GOOD l-FILTRATIONS FOR $q - \mathrm{GL}_3(k)$

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ABSTRACT. Let k be an algebraically closed field of characteristic p, possibly zero, and $G = q$-$\mathrm{GL}_3(k)$, the quantum group of three by three matrices as defined by Dipper and Donkin. We may also take G to be $\mathrm{GL}_3(k)$. We first determine the extensions between simple G-modules for both G and G_1, the first Frobneius kernel of G. We then determine the submodule structure of certain induced modules, $\hat{Z}(\lambda)$, for the infinitesimal group G_1B. We induce this structure to G to obtain a good l-filtration of certain induced modules, $\nabla(\lambda)$, for G. We also determine the homomorphisms between induced modules for G.

INTRODUCTION

Let k be an algebraically closed field of characteristic p, possibly zero. In this paper we study the module category for $G = q$-$\mathrm{GL}_3(k)$, the quantum group of three by three matrices. We use the quantisation of Dipper and Donkin [7]. We may also take G to be $\mathrm{GL}_3(k)$, that is the classical group scheme of three by three invertible matrices.

We want to determine explicitly the structure of two types of modules. First we determine the submodule structure of certain induced modules, $\hat{Z}(\lambda)$, for the infinitesimal group G_1B. We then induce this structure to G to obtain a good l-filtration of certain induced modules, $\nabla(\lambda)$, for G. We also determine the homomorphisms between induced modules for G.

This paper generalises several classical results including the extensions between simple modules for $\mathrm{SL}_3(k)$, [16], the submodule structure of the $\hat{Z}(\lambda)$'s for $\mathrm{SL}_3(k)$, [11], some results about translations, [13], good p-filtrations of the induced modules $\nabla(\lambda)$ for $\mathrm{SL}_3(k)$, [15], and the homomorphisms between induced modules for $\mathrm{SL}_3(k)$, [6]. It also clears up some confusion regarding the validity of results of Irving [11] and Parker [15] for small primes. A large part of this paper produces a quantum version of many results of the PhD thesis of Yehia, [16]. We have reproduced some of his arguments, only applied to the quantum case, as this reference is not that accessible.

1. NOTATION

We first review the basic concepts and most of the notation that we will be using. A very brief introduction to the theory of quantum groups and how it relates to linear algebraic groups may be found in [10, chapter 0]. Some of the cohomological theory of quantum groups and their q-Schur algebras appears in [9]. We will also refer to [2] for many of the basic properties of quantum groups.

Throughout this paper k will be an algebraically closed field of characteristic p which may be zero.

First take G to be $\mathrm{GL}_3(k)$. We take l to be p which we assume for this particular case to be non-zero. We let T be the diagonal matrices in G and B, a Borel subgroup, be the lower triangular matrices. We will write $\mathrm{Mod}(G)$ for the category of dimensional rational G-modules and $\mathrm{mod}(G)$ for the category of finite dimensional rational G-modules. We let D be the one-dimensional determinant module for G.

Now take G to be q-$\mathrm{GL}_3(k)$ the quantum group of Dipper and Donkin, as defined in [10]. We write $\mathrm{Mod}(G)$ for the category of right comodules of $k[G]$, the Hopf algebra of G and $\mathrm{mod}(G)$ for the category of finite dimensional right comodules of $k[G]$. If q is not a root of unity then $\mathrm{mod}(G)$ is semi-simple. We will thus consider the case where q is a primitive lth root of unity with $l \geqslant 2$. We take T, and B as defined in [9]. We let D be the one-dimensional module for G, where G acts by the quantum determinant as defined in [9].

We now consider both cases together.

Let $X(T) = X \cong \mathbb{Z}^3$ be the weight lattice for G with $\mathbb{Z}$-basis $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$. Every module in $\mathrm{mod}(G)$ is semi-simple as a T-module and we define the formal character $\mathrm{ch}(V) \in \mathbb{Z}X$ of V to be the character of V restricted to T. We use $e(\lambda)$ with $\lambda \in X$ as a basis for $\mathbb{Z}X$, so to distinguish characters from the structure of the weight lattice as a $\mathbb{Z}$ vector space. We thus have $e(\lambda)e(\mu) = e(\lambda + \mu)$ in $\mathbb{Z}X$.

We set $R = \{e_i - e_j \mid i \neq j\}$ to be the roots of G. For each $\alpha \in R$ we take $\alpha^\vee = \alpha \in X$ to be the coroot of α. (Here we have identified the weight space with the dual weight space, as we are only considering GL_3, the two are isomorphic.) Let $R^+ = \{e_i - e_j \mid i < j\}$ be the positive roots, (chosen so that B is the negative Borel) and let $S = \{e_i - e_{i+1}\}$ be the set of simple roots. Set $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = (1,0,-1)$.

We have a partial order on X defined by $\mu \leqslant \lambda \Leftrightarrow \lambda - \mu \in \mathbb{N}S$. We also have a bilinear form $\langle -, - \rangle : X \times X \to \mathbb{Z}$ with $\langle e_i, e_j{}^\vee \rangle = \delta_{ij}$ (Kronecker delta). A weight λ is *dominant* if $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in S$ and we let X^+ be the set of dominant weights. In this case $X^+ = \{(a,b,c) \mid a \geqslant b \geqslant c\}$.

Take $\lambda \in X^+$ and let k_λ be the one-dimensional module for B which has weight λ. We define the induced module, $\nabla(\lambda) = \mathrm{Ind}_B^G(k_\lambda)$. This module has formal character given by Weyl's character formula and has simple socle $L(\lambda)$, the irreducible G-module of highest weight λ. These completely exhaust the simple modules in $\mathrm{mod}(G)$. We will denote the socle of a module M by $\mathrm{soc}(M)$.

We return to considering the weight lattice X for G. We consider the affine reflections $s_{\alpha,ml}$ for α a positive root and $m \in \mathbb{Z}$ which act on X as $s_{\alpha,ml}(\lambda) = \lambda - (\langle \lambda, \alpha^\vee \rangle - ml)\alpha$. These generate the affine Weyl group W_l. We let W be the Weyl group of G which is generated by $s_{(1,-1,0),0}$ and $s_{(0,1,-1),0}$. We mostly use the dot action of W_l on X which is the usual action of W_l, with the origin shifted to $-\rho$. So we have $w \cdot \lambda = w(\lambda + \rho) - \rho$. The reason for this is the following, sometimes known as the linkage principle.

Proposition 1.1 ([2, corollary 8.2])**.** *Let $V \in \mathrm{mod}(G)$ and V be indecomposable. If $L(\mu)$ and $L(\lambda)$ are composition factors of V then $\mu \in W_l \cdot \lambda$.*

We now define the quantum version of translation functors. These are defined in [2, section 8]. For any G-module V and any $\mu \in X$, set $\mathrm{pr}_\mu V$ equal to the sum of submodules of V such that all the composition factors have highest weight in $W_p \cdot \mu$. Then $\mathrm{pr}_\mu V$ is the largest submodule of V with this property.

Definition 1.2. Suppose $\lambda,\ \mu \in \bar{C}$. There is a unique $\nu_1 \in X^+ \cap W(\mu - \lambda)$. We define the *translation functor* T^μ_λ from λ to μ via

$$T^\mu_\lambda V = \mathrm{pr}_\mu(L(\nu_1) \otimes \mathrm{pr}_\lambda V)$$

for any G-module V. It is a functor from $\mathrm{mod}(G)$ to itself.

These functors have similar properties to the classical ones, as remarked in [2, section 8].

A *facet* for W_l is a non-empty set of the form

$$F = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \ \mid \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha l \quad \forall\, \alpha \in R_0^+(F),$$
$$(n_\alpha - 1)l < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha l \quad \forall\, \alpha \in R_1^+(F)\}$$

for suitable $n_\alpha \in \mathbb{Z}$ and for a disjoint decomposition $R^+ = R_0^+(F) \cup R_1^+(F)$.

The *closure* $\bar{F}$ of a facet F is similar but with the inequalities replaced with equalities. The *upper closure* $\hat{F}$ of a facet F is defined as

$$\hat{F} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \ \mid \langle \lambda_\rho, \alpha^\vee \rangle = n_\alpha l \quad \forall\, \alpha \in R_0^+(F),$$
$$(n_\alpha - 1)l < \langle \lambda + \rho, \alpha^\vee \rangle \leqslant n_\alpha l \quad \forall\, \alpha \in R_1^+(F)\}$$

A facet F is an *alcove* if $R_0^+(F) = \emptyset$, (or equivalently F is open in $X \otimes_{\mathbb{Z}} \mathbb{R}$). If F is an alcove for W_l then its closure $\bar{F} \cap X$ is a fundamental domain for W_l operating on X. The group W_l permutes the alcoves simply transitively. We set $C = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \ \mid \ 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l \quad \forall\, \alpha \in R^+\}$ and call C the *fundamental alcove*. We have $C \cap X \neq \emptyset$ if and only if $l \geqslant 3$, the Coxeter number of G.

A facet F is a *wall* if there exists a unique $\beta \in R^+$ with $\langle \lambda + \rho, \beta^\vee \rangle = ml$ for some $m \in \mathbb{Z}$ and for all $\lambda \in F$.

The category $\mathrm{Mod}(G)$ has enough injectives and so we may define $\mathrm{Ext}^*_G(-,-)$ as usual by using injective resolutions (see [3], section 2.4 and 2.5).

We let F be the Frobenius morphism from $G \to \mathrm{GL}_3(k)$, and denote by M^{F} the Frobenius twist of a module for $\mathrm{GL}_3(k)$. We will sometimes distinguish modules for $\mathrm{GL}_3(k)$ and G by a bar, $\bar{}$. We set X_1 to be the l-restricted weights. Thus $X_1 = \{(\lambda_1, \lambda_2, \lambda_3) \mid 0 \leqslant \lambda_1 - \lambda_2 < l \text{ and } 0 \leqslant \lambda_2 - \lambda_3 < l\}$. We let G_1 be the kernel of F as a group scheme, (it has defining ideal generated by $c_{ij}{}^l - \delta_{ij}$ where the c_{ij} are the coordinate functions generating the Hopf algebra $k[G]$ and δ_{ij} is the Kronecker delta).

We define λ' and λ'' for $\lambda \in X^+$, $\lambda = l\lambda'' + \lambda'$ with $\lambda'' \in X^+$ and $\lambda' \in X_1$. We will use Steinberg's tensor product theorem: $L(\lambda) \cong \bar{L}(\lambda'')^{\mathrm{F}} \otimes L(\lambda')$, where $\lambda \in X^+$. We define $\nabla_l(\lambda) = \overline{\nabla}(\lambda_1)^{\mathrm{F}} \otimes L(\lambda_0)$.

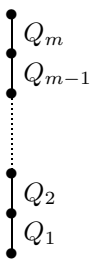
We let $\hat{Z}(\lambda) = \mathrm{Ind}_B^{G_1B} k_\lambda$ and $\hat{L}(\lambda)$ be the simple module for G_1B of highest weight λ. (Note: this is the $\hat{Z}'(\lambda)$ of [13], we have dropped the primes, and so our $\hat{Z}(\lambda)$ is not to be confused with the $\hat{Z}(\lambda)$ of [13]. The subgroup G_1B has defining ideal generated by c_{ij}^l with $i < j$. Our $\hat{Z}(\lambda)$ upon restriction to G_1T, the subgroup with defining ideal generated by c_{ij}^l with $i \neq j$, is the $\hat{\nabla}_1(\lambda)$ of [10] and our "G_1T" is the Janzten subgroup $\hat{G}_1$ of [10]. This reference doesn't consider the case with G_1B. But many properties for G_1B can be deduced from the properties for G_1T.) We have $\hat{L}(\lambda) \cong L(\lambda') \otimes k_{l\lambda''}$. We will often use a hat $\hat{}$ to distinguish modules for G_1B from those for G. Note that we have $\nabla_l(\lambda) \cong \mathrm{Ind}_{G_1B}^G(\hat{L}(\lambda))$. We also note that the $\nabla_l(\lambda)$ are indecomposable with simple socle $L(\lambda)$.

We denote the composition multiplicity of a simple module L in a module M by $[M : L]$.

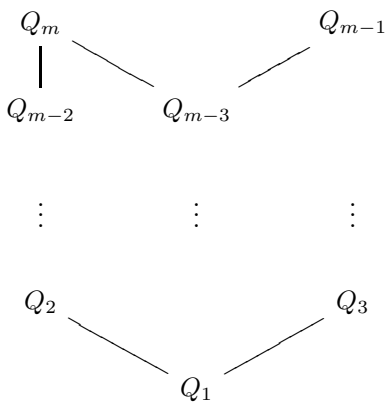
Suppose a G-module M has a filtration:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m,$$

with quotients $Q_i = M_i/M_{i-1}$. This will be depicted graphically as



We will also draw pictures like so



when we have more information about the extensions appearing between the Q_i in the module M. So the above picture represents a module with an indecomposable submodule with Q_1 and Q_2 as factors, etc.

If every quotient Q_i is isomorphic to $\nabla(\mu_i)$ for some $\mu_i \in X^+$ then we say that M has a *good filtration*. If every quotient Q_i is isomorphic to $\nabla_l(\mu_i)$ for some $\mu_i \in X^+$ then we say that M has a *good l-filtration*. We will often abbreviate this to just l-filtration. If every quotient Q_i is isomorphic to dual induced modules $\nabla(\mu_i)^*$ for some $\mu_i \in X^+$ then we say that M has a *Weyl filtration*.

Good filtration multiplicities and Weyl filtration multiplicities, like composition multiplicities are well defined. It is conjectural that the same holds for good l-filtration multiplicities. They are if a conjecture of Donkin holds — this is the subject of [1].

