ISOTROPIC RANDOM WALKS ON AFFINE BUILDINGS

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Abstract. Recently, Cartwright and Woess [4] provided a detailed analysis of *isotropic* random walks on the vertices of thick affine buildings of type $\tilde{A}_n$. Their results generalise results of Sawyer [15] where homogeneous trees are studied (these are $\tilde{A}_1$ buildings), and Lindlbauer and Voit [8], where $\tilde{A}_2$ buildings are studied. In this paper we apply techniques of spherical harmonic analysis to prove a local limit theorem, a rate of escape theorem, and a central limit theorem for isotropic random walks on arbitrary thick regular affine buildings of irreducible type, thus providing a broad generalisation of the $\tilde{A}_n$ case.

Introduction

Let $\mathscr{X}$ be a thick locally finite regular affine building of irreducible type. By *regular* we mean that the number of chambers containing a panel depends only on the cotype of the panel, and by *thick* we mean that this number is always at least 3. The simplest example of such a building is a homogeneous tree with degree $q+1 \geq 3$, where the chambers are the edges of the graph. In this case, Sawyer [15] studied random walks $(Z_k)_{k\geq 0}$ on the vertices of $\mathscr{X}$ which are *isotropic* , meaning that the transition probabilities $p(x,y) = \mathbb{P}(Z_{k+1} = y \mid Z_k = x)$ depend only on the graph distance $d(x,y)$ between x and y. To motivate our results, let us briefly describe these random walks on trees.

Let V be the vertex set of the tree, and for each $x \in V$ and $k \in \mathbb{N} = \{0,1,\ldots\}$, write $V_k(x)$ for the set of all $y \in V$ such that $d(x,y) = k$. It is easily seen that the cardinalities $|V_k(x)|$ are independent of the particular $x \in V$, and we write N_k for this value. For each $k \in \mathbb{N}$ there is a natural operator A_k acting on the space of all functions $f : V \to \mathbb{C}$, where for each $x \in V$, $(A_k f)(x)$ is the average value of f over $V_k(x)$. The operator A_k may be regarded as the transition operator of the isotropic random walk with matrix $(p_k(x,y))_{x,y\in V}$, where $p_k(x,y) = \frac{1}{N_k}$ if $y \in V_k(x)$ and $p_k(x,y) = 0$ otherwise. Indeed it is easily seen that a random walk on V is isotropic if and only if it has a transition operator of the form

$$A = \sum_{k\in\mathbb{N}} a_k A_k$$

where $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k\in\mathbb{N}} a_k = 1$.

Date: November 15, 2005.

2000 *Mathematics Subject Classification.* 20E42 (60G50 33D52).

Key words and phrases. Affine buildings, random walks, Macdonald spherical functions.

The linear span over $\mathbb{C}$ of the operators $\{A_k\}_{k\in\mathbb{N}}$ is a commutative algebra $\mathscr{A}$ with a rich theory of harmonic analysis (see [6]). In particular, the algebra homomorphisms $h:\mathscr{A}\to\mathbb{C}$ may be explicitly described, and local limit theorems, central limit theorems, and rate of escape theorems can be proved as applications.

Now consider a regular affine building $\mathscr{X}$ of irreducible type. Thus $\mathscr{X}$ may be regarded as a simplicial complex made by 'gluing together' many copies of a given *Coxeter complex*, each Coxeter complex called an *apartment* of $\mathscr{X}$ (these are regular tesselations of Euclidean space by simplices). There is an irreducible (but not necessarily reduced) root system R associated to $\mathscr{X}$, as described in Section 1.3, and the *coweight lattice* P of R is a subset of the vertex set of the standard Coxeter complex Σ. We consider random walks on a related subset V_P of the vertices of $\mathscr{X}$, which in most cases is the set of all special vertices of $\mathscr{X}$.

Let P^+ be the set of dominant coweights of R (relative to some fixed base). For each $x\in V_P$ there is a natural partition of V_P into subsets $V_\lambda(x)$, $\lambda\in P^+$, as described in Definition 1.4. Roughly speaking, $y\in V_\lambda(x)$ means that there exists an apartment $\mathcal{A}$ containing x and y and a 'suitable' isomorphism $\psi:\mathcal{A}\to\Sigma$ such that $\psi(x)=0$ and $\psi(y)=\lambda\in P^+$ (in other words, y is *in position* λ *from* x). It is shown in [12, Theorem 5.15] that for all $\lambda\in P^+$ the cardinality of the set $V_\lambda(x)$ is independent of the particular $x\in V_P$, and we write N_λ for this value. Following the tree case, for each $\lambda\in P^+$ let A_λ be the operator acting on the space of functions $f:V_P\to\mathbb{C}$ with $(A_\lambda f)(x)$ being the average value of f over $V_\lambda(x)$. The linear span of these operators over $\mathbb{C}$ is a commutative algebra $\mathscr{A}$, which has been studied extensively in [12]. The algebra homomorphisms $h:\mathscr{A}\to\mathbb{C}$ may be explicitly described both in terms of the *Macdonald spherical functions* and in terms of an integral over the *boundary* of $\mathscr{X}$.

We call a random walk on V_P with transition matrix $(p(x,y))_{x,y\in V_P}$ *isotropic* if $p(x,y)=p(x',y')$ whenever $y\in V_\lambda(x)$ and $y'\in V_\lambda(x')$ for some $\lambda\in P^+$. As in the tree case, the operators A_λ may be regarded as the transition operators of isotropic random walks with matrices $(p_\lambda(x,y))_{x,y\in V}$, where $p_\lambda(x,y)=\frac{1}{N_\lambda}$ if $y\in V_\lambda(x)$ and $p_\lambda(x,y)=0$ otherwise. It is easily seen that a random walk on V_P is isotropic if and only if it has a transition operator of the form

$$A=\sum_{\lambda\in P^+}a_\lambda A_\lambda \tag{0.1}$$

where $a_\lambda\geq 0$ for all $\lambda\in P^+$ and $\sum_{\lambda\in P^+}a_\lambda=1$.

In this paper we apply the spherical harmonic analysis associated to the algebra $\mathscr{A}$ to prove a local limit theorem, a central limit theorem, and a rate of escape theorem for isotropic random walks on V_P. These results generalise the results in [4] where $\tilde{A}_n$ buildings are studied (which in turn generalise the corresponding results for homogeneous trees). While the overall structure of this paper follows [4] quite closely, the general setting is considerably more complicated that the more concrete $\tilde{A}_n$ case.

Let us briefly outline the structure of this paper. In Section 1 we give a summary of some background material, mostly from [12] and [13]. This section includes a discussion of root systems, Coxeter complexes and buildings, the algebra $\mathscr{A}$, and the spherical harmonic analysis associated to this algebra. The main sections of this paper are Sections 2, 3 and 4. In Section 2 we give our local limit theorem for isotropic random walks on V_P, describing the asymptotic behaviour of the k-step transition probabilities $p^{(k)}(x,y) = \mathbb{P}(Z_k = y \mid Z_0 = x)$. We also give necessary and sufficient conditions for irreducibility and aperiodicity of the random walk, and in Remark 2.19 we outline some applications of our local limit theorem to random walks on groups acting on buildings. In Section 3 we prove our rate of escape theorem. For each $k \in \mathbb{N}$, let $\nu_k \in P^+$ be such that $Z_k \in V_{\nu_k}(Z_0)$. We show that $\frac{1}{k}\nu_k$ converges in E (the underlying vector space of R) with probability 1. In Section 4 we prove our central limit theorem, showing that there is a positive definite matrix Γ such that

$$\left(\frac{\langle \nu_k, \alpha_1\rangle - \gamma_1 k}{\sqrt{k}}, \ldots, \frac{\langle \nu_k, \alpha_n\rangle - \gamma_n k}{\sqrt{k}}\right)$$

tends in distribution to the normal distribution $N(0,\Gamma)$. Here $\{\alpha_j\}_{j=1}^n$ is a base of R, and $\langle\cdot,\cdot\rangle$ is the usual inner product on E, and the numbers γ_j, $j = 1,\ldots,n$, are certain constants determined by the underlying building. In Appendix A we determine the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ which are bounded (generalising [9, Theorem 4.7.1]).

1. Affine Buildings and Spherical Harmonic Analysis

1.1. **Root Systems and Weyl Groups.** Root systems play a significant role in this work. We fix the following notations and conventions, generally following [1, Chapter VI].

Let R be an irreducible, but not necessarily reduced, root system in a real vector space E with inner product $\langle\cdot,\cdot\rangle$. The *rank* of R is n, the dimension of E. Let $I_0 = \{1,2,\ldots,n\}$ and $I = \{0,1,2,\ldots,n\}$, and let $B = \{\alpha_i \mid i \in I_0\}$ be a fixed base of R. Write R^+ for the set of positive roots (relative to B), and let $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ be the *dual* root system of R, where for each $\alpha \in R$ we write $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$. Since R is irreducible there is a unique *highest root* $\tilde{\alpha}$ with the property that if

$$\tilde{\alpha} = \sum_{i\in I_0} m_i\alpha_i \tag{1.1}$$

and if $\beta = \sum_{i\in I_0} k_i\alpha_i \in R$, then $m_i \geq k_i$ for each $i \in I_0$ [1, VI, §1, No.8, Proposition 25].

For each $i \in I_0$ define $\lambda_i \in E$ by $\langle\lambda_i,\alpha_j\rangle = \delta_{i,j}$. The elements $\{\lambda_i\}_{i\in I_0}$ are called the *fundamental coweights of* R. The *coweight lattice of* R is $P = \sum_{i\in I_0}\mathbb{Z}\lambda_i$, and elements $\lambda \in P$ are called *coweights of* R. A coweight $\lambda \in P$ is said to be *dominant* if $\langle\lambda,\alpha_i\rangle \geq 0$ for all $i \in I_0$, and we write P^+ for the set of all dominant coweights. Let $Q = \sum_{\alpha\in R}\mathbb{Z}\alpha^\vee$ be the *coroot lattice of* R, and let $Q^+ = \sum_{\alpha\in R^+}\mathbb{N}\alpha^\vee$. For $\lambda,\mu \in P$ we write $\mu \preceq \lambda$ if $\lambda - \mu \in Q^+$ (this defines a partial order on P).

For each $\alpha \in R$ and $k \in \mathbb{Z}$ let $H_{\alpha;k} = \{x \in E \mid \langle x, \alpha\rangle = k\}$. We call these sets *affine hyperplanes*, or simply *hyperplanes*. For each $\alpha \in R$ and $k \in \mathbb{Z}$ let $s_{\alpha;k}$ denote the orthogonal reflection in $H_{\alpha;k}$. Thus $s_{\alpha;k}(x) = x - (\langle x, \alpha\rangle - k)\alpha^\vee$ for all $x \in E$. Write s_α in place of $s_{\alpha;0}$, s_i in place of s_{α_i} (for $i \in I_0$), and let $s_0 = s_{\tilde{\alpha};1}$. Let $W_0 = W_0(R)$ be the *Weyl group of* R, and let $W = W(R)$ be the *affine Weyl group of* R. Thus W_0 is the subgroup of $\mathrm{GL}(E)$ generated by $\{s_i\}_{i\in I_0}$, and W is the subgroup of $\mathrm{Aff}(E)$ generated by $\{s_i\}_{i\in I}$. Both (W_0, S_0) and (W, S) are Coxeter systems, where $S_0 = \{s_i\}_{i\in I_0}$ and $S = \{s_i\}_{i\in I}$, and clearly $W_0 \leq W$. Given $w \in W$, we define the *length* $\ell(w)$ of w to be smallest $k \in \mathbb{N}$ such that $w = s_{i_1} \dots s_{i_k}$, with $i_1, \dots, i_k \in I$.

