivial JSJ decomposition then every geometric piece of type $\widetilde{\mathbb{SL}}$ also admits the geometry $\mathbb{H}^2 \times \mathbb{E}^1$. A similar argument shows that if M is an open $(n+2)$-manifold which is the total space of an orbifold bundle with base a complete open hyperbolic 2-orbifold B of finite area, general fibre a flat

n-manifold F and monodromy group a finite subgroup of $Out(\pi_1(F))$ then there is an $\mathbb{H}^2 \times \mathbb{E}^n$-manifold M_1 which is an orbifold bundle with base B and general fibre F and a homotopy equivalence $f : M \to M_1$ which preserves the conjugacy classes of the subgroups corresponding to the cusps. Since the cusps are flat $(n+1)$-manifolds we may assume that f is a homeomorphism off a compact set, and a relative version of the Farrell-Jones curvature argument then shows that f is homotopic to a homeomorphism, if $n \geq 3$. Is there a direct, elementary argument to show that M and M_1 must be fibrewise diffeomorphic (for any $n \geq 1$)?

In [5] it is shown that if an aspherical $(n+2)$-manifold M admits an effective T^n action of hyperbolic type then the higher Whitehead groups $Wh_i(\pi_1(M))$ are trivial for all $i \geq 0$ and $|S_{TOP}(M \times D^k, \partial)| = 1$, whenever $n+k \geq 4$ (or $n+k \geq 3$, if $\partial M = \emptyset$). Their argument for the Whitehead groups extends immediately to the following situation.

Lemma 2. *Let π be a torsion-free group with a virtually poly-Z normal subgroup N such that $\pi/N \cong \pi_1^{orb}(B)$, where B is a compact 2-orbifold. Then $Wh(\pi) = 0$.*

Proof. If B is a closed $\mathbb{E}^2$-orbifold then π is virtually poly-Z and the result is proven in [2]. If B is a closed $\mathbb{H}^2$-orbifold the argument of [5] using hyperelementary induction applies with little change. If π/N is virtually free it is the fundamental group of a graph of groups with all vertex groups finite or 2-ended and all edge groups finite, and so π is the fundamental group of a graph of groups with all vertex groups torsion free and virtually poly-Z. Thus the result follows from [2] and the Waldhausen Mayer-Vietoris sequence [8]. (Note that $c.d.\pi < \infty$ since π is torsion free, $c.d.N < \infty$ and $v.c.d.\pi/N \leq 2$ in all cases.) $\square$

The argument of [5] determining the surgery structure sets for such manifolds appears to use the hypothesis of a toral action in an essential way, to establish an induction on n. We shall rely instead on the curvature argument of [3].

Theorem 3. *Let M be a closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^n$-manifold, where $n \geq 2$, and let $f : M_1 \to M$ be a homotopy equivalence. Then f is homotopic to a homeomorphism.*

Proof. The composite of projection from the model space $\widetilde{SL} \times R^n$ onto the first factor with the fibration of $\widetilde{SL}$ over $\mathbb{H}^2$ induces an orbifold bundle fibration $p : M \to Q$, with base Q a closed $\mathbb{H}^2$-orbifold and general fibre F a flat n-manifold. Let $p:\pi=\pi_1(M)\to\pi_1^{orb}(Q)$ be the induced epimorphism. In Theorem 9.3 of [4] it is shown that when $n=1$ the fundamental group of a closed $\widetilde{\mathbb{SL}}\times\mathbb{E}^n$-manifold has a subgroup of finite index which is a direct product, and the argument extends immediately to the general case. It follows that $A=\sqrt{\pi_1(F)}\cong Z^{n+1}$ is centralized by a subgroup of finite index in π.

Suppose first that there is an epimorphism $q:\pi_1^{orb}(Q)\to Z$. Let $\hat{Q}$ and $\hat{M}$ be the induced covering spaces and $\hat{p}:\hat{M}\to\hat{Q}$ be the corresponding fiber bundle projection. Then $\hat{Q}$ is noncompact, and is the increasing union $\hat{Q}=\cup_{k\geq1}Q_k$ of compact suborbifolds with nontrivial boundary. We may assume that for each $k\geq0$ the boundary of Q_k does not contain any corner points, $G_k=\pi_1^{orb}(Q_k)$ is not virtually abelian, and G_k maps injectively to $G=\pi_1^{orb}(\hat{Q})$. Let DQ_k be the closed orbifold obtained by doubling Q_k along its boundary. Since $\pi_1^{orb}(DQ_k)$ is not virtually abelian there is a monomorphism $\psi:\pi_1^{orb}(DQ_k)\to Isom(\mathbb{H}^2)$ with image a discrete, cocompact subgroup. (See page 248 of [9].)