We say a module is a *tilting module* if it has both a good filtration and a Weyl filtration. For each $\lambda \in X^+$ there is a unique indecomposable tilting module $T(\lambda)$ with $[T(\lambda) : L(\lambda)] = 1$.

Important convention: All weights (a, b, c) will be denoted $(a - b, b - c)$.

Normally we would label the highest weight modules by $\lambda \in X^+$. However we don't want to have to keep track of the degree of the representation. That is, we really want to pretend we are looking at modules for $\mathrm{SL}_3(k)$, even though such an object does not exist for the Dipper-Donkin quantisation, as the determinant is not central. Since, however, we only need to consider polynomial modules and this category splits up into a direct sum of homogeneous ones, we may assume that we are always looking at modules of the same degree. Also we have the isomorphisms $\nabla(a+d, b+d, c+d) \cong \nabla(a, b, c) \otimes D^{\otimes d}$, $L(a+d, b+d, c+d) \cong L(a, b, c) \otimes D^{\otimes d}$ and $T(a+d, b+d, c+d) \cong T(a, b, c) \otimes D^{\otimes d}$. Thus we will label modules by the equivalent $\mathrm{SL}_3(k)$ weights. Thus all the results in this paper will be in $\mathrm{SL}_3(k)$ notation (i.e. our weights are in $\mathbb{N}^{\oplus 2}$). We may convert back by adding an appropriate power of the determinant so that the modules all have the same degree.

2. Preliminaries

We first start off by noting the composition series of small induced modules.

Lemma 2.1. (i) *Suppose $\lambda = (r, s)$ with $(r, s) \in \hat{C}$, or $\lambda = (l-1, r)$ or $(r, l-1)$ with $0 \leqslant r \leqslant l-1$. Then $\nabla(\lambda) = L(\lambda)$.*

(ii) *Suppose $\lambda = (l-s-2, l-r-2)$ with $(r, s) \in C$. Then $\nabla(\lambda)$ has two composition factors with $L(\lambda)$ as its socle and $L(r, s)$ as its head.*

(iii) *Suppose $\lambda = l(1, 0) + (r, s)$ with $(r, s) \in \hat{C}$. Then $\nabla(\lambda)$ has two composition factors with $L(\lambda)$ as its socle and $L(l-r-2, r+s+1)$ as its head.*

(iv) *Suppose $\lambda = l(0, 1) + (r, s)$ with $(r, s) \in \hat{C}$. Then $\nabla(\lambda)$ has two composition factors with $L(\lambda)$ as its socle and $L(r+s+1, l-r-2)$ as its head.*

This may be proved as in the classical case using Jantzen's sum formula and translation functors.

3. Translating the ∇_l's

We start by considering the action of the translation functors on the ∇_l's.

Lemma 3.1. *The translate of a G-module with a good l-filtration also has a good l-filtration.*

Proof. This follows using the results of [1] and the definition of translation functors. □

We start by translating "onto the walls".

Proposition 3.2. *Let λ, $\mu \in \bar{C}$ such that μ belongs to the closure of the facet containing λ. Let $w \in W_l$ with $w \cdot \lambda \in X^+$ and denote by F the facet with $w \cdot \lambda \in F$. Then*

$$T_\lambda^\mu \nabla_l(w \cdot \lambda) \cong \begin{cases} \nabla_l(w \cdot \mu), & \text{if } w \cdot \mu \in \hat{F}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Now by definition

$$T_\lambda^\mu \nabla_l(w \cdot \lambda) \cong \mathrm{pr}_\mu(\nabla_l(w \cdot \lambda) \otimes L(\nu))$$

where ν is the unique element in $X^+ \cap W(\mu - \lambda)$, (since $\nabla_l(w \cdot \lambda)$ is indecomposable).

We may use the tensor identity,

$$\begin{aligned} T_\lambda^\mu \nabla_l(w \cdot \lambda) &\cong T_\lambda^\mu \operatorname{Ind}_{G_1 B}^G \hat{L}(w \cdot \lambda) \\ &\cong \mathrm{pr}_\mu(\operatorname{Ind}_{G_1 B}^G(\hat{L}(w \cdot \lambda)) \otimes L(\nu)) \\ &\cong \mathrm{pr}_\mu(\operatorname{Ind}_{G_1 B}^G(\hat{L}(w \cdot \lambda) \otimes L(\nu))) \\ &\cong \operatorname{Ind}_{G_1 B}^G(\hat{\mathrm{pr}}_\mu(\hat{L}(w \cdot \lambda) \otimes L(\nu))) \\ &\cong \operatorname{Ind}_{G_1 B}^G(\hat{T}_\lambda^\mu(\hat{L}(w \cdot \lambda))) \\ &\cong \begin{cases} \operatorname{Ind}_{G_1 B}^G \hat{L}(w \cdot \mu), & \text{if } w \cdot \mu \in \hat{F}, \\ 0, & \text{otherwise} \end{cases} \\ &\cong \begin{cases} \nabla_l(w \cdot \mu), & \text{if } w \cdot \mu \in \hat{F}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

where we use $\hat{}$'s to distinguish modules and functors for $G_1 B$ from those for G. We also use the quantum version of [13, II, remark 7.6 (1)] to identify $\hat{\mathrm{pr}}_\mu(- \otimes L(\nu))$ with the translation functor $\hat{T}_\lambda^\mu$ on $\mathrm{mod}(G_1 B)$. □

Remark 3.3. We did not use the assumption that $G = q\text{-}\mathrm{GL}_3(k)$ or $\mathrm{GL}_3(k)$ thus the above proposition is true for any quantum group or linear algebraic group G where we have the appropriate theory of $G_1 B$-modules and translation functors.

It will also be useful to know what happens when we translate back the other way. This is not as nice however and we will work it out on a case by case basis.

Proposition 3.4. *Suppose $l \geqslant 3$. Let λ, $\mu \in X^+$ with μ in the lower closure of the alcove containing λ. Then we have the following.*

(i) *Suppose $\mu' = (l-1, r)$ with $0 \leqslant r \leqslant l-2$, and $\lambda' = (a, b)$ with $0 \leqslant a \leqslant l-3$, and $0 \leqslant a+b \leqslant l-3$. Then $T^\lambda_\mu \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu')$ has a good l-filtration with factors as shown.*

$$\begin{array}{l}
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-a-2, a+b+1) \\
\overline{\nabla}(\mu''+(1,0))^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu''+(-1,1))^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu''+(0,-1))^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-a-b-3, a) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-a-b-3, a) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-a-2, a+b+1)
\end{array}$$

(ii) *Suppose $\mu' = (s, l-1)$ with $0 \leqslant s \leqslant l-2$, and $\lambda' = (a, b)$ with $0 \leqslant a \leqslant l-3$, and $0 \leqslant a+b \leqslant l-3$. Then $T^\lambda_\mu \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu')$ has a good l-filtration with factors as shown.*

$$\begin{array}{l}
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(a+b+1, l-b-2) \\
\overline{\nabla}(\mu''+(0,1))^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu''+(1,-1))^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu''+(-1,0))^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(b, l-a-b-3) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(a+b+1, l-b-2)
\end{array}$$

(iii) *Suppose $\mu' = (r, s)$ with $0 \leqslant r \leqslant l-2$ and $r+s = l-2$ and λ' is in an up alcove. Then $T^\lambda_\mu \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu')$ has a good l-filtration with factors as shown.*

$$\begin{array}{l}
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L((l-2)\rho + w_0\lambda') \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\lambda') \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L((l-2)\rho + w_0\lambda')
\end{array}$$

Proof. Case (i).

$$T^\lambda_\mu \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu') \cong \mathrm{pr}_\lambda \, \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \nabla(\mu') \otimes \nabla(\nu)$$

We may use translation to assume that λ' is such that $\nu = (1, 0)$.

Now $\nabla(l-1, r) \otimes \nabla(1, 0)$ has a good filtration with factors (starting at the top) $\nabla(l, r)$, $\nabla(l-2, r+1)$, $\nabla(l-1, r-1)$.

Thus the module $\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \nabla(\mu') \otimes \nabla(\nu)$ has a filtration as shown,

$$\begin{array}{l}
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-2, r+1) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l, r) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-r-3, 0) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-2, r+1) \\
\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l-1, r-1)
\end{array}$$

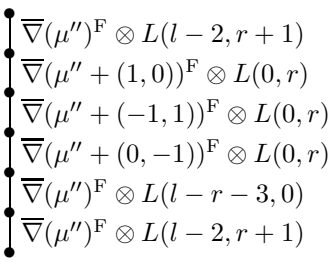
using lemma 2.1. All the simples are l-restricted except for $L(l,r)$.

Now

$$\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(l,r) \cong \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \overline{\nabla}(1,0)^{\mathrm{F}} \otimes L(0,r)$$

using Steinberg's tensor product theorem. Also $\overline{\nabla}(\mu'') \otimes \overline{\nabla}(1,0)$ has a good filtration with factors (starting at the top) $\overline{\nabla}(\mu''+(1,0))$, $\overline{\nabla}(\mu''+(-1,1))$, $\overline{\nabla}(\mu''+(0,-1))$, where the modules $\overline{\nabla}(\mu''+(-1,1))$ and $\overline{\nabla}(\mu''+(0,-1))$ are understood to be zero if the weight isn't dominant.

Now the weight $(l-1,r-1)$ is either not dominant or lies on a wall. So after applying pr_λ to our filtration of $\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \nabla(\mu') \otimes \nabla(\nu)$ we get a module with good l-filtration as shown.



We can use translation again to get the result as stated.

Case (ii). This is the dual case to case (i).

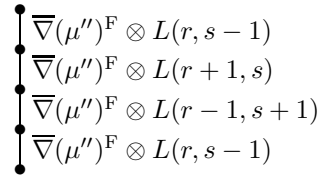
Case (iii).

$$T^{\lambda}_{\mu}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu') \cong \mathrm{pr}_\lambda\, \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \nabla(\mu') \otimes \nabla(\nu)$$

We may use translation to assume that λ' is such that $\nu = (1,0)$.

Now $\nabla(r,s) \otimes \nabla(1,0)$ has a good filtration with factors (starting at the top) $\nabla(r+1,s)$, $\nabla(r-1,s+1)$, $\nabla(r,s-1)$.

Thus the module $\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \nabla(\mu') \otimes \nabla(\nu)$ has good l-filtration as shown,



using lemma 2.1. The weight $(r-1,s+1)$ is either not dominant or lies on a wall, the other simples are all l-restricted. So after applying pr_λ we get a module with good l-filtration as above but without the $\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(r-1,s+1)$.

We can use translation again to get the result as stated. □

A similar proof shows for $l=2$ that

Proposition 3.5. *Assume that $l=2$. Let λ, $\mu \in X^+$ with μ in the lower closure of the alcove for which λ is in the upper closure. Then we have the following.*

(i) *Suppose* $\mu' = (1,0)$, *and* $\lambda' = (0,0)$ *Then* $T_\mu^{\lambda'}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu')$ *has a good* l*-filtration with factors as shown.*

$$\begin{array}{l}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(0,1)\\ \overline{\nabla}(\mu'' + (1,0))^{\mathrm{F}}\\ \overline{\nabla}(\mu'' + (-1,1))^{\mathrm{F}}\\ \overline{\nabla}(\mu'' + (0,-1))^{\mathrm{F}}\\ \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(0,1)\end{array}$$

(ii) *Suppose* $\mu' = (0,1)$, *and* $\lambda' = (0,0)$. *Then* $T_\mu^{\lambda'}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu')$ *has a good* l*-filtration with factors as shown.*

$$\begin{array}{l}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(1,0)\\ \overline{\nabla}(\mu'' + (0,1))^{\mathrm{F}}\\ \overline{\nabla}(\mu'' + (1,-1))^{\mathrm{F}}\\ \overline{\nabla}(\mu'' + (-1,0))^{\mathrm{F}}\\ \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(1,0)\end{array}$$

(iii) *Suppose* $\mu' = (0,0)$ *and* $\lambda' = (1,0)$ *or* $(0,1)$. *Then*

$$T_\mu^{\lambda'}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu') \cong \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\lambda').$$

We will also need.

Proposition 3.6. *Assume that* $l = 2$. *Let* λ, $\mu \in X^+$ *with* λ *and* μ *in the lower closure of the same alcove but on different walls. Then* $\mu' = (1,0)$, *and* $\lambda' = (0,1)$, *or* $\mu' = (0,1)$, *and* $\lambda' = (1,0)$. *We have*

$$T_\mu^{\lambda'}\overline{\nabla}(\mu'')^{\mathrm{F}} \otimes L(\mu') \cong \overline{\nabla}(\mu'')^{\mathrm{F}}.$$

Proof. Now $L(\mu') \otimes \nabla(1,0)$ has a good filtration with factors $\nabla(1,1)$ and $\nabla(0)$. This splits as $\nabla(1,1)$ is the Steinberg module. Thus

$$\mathrm{pr}_\lambda\, \overline{\nabla}(\mu'')^{\mathrm{F}} \otimes \nabla(\mu') \otimes \nabla(1,0) \cong \overline{\nabla}(\mu'')^{\mathrm{F}} \qquad \square$$

4. Characters

Each $\nabla(\lambda)$ has an l-filtration (we may use the quantum version of the argument of Jantzen [12, 3.13]) but we would like to know what the composition factors of $\hat{Z}(\lambda)$ are for $\lambda \in X$.

To do this we will work backwards - and use the formula

$$\mathrm{ch}\,\mathrm{Ind}_{G_1 B}^{G} M = \sum_{\mu \in X} [M : \hat{L}(\mu)]\chi_l(\mu) \tag{1}$$

where $\chi_l(\mu) = \mathrm{ch}\,\nabla_l(\mu) = \chi(\mu'')^{\mathrm{F}}\phi(\mu')$ where we put $\phi(\mu') = \mathrm{ch}\,L(\mu')$. This is the quantum version of [12, section 3].