The *extended affine Weyl group of* R is $\tilde{W}(R) = \tilde{W} = W_0 \ltimes P$. Since $W \cong W_0 \ltimes Q$ (see [1, VI, §2, No.1, Proposition 1]) and $Q \subseteq P$, we may regard W as a subgroup of $\tilde{W}$. Note that $\tilde{W}$ contains all translations by elements of P, while W only contains those translations by elements of Q.

Remark 1.1. We make the following comments for those readers not so familiar with the non-reduced root systems. For each $n \geq 1$ there is exactly one irreducible non-reduced root system (up to isomorphism) of rank n, denoted by BC_n. To describe this root system we may take $E = \mathbb{R}^n$ with the usual inner product, and let R consist of the vectors $\pm e_i, \pm 2e_i$ and $\pm e_j \pm e_k$ for $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_n$. Then $B = \{\alpha_i\}_{i\in I_0}$ is a base of R, and R^+ consists of the vectors $e_i, 2e_i$ and $e_j \pm e_k$ for $1 \leq i \leq n$ and $1 \leq j < k \leq n$. The fundamental coweights are $\lambda_i = e_1 + \cdots + e_i$ for each $1 \leq i \leq n$. Note that $R^\vee = R$ and $Q = P$. The subsystem $R_1 = \{\alpha \in R \mid 2\alpha \notin R\}$ is a root system of type C_n, and the subsystem $R_2 = \{\alpha \in R \mid \frac{1}{2}\alpha \notin R\}$ is a root system of type B_n (with the convention that $C_1 = B_1 = A_1$). We have $Q(R) = P(R) = Q(R_1) \subsetneq P(R_1)$, and so $W(R) = W(R_1)$ but $\tilde{W}(R) \neq \tilde{W}(R_1)$.

1.2. **The Coxeter Complex.** There is a natural *geometric realisation* $\Sigma = \Sigma(R)$ of the *Coxeter complex of* $W = W(R)$. Let $\mathcal{H}$ denote the family of the hyperplanes $H_{\alpha;k}$, $\alpha \in R$, $k \in \mathbb{Z}$, and define *chambers* of Σ to be open connected components of $E \setminus \bigcup_{H\in\mathcal{H}} H$. Since R is irreducible each chamber is an open (geometric) simplex [1, V, §3, No.9, Proposition 8], and we call the extreme points of the closure of chambers *vertices* of Σ, and we write $V(\Sigma)$ for the set of all vertices of Σ.

The set P of coweights of R is a subset of $V(\Sigma)$, and we call elements of P the *good vertices of* Σ. When R is reduced, P is the set of more familiar *special vertices* of Σ [1, VI, §2, No.2, Proposition 3].

The choice of the base $B = \{\alpha_i\}_{i=1}^n$ gives a natural choice of a *fundamental chamber*

$$C_0 = \{x \in E \mid \langle x, \alpha_i\rangle > 0 \text{ for all } i \in I_0 \text{ and } \langle x, \tilde{\alpha}\rangle < 1\}. \tag{1.2}$$

In the notation of (1.1), the vertices of C_0 are the points $\{0\} \cup \{\lambda_i/m_i\}_{i\in I_0}$ [1, VI, §2, No.2]. There is a natural simplicial complex structure on Σ with maximal simplices being the vertex sets of chambers of Σ, and simplices being subsets of the maximal simplices. We define $\tau : V(\Sigma) \to I$ to be the unique labelling of Σ (as a simplicial complex) such that $\tau(0) = 0$ and $\tau(\lambda_i/m_i) = i$ for each $i \in I_0$.

We write $I_P = \{\tau(\lambda) \mid \lambda \in P\} \subseteq I$. Let $\{m_i\}_{i\in I_0}$ be as in (1.1), and define $m_0 = 1$. We have $I_P = \{i \in I \mid m_i = 1\}$, which shows that $0 \in I_P$ for all root systems, and that $I_P = \{0\}$ if R is non-reduced [12, Lemma 4.3]. This also shows that in the non-reduced case, and only in the non-reduced case, there are special vertices which are not good vertices.

We define the *fundamental sector* of Σ to be the open simplicial cone

$$\mathcal{S}_0 = \{x \in E \mid \langle x, \alpha_i\rangle > 0 \text{ for all } i \in I_0\}. \tag{1.3}$$

The *sectors* of Σ are then the sets $\lambda + w\mathcal{S}_0$, where $w \in W_0$ and $\lambda \in P$ (equivalently, the sectors are the sets $\tilde{w}\mathcal{S}_0$, $\tilde{w} \in \tilde{W}$).

An *automorphism of* Σ is a bijection ψ of E which maps chambers, and only chambers, to chambers, with the property that chambers C and D are adjacent if and only if $\psi(C)$ is adjacent to $\psi(D)$. We write $\mathrm{Aut}(\Sigma)$ for the automorphism group of Σ.

An automorphism ψ of Σ is called *type preserving* if $\tau(v) = \tau(\psi(v))$ for all $v \in V(\Sigma)$. By [14, Lemma 2.2] $\psi \in \mathrm{Aut}(\Sigma)$ is type preserving if and only if $\psi \in W$. Generally we have $W_0 \leq W \leq \tilde{W} \leq \mathrm{Aut}(\Sigma)$.

1.3. **Buildings and Regularity.** Recall ([2]) that a *building of type* W is a nonempty simplicial complex $\mathscr{X}$ which contains a family of subcomplexes called *apartments* such that

(i) each apartment is isomorphic to the (simplicial) Coxeter complex of W,
(ii) given any two chambers of $\mathscr{X}$ there is an apartment containing both, and
(iii) given any two apartments $\mathcal{A}$ and $\mathcal{A}'$ that contain a common chamber, there exists an isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ fixing $\mathcal{A} \cap \mathcal{A}'$ pointwise.

Since W is an affine Weyl group, $\mathscr{X}$ is called an *affine building*.

It is an easy consequence of this definition that $\mathscr{X}$ is a labellable simplicial complex, and all the isomorphisms in the above definition may be taken to be type preserving (this ensures that the labellings of $\mathscr{X}$ and Σ are *compatible*).

Let V and $\mathcal{C}$ be the vertex and chamber sets of $\mathscr{X}$, respectively (with chambers being maximal simplices of $\mathscr{X}$). Chambers c and d are declared to be *i-adjacent* (written $c \sim_i d$) if and only if either $c = d$, or if all the vertices of c and d are the same except for those of type i.

Throughout this paper we assume that our buildings are

(i) *locally finite*, meaning that $|I| < \infty$ and $|\{d \in \mathcal{C} \mid d \sim_i c\}| < \infty$ for each $c \in \mathcal{C}$ and $i \in I$,

(ii) *regular*, meaning that $|\{d \in \mathcal{C} \mid d \sim_i c\}|$ is independent of $c \in \mathcal{C}$ for each $i \in I$, and
(iii) *thick*, meaning that $|\{d \in \mathcal{C} \mid d \sim_i c\}| \geq 3$ for each $c \in \mathcal{C}$ and each $i \in I$.

By [12, Theorem 2.4] we see that thickness and regularity are intimately connected. Indeed, the only thick affine buildings of irreducible type which may fail to be regular are those of rank 2 (thus regularity is a very weak hypothesis).

Since $\mathscr{X}$ is assumed to be regular, we define numbers q_i, $i \in I$, called the *parameters* of the building, by $q_i + 1 = |\{d \in \mathcal{C} \mid d \sim_i c\}|$. These numbers satisfy $q_j = q_i$ if $s_j = ws_iw^{-1}$ for some $w \in W$ (see [12, Corollary 2.2]), and by thickness $q_i > 1$ for all $i \in I$. If $w = s_{i_1} \cdots s_{i_k} \in W$ is a reduced expression (that is, $\ell(w) = k$) we define $q_w = q_{i_1} \cdots q_{i_k}$, which is independent of the particular reduced expression for w (see [12, Proposition 2.1(i)]).

To each locally finite regular affine building (of irreducible type) we associate an irreducible root system R (depending on the parameter system of the building) as follows:

(i) If $\mathscr{X}$ is a regular building of type $\tilde{X}_n$, where $X = A$ and $n \geq 2$, or $X = B$ and $n \geq 3$, or $X = D$ and $n \geq 4$, or $X = E$ and $n = 6, 7$ or 8, or $X = F$ and $n = 4$, or $X = G$ and $n = 2$, then we take $R = X_n$.
(ii) If $\mathscr{X}$ is a regular $\tilde{A}_1$ building with $q_0 \neq q_1$, then we take $R = BC_1$.
(iii) If $\mathscr{X}$ is a regular $\tilde{C}_n$ building with $n \geq 2$ and $q_0 \neq q_n$, then we take $R = BC_n$.
(iv) If $\mathscr{X}$ is a regular $\tilde{A}_1$ building with $q_0 = q_1$, then we take $R = A_1$.
(v) If $\mathscr{X}$ is a regular $\tilde{C}_n$ building with $n \geq 2$ and $q_0 = q_n$, then we take $R = C_n$.

The choices above are made to ensure that the coweight lattice P of R preserves the parameter system of $\mathscr{X}$ in the sense that if $v \in V(\Sigma)$ then $q_{\tau(v)} = q_{\tau(v+\lambda)}$ for all $\lambda \in P$. Thus, for example, (iii) above is motivated by the general parameter system of a $\tilde{C}_n$ building (embodied in the Coxeter graph):

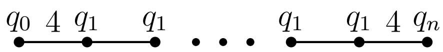

FIGURE 1.1

(see [12, Appendix]). If we take $R = C_n$, then by the definition of the type map (see Section 1.2) and the fact that $m_n = 1$ (see (1.1) and [1, Plate III]) we have $\tau(\lambda_n) = n$. Thus in general we have $q_{\tau(\lambda_n)} = q_n \neq q_0 = q_{\tau(0)}$. If we instead choose $R = BC_n$, then $P = Q$, and so $\tau(v + \lambda) = \tau(v)$ for all $v \in V(\Sigma)$ and $\lambda \in P$ (and hence $q_{\tau(v)} = q_{\tau(v+\lambda)}$).

If R is the root system associated to $\mathscr{X}$, we sometimes say that $\mathscr{X}$ is an *affine building of type R*.

Definition 1.2. Let $\mathscr{X}$ be an affine building of type R with vertex set V. A vertex $x \in V$ is said to be *good* if and only if $\tau(x) \in I_P$ (recall that $I_P = \{\tau(\lambda) \mid \lambda \in P\}$). Write V_P for the set of all good vertices of $\mathscr{X}$.

Clearly V_P is a subset of the more familiar *special vertices* of $\mathscr{X}$. In fact if R is reduced then V_P equals the set of special vertices. If R is non-reduced (so R is of type BC_n for some $n \geq 1$), then V_P is the set of all type 0 vertices of $\mathscr{X}$ (whereas the special vertices are those with types 0 or n).

1.4. **The Algebra $\mathscr{A}$.** In this section we describe an algebra $\mathscr{A}$ of *vertex set averaging operators*. This algebra has been studied in detail in [12], where it is shown that $\mathscr{A}$ is isomorphic to the center of an appropriate affine Hecke algebra.