Let $M_k=\hat{p}^{-1}(Q_k)$. Then M_k is a compact bounded $(n+3)$-manifold and $\hat{p}:M_k\to Q_k$ is an orbifold fibration with general fibre F. Doubling M_k gives a closed $(n+3)$-manifold DM_k with an orbifold fibration over DQ_k, and $\pi(k)=\pi_1(DM_k)$ is an extension of $\pi_1^{orb}(DQ_k)$ by $\pi_1(F)$. As $\pi_1^{orb}(Q_k)$ acts on A through a finite subgroup the centralizer of A in $\pi(k)$ again has finite index. Let N be a characteristic subgroup of finite index in $\pi(k)$ which centralizes A and such that N/A is a PD_2^+-group, and let $e\in H^2(N/A;A)$ be the cohomology class of the extension $0\to A\to N\to N/A\to1$. The reflection which interchanges the copies of M_k leaves the boundary pointwise fixed, and projects to the corresponding reflection of DQ_k. Thus it induces an automorphism of N which is the identity on A and reverses the orientation of N/A. It follows that $e=-e$ and so the extension splits: $N\cong A\times N/A$. Therefore there is a homomorphism $f:\pi(k)\to E(n+1)$ which is injective on A, by Lemma 1. The homomorphism $(\psi_k p|_{\pi(k)},f):\pi(k)\to Isom(\mathbb{H}^2\times\mathbb{E}^{n+1})$ has finite kernel, and so is injective, since π is torsion free. The quotient $P_k=H^2\times R^{n+1}/\pi(k)$ is closed and nonpositively curved, and is Seifert fibred over DQ_k. Moreover $DM_k\simeq P_k$ since each is aspherical, and so M_k is a homotopy retract of P_k.

Now the structure set of P_k is trivial, by the Topological Rigidity theorem of Farrell and Jones [3]. Since M_k is a homotopy retract of P_k, the structure set of M_k is also trivial. Equivalently, the assembly maps $H_j(M_k;\mathbb{L}_o^w) \to L_j(\pi_1(M_k),w)$ are isomorphisms for j large, where $w = w_1(M)$. (Note that no decorations are needed on the surgery obstruction groups as $Wh(\pi) = 0$, by Lemma 2.) Since homology and L-theory commute with direct limits we conclude that $H_j(\hat{M};\mathbb{L}_o^w) \to L_j(\pi_1(\hat{M}),w)$ is an isomorphism for j large. Using the Wang sequence for homology, naturality of the assembly maps and Ranicki's algebraic version of Cappell's Mayer-Vietoris sequence for square root closed HNN extensions it follows that the same is true for M. (See [7] for more details.)

If $\beta_1(\pi_1^{orb}(Q)) = 0$ we may use hyperelementary induction, as in [5], to reduce to the case already treated. □

A similar curvature argument could be used to show that $Wh(\pi) = 0$, for $\pi = \pi_1(M)$ as in the theorem.

We may adapt this result to obtain a somewhat weaker result for the case $n = 1$ by taking products with S^1.

Corollary. *Let N be a closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold. If N_1 is homotopy equivalent to N then it is s-cobordant to N.*

Proof. Let $M = N \times S^1$, $M_1 = N_1 \times S^1$, and $f = g \times id_{S^1}$, where $g : N_1 \to N$ is a homotopy equivalence. Then M is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^2$-manifold. Hence f is homotopic to a homeomorphism $h : M_1 \cong M$, by the theorem. Since $h \sim g \times id_{S^1}$ it lifts to a homeomorphism $N_1 \times R \cong N \times R$. The submanifold of $N \times R$ bounded by $N \times \{0\}$ and a disjoint copy of N_1 is an h-cobordism. It is in fact an s-cobordism, since $Wh(\pi_1(N)) = 0$, by Lemma 2. □

This result complements Theorem 9.11 of [4], where a similar result is proven for all 4-manifolds admitting a nonpositively curved geometry.

Is there a corresponding result for manifolds with a proper geometric decomposition? The argument for Theorem 3.3 of [6] extends readily to show that if M is a n-manifold with a finite collection of disjoint flat hypersurfaces $\mathcal{S}$ such that the components of $M - \cup\mathcal{S}$ all have complete finite volume geometries of type $\mathbb{H}^n$ or $\mathbb{H}^{n-1} \times \mathbb{E}^1$, and if there is at least one piece of type $\mathbb{H}^n$ then M admits a Riemannian metric of nonpositive sectional curvature (see [1]). Such manifolds are topologically rigid if $n \geq 5$, by [3], and we again deduce rigidity up to s-cobordism when $n = 4$, as in the above corollary.

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