Theorem 4.1. (i) *Suppose* $\lambda = l(a,b) + (l-1, l-1)$ *with* $(a,b) \in X^+$. *Then* $\chi(\lambda) = \chi(a,b)^{\mathrm{F}}\phi(l-1,l-1)$.

(ii) *Suppose* $\lambda = l(a,b) + (l-1,r)$ *with* $(a,b) \in X^+$ *and* $(l-1,r) \in X_1$. *If we set* $s = l-r-2$ *then*

$$\chi(\lambda) = \chi(a,b-1)^{\mathrm{F}}\phi(s,l-1) + \chi(a+1,b-1)^{\mathrm{F}}\phi(r,s) + \chi(a-1,b)^{\mathrm{F}}\phi(r,s) + \chi(a,b)^{\mathrm{F}}\phi(l-1,r).$$

These weights are depicted in Figure 1(a).

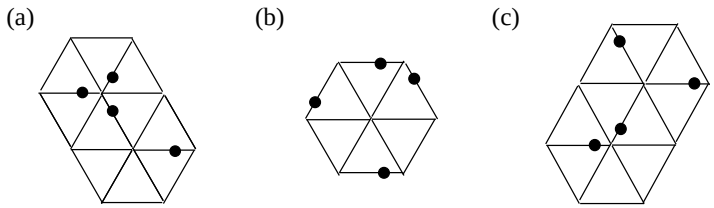


FIGURE 1. Diagram showing weights for λ on (a) a right hand wall, (b) a left hand wall and (c) a horizontal wall

(iii) *Suppose* $\lambda = l(a,b) + (s,l-1)$ *with* $(a,b) \in X^+$ *and* $(s,l-1) \in X_1$. *If we set* $r = l-s-2$ *then*

$$\chi(\lambda) = \chi(a-1,b)^{\mathrm{F}}\phi(l-1,r) + \chi(a-1,b+1)^{\mathrm{F}}\phi(r,s) + \chi(a,b-1)^{\mathrm{F}}\phi(r,s) + \chi(a,b)^{\mathrm{F}}\phi(s,l-1).$$

These weights are depicted in Figure 1(b).

(iv) *Suppose* $\lambda = l(a,b) + (r,s)$ *with* $(a,b) \in X^+$, $(r,s) \in X_1$ *and* $r+s = l-2$. *Then*

$$\chi(\lambda) = \chi(a-1,b-1)^{\mathrm{F}}\phi(r,s) + \chi(a,b-1)^{\mathrm{F}}\phi(l-1,r) + \chi(a-1,b)^{\mathrm{F}}\phi(s,l-1)\chi(a,b)^{\mathrm{F}}\phi(r,s).$$

These weights are depicted in Figure 1(c).

(v) *Suppose* $\lambda = l(a,b) + (r,s)$ *with* $(a,b) \in X^+$ *and* $(r,s) \in C$. *We let*

$$\begin{aligned}
\mu_1 &= \lambda, & \mu_2 &= (la{+}r{+}s{+}1, lb{-}s{-}2),\\
\mu_3 &= (la{+}l{-}r{-}s{-}3, lb{-}2l{+}r), & \mu_4 &= (la{-}r{-}2, lb{+}r{+}s{+}1),\\
\mu_5 &= (la{-}2l{+}s, lb{+}l{-}r{-}s{-}3), & \mu_6 &= (la{+}s, lb{-}r{-}s{-}3),\\
\mu_7 &= (la{-}l{+}r, lb{-}l{+}s), & \mu_8 &= (la{-}r{-}s{-}3, lb{+}r),\\
\mu_9 &= (la{-}s{-}2, lb{-}r{-}2).
\end{aligned}$$

These weights are depicted in Figure 2(a), where the number corresponds to the subscript of μ.

Then $\chi(\lambda) = \sum_{i=0}^{9} \chi_l(\mu_i)$.

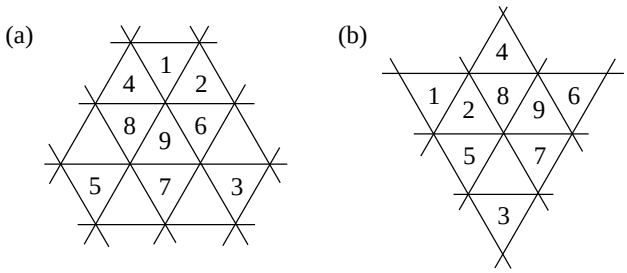


FIGURE 2. Diagram showing weights for λ inside (a) a lower alcove and (b) an upper alcove

(vi) *Suppose* $\lambda = l(a,b) + (l-s-2, l-r-2)$ *with* $(a,b) \in X^+$, *and* $(r,s) \in C$. *We let*

$$\begin{aligned}
\mu_1 &= (la{-}l{+}s, lb{+}2l{-}r{-}s{-}3), & \mu_2 &= (la{-}r{-}2, lb{+}r{+}s{+}1),\\
\mu_3 &= (la{-}l{+}r, lb{-}l{+}s), & \mu_4 &= \lambda,\\
\mu_5 &= (la{-}r{-}s{-}3, lb{+}r), & \mu_6 &= (la{+}2l{-}r{-}s{-}3, lb{-}l{+}r),\\
\mu_7 &= (la{+}s, lb{-}r{-}s{-}3), & \mu_8 &= (la{+}r, lb{+}s)\\
\mu_9 &= (la{+}r{+}s{+}1, lb{-}s{-}2).
\end{aligned}$$

These weights are depicted in Figure 2(b).

Then $\chi(\lambda) = \sum_{i=0}^{9} \chi_l(\mu_i)$.

Proof. This is easily verified using induction and translation functors and the previous propositions.

If $\lambda \in C$ then $\chi_l(\mu_i) = 0$ for $2 \leqslant i \leqslant 9$. For these μ_i, $\chi(\mu_i'') = 0$, as μ_i'' is fixed by one of the elements of W under the dot action. Thus

$$\sum_{i=0}^{9} \chi_l(\mu_i) = \chi_l(\mu_1) = \chi(\lambda)$$

using 2.1. We may use a similar argument for $\lambda \in \bar{C} \cap X^+$.

Now let $\lambda \in X^+$. If λ lies on a vertex then we have the well known result that $\nabla(\lambda) \cong \overline{\nabla}(\lambda'')^{\mathrm{F}} \otimes L(l-1, l-1)$ and thus have the required character formulae.

Suppose λ lies on a wall and $l \geqslant 3$ - then we may translate an induced module corresponding to a weight inside the alcove lying below it (μ say) onto the wall. Since $T_\mu^\lambda \nabla(\mu) = \nabla(\lambda)$ we have

$$\chi(\lambda) = \sum_i \mathrm{ch}(T_\mu^\lambda(\nabla_l(\mu_i)))$$

where μ_i are as in Figure 2. We may now use proposition 3.4 to deduce the desired character, noting that $\chi_l(\lambda_i)$ will be zero if one of the parts of λ_i'' is -1.

If λ lies inside an alcove (or lies on a wall and $l = 2$) then we may take a weight μ lying on a wall in the lower closure of (the closure of) the alcove containing λ. Then $\mathrm{ch}(T_\mu^\lambda \nabla(\mu)) = \mathrm{ch}(\nabla(\lambda)) + \mathrm{ch}(\nabla(w \cdot \lambda))$, where w is the unique reflection of W_l that fixes μ. So

$$\chi(\lambda) = \sum_i \mathrm{ch}(T_\mu^\lambda(\nabla_l(\mu_i))) - \chi(w \cdot \lambda)$$

where the μ_i will be (at most) four weights in the good l-filtration of $\nabla(\mu)$. The $\chi(w\cdot\lambda)$ is known by induction and the characters of the translated $\nabla_l(\mu_i)$ may be deduced from proposition 3.4 if $l \geqslant 3$ or propositions 3.5 and 3.6 if $l = 2$. Note that for generic μ and $l \geqslant 3$ the translate will have $6+6+2\times 3 = 18$ factors as one would expect from adding the factors of $\nabla(\lambda)$ and $\nabla(w\cdot\lambda)$. For generic μ and $l = 2$ then the translate has $5+1+1+1 = 8$ factors.

Also note that if λ is in a down alcove and is right on the edge of the dominant region (that is $\lambda'' = (a,0)$ or $(0,a)$ for some $a \in \mathbb{N}$), then $\chi_l(\mu_3) = -\chi_l(\mu_8)$ so these cancel in the sum. □

Corollary 4.2. *We have the following characters for $\hat{Z}(l\lambda'' + \lambda')$ with $\lambda'' \in X$ and $\lambda' \in X_1$.*

(i) *Suppose $\lambda' = (l-1, l-1)$, then*

$$\operatorname{ch}\hat{Z}(l\lambda'' + (l-1,l-1)) = \operatorname{ch}\hat{L}(l\lambda'' + (l-1,l-1)).$$

(ii) *Suppose $\lambda' = (l-1, r)$ with $0 \leqslant r \leqslant l-2$, then*

$$\begin{aligned}\operatorname{ch}\hat{Z}(l\lambda'' + (l-1,r)) = \operatorname{ch}\hat{L}((l-1,r) + l\lambda'') + \operatorname{ch}\hat{L}((r-l,s) + l\lambda'')\\ + \operatorname{ch}\hat{L}((r+l,s-l) + l\lambda'') + \operatorname{ch}\hat{L}((s,-1) + l\lambda'').\end{aligned}$$

(iii) *Suppose $\lambda' = (s, l-1)$ with $0 \leqslant s \leqslant l-2$, then*

$$\begin{aligned}\operatorname{ch}\hat{Z}(l\lambda'' + (s,l-1)) = \operatorname{ch}\hat{L}((s,l-1) + l\lambda'') + \operatorname{ch}\hat{L}((r,s-l) + l\lambda'')\\ + \operatorname{ch}\hat{L}((r-l,s+l) + l\lambda'') + \operatorname{ch}\hat{L}((-1,r) + l\lambda'').\end{aligned}$$

(iv) *Suppose $\lambda' = (r,s)$ with $0 \leqslant r \leqslant l-2$ and $r+s = l-2$, then*

$$\begin{aligned}\operatorname{ch}\hat{Z}(l\lambda'' + (r,s)) = \operatorname{ch}\hat{L}((r,s) + l\lambda'') + \operatorname{ch}\hat{L}((s-l,l-1) + l\lambda'')\\ + \operatorname{ch}\hat{L}((l-1,r-l) + l\lambda'') + \operatorname{ch}\hat{L}((r-l,s-l) + l\lambda'').\end{aligned}$$

(v) *Suppose $\lambda' = (r,s) \in C$, then*

$$\operatorname{ch}\hat{Z}(l\lambda'' + (r,s)) = \sum_i \operatorname{ch}\hat{L}(\mu_i)$$

where the μ_i are as in Figure 2(a).

(vi) *Suppose $\lambda' = (l-s-2, l-r-2)$ with $(r,s) \in C$, then*

$$\operatorname{ch}\hat{Z}(l\lambda'' + (l-s-2,l-r-2)) = \sum_i \operatorname{ch}\hat{L}(\mu_i)$$

where the μ_i are as in Figure 2(b).

Proof. We have $\operatorname{Ind}_{G_1B}^{G}\hat{Z}(\lambda) \cong \nabla(\lambda)$ so this follows using the character formula (1), the previous theorem and the identity

$$\hat{Z}(\lambda' + l\lambda'') \cong \hat{Z}(\lambda') \otimes k_{l\lambda''}$$

which is the quantum version of [13, II 9.2 (5)]. The quantum result follows as in the classical case. □

5. Extensions for simple modules

We will need to be able to work out the G_1B extensions between simple modules for G_1B. To do this we will need to generalise the extension results of Yehia [16]. We will essentially reproduce his proofs but in the quantum case, as the reference is not widely accessible.

Lemma 5.1. *Let $\lambda \in X_1$ then $L(\lambda) \otimes \mathrm{St}$ has a good filtration.*

Proof. If $l \geqslant 4 = 2h-2$ then this is the quantum version of [1, 2.5 corollary].

If λ is not in an up alcove then $L(\lambda) \cong \nabla(\lambda)$ and we are done by [14] and [2, corollary 5.14].

So the only case left is if $l = 3$ and $\lambda = (1,1)$. But now $\mathrm{ch}(L(1,1)) = \mathrm{ch}(\nabla(1,1)) - \mathrm{ch}(\nabla(0,0)) = e(1,1) + e(2,-1) + e(1,-2) + e(-1,-1) + e(-2,1) + e(-1,2) + e(0,0)$. So all the weights of $L(1,1)|_B \otimes k_{(2,2)}$ are dominant and so $\mathrm{Ind}_B^G L(1,1) \otimes k_{(2,2)} = L(1,1) \otimes \mathrm{St}$ has a good filtration. □

Proposition 5.2. *Let $\lambda \in X_1$. There is an indecomposable G-module $Q(\lambda)$ which restricts to the G_1 injective hull of $L(\lambda)$ and this module is a tilting module for G. Moreover $Q(\lambda)$ is the tilting module $T(2(l-1)\rho + w_0\lambda)$ and this module is a direct summand of the module $L((l-1)\rho + w_0\lambda) \otimes \mathrm{St}$.*

Proof. If $l \geqslant 4$ then this is the result [1, proposition 5.7].