Definition 1.3. Let $\mathcal{A}$ be an apartment of $\mathscr{X}$. An isomorphism $\psi : \mathcal{A} \to \Sigma$ is called *type rotating* if and only if it is of the form $\psi = w \circ \psi_0$, where $\psi_0 : \mathcal{A} \to \Sigma$ is a type preserving isomorphism, and $w \in \tilde{W}$.

Definition 1.4. Given $x \in V_P$ and $\lambda \in P^+$, let $V_\lambda(x)$ be the set of all $y \in V_P$ such that there exists an apartment $\mathcal{A}$ containing x and y, and a type rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$. Equivalently, $y \in V_\lambda(x)$ if and only if there exists an apartment $\mathcal{A}$ containing x and y and a type rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(x) = 0$ and $\psi(y) \in W_0\lambda$.

The requirement that ψ is type rotating in Definition 1.4 ensures that $y \in V_\lambda(x) \cap V_\mu(x)$ implies that $\lambda = \mu$. Indeed, in [12, Proposition 5.6] we showed that for each $x \in V_P$, $\{V_\lambda(x)\}_{\lambda \in P^+}$ forms a partition of V_P.

Remark 1.5. To get a feel for Definition 1.4 in a special case, suppose that R is of type A_1, and so $\mathscr{X}$ is a homogeneous tree with degree $q + 1$ (where $q = q_0 = q_1$). We may take $R = \{\alpha, -\alpha\}$ where $\alpha = e_1 - e_2$ (the underlying vector space here is $E = \{\xi \in \mathbb{R}^2 \mid \langle \xi, e_1 + e_2 \rangle = 0\}$). Taking $B = \{\alpha\}$ we have $\lambda_1 = \frac{\alpha}{2}$ and $P^+ = \{k\lambda_1 \mid k \in \mathbb{N}\}$. We have $V_P = V$ (all vertices are 'good'), and writing $V_k(x)$ in place of $V_{k\lambda_1}(x)$, we have

$$V_k(x) = \{y \in V \mid d(x, y) = k\},$$

where $d : V \times V \to \mathbb{N}$ is the usual graph metric.

For $\lambda \in P^+$ let $\lambda^* = -w_0\lambda$, where w_0 is the unique longest element of W_0. In [12, Proposition 5.8] we showed that $\lambda^* \in P^+$ for all $\lambda \in P^+$, and that $y \in V_\lambda(x)$ if and only if $x \in V_{\lambda^*}(y)$. Note that $*$ is trivial unless $w_0 \neq -1$, that is, unless $R = A_n, D_{2n+1}$ or E_6 for some $n \geq 2$ (see [1, Plates I-IX]).

In [12, Theorem 5.15] we showed that $|V_\lambda(x)| = |V_\lambda(y)|$ for all $x, y \in V_P$ and $\lambda \in P^+$, and we denote this common value by N_λ (we give a formula for N_λ in the next section). For each $\lambda \in P^+$ define an operator A_λ, acting on the space of functions $f : V_P \to \mathbb{C}$, by

$$(A_\lambda f)(x) = \frac{1}{N_\lambda} \sum_{y \in V_\lambda(x)} f(y) \quad \text{for all } x \in V_P$$

(thus $(A_\lambda f)(x)$ is the average value of f over the set $V_\lambda(x)$). The linear span $\mathscr{A}$ of $\{A_\lambda\}_{\lambda\in P^+}$ over $\mathbb{C}$ is a commutative algebra [12, Theorem 5.24].

Remark 1.6. (i) In the situation of Remark 1.5, writing N_k in place of $N_{k\lambda_1}$ we have $N_0 = 1$ and $N_k = (q+1)q^{k-1}$ for $k \geq 1$. In this case the operators $A_k = A_{k\lambda_1}$ have been studied by many authors (see [6, p.57], [15] or [17, §III.19.C]). They satisfy the simple recurrence

$$A_kA_1 = \frac{q}{q+1}A_{k+1} + \frac{1}{q+1}A_{k-1} \qquad \text{for } k \geq 1,$$

although for general affine buildings such a formula is not readily available.

(ii) Let $\mathscr{A}_Q$ denote the linear space (over $\mathbb{C}$) of $\{A_\lambda \mid \lambda \in Q \cap P^+\}$. It is easily seen that $\mathscr{A}_Q$ is a subalgebra of $\mathscr{A}$, and in the case when $\mathscr{X}$ is the Bruhat-Tits building of a group G of p-adic type with maximal compact subgroup K (as in [9, §2.4–2.7]), $\mathscr{A}_Q$ is isomorphic to $\mathscr{L}(G,K)$, the space of continuous, compactly supported bi-K-invariant functions on G.

1.5. **Isotropic Random Walks.** We call a random walk on V_P with transition probability matrix $A = (p(x,y))_{x,y\in V_P}$ *isotropic* if $p(x,y) = p(x',y')$ whenever $y \in V_\lambda(x)$ and $y' \in V_\lambda(x')$ for some $\lambda \in P^+$. In particular, each operator A_λ, $\lambda \in P^+$, represents an isotropic random walk with transition matrix (also called A_λ) given by $A_\lambda = (p_\lambda(x,y))_{x,y\in V_P}$, where $p_\lambda(x,y) = N_\lambda^{-1}$ if $y \in V_\lambda(x)$ and $p_\lambda(x,y) = 0$ otherwise.

It is easily seen that a random walk is isotropic if and only it its transition matrix (operator) A is as in (0.1). To avoid triviality we always assume that $a_\lambda > 0$ for at least one $\lambda \neq 0$ (so that A is not simply the identity). In this paper we will prove a local limit theorem, a central limit theorem, and a rate of escape theorem for such random walks, generalising the work of [15] (where homogeneous trees are studied) and [4] (where $\tilde{A}_n$ buildings are studied). The main techniques we use are those of *spherical harmonic analysis*, as recalled in the following sections. We note that isotropic random walks on $\tilde{A}_2$ buildings have been studied by Lindlbauer and Voit [8] where more hypergroup oriented techniques are used (see [12, §7] for a discussion of the hypergroups that arise in the setting of general affine buildings).

In the case of Remark 1.6(ii), the theorems we prove in this paper can be translated into theorems concerning probability measures on groups of p-adic type. We briefly discuss this in Remark 2.19.

1.6. **The Algebra Homomorphisms** $h : \mathscr{A} \to \mathbb{C}$**.** In this section we recall two formulae for the algebra homomorphisms $h : \mathscr{A} \to \mathbb{C}$ from [13].

To simultaneously deal with the reduced and non-reduced cases we introduce the following notation. Let $R_1 = \{\alpha \in R \mid 2\alpha \notin R\}$, $R_2 = \{\alpha \in R \mid \frac{1}{2}\alpha \notin R\}$ and $R_3 = R_1 \cap R_2$. Notice that $R_1 = R_2 = R_3 = R$ if R is reduced. For $\alpha \in R_2$, write $q_\alpha = q_i$ if $|\alpha| = |\alpha_i|$ (if

$|\alpha| = |\alpha_i|$ then necessarily $\alpha \in R_2$). Since $q_j = q_i$ whenever $s_j = ws_iw^{-1}$ for some $w \in W$ (see [12, Corollary 2.2]), it follows that $q_i = q_j$ whenever $|\alpha_i| = |\alpha_j|$, and so the definition of q_α is unambiguous.

Note that $R = R_3 \cup (R_1 \backslash R_3) \cup (R_2 \backslash R_3)$ where the union is disjoint. Define a set of numbers $\{\tau_\alpha\}_{\alpha \in R}$ by

$$\tau_\alpha = \begin{cases} q_\alpha & \text{if } \alpha \in R_3 \\ q_0 & \text{if } \alpha \in R_1 \backslash R_3 \\ q_\alpha q_0^{-1} & \text{if } \alpha \in R_2 \backslash R_3. \end{cases}$$

It is convenient to define $\tau_\alpha = 1$ if $\alpha \notin R$. Note that $\tau_\alpha = q_\alpha$ if R is reduced (and many subsequent formulae will simplify in this case).

If $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ we write u^λ in place of $u(\lambda)$. The homomorphism $r \in \mathrm{Hom}(P, \mathbb{C}^\times)$ defined by

$$r^\lambda = \prod_{\alpha \in R^+} \tau_\alpha^{\frac{1}{2}\langle \lambda, \alpha \rangle} \qquad \text{for all } \lambda \in P \tag{1.4}$$

plays an important role. By [13, Proposition 1.5] and [13, Proposition A.1] we have

$$N_\lambda = N_{\lambda^*} = \frac{W_0(q^{-1})}{W_{0\lambda}(q^{-1})} r^{2\lambda} \tag{1.5}$$

where $W_{0\lambda} = \{w \in W_0 \mid w\lambda = \lambda\}$ and where $X(q^{-1}) = \sum_{w \in X} q_w^{-1}$ for subsets $X \subset W_0$.

For $w \in W_0$ and $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ we write $wu \in \mathrm{Hom}(P, \mathbb{C}^\times)$ for the homomorphism with $(wu)^\lambda = u^{w\lambda}$ for all $\lambda \in P$. For $\lambda \in P^+$ and $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ we define

$$P_\lambda(u) = \frac{r^{-\lambda}}{W_0(q^{-1})} \sum_{w \in W_0} c(wu) u^{w\lambda} \quad \text{where} \quad c(u) = \prod_{\alpha \in R^+} \frac{1 - \tau_\alpha^{-1} \tau_{\alpha/2}^{-1/2} u^{-\alpha^\vee}}{1 - \tau_{\alpha/2}^{-1/2} u^{-\alpha^\vee}}, \tag{1.6}$$

provided that the denominators of the $c(wu)$ functions do not vanish. Since $P_\lambda(u)$ is a Laurent polynomial (see [13, (1.8)]), these *singular* cases can be obtained from the general formula by taking an appropriate limit (see Lemma 2.9 for one example).

For $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, let $h_u : \mathscr{A} \to \mathbb{C}$ be the linear map with $h_u(A_\lambda) = P_\lambda(u)$ for each $\lambda \in P^+$. By [13, Proposition 2.1] every algebra homomorphism $h : \mathscr{A} \to \mathbb{C}$ is of the form $h = h_u$ for some $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, and $h_{u'} = h_u$ if and only if $u' = wu$ for some $w \in W_0$. We call the formula $h_u(A_\lambda) = P_\lambda(u)$ the *Macdonald formula for the algebra homomorphisms* $h : \mathscr{A} \to \mathbb{C}$.

Remark 1.7. (i) In the situation of Remarks 1.5 and 1.6(i), if $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, then writing $z = u^{\lambda_1} \in \mathbb{C}^\times$ we have

$$h_u(A_k) = \frac{q^{-k/2}}{1 + q^{-1}} \left(\frac{1 - q^{-1} z^{-2}}{1 - z^{-2}} z^k + \frac{1 - q^{-1} z^2}{1 - z^2} z^{-k} \right)$$

provided that $z \neq \pm 1$ (with the values at $z = \pm 1$ found by taking appropriate limits).

(ii) At times the BC_n case (see Remark 1.1) requires separate treatment. Recall the description of the parameter system from Figure 1.1. For $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, by writing $t_i = u^{e_i}$ for $1 \le i \le n$ (noting that in this case $e_i \in P$ for each $1 \le i \le n$), we have

$$c(u) = \left\{ \prod_{i=1}^n \frac{(1-a^{-1}t_i^{-1})(1+b^{-1}t_i^{-1})}{1-t_i^{-2}} \right\} \left\{ \prod_{1 \le j < k \le n} \frac{(1-q_1^{-1}t_j^{-1}t_k)(1-q_1^{-1}t_j^{-1}t_k^{-1})}{(1-t_j^{-1}t_k)(1-t_j^{-1}t_k^{-1})} \right\},$$

where $a = \sqrt{q_n q_0}$ and $b = \sqrt{q_n/q_0}$ (see [13, Section 5.2]).