Let $\nu = (l-1)\rho + w_0\lambda \in X_1$. So $L(\nu)^* \cong L((l-1)\rho - \lambda)$. If $l \leqslant 3$ and λ lies on a left or right hand wall then the tilting module $T(2(l-1)\rho + w_0\lambda)$ is $T^{\lambda}_{(1,1)}\mathrm{St} \cong \mathrm{pr}_{\lambda} L(\nu) \otimes \mathrm{St}$. This then has simple G-socle $L(\lambda)$ and is injective as a G_1-module. Let $\mu \in X_1$. We have

$$\mathrm{Hom}_{G_1}(L(\mu), L(\nu) \otimes \mathrm{St}) \cong \mathrm{Hom}_{G_1}(L(\mu) \otimes L((l-1)\rho - \lambda), \mathrm{St})$$

and the latter group has dimension $[L(\mu) \otimes L((l-1)\rho - \lambda) : \mathrm{St}]_{G_1}$, the G_1 composition multiplicity of St in $L(\mu) \otimes L((l-1)\rho - \lambda)$, as St is the G_1 injective hull of St. We may check that

$$[L(\mu) \otimes L((l-1)\rho - \lambda) : \mathrm{St}]_{G_1} \cong \begin{cases} 1, & \text{if } \mu = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $L(\nu) \otimes \mathrm{St} \cong T(2(l-1)\rho + w_0\lambda$ and is the G_1 injective hull of $L(\lambda)$.

If $\lambda = (0,0)$ and $l = 2$ then $\nu = (1,1)$. We may check that

$$[L(\mu) \otimes \mathrm{St} : \mathrm{St}]_{G_1} \cong \begin{cases} 1, & \text{if } \mu = (0,0), \\ 3, & \text{if } \mu = (1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Thus the module $\mathrm{St} \otimes \mathrm{St}$ is the direct sum of three copies of the Steinberg module and one copy of the G_1 injective hull of $L(0,0)$ which is $\mathrm{pr}_{(0,0)}(\mathrm{St} \otimes \mathrm{St}) \cong T(2,2)$.

If $\lambda = (1,1)$ and $l = 3$ then the translate $T^{\lambda}_{(2,2)}\mathrm{St} = \mathrm{pr}_{\lambda} L(1,1) \otimes \mathrm{St}$. We may check that

$$[L(\mu) \otimes L(1,1) : \mathrm{St}]_{G_1} \cong \begin{cases} 1, & \text{if } \mu = (1,1) \text{ or } \mu = (2,2), \\ 0, & \text{otherwise.} \end{cases}$$

Thus $L(1,1)\otimes \mathrm{St}$ is the direct sum of the Steinberg module and the G_1-injective hull of $L(1,1)$ which is $\mathrm{pr}_{(1,1)}(L(1,1)\otimes \mathrm{St}) \cong T(3,3)$.

We may now get the G_1 injective hull of $L(0,1)$ or $L(1,0)$ by translating the $T(3,3)$ onto the wall. This translate is $T(3,4)$ or $T(4,3)$ respectively. A similar argument to above shows that this module is injective as a G_1 module and has G_1 socle $L(0,2)$ or $L(2,0)$ respectively. Also the module $L(l-2,l-1)\otimes \mathrm{St}$ is a tilting module, a character calculation shows that $T(3,4)$ is a direct summand of this module.

If $\lambda=(0,0)$ and $l=3$ then the translate $T^{(0,0)}_{(0,1)}T(4,3) = \mathrm{pr}_{(0,0)}\, L(0,1)\otimes T(4,3)$ is injective as a G_1-mod as it is a direct summand of a tensor product of an injective G_1-module. As a G-module $L(0,1)\otimes T(4,3)$ is isomorphic to $T(5,5)\oplus T(5,2)\oplus T(5,2)$. We have

$$\mathrm{Hom}_{G_1}(L(\mu), L(0,1)\otimes T(4,3)) \cong \mathrm{Hom}_{G_1}(L(\mu)\otimes L(1,0), T(4,3))$$

the latter group has dimension equal to the G_1 composition multiplicity of $L(1,0)$ in $L(\mu)\otimes L(1,0)$ as $T(4,3)$ is the G_1 injective hull of $L(1,0)$. We may check that for $\mu\in X_1$

$$[L(\mu)\otimes L(1,0) : L(1,0)]_{G_1} \cong \begin{cases} 1, & \text{if } \mu=(0,0)\\ 6, & \text{if } \mu=(2,2)\\ 0, & \text{otherwise.}\end{cases}$$

Since $T(5,2)\cong \nabla(1,0)^{\mathrm{F}}\otimes \mathrm{St}$ we have $\mathrm{Hom}_{G_1}(L(\mu),T(5,2)) \cong \nabla(1,0)^{\mathrm{F}}$ if $\mu=(2,2)$ and zero otherwise. Thus $\mathrm{Hom}_{G_1}(L(\mu),T(4,3))$ is k if $\mu=(0,0)$ and zero otherwise and hence $T(4,3)$ is the G_1-injective hull of $L(0,0)$.

For $l=3$ the module $\mathrm{St}\otimes\mathrm{St}$ is a tilting module and it has summands $T(4,4)$, $T(3,3)$, $T(5,2)$, $T(2,5)$ and three copies of the Steinberg module, by characters. □

Corollary 5.3. *The G-head of $\nabla(2(l-1)\rho+w_0\lambda)$ is simple and is isomorphic to $L(\lambda)$.*

Proof. We have that $\mathrm{hd}(\nabla(2(l-1)\rho+w_0\lambda)\subseteq \mathrm{hd}\, T(2(l-1)\rho+w_0\lambda)\cong L(\lambda)$ by the previous proposition. □

The following four results follow as in the classical case [15, 4.8-4.11], see also [12].

Corollary 5.4. *If $\lambda\in X^+$ and $\mu\in X_1$ then $\overline{\nabla}(\lambda)^{\mathrm{F}}\otimes T(2(l-1)\rho+w_0\mu)$ has a good filtration.*

Corollary 5.5. *If $\lambda\in X^+$ and $\mu\in X_1$ then $\nabla(l\lambda+2(l-1)\rho+w_0\mu)$ is a quotient of $\overline{\nabla}(\lambda)^{\mathrm{F}}\otimes T(2(l-1)\rho+w_0\mu)$ and $\nabla(l\lambda+\mu)$ as a submodule.*

Corollary 5.6. *For all $\lambda\in X^+$ and $\mu\in X_1$ we have*

$$\mathrm{hd}_{G_1}\nabla(l\lambda+2(l-1)\rho+w_0\mu)\cong \overline{\nabla}(\lambda)^{\mathrm{F}}\otimes L(\mu)$$

and

$$\mathrm{soc}_{G_1}\nabla(l\lambda+\mu)\cong \overline{\nabla}(\lambda)^{\mathrm{F}}\otimes L(\mu).$$

Corollary 5.7. *For all $\lambda \in X^+$ the module $\nabla(\lambda)$ has simple head.*

To determine $\mathrm{Ext}^1_{G_1}(L(\mu), L(\lambda))$ we need to determine $\mathrm{Ext}^1_G(L(\mu), L(\lambda))$ for small μ and λ. "Small" in this case means that $\lambda \leqslant 2(l-1)\rho$ and $\mu \in X_1$.

The idea is to use the quantum version of the short exact sequence [8]

$$0 \to \mathrm{Ext}^1_{G/G_1}(k, \mathrm{Hom}_{G_1}(L(\mu), L(\lambda))) \to \mathrm{Ext}^1_G(L(\mu), L(\lambda)) \\ \to \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(L(\mu), L(\lambda))) \to 0 \quad (2)$$

Note that

$$\mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(L(\mu), L(\lambda))) \cong \mathrm{Hom}_{G/G_1}(L(\mu'')^{\mathrm{F}}, \mathrm{Ext}^1_{G_1}(L(\mu'), L(\lambda))).$$

Also $\mathrm{Ext}^1_{G_1}(L(\mu'), L(\lambda)) \cong \mathrm{Hom}_{G_1}(L(\mu'), Q(\lambda)/L(\lambda))$ so determining $\mathrm{Ext}^1_G(L(\mu), L(\lambda))$ for enough μ determines the G_1 socle of $Q(\lambda)/L(\lambda)$ which in turn determines $\mathrm{Ext}^1_{G_1}(L(\mu'), L(\lambda))$. We thus only need to calculate the Ext groups for μ a composition factor of $Q(\lambda)$. I.e., it is enough to determine the Ext's for $\mu \leqslant 2(l-1)\rho$ and μ in the same block as λ.

Lemma 5.8. *Suppose $0 \leqslant r \leqslant l-2$ and $r+s = l-2$ then*

$$\mathrm{Ext}^1_G(L(r,s), L(2l-1, r)) \cong \mathrm{Ext}^1_G(L(r,s), L(s, 2l-1)) \cong k.$$

If $l \neq 3$ then

$$\mathrm{Ext}^1_G(L(r,s), L(l+r, l+s)) \cong 0.$$

If $l = 3$ then

$$\mathrm{Ext}^1_G(L(r,s), L(l+r, l+s)) \cong k.$$

Proof. Since if $\mu \not> \lambda$ we have $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \cong \mathrm{Hom}_G(L(\mu), \nabla(\lambda)/L(\lambda))$, this lemma will follow if we know what the socle of $\nabla(\lambda)/L(\lambda)$ is.

Now if $\lambda = (2l-1, r)$ or $(s, 2l-1)$ then $\nabla(\lambda)$ only has two composition factors $L(\lambda)$ and $L(r,s)$. Thus $\nabla(\lambda)/L(\lambda) \cong L(r,s)$ and the result follows.

If $\lambda = (l+r, l+s)$ and $l \neq 3$ then $\nabla(\lambda)$ has four composition factors: $L(\lambda)$, $L(l-1, r)$, $L(l-1, s)$ and $L(r,s)$. The previous corollary says that $L(r,s)$ is the head of $\nabla(\lambda)$. We also know that $\mathrm{Ext}^1_G(L(s, l-1), L(l-1, r)) \cong \mathrm{Ext}^1_G(L(l-1, r), L(s, l-1)) \cong 0$ thus the socle of $\nabla(\lambda)/L(\lambda)$ is $L(s, l-1) \oplus L(l-1, r)$. Thus $\mathrm{Ext}^1_G(L(r,s), L(l+r, l+s) \cong 0$.

If $l = 3$ then $[\nabla(l+r, l+s) : L(r,s)] = 2$. The module $\nabla(l+r, l+s)$ has simple head $L(r,s)$. Since $\nabla(l+r, l+s)$ has five composition factors in total and is indecomposable the multiplicity of $L(r,s)$ in socle of $\nabla(l+r, l+s)/L(l+r, l+s)$ is at most one. Thus the dimension of $\mathrm{Ext}^1_G(L(r,s), L(l+r, l+s))$ is at most one. But there is at least one non-split extension - it is the indecomposable module $\overline{\nabla}(1,1)^{\mathrm{F}} \otimes L(r,s)$. □

We similarly get:

Lemma 5.9. *Suppose* $0 \leqslant r \leqslant l-2$ *and* $r+s=l-2$ *then*

$$\mathrm{Ext}^1_G(L(l-1,r),L(r,l+s)) \cong \mathrm{Ext}^1_G(L(s,l-1),L(l+r,s) \cong k.$$

and

$$\mathrm{Ext}^1_G(L(l-1,r),L(l+s,l-1)) \cong \mathrm{Ext}^1_G(L(s,l-1),L(l-1,l+r)) \cong 0.$$

Lemma 5.10. *Suppose* $0 \leqslant r \leqslant l-3$ *and* $0 \leqslant r+s \leqslant l-3$ *then*

$$\mathrm{Ext}^1_G(L(l-s-2,l-r-2),L(\nu)) \cong k$$

if $\nu \in \{(r,s),(l+s,l-r-s-3),(l-r-s-3,l+r)\}$ *and*

$$\mathrm{Ext}^1_G(L(l-s-2,l-r-2),L(\nu)) \cong 0$$

if $\nu \in \{(l-s-2,l-r-2),(2l-s-2,l-r-2),(l-s-2,2l-r-2),(l+r,l+s)\}$.

Proof. The result for the first Ext group follows from the fact that there are only two composition factors of $\nabla(\nu)$ and $\nabla(l-s-2,l-r-2)$.

For the second Ext group we use that fact that $\nabla(\nu)$ (if $\nu \neq (l-s-2,l-r-2)$) has simple head $L(l-s-2,l-r-2)$ and this is the only occurrence of this simple module in $\nabla(\nu)$. We may deduce that $\nabla(\nu)$ has simple head $L(l-s-2,l-r-2)$ by either using corollary 5.3 or by translating an induced module off the wall. □

Lemma 5.11. *Suppose* $0 \leqslant r \leqslant l-3$ *and* $0 \leqslant r+s \leqslant l-3$ *then*

$$\mathrm{Ext}^1_G(L(r,s),L(\nu)) \cong k$$

if $\nu \in \{(l-s-2,l-r-2),(l-r-2,l+r+s+1),(l+r+s+1,l-s-2)\}$ *and*

$$\mathrm{Ext}^1_G(L(r,s),L(\nu)) \cong 0$$

if $\nu \in \{(r,s),(l+s,l-r-s-3),(l-r-s-3,l+r),(s,3l-r-s-3),(3l-r-s-3,r),(2l-s-2,2l-r-2)\}$.
If $l \neq 3$ *then*

$$\mathrm{Ext}^1_G(L(r,s),L(l+r,l+s)) \cong 0.$$

If $l=3$ *then*

$$\mathrm{Ext}^1_G(L(r,s),L(l+r,l+s)) \cong k.$$

Proof. We first observe that $\nabla(r+s+1,2l-s-2)$ is a quotient of $\nabla(l+r,l+s)$ (and dually so is $\nabla(2l-r-2,r+s+1)$). These modules all have the same simple head — namely $L(l-s-2,l-r-2)$. Also there is a unique homomorphism from $\nabla(l+r,l+s)$ to $\nabla(r+s+1,2l-s-2)$. (Quantum version [2, section 7] of [13, II, 7.19(d)].) Since this homomorphism must be non-zero on the head of $\nabla(l+r,l+s)$ and this head is the same as the head of $\nabla(r+s+1,2l-s-2)$ and this simple module only occurs once in $\nabla(r+s+1,2l-s-2)$ this map must be onto.

Thus by considering the composition factors of the kernel of this homomorphism, the socle of the quotient $\nabla(l+r,l+s)/L(l+r,l+s)$ is contained in $L(2l-r-2,r+s+1)\oplus L(r+s+1,2l-r-2)$ if $l \neq 3$ and $L(4,1) \oplus L(1,4) \oplus L(0,0)$ if $l = 3$.

Thus $\mathrm{Ext}^1_G(L(r,s), L(l+r,l+s)$ is zero if $l \neq 3$. If $l = 3$ then $\mathrm{Ext}^1_G(L(0,0), L(3,3))$ is at most one-dimensional. But there is a non-split extension — namely the module $\nabla(1,1)^{\mathrm{F}}$.

If $\nu \in \{(r,s), (l+s, l-r-s-3), (l-r-s-3, l+r)\}$ then $L(r,s)$ is not a composition factor of $\nabla(\nu)$ so $\mathrm{Ext}^1_G(L(r,s), L(\nu)) \cong 0$.

We may now deduce that the socle of the quotient $\nabla(r+s+1, 2l-s-2)/L(r+s+1, 2l-s-2)$ is $L(r,s) \oplus L(l+s, l-r-s-3) \oplus L(l-r-s-3, l+r)$ as these cannot extend each other and the only other composition factor of $\nabla(r+s+1, 2l-s-2)$ is its head $L(l-s-2, l-r-2)$. Thus $\mathrm{Ext}^1_G(L(r,s), L(r+s+1, 2l-s-2)) \cong k$. Dually we have $\mathrm{Ext}^1_G(L(r,s), L(2l-r-2, r+s+1)) \cong k$.

If $\nu = (l-s-2, l-r-2)$ then this extension is the module $\nabla(l-s-2, l-r-2)$.

If $\nu \in (s, 3l-r-s-3), (3l-r-s-3, r), (2l-s-2, 2l-r-2)\}$ and $l \neq 3$ then $L(r,s)$ is the head of $\nabla(\nu)$. Since $\nabla(\nu)$ has both simple head and socle and has at least three composition factors and $L(r,s)$ occurs with multiplicity one, it cannot be in the socle of the quotient $\nabla(\nu)/L(\nu)$ thus $\mathrm{Ext}^1_G(L(r,s), L(\nu))$ is zero.

If $l = 3$ the only case that the above paragraph does not work is for $\nu = (4,4)$ when $L(0,0)$ occurs with multiplicity two. If $\mathrm{Ext}^1_G(L(0,0), \bar{L}(1,1)^{\mathrm{F}} \otimes L(1,1))$ is non-zero then using the five term exact sequence $\bar{L}(1,1)^{\mathrm{F}}$ must be a composition factor of $\mathrm{Ext}^1_{G_1}(L(0,0), L(1,1))$. The following lemma will show that this is not the case and so $\mathrm{Ext}^1_G(L(0,0), L(4,4))$ is zero. □

Lemma 5.12. *If $l = 3$ then*

$$\mathrm{Ext}^1_{G_1}(L(0,0), L(1,1)) \cong \overline{\nabla}(1,0)^{\mathrm{F}} \oplus \overline{\nabla}(0,1)^{\mathrm{F}} \oplus k.$$

Proof. The G_1 injective hull of $L(1,1)$ is $T(3,3)$. We apply $\mathrm{Hom}_{G_1}(k, -)$ to the short exact sequence

$$0 \to L(1,1) \to T(3,3) \to Q \to 0$$

to get

$$0 \to \mathrm{Hom}_{G_1}(k, L(1,1)) \to \mathrm{Hom}_{G_1}(k, T(3,3)) \to \mathrm{Hom}_{G_1}(k, Q) \to \mathrm{Ext}^1_{G_1}(k, L(1,1)) \to 0$$

The first two Hom groups are zero so the last two groups are isomorphic. Thus $Q^{G_1} \cong \mathrm{Ext}^1_{G_1}(k, L(1,1))$.

Now the G_1 fixed points of Q are contained in the G_1 fixed points of the induced modules appearing in a good filtration of $T(3,3)/\nabla(1,1)$ together with the G_1 fixed points of $\nabla(1,1)/L(1,1)$. We thus have

$$Q^{G_1} \subseteq k \oplus \overline{\nabla}(1,0)^{\mathrm{F}} \oplus \overline{\nabla}(0,1)^{\mathrm{F}} \oplus \bar{L}(1,1)^{\mathrm{F}}$$

But $L(1,1)^{\mathrm{F}}$ can't be in the G_1 socle of Q as then it would also be in the G_1 head of the Q^*. The G_1 head of Q^* is contained in the G_1 heads of the induced modules appearing in a good filtration

of $T(3,3)$ as $T(3,3)$ is self dual. Thus

$$\mathrm{hd}_{G_1}(Q^*) \subseteq L(1,1)^{\oplus 5} \oplus k^{\oplus 2} \oplus \overline{\nabla}(1,0)^{\mathrm{F}} \oplus \overline{\nabla}(0,1)^{\mathrm{F}}.$$

Hence

$$Q^{G_1} \subseteq k \oplus \overline{\nabla}(1,0)^{\mathrm{F}} \oplus \overline{\nabla}(0,1)^{\mathrm{F}}.$$

We now observe from the good filtration of $T(3,3)$ that $\overline{\nabla}(1,0)^{\mathrm{F}} \oplus \overline{\nabla}(0,1)^{\mathrm{F}}$ must occur directly above k in $T(3,3)/L(1,1)$. The previous lemma tells us that k cannot extend either $\overline{\nabla}(1,0)^{\mathrm{F}}$ nor $\overline{\nabla}(0,1)^{\mathrm{F}}$ so this is indeed the G_1 fixed points of Q. □

We may now prove the following.

Theorem 5.13. *The* $\mathrm{Ext}^1_{G_1}\big(L(\alpha), L(\beta)\big)$ *for* α, $\beta \in X_1$ *are given by the following tables. (i) For* $(r,s) \in X_1$ *with* $r+s = l-2$, *we have*

$\alpha\downarrow,\ \beta\rightarrow$	(r,s)	$(l-1,r)$	$(s,l-1)$
(r,s)	0	$\overline{\nabla}(0,1)^{\mathrm{F}}$	$\overline{\nabla}(1,0)^{\mathrm{F}}$
$(l-1,r)$	$\overline{\nabla}(1,0)^{\mathrm{F}}$	0	0
$(s,l-1)$	$\overline{\nabla}(0,1)^{\mathrm{F}}$	0	0

(ii) For $(r,s) \in C$ *and* $l \geqslant 4$, *the only non-zero entries we have*

$\alpha\downarrow,\ \beta\rightarrow$	$(l-s-2,l-r-2)$	$(r+s+1,l-s-2)$	$(l-r-2,r+s+1)$
(r,s)	k	$\overline{\nabla}(0,1)^{\mathrm{F}}$	$\overline{\nabla}(1,0)^{\mathrm{F}}$

$\alpha\downarrow,\ \beta\rightarrow$	(r,s)	$(s,l-r-s-3)$	$(l-r-s-3,r)$
$(l-s-2,l-r-2)$	k	$\overline{\nabla}(0,1)^{\mathrm{F}}$	$\overline{\nabla}(1,0)^{\mathrm{F}}$

If $l=3$ *then all the entries in the two tables above are replaced by* $k \oplus \overline{\nabla}(0,1)^{\mathrm{F}} \oplus \overline{\nabla}(1,0)^{\mathrm{F}}$.

Proof. We use the sequence (2) and the previous results to show that the $\mathrm{Ext}^1_{G_1}$ are as described. We have to argue as in the previous lemma to do the case $l=3$. □

To now determine $\mathrm{Ext}^1_G(L(\mu), L(\lambda))$ for μ and $\lambda \in X^+$ we need to know the G_1 socle of the tensor products $L(1,0)\otimes L(\lambda)$ and $L(0,1)\otimes L(\lambda)$ for $\lambda \in X_1$. We essentially determined the tensor product in the proofs of propositions 3.2, 3.4, 3.5 and 3.6. We just need to determine the socles of these tensor products. These are not hard to compute using translation functors and follow exactly as in the classical case so we will just state the result.

Proposition 5.14. *The G_1 socle of the tensor product $L(1,0) \otimes L(\lambda)$ for $\lambda \in X_1$ is the same as its G socle and is given by the following table.*

l	λ	$\mathrm{soc}_G\, L(1,0) \otimes L(\lambda)$
all l	$(0,0)$	$L(1,0)$
$l \geqslant 4$	$(0,s)$, $1 \leqslant s \leqslant l-3$	$L(1,s) \oplus L(0,s-1)$
$l \geqslant 3$	$(0,l-2)$	$L(0,l-3)$
$l \geqslant 4$	(r,s), $1 \leqslant r \leqslant l-3$ *and* $r+s=l-2$	$L(r,s-1) \oplus L(r-1,s+1)$
$l \geqslant 3$	$(r,0)$, $1 \leqslant r \leqslant l-2$	$L(r+1,0) \oplus L(r-1,1)$
$l \geqslant 4$	(r,s) *deep inside* C	$L(r+1,s) \oplus L(r-1,s+1) \oplus L(r,s-1)$
all l	$(0,l-1)$	$L(0,1,l-1) \oplus L(0,l-2)$
$l \geqslant 3$	$(r,l-1)$, $1 \leqslant r \leqslant l-2$	$L(r+1,l-1) \oplus L(r,l-2)$
all l	$(l-1,l-1)$	$L(l-1,l-2)$
$l \geqslant 3$	$(1,l-2)$	$L(2,l-2) \oplus L(0,l-1)$
$l \geqslant 4$	$(r,l-2)$, $2 \leqslant r \leqslant l-2$	$L(r+1,l-2) \oplus L(r,l-3) \oplus L(r-1,l-3)$
$l \geqslant 4$	(r,s), $2 \leqslant r \leqslant l-3$ *and* $r+s=l-1$	$L(r+1,s) \oplus L(r-1,s+1)$
$l \geqslant 4$	$(l-2,1)$	$L(l-1,1) \oplus L(l-3,2)$
all l	$(l-1,0)$	$L(l-2,1)$
$l \geqslant 3$	$(l-1,s)$, $1 \leqslant s \leqslant l-2$	$L(l-2,s+1) \oplus L(l-1,s-1)$
$l \geqslant 4$	$(l-2,s)$, $2 \leqslant s \leqslant l-2$	$L(l-1,s) \oplus L(l-2,s-1) \oplus L(l-3,s+1)$
$l \geqslant 4$	(r,s) *deep inside upper alcove*	$L(r+1,s) \oplus L(r-1,s+1) \oplus L(r,s-1)$

We may use the dual of the above table to determine $L(0,1) \otimes L(\lambda)$ for $\lambda \in X_1$.

Corollary 5.15. *Let $\lambda \in X^+$. Then*

$$\mathrm{soc}_G\, L(1,0) \otimes L(\lambda) = (\mathrm{soc}_G\, L(1,0) \otimes L(\lambda')) \otimes L(\lambda'')^{\mathrm{F}}$$

and

$$\mathrm{soc}_G\, L(0,1) \otimes L(\lambda) = (\mathrm{soc}_G\, L(0,1) \otimes L(\lambda')) \otimes L(\lambda'')^{\mathrm{F}}$$

Proof. We have

$$\mathrm{soc}_G\, L(1,0) \otimes L(\lambda) = \mathrm{soc}_G(\mathrm{soc}_{G_1}(L(1,0) \otimes L(\lambda')) \otimes L(\lambda'')^{\mathrm{F}}),$$

but the G_1 socle of $L(1,0) \otimes L(\lambda')$ is the same as its G socle. Steinberg's tensor product theorem then tells us that that $\mathrm{soc}_{G_1}(L(1,0) \otimes L(\lambda')) \otimes L(\lambda'')^{\mathrm{F}}$ is semi-simple as a G-module. □

We may now deduce the following theorem.

Theorem 5.16. *Let μ, $\lambda \in X^+$. If $\mu' = \lambda'$ then $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \cong \mathrm{Ext}^1_G(L(\mu''), L(\lambda'')$.*

If $\mu' \neq \lambda'$ then $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \cong \mathrm{Hom}_G(L(\mu''), \mathrm{Ext}^1_{G_1}(L(\mu'), L(\lambda'))^{(-1)} \otimes L(\lambda'')$.

We have $\dim \mathrm{Ext}^1_G(L(\mu), L(\lambda) \leqslant 1$.

Proof. This follows using sequence (2) and the previous results. □

We may determine exactly the value of the right hand side of both equations using induction and the previous lemmas.

6. G_1B extensions between the simples

We now use the G_1 results to classify the G_1B and the G_1T extensions between the simple G_1B modules. We use the following.

Proposition 6.1. *Let* λ, $\mu \in X$.
(i) If $\mu'' - \lambda'' \in X^+$, *then*

$$\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda), \hat{L}(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda'), L(\mu') \otimes \nabla(\mu'' - \lambda'')^{\mathrm{F}})$$

(ii) Suppose $\mu'' - \lambda'' \notin X^+$. *If* $\lambda' = \mu'$ *and there exists* $\alpha \in S$ *and* $i \in \mathbb{N}$ *with* $\mu'' - \lambda'' = -l^i\alpha$, *then* $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda), \hat{L}(\mu)) \cong k$. *Otherwise* $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda), \hat{L}(\mu)) = 0$.

Proof. The proof of this proposition follows exactly as in the classical case [13, proposition 9.21] □

Lemma 6.2. *Let* $\eta \in X_1$, $\mu \in X^+$. *Then* $\mathrm{Ext}^1_G(L(\eta), L(\eta) \otimes \nabla(\mu)^{\mathrm{F}}) \cong 0$.