A *sector* of $\mathscr{X}$ is a subcomplex $\mathcal{S} \subset \mathscr{X}$ such that there exists an apartment $\mathcal{A}$ with $\mathcal{S} \subset \mathcal{A}$, and a type preserving isomorphism $\psi : \mathcal{A} \to \Sigma$ such that $\psi(\mathcal{S})$ is a sector of Σ. The *base vertex* of $\mathcal{S}$ is $\psi^{-1}(\lambda)$, where $\lambda \in P$ is the base vertex of $\psi(\mathcal{S})$. If $\mathcal{S}$ and $\mathcal{S}'$ are sectors of $\mathscr{X}$ with $\mathcal{S}' \subseteq \mathcal{S}$, then we say that $\mathcal{S}'$ is a *subsector* of $\mathcal{S}$. The *boundary* Ω of $\mathscr{X}$ is the set of equivalence classes of sectors, where we declare two sectors to be equivalent if and only if they contain a common subsector. Given $x \in V_P$ and $\omega \in \Omega$ there exists a unique sector, denoted $\mathcal{S}^x(\omega)$, in the class ω with base vertex x [14, Lemma 9.7].

For each $x \in V_P$, $\omega \in \Omega$ and $\lambda \in P^+$, the intersection $V_\lambda(x) \cap \mathcal{S}^x(\omega)$ contains a unique vertex, denoted $v_\lambda^x(\omega)$. By [13, Theorem 3.4], for each $\omega \in \Omega$ and $x, y \in V_P$ there exists a coweight $h(x, y; \omega) \in P$ such that

$$v_\mu^x(\omega) = v_{\mu - h(x,y;\omega)}^y(\omega) \tag{1.7}$$

for $\mu \in P^+$ with each $\langle \mu, \alpha_i \rangle$, $i \in I_0$, sufficiently large. Indeed, if $y \in V_\lambda(x)$ then (1.7) holds, for all $\omega \in \Omega$, whenever $\mu - \Pi_\lambda \subset P^+$ (see [13, Theorem 3.6]). Here $\Pi_\lambda \subset P$ is the *saturated* set with highest coweight λ, which can be explicitly described by

$$\Pi_\lambda = \{ w\nu \mid \nu \in P^+, \nu \preceq \lambda, w \in W_0 \}$$

(see [7, Lemma 13.4B] for example).

By [13, Proposition 3.5], for all $\omega \in \Omega$ and all $x, y, z \in V_P$ we have the *cocycle relation*

$$h(x, y; \omega) = h(x, z; \omega) + h(z, y; \omega). \tag{1.8}$$

Thus $h(x, x; \omega) = 0$ and $h(x, y; \omega) = -h(y, x; \omega)$ for all $\omega \in \Omega$ and all $x, y \in V_P$.

There is a natural topology on Ω (discussed in [13]) in which for each $x \in V_P$ the sets $\Omega_x(y) = \{\omega \in \Omega \mid y \in \mathcal{S}^y(\omega)\}$, $y \in V_P$, form a basis of open and closed sets (this topology is independent of the particular $x \in V_P$ chosen). For each $x \in V_P$ there is a unique regular Borel probability measure ν_x on Ω such that $\nu_x(\Omega_x(y)) = N_\lambda^{-1}$ if $y \in V_\lambda(x)$. For $x, y \in V_P$ the measures ν_x and ν_y are mutually absolutely continuous with Radon-Nikodym derivative $(d\nu_y/d\nu_x)(\omega) = r^{2h(x,y;\omega)}$ (see [13, Theorem 3.17]).

The *integral formula for the algebra homomorphisms* $h : \mathscr{A} \to \mathbb{C}$ is

$$P_\lambda(u) = h_u(A_\lambda) = \int_\Omega (ur)^{h(x,y;\omega)} d\nu_x(\omega) \tag{1.9}$$

for any $x, y \in V_P$ with $y \in V_\lambda(x)$ (see [13, Corollary 3.23 and Theorem 6.2]).

1.7. **The Plancherel measure.** It is easy to see that each $A \in \mathscr{A}$ maps $\ell^2(V_P)$ into itself, and for $\lambda \in P^+$ and $f \in \ell^2(V_P)$ we have $\|A_\lambda f\|_2 \leq \|f\|_2$ (see [3, Lemma 4.1] for a proof in a similar context). So we may regard $\mathscr{A}$ as a subalgebra of the C^*-algebra $\mathscr{L}(\ell^2(V_P))$ of bounded linear operators on $\ell^2(V_P)$. The facts that $y \in V_\lambda(x)$ if and only if $x \in V_{\lambda^*}(y)$, and $N_{\lambda^*} = N_\lambda$, imply that $A_\lambda^* = A_{\lambda^*}$, and so the adjoint A^* of any $A \in \mathscr{A}$ is also in $\mathscr{A}$.

Let $\mathscr{A}_2$ denote the completion of $\mathscr{A}$ with respect to $\|\cdot\|$, the ℓ^2-operator norm. So $\mathscr{A}_2$ is a commutative C^*-algebra. The algebra homomorphisms $h : \mathscr{A}_2 \to \mathbb{C}$ are precisely the extensions $h = \tilde{h}_u$ of those algebra homomorphisms $h_u : \mathscr{A} \to \mathbb{C}$ which are continuous with respect to the ℓ^2-operator norm. Let us describe the latter homomorphisms.

The analysis here splits into two cases. Following [9, Chapter V] we call the situation where $\tau_\alpha \geq 1$ for all $\alpha \in R$ the *standard case*, and the situation where $\tau_\alpha < 1$ for some $\alpha \in R$ the *exceptional case* (the use of the word "exceptional" here is unrelated to the so called *exceptional root systems*). It is immediate from the definition of the numbers τ_α that the exceptional case occurs exactly when $R = BC_n$ for some $n \geq 1$ and $q_n < q_0$ (see [13, Lemma 5.1]). In particular , if R is reduced then we are in the standard case.

Let us consider the standard case first. Let

$$\mathbb{U} = \{u \in \mathrm{Hom}(P, \mathbb{C}^\times) : |u^\lambda| = 1 \text{ for all } \lambda \in P\}.$$

In the standard case the algebra homomorphism $h_u : \mathscr{A} \to \mathbb{C}$ is continuous with respect to the ℓ^2-operator norm if and only if $u \in \mathbb{U}$ (see [13, Corollary 5.4]). If $h = \tilde{h}_u$, $u \in \mathbb{U}$, we write $\widehat{A}(u) = h(A)$ for $A \in \mathscr{A}_2$. In particular, $\widehat{A}_\lambda(u) = P_\lambda(u)$ for $u \in \mathbb{U}$.

In the standard case, let π be the measure on $\mathbb{U}$ given by $d\pi(u) = \frac{W_0(q^{-1})}{|W_0|}|c(u)|^{-2}du$, where du is normalised Haar measure on $\mathbb{U}$ (note that in [13] we write π_0 instead of π). Then for $A \in \mathscr{A}_2$ we have

$$(A\delta_y)(x) = \int_{\mathbb{U}} \widehat{A}(u)\overline{P_\lambda(u)}d\pi(u) \qquad \text{whenever } y \in V_\lambda(x)$$

where $\delta_y(x) = 1$ if $x = y$ and $\delta_y(x) = 0$ otherwise (see [13, Theorem 5.2 and Corollary 5.5]). The measure π is essentially the Plancherel measure of $\mathscr{A}$ (more precisely, the Plancherel measure is the image of the measure π under the homeomorphism $\varpi : \mathbb{U}/W_0 \to \mathrm{Hom}(\mathscr{A}_2, \mathbb{C})$, $u \mapsto \tilde{h}_u$).

Let us consider the exceptional case, and so $R = BC_n$ for some $n \geq 1$ and $q_n < q_0$. For $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, recall the definition of the numbers $t_i = t_i(u)$, $1 \leq i \leq n$, from Remark 1.7(ii). We use the isomorphism $\mathbb{U} \to \mathbb{T}^n$, $u \mapsto (t_1, \ldots, t_n)$ to identify $\mathbb{U}$ with $\mathbb{T}^n$ (here $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$). Define $\mathbb{U}' = \{-b\} \times \mathbb{T}^{n-1}$, and write $U = \mathbb{U} \cup \mathbb{U}'$ (recall from Remark 1.7(ii) that $b = \sqrt{q_n/q_0}$).

Let $dt = dt_1 \cdots dt_n$, where dt_i is normalised Haar measure on $\mathbb{T}$. Let $\phi_0(u) = c(u)c(u^{-1})$, and let

$$\phi_1(u) = \lim_{t_1 \to -b} \frac{\phi_0(u)}{1 + b^{-1}t_1} \qquad \text{and} \qquad dt' = d\delta_{-b}(t_1)dt_2 \cdots dt_n.$$

Note that this limit exists since there is a factor $1+b^{-1}t_1$ in $c(u^{-1})$ (see Remark 1.7(ii)).

In the exceptional case, let π be the measure on $U = \mathbb{U} \cup \mathbb{U}'$ given by $d\pi(u) = \frac{W_0(q^{-1})}{|W_0|}\frac{dt}{\phi_0(u)}$ on $\mathbb{U}$ and $d\pi(u) = \frac{W_0(q^{-1})}{|W_0'|}\frac{dt'}{\phi_1(u)}$ on $\mathbb{U}'$, where W_0' is the Coxeter group C_{n-1} (with $C_1 = A_1$ and $C_0 = \{1\}$). Then for all $A \in \mathscr{A}_2$,

$$(A\delta_y)(x) = \int_U \widehat{A}(u)\overline{P_\lambda(u)}d\pi(u) \quad \text{whenever } y \in V_\lambda(x)$$

(see [13, Theorem 5.7 and Corollary 5.8]).

Remark 1.8. The form of the Plancherel measure in the exceptional case requires that $q_1 b \geq 1$, which follows from a theorem of D. Higman since the numbers q_i, $i \in I$, are the parameters of a building (see [13, Lemma 5.6]). We note that for the hypergroups associated to the BC_n case the Plancherel measure is supported on $U = \mathbb{U} \cup \mathbb{U}' \cup \mathbb{U}'' \cup \cdots$, where there are k components, with k defined by $q_1^{k-1}b \geq 1 > q_1^{k-2}b$. See [9, Theorem 5.2.10].

To conveniently state formulae in the standard and exceptional cases simultaneously, we write $U = \mathbb{U}$ in the standard case and (as above) $U = \mathbb{U} \cup \mathbb{U}'$ in the exceptional case. Thus (in all cases), for $A \in \mathscr{A}_2$,

$$(A\delta_y)(x) = \int_U \widehat{A}(u)\overline{P_\lambda(u)}d\pi(u) \quad \text{whenever } y \in V_\lambda(x). \tag{1.10}$$

2. The Local Limit Theorem

The basic approach for the local limit theorem is as follows. Let A be the transition operator for an isotropic random walk with matrix $(p(x,y))_{x,y\in V_P}$, as in (0.1). Then

$$p^{(k)}(x,y) = (A^k\delta_y)(x) \qquad \text{for all } x,y \in X \text{ and } k \in \mathbb{N}. \tag{2.1}$$

Since $\|A\| \leq 1$, we may regard A as in $\mathscr{A}_2$ and so $h_u(A)$, $u \in U$, is defined. Writing $\widehat{A}(u) = h_u(A)$ for $u \in U$, we have $\widehat{A}_\lambda(u) = P_\lambda(u)$ and so

$$\widehat{A}(u) = \sum_{\lambda \in P^+} a_\lambda P_\lambda(u). \tag{2.2}$$

By (2.1) and (1.10) we have

$$p^{(k)}(x,y) = \int_U \big(\widehat{A}(u)\big)^k\, \overline{P_\lambda(u)}d\pi(u) \qquad \text{whenever } y \in V_\lambda(x). \tag{2.3}$$

We will prove the local limit theorem by determining the asymptotic behaviour of the integral in (2.3) as $k \to \infty$.