Proof. We apply the Lyndon-Hochschild-Serre five term exact sequence to this group. Since $\mathrm{Ext}^1_{G_1}(L(\eta), L(\eta)) \cong 0$ we have $\mathrm{Ext}^1_G(L(\eta), L(\eta) \otimes \nabla(\mu)^{\mathrm{F}}) \cong \mathrm{Ext}^1_{G/G_1}(k, \nabla(\mu)^{\mathrm{F}}) \cong \mathrm{Ext}^1_G(k, \nabla(\mu)) \cong 0$. □

Lemma 6.3. *Let* η, $\zeta \in X_1$, *with* $\eta \neq \zeta$ *and* $\mu \in X^+$. *Then* $\mathrm{Ext}^1_G(L(\eta), L(\zeta) \otimes \nabla(\mu)^{\mathrm{F}}) \cong \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(L(\eta), L(\zeta)) \otimes \nabla(\mu)^{\mathrm{F}})$.

Proof. We apply the Lyndon-Hochschild-Serre five term exact sequence to this group. Since $\mathrm{Hom}^1_{G_1}(L(\eta), L(\zeta)) \cong 0$ we have $\mathrm{Ext}^1_G(L(\eta), L(\eta) \otimes \nabla(\mu)^{\mathrm{F}}) \cong \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(L(\eta), L(\zeta)) \otimes \nabla(\mu)^{\mathrm{F}})$. □

We now apply these results to our case with $G = q\text{-}\mathrm{GL}_3(k)$ or $G = \mathrm{GL}_3(k)$. We wish to determine all the extensions between the simples that appear in a $\hat{Z}(\mu)$. Note that the tables below will not be symmetric, we do not have $\mathrm{Ext}^i_{G_1B}(\hat{L}(\mu), \hat{L}(\lambda)) \cong \mathrm{Ext}^i_{G_1B}(\hat{L}(\lambda), \hat{L}(\mu))$ in general.

Theorem 6.4. *(i) Let* $(r, s) \in X_1$ *with* $r+s = l-2$. *If* $\mu = l(a, b)+(l-1, r)$ *then* $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda), \hat{L}(\eta))$ *with* $\hat{L}(\lambda)$ *and* $\hat{L}(\eta)$ *composition factors of* $\hat{Z}(\mu)$ *is given by the following table.*

$\lambda \setminus \eta$	μ	$l(a-1,b)+(r,s)$	$l(a+1,b-1)+(r,s)$	$l(a,b-1)+(s,l-1)$
μ	0	0	0	0
$l(a-1,b)+(r,s)$	k	0	0	0
$l(a+1,b-1)+(r,s)$	0	k	0	0
$l(a,b-1)+(s,l-1)$	0	0	k	0

(ii) Let $(r,s)\in X_1$ with $r+s=l-2$. If $\mu=l(a,b)+(s,l-1)$ then $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda),\hat{L}(\eta))$ with $\hat{L}(\lambda)$ and $\hat{L}(\eta)$ composition factors of $\hat{Z}(\mu)$ is given by the following table.

$\lambda\setminus\eta$	μ	$l(a,b-1)+(r,s)$	$l(a-1,b+1)+(r,s)$	$l(a-1,b)+(l-1,r)$
μ	0	0	0	0
$l(a-1,b)+(r,s)$	k	0	0	0
$l(a+1,b-1)+(r,s)$	0	k	0	0
$l(a,b-1)+(s,l-1)$	0	0	k	0

(iii) Let $(r,s)\in X_1$ with $r+s=l-2$. If $\mu=l(a,b)+(r,s)$ then $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda),\hat{L}(\eta))$ with $\hat{L}(\lambda)$ and $\hat{L}(\eta)$ composition factors of $\hat{Z}(\mu)$ is given by the following table.

$\lambda\setminus\eta$	μ	$l(a,b-1)+(l-1,r)$	$l(a-1,b)+(s,l-1)$	$l(a-1,b-1)+(r,s)$
μ	0	0	0	0
$l(a-1,b)+(r,s)$	k	0	0	0
$l(a+1,b-1)+(r,s)$	k	0	0	0
$l(a,b-1)+(s,l-1)$	0	k	k	0

(iv) For $(r,s)\in C$ and if $\mu=l(a,b)+(r,s)$ then $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda),\hat{L}(\eta))$ with $\hat{L}(\lambda)$ and $\hat{L}(\eta)$ composition factors of $\hat{Z}(\mu)$ is given by the following table.

$\lambda\setminus\eta$	μ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
μ	0	0	0	0	0	0	0	0	0
μ_2	k	0	0	0	0	k	0	0	0
μ_3	0	k	0	0	0	0	0	0	0
μ_4	k	0	0	0	0	0	0	k	0
μ_5	0	0	0	k	0	0	0	0	0
μ_6	0	k	0	0	k	0	0	0	0
μ_7	0	k	0	k	0	0	0	0	k
μ_8	0	0	k	k	0	0	0	0	0
μ_9	0	0	0	0	0	k	k	k	0

(v) For $(r,s)\in C$ and if $\mu=l(a,b)+(l-s-2,l-r-2)$ then $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda),\hat{L}(\eta))$ with $\hat{L}(\lambda)$ and $\hat{L}(\eta)$ composition factors of $\hat{Z}(\mu)$ is given by the following table.

$\lambda \setminus \eta$	μ_1	μ_2	μ_3	μ	μ_5	μ_6	μ_7	μ_8	μ_9
μ_1	0	0	0	0	0	0	k	0	0
μ_2	k	0	0	0	k	0	0	k	0
μ_3	0	k	0	0	0	0	0	0	k
μ	k	0	0	0	0	0	0	k	0
μ_5	0	k	0	k	0	0	0	0	0
μ_6	0	0	0	0	k	0	0	0	0
μ_7	0	0	0	k	0	0	0	0	k
μ_8	0	0	0	k	0	0	0	0	0
μ_9	0	0	0	0	0	k	k	k	0

Proof. Most of the Ext groups above can be computed in a straight-forward manner using the previous results.

We do sometimes need to argue as in the following case for $l = 3$.

Suppose we are considering case (iv). If $\lambda = \mu_9 = l(a-1, b-1) + (l-s-2, l-r-2)$ then $\mu'' - (a-1, b-1) = (1, 1)$ and so $\mathrm{Ext}^1_{G_1B}(\hat{L}(\lambda), \hat{L}(\mu)) \cong \mathrm{Ext}^1_G(L(l-s-2, l-r-2), L(r, s) \otimes \nabla(1, 1)^{\mathrm{F}}) \cong 0$ using lemma 5.11 if $l \geqslant 4$.

If $l = 3$ we then use the Lyndon-Hochschild-Serre five term exact sequence. We get

$$\begin{aligned}
0 &\to \mathrm{Ext}^1_{G/G_1}(k, \mathrm{Hom}_{G_1}(L(l-s-2, l-r-2), L(r, s)) \otimes \overline{\nabla}(1, 1)^{\mathrm{F}}) \\
&\to \mathrm{Ext}^1_G(L(l-s-2, l-r-2), L(r, s) \otimes \overline{\nabla}(1, 1)^{\mathrm{F}}) \\
&\to \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(L(l-s-2, l-r-2), L(r, s)) \otimes \overline{\nabla}(1, 1)^{\mathrm{F}}) \\
&\to \mathrm{Ext}^2_{G/G_1}(k, \mathrm{Hom}_{G_1}(L(l-s-2, l-r-2), L(r, s)) \otimes \overline{\nabla}(1, 1)^{\mathrm{F}}) \\
&\to \mathrm{Ext}^2_G(L(l-s-2, l-r-2), L(r, s) \otimes \overline{\nabla}(1, 1)^{\mathrm{F}})
\end{aligned}$$

Since $\mathrm{Hom}_{G_1}(L(l-s-2, l-r-2), L(r, s))$ is zero we have using theorem 5.13

$$\begin{aligned}
&\mathrm{Ext}^1_G(L(l-r-2, l-s-2), L(r, s) \otimes \overline{\nabla}(1, 1)^{\mathrm{F}}) \\
&\quad\cong \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^1_{G_1}(L(l-r-2, l-s-2), L(r, s)) \otimes \nabla(1, 1)^{\mathrm{F}}) \\
&\quad\cong \mathrm{Hom}_{G/G_1}(k, \overline{\nabla}(1, 1)^{\mathrm{F}} \oplus \overline{\nabla}(0, 1)^{\mathrm{F}} \otimes \overline{\nabla}(1, 1)^{\mathrm{F}} \oplus \overline{\nabla}(1, 0)^{\mathrm{F}} \otimes \overline{\nabla}(1, 1)^{\mathrm{F}}) \\
&\quad\cong \mathrm{Hom}_{\mathrm{SL}_3}(k, \overline{\nabla}(1, 1) \oplus \overline{\nabla}(0, 1) \otimes \overline{\nabla}(1, 1) \oplus \overline{\nabla}(1, 0) \otimes \overline{\nabla}(1, 1)) \\
&\quad\cong \mathrm{Hom}_{\mathrm{SL}_3}(k, \overline{\nabla}(1, 1)) \oplus \mathrm{Hom}_G(\overline{\nabla}(1, 0), \overline{\nabla}(1, 1)) \oplus \mathrm{Hom}_G(\overline{\nabla}(0, 1), \overline{\nabla}(1, 1)) \\
&\quad\cong 0
\end{aligned}$$

□

7. The composition series of induced modules for G_1B.

Before deducing the G_1B structure of the $\hat{Z}(\mu)$'s we need some more propositions.

Proposition 7.1. *Suppose $\lambda \in C$ and $\mu \in \bar{C}$ lies on a wall. Suppose also that s is a simple reflection that fixes μ. and that $w \cdot \lambda < ws \cdot \lambda$. We have the following properties.*

(i) $T^{\mu}_{\lambda} L(w \cdot \lambda) \cong L(w \cdot \mu)$ *and* $T^{\mu}_{\lambda} L(ws \cdot \lambda) \cong 0$

(ii) $\hat{T}^{\mu}_{\lambda} \hat{Z}(w \cdot \lambda) \cong \hat{T}^{\mu}_{\lambda} \hat{Z}(ws \cdot \lambda) \cong \hat{Z}(w \cdot \mu)$

(iii) *We have a short exact sequence*

$$0 \to \hat{Z}(w \cdot \lambda) \to \hat{T}^{\lambda}_{\mu} \hat{Z}(w \cdot \mu) \to \hat{Z}(ws \cdot \lambda) \to 0$$

The socle of $\hat{T}^{\lambda}_{\mu} \hat{Z}(w \cdot \mu)$ is $\hat{L}(w \cdot \lambda)$.

This is the quantum version of [13, II 9.22 (4), (2), (3)] and may be proved as in the classical case using the results of [2] and [10].

Proposition 7.2. *Let λ, μ , w and s be as in the previous proposition. We have* $\mathrm{Hom}_{G_1 B}(\hat{Z}(ws \cdot \lambda), \hat{Z}(w \cdot \lambda)) \cong \mathrm{Hom}_{G_1 B}(\hat{Z}(w \cdot \mu), \hat{Z}(w \cdot \mu)) \cong k$

Proof. Firstly, we have $\mathrm{Hom}_{G_1 B}(\hat{Z}(w \cdot \mu), \hat{Z}(w \cdot \mu)) \cong \mathrm{Hom}_{B}(\hat{Z}(w \cdot \mu), k_{w \cdot \mu})$ by Frobenious reciprocity. The latter group is at most one dimensional, as the dimension of the $w \cdot \mu$ weight space in $\hat{Z}(w \cdot \mu)$ is one. On the other hand $\mathrm{Hom}_{G_1 B}(\hat{Z}(w \cdot \mu), \hat{Z}(w \cdot \mu))$ is certainly non-zero. Thus there is unique homomorphism (upto scalars), the identity homomorphism.

We may now argue as in the proof of [13, II, proposition 7.19] to show that the map ϕ in the following long exact sequence is zero,

$$0 \to \mathrm{Hom}_{G_1 B}(\hat{Z}(ws \cdot \lambda), \hat{Z}(w \cdot \lambda)) \to \mathrm{Hom}_{G_1 B}(\hat{Z}(ws \cdot \lambda), \hat{T}^{\lambda}_{\mu} \hat{Z}(w \cdot \mu)) \xrightarrow{\phi} \mathrm{Hom}_{G_1 B}(\hat{Z}(ws \cdot \lambda), \hat{Z}(ws \cdot \lambda))$$

and we thus get the isomorphism as claimed. □

We may now prove the following theorem, We use the following various facts about $\hat{Z}(\lambda)$ for $\lambda \in X^+$:

(i) $\hat{Z}(\lambda)$ has simple $G_1 B$ socle $\hat{L}(\lambda)$ (see [13, II, 9.6 (1)] and [10, 3.1 (13) (i)])

(ii) $\hat{Z}(\lambda)$ has simple $G_1 B$ head $\hat{L}(2(l-1)\rho - \lambda)^* \cong \hat{L}(2(l-1)\rho + w_0 \lambda + l(w_0 \lambda'' - \lambda'))$ (see [13, II, 9.6 (2)] and [10, 3.1 (22)])

(iii) $\hat{Z}(\lambda)^* \cong \hat{Z}(2(l-1)\rho - \lambda)$ (see [13, II, 9.2 (2)] and [10, 3.1 (21)])

(iv) $\hat{Z}(\lambda + l\mu) \cong \hat{Z}(\lambda) \otimes k_{l\mu}$ (see [13, II, 9.2 (5)], also follows in the quantum case using the tensor identity).

Strictly speaking the results in the quantum case using [10] are only $G_1 T$ results. But the above properties clearly lift to $G_1 B$.