Lemma 2.1. *Let $\lambda \in P^+$, $\lambda \neq 0$, $x \in V_P$, and $y \in V_\lambda(x)$. Then*

(i) *there exists $z \in V_\lambda(x) \cap V_{\tilde{\alpha}^\vee}(y)$, and*

(ii) *with z as in* (i), *there exists $\omega \in \Omega$ such that $h(y,z;\omega) = \tilde{\alpha}^\vee$.*

Proof. Note first that if c and d are distinct i-adjacent chambers, $i \in I_P$, with type i vertices u and v respectively, then $v \in V_{\tilde{\alpha}^\vee}(u)$ (and $u \in V_{\tilde{\alpha}^\vee}(v)$). To see this, let $\mathcal{A}$ be any apartment containing c and d, and let $\psi : \mathcal{A} \to \Sigma$ be a type rotating isomorphism such that $\psi(u) = 0$ and $\psi(c) = C_0$. Since $\psi(d)$ is 0-adjacent to $\psi(c)$ we have $\psi(d) = s_{\tilde{\alpha};1}(C_0)$, and so $\psi(v) = s_{\tilde{\alpha};1}(0) = \tilde{\alpha}^\vee$. Thus $v \in V_{\tilde{\alpha}^\vee}(u)$.

Part (i) now follows exactly as in [4, Lemma 5.1]; we include the proof for completeness. Let $\mathcal{A}$ be an apartment containing x and y, and let $c_0, c_1, \ldots, c_m$ be a gallery (that is, a sequence of adjacent chambers with $c_{i-1} \neq c_i$ for $1 \leq i \leq m$) with $x \in c_0$ and $y \in c_m$ and m minimal. Let π be the panel $c_m \backslash \{y\}$, and let c' be the chamber of $\mathcal{A}$ with $c' \neq c_m$ and $\pi \subset c'$ (so $c' = c_{m-1}$ if $m \geq 1$). Let H be the wall of $\mathcal{A}$ determined by π, and let $\mathcal{A}^+$ be the half apartment of $\mathcal{A}$ determined by H containing c'. By thickness there exists a chamber $d \neq c', c_m$ with $\pi \subset d$, and writing z for the vertex in $d \backslash \pi$ we have $z \in V_{\tilde{\alpha}^\vee}(y)$ by the above discussion. We now show that $z \in V_\lambda(x)$. By the proof of [14, Lemma 9.4] there exists an apartment $\mathcal{B}$ containing $\mathcal{A}^+ \cup d$. Let $\rho_{\mathcal{A},c'}$ be the retraction of $\mathscr{X}$ onto $\mathcal{A}$ with center c' (see [2, §IV.3]), and so the map $\varphi = \rho_{\mathcal{A},c'}|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is a type preserving isomorphism with $\varphi(x) = x$ and $\varphi(z) = y$ (since $\varphi(d) = c_m$). Since $y \in V_\lambda(x)$ there exists a type rotating isomorphism $\psi : \mathcal{A} \to \Sigma$ with $\psi(x) = 0$ and $\psi(y) = \lambda$ (see [12, Proposition 5.6(iii)]), and so the map $\phi = \psi \circ \varphi : \mathcal{B} \to \Sigma$ is a type rotating isomorphism with $\phi(x) = 0$ and $\phi(z) = \lambda$. Thus $z \in V_\lambda(x)$.

Part (ii) is a consequence of the following fact. Let $u, v \in V_P$ with $v \in V_\lambda(u)$. Then there exists $\omega \in \Omega$ such that $h(u, v; \omega) = \lambda$. To see this, let $\mathcal{A}$ be any apartment containing u and v, and let $\psi : \mathcal{A} \to \Sigma$ be a type rotating isomorphism such that $\psi(u) = 0$ and $\psi(v) = \lambda$. Let ω be the class of $\psi^{-1}(\mathcal{S}_0)$. Since $\psi^{-1}(\mathcal{S}_0) = \mathcal{S}^u(\omega)$ and $\psi^{-1}(\lambda + \mathcal{S}_0) = \mathcal{S}^v(\omega)$, we have $\psi^{-1}(\mu) = v^u_\mu(\omega) = v^v_{\mu-\lambda}(\omega)$ for sufficiently large $\mu \in P^+$, and so $h(u, v; \omega) = \lambda$. ☐

Recall that $\mathbb{U} = \{u \in \mathrm{Hom}(P, \mathbb{C}^\times) : |u^\lambda| = 1 \text{ for all } \lambda \in P\}$. Let

$$\mathbb{U}_Q = \{u \in \mathrm{Hom}(P, \mathbb{C}^\times) \mid u^\gamma = 1 \text{ for all } \gamma \in Q\}.$$

Thus $\mathbb{U}_Q$ is isomorphic to the dual of the finite abelian group P/Q, and so $\mathbb{U}_Q \cong P/Q$. Hence $\mathbb{U}_Q$ is finite, and $\mathbb{U}_Q \subset \mathbb{U}$.

While the following proposition can easily be verified on a case by case basis using the classification of irreducible root systems, we provide a proof independent of the classification theorem.

Proposition 2.2. *The set $W_0\tilde{\alpha}^\vee$ spans Q over $\mathbb{Z}$. Thus if $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ and $u^{w\tilde{\alpha}^\vee} = 1$ for all $w \in W_0$, then $u \in \mathbb{U}_Q$.*

Proof. Let Q' denote the $\mathbb{Z}$-span of $W_0\tilde{\alpha}^\vee$. We show that $R^\vee \subset Q'$, from which it follows that $Q = Q'$, hence the result. Suppose first that R is reduced, and let $\beta \in R$. By [1, VI, §1, No.3, Proposition 11] all roots of a given length are conjugate under W_0, and so if $|\beta| = |\tilde{\alpha}|$ then $\beta^\vee \in Q'$. Suppose that $|\beta| \neq |\tilde{\alpha}|$. Since at most 2 root lengths occur in R (see [7, Lemma 10.4.C]), and since R is irreducible, there exists $v, v' \in W_0$ such that $\langle v\tilde{\alpha}, v'\beta\rangle \neq 0$ (for otherwise $W_0\tilde{\alpha} \cup W_0\beta$ is a partition of R into pairwise orthogonal sets). Thus $\langle w\tilde{\alpha}, \beta\rangle \neq 0$, where $w = v'^{-1}v$, and so by [1, VI, §1, No.8, Proposition 25(iv)] we have $\langle w\tilde{\alpha}^\vee, \beta\rangle = \pm 1$ depending on if $w^{-1}\beta \in R^+$ or $w^{-1}\beta \in R^-$. Since $s_\beta(w\tilde{\alpha}^\vee) = w\tilde{\alpha}^\vee - \langle w\tilde{\alpha}^\vee, \beta\rangle\beta^\vee$ we have $\beta^\vee \in Q'$.

Finally, if R is non-reduced, then $Q(R) = Q(R_1)$ and $W_0(R) = W_0(R_1)$ (these facts can be seen without the classification theorem). Since $\tilde{\alpha}$ is also the highest root of the reduced root system R_1 (with respect to the natural base), we have $Q'(R) = Q'(R_1)$, and so $Q' = Q$ in all cases. $\square$

Remark 2.3. Since $\tilde{\alpha}$ is a *long root* [7, Lemma 10.4.D] (with the convention that all roots are long if there is only one root length), Proposition 2.2 is true whenever $\tilde{\alpha}$ is replaced with a long root α (for $W_0\alpha = W_0\tilde{\alpha}$). However for general $\alpha \in R$ the proposition fails, even though (by [7, Lemma 10.4.B]) $W_0\alpha^\vee$ spans E. For example let R be the standard B_2 root system, and take $\alpha = e_2$. Then the $\mathbb{Z}$-span of $W_0\alpha^\vee$ is $2\mathbb{Z}^2$, whereas $Q = \mathbb{Z}^2$.

As usual, if $u, v \in \mathrm{Hom}(P, \mathbb{C}^\times)$, define $uv \in \mathrm{Hom}(P, \mathbb{C}^\times)$ by $(uv)^\lambda = u^\lambda v^\lambda$ for all $\lambda \in P$.

Lemma 2.4. *Let $u \in \mathbb{U}$ and $\lambda \in P^+$. Then $|P_\lambda(u)| \leq P_\lambda(1)$, and equality holds for $\lambda \neq 0$ if and only if $u \in \mathbb{U}_Q$. Moreover, if $u_0 \in \mathbb{U}_Q$ then $P_\lambda(u_0u) = u_0^\lambda P_\lambda(u)$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$.*

Proof. (cf. [4, Lemma 5.3]) Let $x, y \in V_P$ be any vertices with $y \in V_\lambda(x)$. The inequality is clear from the integral formula (1.9). Suppose equality holds for some $\lambda \neq 0$. Write $f(\omega)$ for the integrand in (1.9). Then f is a continuous function on Ω and $f(\omega) \neq 0$ for all $\omega \in \Omega$. So $|\int_\Omega f(\omega)d\nu_x(\omega)| = \int_\Omega |f(\omega)|d\nu_x(\omega)$ implies that $f(\omega)/|f(\omega)|$ is constant, since $\nu_x(O) > 0$ for all non-empty open sets $O \subset \Omega$. Thus $u^{h(x,y;\omega)}$ takes the constant value $P_\lambda(u)/P_\lambda(1)$ for all $\omega \in \Omega$. Let z be as in Lemma 2.1(i). Since the value of the integral in (1.9) is unchanged if y is replaced by z, it follows that $u^{h(x,y;\omega)} = u^{h(x,z;\omega)}$ for all $\omega \in \Omega$. Choosing $\omega \in \Omega$ as in Lemma 2.1(ii) and using the cocycle relations we have $u^{\tilde{\alpha}^\vee} = u^{h(y,z;\omega)} = 1$. Furthermore, since the value of the integral in (1.9) is unchanged if u is replaced by wu for any $w \in W_0$, then $u^{w\tilde{\alpha}^\vee} = 1$ for all $w \in W_0$. It follows from Proposition 2.2 that $u \in \mathbb{U}_Q$.

Conversely, if $u_0 \in \mathbb{U}_Q$ and $y \in V_\lambda(x)$, then $u_0^{h(x,y;\omega)} = u_0^\lambda$ for all $\omega \in \Omega$, because $\lambda - h(x, y; \omega) \in Q$ (see [13, Theorem 3.4(ii)]). Thus it follows from (1.9) that $P_\lambda(u_0u) = u_0^\lambda P_\lambda(u)$ for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$. In particular, $|P_\lambda(u_0)| = P_\lambda(1)$. $\square$

In the following series of estimates we will write C for a positive constant, whose value may vary from line to line.

For each $\omega \in \Omega$, $x, y \in V_P$ and $1 \leq j \leq n$, define $h_j(x, y; \omega) = \langle h(x, y; \omega), \alpha_j\rangle$.

Lemma 2.5. *Let $x \in V_P$ and $\lambda \in P^+$. Then $|h(x,y;\omega)| \leq |\lambda|$ and $|h_j(x,y;\omega)| \leq C|\lambda|$ for all $\omega \in \Omega$, all $y \in V_\lambda(x)$, and all $j = 1, \ldots, n$.*

Proof. Recall from [13, Theorem 3.4(ii)] that $h(x,y;\omega) \in \Pi_\lambda$ for all $\omega \in \Omega$ and $y \in V_\lambda(x)$. By [10, (2.6.2)] we have that $\Pi_\lambda \subset \mathrm{conv}(W_0\lambda)$ (the usual convex hull in E here), and since $|w\lambda| = |\lambda|$ for all $w \in W_0$, this implies that $|h(x,y;\omega)| \leq |\lambda|$ for all $\omega \in \Omega$ and for all $y \in V_\lambda(x)$.

We have $|\langle h(x,y;\omega), \alpha_j\rangle| \leq |h(x,y;\omega)||\alpha_j|$, proving the final claim. □

Remark 2.6. There is a natural graph with vertex set V_P and vertices $x, y \in V_P$ joined by an edge if and only if $y \in V_{\lambda_i}(x)$ for some $i \in I_0$. In this graph we have $d(x,y) = \sum_{i=1}^n \langle \lambda, \alpha_i\rangle$ if $y \in V_\lambda(x)$. Lemma 2.5 shows that $|h(x,y;\omega)|$ and $|h_j(x,y;\omega)|$ are bounded by $Cd(x,y)$.

Notation. Let $\theta_1, \ldots, \theta_n \in \mathbb{R}$ and write $\theta = \theta_1\alpha_1 + \cdots + \theta_n\alpha_n$ (so $\theta \in E$). Write $e^{i\theta}$ for the element of $\mathrm{Hom}(P, \mathbb{C}^\times)$ with $(e^{i\theta})^\lambda = e^{i\langle\lambda,\theta\rangle}$ for all $\lambda \in P^+$. With this notation (1.9) gives

$$P_\lambda(e^{i\theta}) = \int_\Omega r^{h(x,y;\omega)} e^{i\langle h(x,y;\omega),\theta\rangle} d\nu_x(\omega) \quad \text{for all } y \in V_\lambda(x), \tag{2.4}$$

and since $P_\lambda(w^{-1}e^{i\theta}) = P_\lambda(e^{i\theta})$ for all $w \in W_0$, it follows that

$$P_\lambda(e^{i\theta}) = \int_\Omega r^{h(x,y;\omega)} e^{i\langle h(x,y;\omega),w\theta\rangle} d\nu_x(\omega) \quad \text{for all } w \in W_0,\ y \in V_\lambda(x). \tag{2.5}$$

Corollary 2.7. *For all $\lambda \in P^+$, $P_\lambda(e^{i\theta}) = P_\lambda(1)(1 + E_\lambda(\theta))$, where $|E_\lambda(\theta)| \leq |\lambda||\theta|$.*

Proof. We have

$$|P_\lambda(e^{i\theta}) - P_\lambda(1)| \leq \int_\Omega r^{h(x,y;\omega)} |e^{i\langle h(x,y;\omega),\theta\rangle} - 1| d\nu_x(\omega),$$

and the result follows from Lemma 2.5 since $|e^{iz} - 1| \leq |z|$ for all $z \in \mathbb{R}$. □

Let $\lambda \in P^+$ and $y \in V_\lambda(x)$. For each $1 \leq j,k \leq n$ define

$$b_{j,k}^\lambda = \frac{1}{2}\int_\Omega h_j(x,y;\omega)h_k(x,y;\omega) r^{h(x,y;\omega)} d\nu_x(\omega). \tag{2.6}$$

This is independent of the particular pair $x, y \in V_P$ with $y \in V_\lambda(x)$, for by (2.4)

$$\frac{\partial^2}{\partial\theta_j\partial\theta_k} P_\lambda(e^{i\theta})\bigg|_{\theta=0} = -\int_\Omega h_j(x,y;\omega)h_k(x,y;\omega) r^{h(x,y;\omega)} d\nu_x(\omega).$$

(Indeed any expression $\int_\Omega p(h_1(x,y;\omega), \ldots, h_n(x,y;\omega)) r^{h(x,y;\omega)} d\nu_x(\omega)$, where p is a polynomial, is independent of the particular pair $x, y \in V_P$ with $y \in V_\lambda(x)$).

Lemma 2.8. *Let $\lambda \in P^+$, and $\theta_1, \ldots, \theta_n \in \mathbb{R}$, and as usual write $\theta = \theta_1\alpha_1 + \cdots + \theta_n\alpha_n$. Then*

$$P_\lambda(e^{i\theta}) = P_\lambda(1) - \sum_{j,k=1}^{n} b_{j,k}^{\lambda}\theta_j\theta_k + R_\lambda(\theta) \tag{2.7}$$

where $|R_\lambda(\theta)| \leq C|\lambda|^3|\theta|^3 P_\lambda(1)$. Furthermore, $\sum_{j,k=1}^{n} b_{j,k}^{\lambda}\theta_j\theta_k \geq 0$, and when $\lambda \neq 0$, equality holds if and only if $\theta = 0$.

Proof. For $\varphi \in \mathbb{R}$ we have $e^{i\varphi} = 1 + i\varphi - \frac{1}{2}\varphi^2 + R(\varphi)$ where $|R(\varphi)| \leq \frac{1}{6}|\varphi|^3$. Applying this to $\varphi = \langle h(x, y; \omega), \theta\rangle$ and using (2.4) we have

$$\begin{aligned} P_\lambda(e^{i\theta}) &= P_\lambda(1) + i\int_\Omega \langle h(x, y; \omega), \theta\rangle r^{h(x,y;\omega)} d\nu_x(\omega) \\ &\quad - \frac{1}{2}\int_\Omega \langle h(x, y; \omega), \theta\rangle^2 r^{h(x,y;\omega)} d\nu_x(\omega) + R_\lambda(\theta), \end{aligned}$$

where $|R_\lambda(\theta)| \leq \frac{1}{6}|\langle h(x, y; \omega), \theta\rangle|^3 P_\lambda(1) \leq \frac{1}{6}|h(x, y; \omega)|^3|\theta|^3 P_\lambda(1)$. The bound for $|R_\lambda(\theta)|$ follows from Lemma 2.5.

We claim that for all $j = 1, \ldots, n$ and for all $y \in V_P$,

$$\int_\Omega h_j(x, y; \omega) r^{h(x,y;\omega)} d\nu_x(\omega) = 0.$$

To see this, let $j \in \{1, \ldots, n\}$ and set $\theta = \theta_j\alpha_j$ (that is, $\theta_k = 0$ for all $k \neq j$). By differentiating (2.5) with respect to θ_j, and then evaluating at $\theta_j = 0$, firstly with $w = 1$ and secondly with $w = s_j$, we see that

$$\int_\Omega h_j(x, y; \omega) r^{h(x,y;\omega)} d\nu_x(\omega) = -\int_\Omega h_j(x, y; \omega) r^{h(x,y;\omega)} d\nu_x(\omega),$$

proving the claim.

It is now clear that (2.7) holds, and that $\sum_{j,k=1}^{n} b_{j,k}^{\lambda}\theta_j\theta_k \geq 0$. If equality holds, then

$$\int_\Omega \langle h(x, y; \omega), \theta\rangle^2 r^{h(x,y;\omega)} d\nu_x(\omega) = 0.$$

Thus $\langle h(x, y; \omega), \theta\rangle = 0$ for almost all $\omega \in \Omega$, and thus for all $\omega \in \Omega$. Thus, since $\langle h(x, y; \omega), t\theta\rangle = 0$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$, we have $P_\lambda(e^{i(t\theta)}) = P_\lambda(1)$ for all $t \in \mathbb{R}$ by (2.4), and so $e^{i(t\theta)} \in \mathbb{U}_Q$ for all $t \in \mathbb{R}$ by Lemma 2.4. Thus $\theta = 0$ since $|\mathbb{U}_Q| < \infty$. ☐

Lemma 2.9. *There exists a polynomial $p(x_1, \ldots, x_n)$ of degree at most M such that*

$$P_\lambda(1) = r^{-\lambda} p(\langle\lambda, \alpha_1\rangle, \ldots, \langle\lambda, \alpha_n\rangle) \tag{2.8}$$

for all $\lambda \in P^+$, where $M > 0$ is some integer depending only on the underlying root system. Furthermore, (by thickness) there exists some $q > 1$ such that

$$P_\lambda(1) \leq C(|\lambda| + 1)^M q^{-|\lambda|}. \tag{2.9}$$

Proof. Assuming that $u^{-\alpha^\vee}\neq 1$ for all $\alpha\in R_2^+$, by (1.6) and the definition of the numbers τ_α we have

$$c(u)=\prod_{\alpha\in R_2^+}\frac{(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}u^{-\alpha^\vee/2})(1+\tau_\alpha^{-1/2}u^{-\alpha^\vee/2})}{1-u^{-\alpha^\vee}}. \tag{2.10}$$

Write $\sigma=\lambda_1+\cdots+\lambda_n$. It follows from [1, VI, §3, No.3, Proposition 2] that

$$\prod_{\alpha\in R_2^+}(1-u^{-w\alpha^\vee})=(-1)^{\ell(w)}u^{\sigma-w\sigma}\prod_{\alpha\in R_2^+}(1-u^{-\alpha^\vee})$$

for all $w\in W_0$, and so by (1.6) and (2.10) we have

$$P_\lambda(u)=r^{-\lambda}\frac{F(\lambda)}{\prod_{\alpha\in R_2^+}(1-u^{-\alpha^\vee})} \tag{2.11}$$

where $F(\lambda)$ equals

$$\frac{1}{W_0(q^{-1})}\sum_{w\in W_0}\bigg\{(-1)^{\ell(w)}u^{w\lambda+w\sigma-\sigma}\prod_{\alpha\in R_2^+}(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}u^{-w\alpha^\vee/2})(1+\tau_\alpha^{-1/2}u^{-w\alpha^\vee/2})\bigg\}.$$

We know that $P_\lambda(u)$ is a Laurent polynomial in $u_1,\ldots,u_n$, and so (2.8) follows from (2.11) by repeated applications of L'Hôpital's rule. The inequality (2.9) follows from Proposition 3.4(ii) and the proof of Proposition 3.4(iv) in Section 3. □

Let A be as in (0.1) and $\widehat{A}(u)=h_u(A)$ be as in (2.2). It follows from Lemma 2.5 that $|b_{j,k}^\lambda|\leq C|\lambda|^2P_\lambda(1)$, and thus the inequality (2.9) implies that $\sum_{\lambda\in P^+}a_\lambda b_{j,k}^\lambda$ is absolutely convergent for each $1\leq j,k\leq n$. We define

$$b_{j,k}=\frac{1}{\widehat{A}(1)}\sum_{\lambda\in P^+}a_\lambda b_{j,k}^\lambda. \tag{2.12}$$

Corollary 2.10. *Let A be as in (0.1), and let $\theta_1,\ldots,\theta_n\in\mathbb{R}$. Then*

$$\widehat{A}(e^{i\theta})=\widehat{A}(1)\bigg(1-\sum_{j,k=1}^n b_{j,k}\theta_j\theta_k+R(\theta)\bigg),$$

where $\sum_{j,k=1}^n b_{j,k}\theta_j\theta_k>0$ unless $\theta=0$, and where $|R(\theta)|\leq C|\theta|^3$.