Item (iii) above implies that the submodule structure of $\hat{Z}(\lambda)$ for λ in a down alcove and the structure of $\hat{Z}(\mu)$ for μ in an up alcove are inversions of each other. Item (iv) above implies that the structure for a weight of a particular G_1 type is always the same.

Theorem 7.3. *The submodule structure of the $\hat{Z}(\lambda)$ for $\lambda \in X^+$ is as follows.*

(i) *Suppose* $\lambda = l(a,b) + (l-1, l-1)$ *with* $(a,b) \in X^+$. *Then*

$$\hat{Z}(\lambda) = \hat{L}(\lambda).$$

(ii) *Suppose* $\lambda = l(a,b) + (l-1, r)$ *with* $(a,b) \in X^+$ *and* $(l-1,r) \in X_1$. *If we set* $s = l - r - 2$ *then the module* $\hat{Z}(\lambda)$ *has filtration*

$$\begin{array}{c} \hat{L}(l(a,b-1) + (s, l-1)) \\ | \\ \hat{L}(l(a+1,b-1) + (r,s)) \\ | \\ \hat{L}(l(a-1,b) + (r,s)) \\ | \\ \hat{L}(l(a,b) + (l-1,r)). \end{array}$$

(iii) *Suppose* $\lambda = l(a,b) + (s, l-1)$ *with* $(a,b) \in X^+$ *and* $(s,l-1) \in X_1$. *If we set* $r = l - s - 2$ *then the module* $\hat{Z}(\lambda)$ *has filtration*

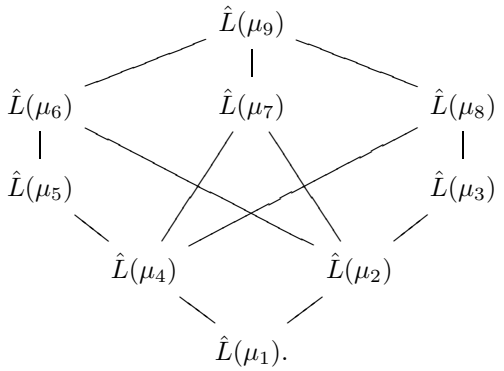
$$\begin{array}{c} \hat{L}(l(a-1,b) + (l-1, r)) \\ | \\ \hat{L}(l(a-1,b+1) + (r,s)) \\ | \\ \hat{L}(l(a,b-1) + (r,s)) \\ | \\ \hat{L}(l(a,b) + (s,l-1)). \end{array}$$

(iv) *Suppose* $\lambda = l(a,b) + (r,s)$ *with* $(a,b) \in X^+$, $(r,s) \in X_1$ *and* $r + s = l - 2$. *Then the module* $\hat{Z}(\lambda)$ *has filtration*

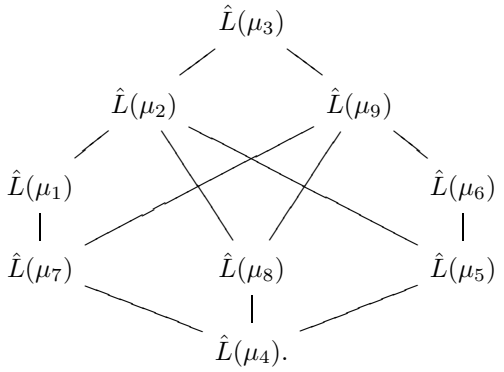
$$\begin{array}{ccc} & \hat{L}(l(a-1,b-1) + (r,s)) & \\ \swarrow & & \searrow \\ \hat{L}(l(a,b-1) + (l-1,r)) & & \hat{L}(l(a-1,b) + (s,l-1)) \\ \searrow & & \swarrow \\ & \hat{L}(l(a,b) + (r,s)). & \end{array}$$

(v) *Suppose* $\lambda = l(a,b) + (r,s)$ *with* $(a,b) \in X^+$, *and* $(r,s) \in C$. *We let* μ_1 *upto* μ_9 *be as before, depicted in Figure 1 (a), where the number corresponds to the subscript of* μ.

Then $\hat{Z}(\lambda)$ has filtration



(vi) *Suppose $\lambda = l(a,b) + (l-s-2, l-r-2)$ with $(a,b) \in X^+$, and $(r,s) \in C$. We let μ_1 upto μ_9 be as before, depicted in Figure 1 (b), where the number corresponds to the subscript of μ. Then $\hat{Z}(\lambda)$ has filtration*

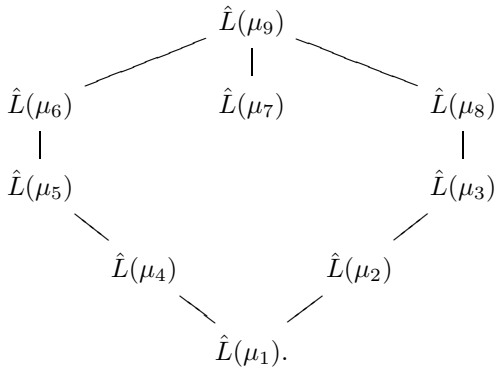


Proof. The structures for (i)-(iv) are the only possible ones using the fact that $\hat{Z}(\lambda)$ has simple head and socle as described above and the possible extensions that exist between the composition factors.

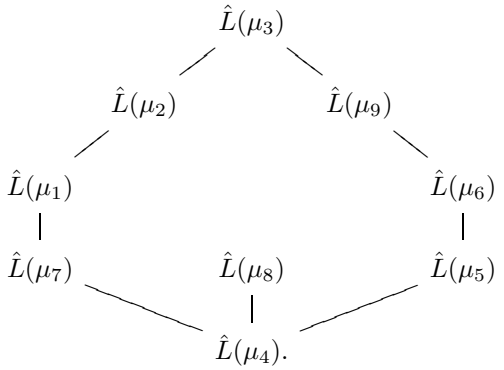
Cases (v) and (vi). The structure depicted has all the possible extensions drawn in. We need to prove that all these extensions do actually appear. The simples must be in the layers as described, for otherwise it would contradict the $\hat{Z}(\lambda)$ having simple socle $\hat{L}(\lambda)$ and simple head $L(\mu_9)$ ($L(\mu_3)$) if λ is a down (up) alcove respectively.

For instance, in case (v) we must have a uniserial subquotient of $\hat{L}(\mu_4)$, $\hat{L}(\mu_5)$ and $\hat{L}(\mu_6)$, since $L(\hat{\mu_5})$ can only extend one simple below it (namely $\hat{L}(\mu_4)$) and one simple above it, (namely $\hat{L}(\mu_6)$). Otherwise $\hat{L}(\mu_5)$ would either be in the head or socle of $\hat{Z}(\lambda)$.

So for case (v) we can deduce the following structure so far:



We get a similar picture (only inverted) for case (vi).

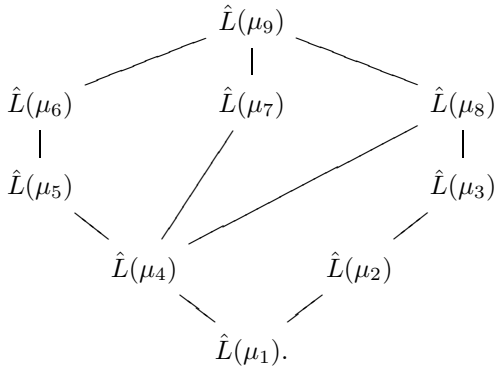


Consider the structure for case (v) so far. The $\hat{L}(\mu_7)$ must extend at least one of $\hat{L}(\mu_4)$ or $\hat{L}(\mu_2)$. Suppose that it extends $\hat{L}(\mu_4)$. Now the existence of a homomorphism from $\hat{Z}(\mu_1)$ to $\hat{Z}(\mu_4)$ (using proposition 7.2) implies that there is an extension of $\hat{L}(\mu_7)$ by $\hat{L}(\mu_9)$ in $\hat{Z}(\mu_4)$, as the image of the homorphism must contain at least $\hat{L}(\mu_4)$, $\hat{L}(\mu_5)$, $\hat{L}(\mu_6)$, $\hat{L}(\mu_7)$ and $\hat{L}(\mu_9)$, and it has simple head $\hat{L}(\mu_9)$.

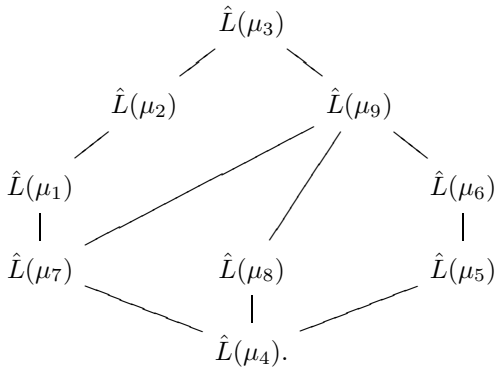
Now consider the module $\hat{Z}(\eta)$ defined to be $\hat{Z}(\mu_4)^* \otimes k_{l(2a-1,2b-1)}$. The weight η is in the same (down) alcove as the μ_8 from $\hat{Z}(\mu_4)$. We now consider the dual of the extension of $\hat{L}(\mu_7)$ by $\hat{L}(\mu_9)$ and tensor it by $k_{l(2a-1,2b-1)}$. This extension then appears in $\hat{Z}(\eta)$ and working out what the duals of the simples are gives us an extension of $\hat{L}(\eta_4)$ by $\hat{L}(\eta_8)$. Translation principle then tells us that our original $\hat{Z}(\mu_1)$ has an extension of $\hat{L}(\mu_4)$ by $\hat{L}(\mu_8)$.

Considering the homomorphism from $\hat{Z}(\mu_1)$ to $\hat{Z}(\mu_4)$ again implies that there is an extension of $\hat{L}(\mu_8)$ by $\hat{L}(\mu_9)$ in $\hat{Z}(\mu_4)$.

So we now have for case (v) (assuming that $\hat{L}(\mu_4)$ extends $\hat{L}(\mu_7)$)



For case (vi) we get:



Now the image of the homomorphism from $\hat{Z}(\mu_4)$ to $\hat{Z}(\mu_8)$ (which exists using proposition 7.2) contains an extension of $\hat{L}(\mu_9)$ and $\hat{L}(\mu_3)$. Thus there is also an an extension of $\hat{L}(\mu_2)$ and $\hat{L}(\mu_7)$ in the original $\hat{Z}(\mu_1)$ for case (v).

Repeating the above argument with μ_2 in place of μ_4 thus gives us the result. □

8. The good l-filtrations of the induced modules for G

Theorem 8.1. *Each $\nabla(\lambda)$ has a l-filtration. This filtration takes the following form:*

(i) *Suppose $\lambda = l(a,b) + (l-1, l-1)$ with $(a,b) \in X^+$. Then*

$$\nabla(\lambda) = \overline{\nabla}(a,b)^{\mathrm{F}} \otimes L(l-1, l-1).$$

(ii) *Suppose* $\lambda = l(a,b) + (l-1,r)$ *with* $(a,b) \in X^+$ *and* $(l-1,r) \in X_1$. *If we set* $s = l-r-2$ *then for* $a \equiv -1 \pmod{l}$, *the module* $\nabla(\lambda)$ *has filtration*

$$\begin{array}{c} \overline{\nabla}(a,b-1)^{\mathrm{F}} \otimes L(s,l-1) \\ | \\ \overline{\nabla}(a+1,b-1)^{\mathrm{F}} \otimes L(r,s) \\ | \\ \overline{\nabla}(a-1,b)^{\mathrm{F}} \otimes L(r,s) \\ | \\ \overline{\nabla}(a,b)^{\mathrm{F}} \otimes L(l-1,r) \end{array}$$

while for $a \not\equiv -1 \pmod{l}$, $\nabla(\lambda)$ *has filtration*

$$\begin{array}{ccc} & \overline{\nabla}(a,b-1)^{\mathrm{F}} \otimes L(s,l-1) & \\ \swarrow & & \searrow \\ \overline{\nabla}(a+1,b-1)^{\mathrm{F}} \otimes L(r,s) & & \overline{\nabla}(a-1,b)^{\mathrm{F}} \otimes L(r,s) \\ \searrow & & \swarrow \\ & \overline{\nabla}(a,b)^{\mathrm{F}} \otimes L(l-1,r). & \end{array}$$

(iii) *Suppose* $\lambda = l(a,b) + (s,l-1)$ *with* $(a,b) \in X^+$ *and* $(s,l-1) \in X_1$. *If we set* $r = l-s-2$ *then for* $b \equiv -1 \pmod{l}$, *the module* $\nabla(\lambda)$ *has filtration*

$$\begin{array}{c} \overline{\nabla}(a-1,b)^{\mathrm{F}} \otimes L(l-1,r) \\ | \\ \overline{\nabla}(a-1,b+1)^{\mathrm{F}} \otimes L(r,s) \\ | \\ \overline{\nabla}(a,b-1)^{\mathrm{F}} \otimes L(r,s) \\ | \\ \overline{\nabla}(a,b)^{\mathrm{F}} \otimes L(s,l-1) \end{array}$$

while for $b \not\equiv -1 \pmod{l}$, $\nabla(\lambda)$ *has filtration*

$$\begin{array}{ccc} & \overline{\nabla}(a-1,b)^{\mathrm{F}} \otimes L(l-1,r) & \\ \swarrow & & \searrow \\ \overline{\nabla}(a-1,b+1)^{\mathrm{F}} \otimes L(r,s) & & \overline{\nabla}(a,b-1)^{\mathrm{F}} \otimes L(r,s) \\ \searrow & & \swarrow \\ & \overline{\nabla}(a,b)^{\mathrm{F}} \otimes L(s,l-1). & \end{array}$$