Proof. This follows from Lemma 2.8, using (2.9) to bound $R(\theta)$. □

Lemma 2.11. *Let $\theta_1,\ldots,\theta_n\in\mathbb{R}$. Then*

$$\frac{1}{|c(e^{i\theta})|^2}=\prod_{\alpha\in R_2^+}\frac{\langle\alpha^\vee,\theta\rangle^2}{\big(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}\big)^2\big(1+\tau_\alpha^{-1/2}\big)^2}(1+E_\alpha(\theta))$$

where $|E_\alpha(\theta)|\leq C\langle\alpha^\vee,\theta\rangle^2$ for each $\alpha\in R_2^+$.

Proof. Observe that for $x\in\mathbb{R}$ and $p>1$

$$\left|\frac{1-e^{-ix}}{1-p^{-1}e^{-ix}}\right|^2=\frac{x^2}{(1-p^{-1})^2}(1+E_1(x)),\tag{2.13}$$

where $|E_1(x)|\leq Cx^2$, and for $p>0$

$$\left|\frac{1+e^{-ix}}{1+p^{-1}e^{-ix}}\right|^2=\frac{4}{(1+p^{-1})^2}(1+E_2(x))\tag{2.14}$$

where $|E_2(x)|\leq Cx^2$ (indeed, we may take $C=1/4$).

The result follows by using (2.13), (2.14) and the formula (2.10) for $c(e^{i\theta})$. □

Let $\mathbb{U}_A=\{u\in\mathbb{U}:|\widehat{A}(u)|=\widehat{A}(1)\}$.

Lemma 2.12. *We have $\mathbb{U}_A=\{u\in\mathbb{U}_Q\mid u^\mu=u^\nu$ for all $\mu,\nu\in P^+$ with $a_\mu,a_\nu>0\}$. If $u_0\in\mathbb{U}_A$ then $\widehat{A}(u_0u)=u_0^\mu\widehat{A}(u)$ for all $u\in\mathrm{Hom}(P,\mathbb{C}^\times)$, and all $\mu\in P^+$ such that $a_\mu>0$.*

Proof. For $u\in\mathbb{U}$ we have

$$|\widehat{A}(u)|=\Big|\sum_{\lambda\in P^+}a_\lambda P_\lambda(u)\Big|\leq\sum_{\lambda\in P^+}a_\lambda|P_\lambda(u)|\leq\sum_{\lambda\in P^+}a_\lambda P_\lambda(1)=\widehat{A}(1).\tag{2.15}$$

If $u=u_0\in\mathbb{U}_A$, then since equality must hold in the second inequality in (2.15) we have $|P_\lambda(u_0)|=P_\lambda(1)$ whenever $a_\lambda>0$. Since we assume that $a_\lambda>0$ for at least one nonzero $\lambda\in P^+$ we have $u_0\in\mathbb{U}_Q$ by Lemma 2.4. Thus by Lemma 2.4 we have $P_\lambda(u_0)=u_0^\lambda P_\lambda(1)$ for all $\lambda\in P^+$, and so since equality must hold in the first inequality in (2.15) we have $u_0^\mu=u_0^\nu$ whenever $a_\mu,a_\nu>0$, proving that

$$\mathbb{U}_A\subseteq\{u\in\mathbb{U}_Q\mid u^\mu=u^\nu\text{ for all }\mu,\nu\in P^+\text{ with }a_\mu,a_\nu>0\}.$$

Conversely, if $u_0\in\mathbb{U}_Q$ and $u_0^\mu=u_0^\nu$ for all $\mu,\nu\in P^+$ with $a_\mu,a_\nu>0$, then by Lemma 2.4 we see that $\widehat{A}(u_0u)=u_0^\mu\widehat{A}(u)$ for all $u\in\mathbb{U}$ and any $\mu\in P^+$ such that $a_\mu>0$, and so taking $u=1$ we have $|\widehat{A}(u_0)|=\widehat{A}(1)$, so $u_0\in\mathbb{U}_A$. □

For $k\in\mathbb{N}$ and $\lambda\in P^+$ let

$$I_{k,\lambda}=\int_{\mathbb{U}}\big(\widehat{A}(u)\big)^k\,\overline{P_\lambda(u)}\,d\pi(u).$$

If $y \in V_\lambda(x)$, then by (2.3)

$$p^{(k)}(x,y) = \begin{cases} I_{k,\lambda} & \text{in the standard case, and} \\ I_{k,\lambda} + I'_{k,\lambda} & \text{in the exceptional case,} \end{cases} \tag{2.16}$$

where

$$I'_{k,\lambda} = \int_{\mathbb{U}'} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi(u). \tag{2.17}$$

Thus to give an asymptotic formula for $p^{(k)}(x,y)$ we need to give estimates for $I_{k,\lambda}$ and $I'_{k,\lambda}$.

Given $\epsilon > 0$ and $u_0 \in \mathbb{U}$, let $N_\epsilon(u_0) = \{u \in \mathbb{U} : |u^{\lambda_i} - u_0^{\lambda_i}| < \epsilon \text{ for all } i \in I_0\}$. Since $|\mathbb{U}_A| < \infty$ we may choose $\epsilon > 0$ sufficiently small so that

$$N_\epsilon(u_0) \cap N_\epsilon(u_0') = \emptyset \qquad \text{whenever } u_0, u_0' \in \mathbb{U}_A \text{ are distinct.} \tag{2.18}$$

Write $N_\epsilon = N_\epsilon(1)$ and $N_\epsilon(\mathbb{U}_A) = \bigcup_{u_0 \in \mathbb{U}_A} N_\epsilon(u_0)$.

Define $\rho_1 = \rho_1(\epsilon) = \max\{|\widehat{A}(u)|/\widehat{A}(1) : u \in \mathbb{U}\backslash N_\epsilon(\mathbb{U}_A)\}$, and so $0 < \rho_1 < 1$. Let

$$I^\epsilon_{k,\lambda} = \int_{N_\epsilon} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi(u).$$

Lemma 2.13. *Fix $\mu \in P^+$ such that $a_\mu > 0$, and let $\epsilon > 0$ satisfy (2.18). If $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, then*

$$I_{k,\lambda} = |\mathbb{U}_A| I^\epsilon_{k,\lambda} + \mathcal{O}\big(\rho_1^k \widehat{A}(1)^k\big).$$

Otherwise, $I_{k,\lambda} = 0$.

Proof. It is clear from the formula for $c(u)$ that $c(u_0u) = c(u)$ for all $u_0 \in \mathbb{U}_Q$ and $u \in \mathbb{U}$. Thus by Lemmas 2.4 and 2.12, if $u_0 \in \mathbb{U}_A$ we have

$$I_{k,\lambda} = u_0^{k\mu-\lambda} \int_{\mathbb{U}} \big(\widehat{A}(u_0^{-1}u)\big)^k \overline{P_\lambda(u_0^{-1}u)} d\pi(u_0^{-1}u) = u_0^{k\mu-\lambda} I_{k,\lambda}. \tag{2.19}$$

This shows that $I_{k,\lambda} = 0$ if there exists $u_0 \in \mathbb{U}_A$ such that $u_0^{k\mu-\lambda} \neq 1$.

Suppose now that $u_0^{k\mu-\lambda} = 1$ for all $u_0 \in \mathbb{U}_A$. It is clear that

$$I_{k,\lambda} = \int_{N_\epsilon(\mathbb{U}_A)} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi(u) + \mathcal{O}\big(\rho_1^k \widehat{A}(1)^k\big), \tag{2.20}$$

and since $N_\epsilon(u_0) = u_0 N_\epsilon$, the calculation in (2.19) shows that for each $u_0 \in \mathbb{U}_A$,

$$\int_{N_\epsilon(u_0)} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi(u) = u_0^{k\mu-\lambda} \int_{N_\epsilon} \big(\widehat{A}(u)\big)^k \overline{P_\lambda(u)} d\pi(u) = I^\epsilon_{k,\lambda},$$

since $u_0^{k\mu-\lambda} = 1$. The result follows from (2.20) by the choice of ϵ. ☐

It is clear from Corollary 2.10 that if each $|\theta_j|$, $j=1,\ldots,n$, is sufficiently small, then

$$\widehat{A}(e^{i\theta})=\widehat{A}(1)e^{-\sum_{i,j=1}^n b_{i,j}\theta_i\theta_j+G(\theta)}\quad\text{where}\quad G(\theta)=o\bigg(\sum_{i,j=1}^n b_{i,j}\theta_i\theta_j\bigg).\tag{2.21}$$

Writing $\delta=2\sin^{-1}(\epsilon/2)$ we have $N_\epsilon=\{e^{i\theta}:|\theta_j|<\delta\text{ for }j=1,\ldots,n\}$, and so we may choose $\epsilon>0$ sufficiently small so that

$$|G(\theta)|\leq\frac{1}{2}\sum_{i,j=1}^n b_{i,j}\theta_i\theta_j\tag{2.22}$$

whenever $e^{i\theta}\in N_\epsilon$ and $|\theta_j|\leq\pi$ for $j=1,\ldots,n$.

Define constants K_1, K_2 and K_3 by $K_1=W_0(q^{-1})|W_0|^{-1}(2\pi)^{-n}$,

$$\begin{aligned}K_2&=\prod_{\alpha\in R_2^+}\big(1-\tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}\big)^{-2}\big(1+\tau_\alpha^{-1/2}\big)^{-2}\\ K_3&=\int_{\mathbb{R}^n}e^{-\sum_{i,j=1}^n b_{i,j}\varphi_i\varphi_j}\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2d\varphi_1\cdots d\varphi_n,\end{aligned}\tag{2.23}$$

where $\varphi=\varphi_1\alpha_1+\cdots+\varphi_n\alpha_n$.

Lemma 2.14. *Let $\epsilon>0$ be such that (2.18) and (2.22) hold. Then*

$$I_{k,\lambda}^\epsilon=KP_\lambda(1)\widehat{A}(1)^k\,k^{-|R_2^+|-n/2}\left(1+\mathcal{O}\big(k^{-1/2}\big)\right),$$

where $K=K_1K_2K_3$.

Proof. Let $\delta=2\sin^{-1}(\epsilon/2)$ as above. We have

$$I_{k,\lambda}^\epsilon=K_1\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}\big(\widehat{A}(e^{i\theta})\big)^k\frac{P_\lambda(e^{-i\theta})}{|c(e^{i\theta})|^2}\,d\theta_1\cdots d\theta_n,$$

and so by making the change of variable $\varphi_j=\sqrt{k}\theta_j$ for each $j=1,\ldots,n$ we see that

$$I_{k,\lambda}^\epsilon=K_1k^{-n/2}\int_{-\sqrt{k}\delta}^{\sqrt{k}\delta}\cdots\int_{-\sqrt{k}\delta}^{\sqrt{k}\delta}\big(\widehat{A}(e^{i\varphi/\sqrt{k}})\big)^k\frac{P_\lambda(e^{-i\varphi/\sqrt{k}})}{|c(e^{i\varphi/\sqrt{k}})|^2}\,d\varphi_1\cdots d\varphi_n,\tag{2.24}$$

where $\varphi=\varphi_1\alpha_1+\cdots+\varphi_n\alpha_n$.