(iv) *Suppose* $\lambda = l(a,b) + (r,s)$ *with* $(a,b) \in X^+$, $(r,s) \in X_1$ *and* $r+s = l-2$. *Then the module* $\nabla(\lambda)$ *has filtration*

$$\begin{array}{ccc} & \overline{\nabla}(a-1,b-1)^{\mathrm{F}} \otimes L(r,s) & \\ \swarrow & & \searrow \\ \overline{\nabla}(a,b-1)^{\mathrm{F}} \otimes L(l-1,r) & & \overline{\nabla}(a-1,b)^{\mathrm{F}} \otimes L(s,l-1) \\ \searrow & & \swarrow \\ & \overline{\nabla}(a,b)^{\mathrm{F}} \otimes L(r,s). & \end{array}$$

(v) *Suppose* $\lambda = l(a,0) + (r,s)$ *with* $(a,0) \in X^+$, $a \geqslant 1$ *and* $(r,s) \in C$ *then the module* $\nabla(\lambda)$ *has filtration*

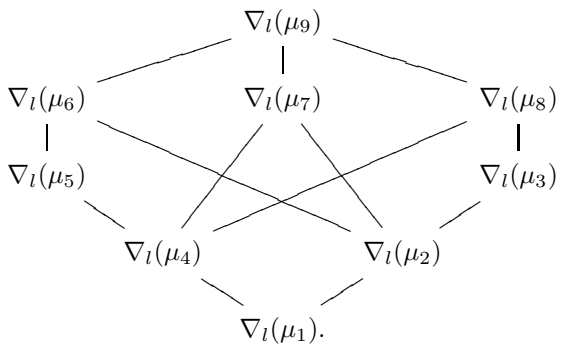
$$\begin{array}{c} \overline{\nabla}(a-2,0)^{\mathrm{F}} \otimes L(s, l-r-s-3) \\ | \\ \overline{\nabla}(a-1,0)^{\mathrm{F}} \otimes L(l-r-2, r+s+1) \\ | \\ \overline{\nabla}(a,0)^{\mathrm{F}} \otimes L(r,s). \end{array}$$

(vi) *Suppose* $\lambda = l(0,b) + (r,s)$ *with* $(0,b) \in X^+$, $b \geqslant 1$ *and* $(r,s) \in C$. *Then the module* $\nabla(\lambda)$ *has filtration*

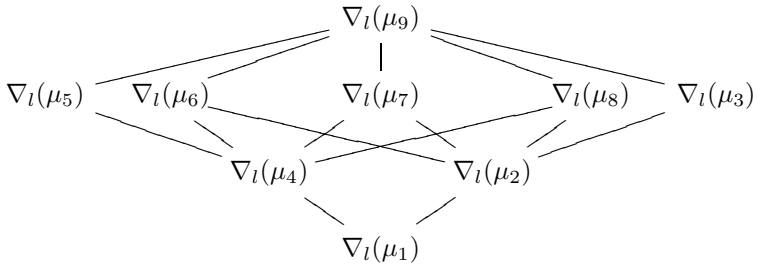
$$\begin{array}{c} \overline{\nabla}(0,b-2)^{\mathrm{F}} \otimes L(l-r-s-3, r) \\ | \\ \overline{\nabla}(0,b-1)^{\mathrm{F}} \otimes L(r+s+1, l-s-2) \\ | \\ \overline{\nabla}(0,b)^{\mathrm{F}} \otimes L(r,s). \end{array}$$

(vii) *Suppose* $\lambda = l(a,b) + (r,s)$ *with* $(a,b) \in X^+$, a *and* $b \geqslant 1$, *and* $(r,s) \in C$. *We let* μ_1 *upto* μ_9 *be as before, depicted in Figure 1 (a), where the number corresponds to the subscript of* μ.

Then for a *and* $b \equiv 0 \pmod{l}$, $\nabla(\lambda)$ *has filtration*

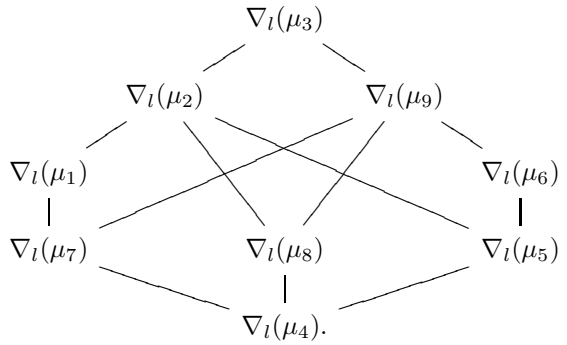


For $a \not\equiv 0 \pmod{l}$ *there is no extension of* $\nabla_l(\mu_5)$ *by* $\nabla_l(\mu_6)$. *For* $b \not\equiv 0 \pmod{l}$ *there is no extension of* $\nabla_l(\mu_3)$ *by* $\nabla_l(\mu_8)$. *So for* a *and* $b \not\equiv 0 \pmod{l}$ *we have:*

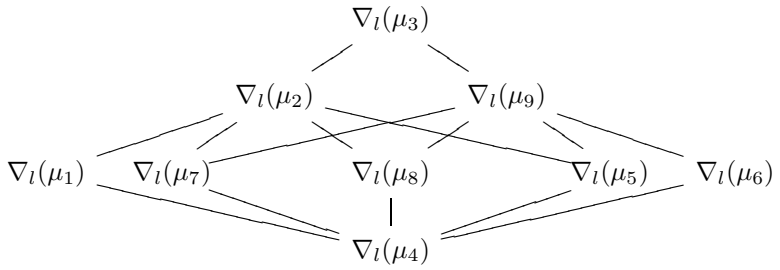


and similarly for the other cases for a *and* b.

(viii) *Suppose* $\lambda = l(a,b) + (l-s-2, l-r-2)$ *with* $(a,b) \in X^+$, *and* $(r,s) \in C$. *We let* μ_1 *upto* μ_9 *be as before, depicted in Figure 1 (b), where the number corresponds to the subscript of* μ. *Then for* a *and* $b \equiv -1 \pmod{l}$, $\nabla(\lambda)$ *has filtration*



For $a \not\equiv -1 \pmod{l}$ *there is no extension of* $\nabla_l(\mu_5)$ *by* $\nabla_l(\mu_6)$. *For* $b \not\equiv -1 \pmod{l}$ *there is no extension of* $\nabla_l(\mu_7)$ *by* $\nabla_l(\mu_1)$. *So for* a *and* $b \not\equiv -1 \pmod{l}$ *we have:*



and similarly for the other cases for a *and* b.

Proof. This may now be proved as in the classical case [15]. □

9. Homorphisms between induced modules for q-$\mathrm{GL}_3(k)$

We now show how to generalise the results of [6] to the quantum case. As noted in that paper, there were two obstacles to this. The first was that we needed an l-filtration of the induced modules, and the second was that we needed a quantum version of main result of [4]. We can now prove that this result ([4]) holds for q-$\mathrm{GL}_3(k)$, but unfortunately not in general. We will assume that $p \neq 0$. The case with $p = 0$ is easier.

We define a *lp^e-wall* for $e \in \mathbb{N}$ to be a wall for X^+ that is fixed by a reflection of the form s_{β, mlp^e} for some $m \in \mathbb{Z}$ and $\beta \in R$.

Theorem 9.1. *Suppose that* $\lambda, \mu \in X^+$ *satisfy the following conditions:*

(i) $\mu < \lambda$.

(ii) *There exists some* $e \in \mathbb{N}$ *such that:*

(a) λ *and* μ *are mirror images in some* lp^e*-wall* L *and*

(b) L *is the unique* lp^e*-wall between* λ *and* μ *(possibly containing* λ *or* μ*) parallel to* L.

Then $\mathrm{Hom}_G(\nabla(\lambda),\nabla(\mu)) \neq 0$.

Proof. We may assume that λ is not a Steinberg weight as then the result follows by twisting the corresponding map for the classical case.

Suppose L is fixed by s_{β,mlp^e} for some $m \in \mathbb{N}$ and $\beta \in \mathbb{R}^+$. There are two cases to consider.

Case (1): β is a simple root. In this case the theorem reduces to the analogous one for q-$\mathrm{GL}_2(k)$ using Levi subgroups and the results of Donkin [10]. See [5, theorem 5.1 and 7.1].

Case (2): $\beta = \rho$. In this case, we construct the homomorphism directly.

We first suppose that $e = 0$ and that λ doesn't lie in an up alcove. We then claim that the required map is the one obtained by inducing the map $\hat{Z}(\lambda) \to \mathrm{hd}(\hat{Z}(\lambda))$ from G_1B upto G.

We claim that the head of $\hat{Z}(\lambda)$ is $\hat{L}(\mu)$. We write $\lambda = l(a,b) + (r,s)$ with $(a,b) \in X^+$ and $(r,s) \in X_1$. Now

$$\begin{aligned}
\mathrm{hd}(\hat{Z}(\lambda)) &= \hat{L}(2(l-1)\rho - \lambda)^* \\
&= \hat{L}(l(2-a,2-b) - (r+2,s+2))^* \\
&\cong L(-w_0(l-r-2,l-s-2)) \otimes k_{-l(1-a,1-b)} \\
&= L(l-s-2,l-r-2)) \otimes k_{-l(1-a,1-b)} \\
&\cong \hat{L}((l-s-2,l-r-2) + l(a-1,b-1))
\end{aligned}$$

Also the condition on L, λ and μ implies that m is the greatest integer such that $\langle \lambda+\rho, \rho^\vee \rangle - ml$ is positive. We thus have $\langle \lambda+\rho, \rho^\vee \rangle = ml + d$, where $1 \leqslant d \leqslant l$. Hence

$$\begin{aligned}
\mu &= s_{\rho,ml} \cdot \lambda \\
&= \lambda - (\langle \lambda+\rho, \rho^\vee \rangle - ml)\rho \\
&= \lambda - d\rho.
\end{aligned}$$

Since $\langle \lambda+\rho, \rho^\vee \rangle = l(a+b) + r + s + 2$, d is then $r+s+2$, as the condition that λ is not in an up alcove implies that $r+s+2$ is at most l. Thus $\mu = (l-s-2,l-r-2) + l(a-1,b-1)$, as required.

We note that the image of this map is $\mathrm{Ind}_{G_1B}^G \hat{L}(\mu) = \nabla_l(\mu)$.

If $e = 0$ and λ lies in an up alcove then the required map is that of 7.2. (It has image the quotient module of $\nabla(\lambda)$ with an l-filtration by $\nabla_l(\lambda_8)$, $\nabla_l(\lambda_2)$, $\nabla_l(\lambda_3)$ and $\nabla_l(\lambda_9)$.)

We now suppose $e > 0$. We let $\nabla_l(\eta)$ be the G_1-head of $\nabla(\lambda)$. We know that $\eta = \lambda - (r+s+2)\rho$, using the same notation as in the previous case. Note that $\eta' = \mu'$, as they are both downward reflections of λ.

We claim that η'', (considered as a weight for $\mathrm{SL}_3(k)$) is $s_{\beta,mp^e} \cdot \mu''$. Thus there is a Carter-Payne map from

$$\phi : \overline{\nabla}(\eta'') \to \overline{\nabla}(\mu'').$$

We then twist the above map:

$$Id \otimes \phi^{\mathrm{F}} : \nabla_l(\eta) \to \nabla_l(\mu).$$

This then induces the required map from $\nabla(\lambda)$ to $\nabla(\mu)$.

We now prove the claim. Consider $s_{\beta,mp^e} \cdot \mu'' = \mu'' - (\langle \mu'' + \rho, \rho^\vee \rangle - mp^e)\rho$. Now the condition on L, λ and μ imply that $\langle \mu + \rho, \rho^\vee \rangle = mlp^e - d$, where $1 \leqslant d \leqslant lp^e$. Thus $\langle \mu'', \rho^\vee \rangle - mp^e = -\frac{1}{l}(d + \langle \mu' + \rho, \rho^\vee \rangle)$. And so

$$\begin{aligned} l(s_{\beta,mp^e} \cdot \mu'') + \mu' &= l\mu'' + (d + \langle \mu' + \rho, \rho^\vee \rangle - 2l)\rho + \mu' \\ &= \mu + d\rho + (\langle \eta' + \rho, \rho^\vee \rangle - 2l)\rho \\ &= \lambda - (2l - \langle \eta' + \rho, \rho^\vee \rangle)\rho \\ &= \lambda - (2l - \langle (l-s-1, l-r-1), (1,1) \rangle)\rho \\ &= \lambda - (s + r + 2)\rho \\ &= \eta. \end{aligned}$$

Thus $\eta'' = s_{\beta,mp^e} \cdot \mu''$ as required. □

As a corollary we get that all the results of [6] regarding homomorphisms between induced modules now generalise to the quantum case if $l \geqslant 3$. We just need to replace the p^{e+1} walls and reflections with lp^e walls and reflections.

In particular we have

Theorem 9.2. *Suppose $l \geqslant 3$, then all the $\mathrm{Hom}_G(\nabla(\lambda), \nabla(\mu))$, with λ, $\mu \in X^+$ are at most one-dimensional.*

The non-zero homomorphisms may be determined by using the appropriate generalisations of the main theorems of [6].

The characteristic zero case is easier. Here, we only get reflections about *l-walls*, that is, a wall fixed by a reflection of the form $s_{\beta,ml}$ for some $m \in \mathbb{Z}$ and $\beta \in R$. In this case, the only maps are the l-good maps. This is because $\nabla_l(\lambda)$ is always isomorphic to $L(\lambda)$, thus any map between induced modules must respect the l-filtration. Hence we have the following.

Theorem 9.3. *Suppose $p = 0$. All the $\mathrm{Hom}_G(\nabla(\lambda), \nabla(\mu))$, with λ, $\mu \in X^+$ are at most one-dimensional. Any non-zero map is an l-good map and is described by the appropriate quantum version of* [6, Lemma 3.1].

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