By Corollary 2.7 we have

$$P_\lambda(e^{-i\varphi/\sqrt{k}})=P_\lambda(1)(1+E_1(\varphi))\quad\text{where}\quad|E_1(\varphi)|\leq\frac{|\lambda||\varphi|}{\sqrt{k}},$$

and it follows from Lemma 2.11 that

$$\frac{1}{|c(e^{i\varphi/\sqrt{k}})|^2}=K_2k^{-|R_2^+|}(1+E_2(\varphi))\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2,$$

where $|E_2(\varphi)| \leq k^{-1}p(\varphi_1, \ldots, \varphi_n)$ for some polynomial $p(x_1, \ldots, x_n)$. Using these estimates (along with (2.21)) in (2.24), we see that $I^{\epsilon}_{k,\lambda}$ equals $K_1K_2P_\lambda(1)\widehat{A}(1)^k\, k^{-|R_2^+|-n/2}$ times

$$\int_{[-\sqrt{k}\delta,\sqrt{k}\delta]^n} e^{-\sum_{i,j=1}^n b_{i,j}\varphi_i\varphi_j + kG(\varphi/\sqrt{k})}\Big(\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2\Big)(1+E_1(\varphi))(1+E_2(\varphi))\,d\varphi_1\cdots d\varphi_n.$$

By (2.22), the above integrand is bounded by

$$e^{-\frac{1}{2}\sum_{i,j=1}^n b_{i,j}\varphi_i\varphi_j}\Big(\prod_{\alpha\in R_2^+}\langle\alpha^\vee,\varphi\rangle^2\Big)\Big(1+\frac{|\lambda||\varphi|}{\sqrt{k}}\Big)\Big(1+\frac{p(\varphi_1,\ldots,\varphi_n)}{k}\Big),$$

and the lemma follows by the Dominated Convergence Theorem. □

Lemma 2.15. *Let $\lambda \in P^+$ and $k \in \mathbb{N}$. In the exceptional case, there exists $0 < \rho_2 < 1$ such that*

$$\int_{\mathbb{U}'}\big(\widehat{A}(u)\big)^k\,\overline{P_\lambda(u)}d\pi(u) = \mathcal{O}\big(\rho_2^k\widehat{A}(1)^k\big).$$

Proof. Let us sketch the proof of this result. The details are given in [11, Appendix B.3]. Since we are in the exceptional case, we have $R = BC_n$ for some $n \geq 1$ and $q_n < q_0$. Use the isomorphism $\mathbb{U} \to (\mathbb{C}^\times)^n$, $u \mapsto (t_1, \ldots, t_n)$, where $t_i = u^{e_i}$, to identify $\mathbb{U}$ with $(\mathbb{C}^\times)^n$. Recall that $\mathbb{U}'$ consists of those $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ such that $t_1 = -\sqrt{q_n/q_0}$ and $t_j \in \mathbb{T}$ for $2 \leq j \leq n$. Write $\xi_t = (-\sqrt{q_n/q_0}, t_2, \ldots, t_n)$, and set $t_j = e^{i\theta_j}$ for $2 \leq j \leq n$.

We claim that $|P_\lambda(\xi_t)| < P_\lambda(1)$ for all $\lambda \neq 0$ and all $t_2, \ldots, t_n \in \mathbb{T}$, from which the result clearly follows (since $\widehat{A}$ is continuous on $\mathbb{U}'$, and $\mathbb{U}'$ is compact). The first step is to explicitly compute $P_{\lambda_1}(u)$ for arbitrary $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$. By [11, Lemma B.3.2] we have

$$P_{\lambda_1}(u) = N_{\lambda_1}^{-1}\left((q_0-1)(1+q_1+\cdots+q_1^{n-1}) + \sqrt{q_0q_n}\,q_1^{n-1}\sum_{j=1}^n(t_j+t_j^{-1})\right)$$

for all $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$. From this formula we deduce that $|P_{\lambda_1}(\xi_t)| < P_{\lambda_1}(1)$ for all $t_2, \ldots, t_n \in \mathbb{T}$ (see [11, Theorem B.3.3]). We now use this fact to show that $|P_\lambda(\xi_t)| < P_\lambda(1)$ for all $\lambda \neq 0$ and all $t_2, \ldots, t_n \in \mathbb{T}$.

Recall from [12, Corollary 5.22] that the operators $\{A_\lambda\}_{\lambda\in P^+}$ satisfy

$$A_\lambda A_\mu = \sum_{\nu\in P^+} a_{\lambda,\mu;\nu}A_\nu,$$

where

$$a_{\lambda,\mu;\nu} = \frac{N_\nu}{N_\lambda N_\mu}|V_\lambda(x)\cap V_{\mu^*}(y)| \geq 0,$$

and where $x, y \in V_P$ is any pair with $y \in V_\nu(x)$. Since $\lambda_1 = \tilde{\alpha}^\vee$ here, an analogous argument to that given in Lemma 2.1(i) shows that $a_{\lambda,\lambda;\lambda_1} > 0$ for all $\lambda \neq 0$ (see [11, Lemma B.3.4]).

Since the algebra homomorphisms $h_{\xi_t} : \mathscr{A} \to \mathbb{C}$ are continuous with respect to the ℓ^2-operator norm, and since $\|A_\mu\| = P_\mu(1)$ for all $\mu \in P^+$ (see [13, Theorem 6.3]), we have $|P_\mu(\xi_t)| \le P_\mu(1)$ for all $\mu \in P^+$. Hence for $\lambda \neq 0$,

$$|P_\lambda(\xi_t)|^2 = |h_{\xi_t}(A_\lambda^2)| \le \sum_{\mu \in P^+} a_{\lambda,\lambda;\mu} |P_\mu(\xi_t)| < \sum_{\mu \in P^+} a_{\lambda,\lambda;\mu} P_\mu(1) = P_\lambda(1)^2,$$

where we have used the facts that $|P_{\lambda_1}(\xi_t)| < P_{\lambda_1}(1)$ and $a_{\lambda,\lambda;\lambda_1} > 0$. □

We now give our local limit theorem.

Theorem 2.16. *Let $y \in V_\lambda(x)$ and $k \in \mathbb{N}$, and suppose that $a_\mu > 0$. If $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, then*

$$p^{(k)}(x,y) = |\mathbb{U}_A| K P_\lambda(1) \widehat{A}(1)^k \, k^{-|R_2^+|-n/2}\big(1 + \mathcal{O}(k^{-1/2})\big),$$

where K is as in Lemma 2.14. If $u_0^{k\mu} \neq u_0^\lambda$ for some $u_0 \in \mathbb{U}_A$, then $p^{(k)}(x,y) = 0$.

Proof. In the standard case the result follows from (2.16) and Lemmas 2.13 and 2.14. In the exceptional case, $Q = P$, and so $\mathbb{U}_Q = \{1\}$, and so $\mathbb{U}_A = \{1\}$. The result now follows from (2.16) and Lemmas 2.13, 2.14, and 2.15. □

A random walk on a state-space X is called *irreducible* if for each pair $x, y \in X$ there exists $k = k(x,y) \in \mathbb{N}$ such that $p^{(k)}(x,y) > 0$. The *period* of an irreducible random walk is $\mathfrak{p} = \gcd\{k \ge 1 \mid p^{(k)}(x,x) > 0\}$, which is independent of $x \in X$ by irreducibility (see [17]). An irreducible random walk is called *aperiodic* if $\mathfrak{p} = 1$.

Corollary 2.17. *Let A be as in (0.1), and suppose that $a_\mu > 0$. Then*

(i) *A is irreducible if and only if for each $\lambda \in P^+$ there exists $k = k(\lambda) \in \mathbb{N}$ such that $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, and*

(ii) *A is irreducible and aperiodic if and only if $|\mathbb{U}_A| = 1$.*

Proof. First let us note that in the exeptional case it is easy to see that any walk with $a_\mu > 0$ for some $\mu \neq 0$ is both aperiodic and irreducible, and since $Q = P$ we have $\mathbb{U}_A = \{1\}$. Consider the standard case. Suppose that $a_\mu > 0$.

Let $y \in V_\lambda(x)$. If A is irreducible, then there exists $k \in \mathbb{N}$ such that $p^{(k)}(x,y) > 0$, and so $u_0^{k\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, by (2.16) and Lemma 2.13. Conversely, if for each $\lambda \in P^+$ there exists $k_0 \in \mathbb{N}$ such that $u_0^{k_0\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$, then writing $r = |\mathbb{U}_A|$ we have $u_0^{(k_0+rl)\mu} = u_0^\lambda$ for all $u_0 \in \mathbb{U}_A$ and all $l \ge 0$. As $k \to \infty$ through the values $k_0 + rl$, Theorem 2.16 implies irreducibility.

If $|\mathbb{U}_A| = 1$ then A is clearly irreducible, and Theorem 2.16 shows that A is aperiodic. Conversely, if A is irreducible and aperiodic, then

$$1 = \gcd\{k \ge 1 \mid p^{(k)}(x,x) > 0\} = \gcd\{k \ge 1 \mid u_0^{k\mu} = 1 \text{ for all } u_0 \in \mathbb{U}_A\},$$

and so $\mathbb{U}_A = \{1\}$. □

Remark 2.18. It is possible to explicitly compute the constant K_3 from (2.23) (at least in most cases). We refer the reader to [11, Section 8.4] for details. A key step in the calculation is to observe that there is a number $b > 0$ such that $b_{j,k} = \langle\alpha_j, \alpha_k\rangle b$ for all $1 \leq j, k \leq n$, and so $K_3 = b^{-|R_2^+|-n/2} J$ where

$$J = \int_{\mathbb{R}^n} e^{-\sum_{j,k=1}^n \langle\alpha_j,\alpha_k\rangle\theta_j\theta_k} \prod_{\alpha\in R_2^+} \langle\alpha^\vee, \theta\rangle^2 \, d\theta_1 \dots d\theta_n$$

and $\theta = \theta_1\alpha_1 + \cdots + \theta_n\alpha_n$. The integral J depends only on the underlying root system.

We have computed this integral using Gram's identity in the cases when $R = B_n, C_n, D_n$ or BC_n. We have

$$J = \begin{cases} \pi^{n/2} 2^{-n(n-1)} \prod_{i=1}^n (2i)! & \text{if } R = B_n \text{ or } R = BC_n \\ \pi^{n/2} 2^{-n^2-n-1} \prod_{i=1}^n (2i)! & \text{if } R = C_n \\ \pi^{n/2} 2^{-n^2+n-1} n! \prod_{i=1}^{n-1} (2i)! & \text{if } R = D_n. \end{cases}$$

When $R = A_n$ the integral J may be written as

$$\int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_{n+1}^2)} \prod_{1\leq i<j\leq n+1} (x_i - x_j)^2 \, dx_1 \cdots dx_n$$

(up to some constant factors), where $x_{n+1} = -(x_1 + \cdots + x_n)$. We have been unable to compute this integral. In principle the integrals for the E, F and G cases could be explicitly computed using a computer package.

Remark 2.19. Let us briefly discuss some applications of our local limit theorem to probability measures on groups acting on $\mathscr{X}$. An automorphism ψ of $\mathscr{X}$ is called *type rotating* if there exists a type rotating automorphism σ of the Coxeter graph of W (in the sense of [12, §4.8]) such that $\tau(\psi(x)) = \sigma(\tau(x))$ for all $x \in V$. Suppose that G is a locally compact group acting on V such that G acts transitively on V_P and such that for each $x \in V_P$ and $g \in G$ the automorphism $x \mapsto gx$ is type rotating. Assume that $K = \{g \in G \mid go = o\}$ acts transitively on each set $V_\lambda(o)$, $\lambda \in P^+$, where $o \in V_P$ is some fixed vertex. Let φ be the density function of a bi-K-invariant probability measure on G. Then, exactly as in [4, Lemma 8.1], setting $p(go, ho) = \varphi(g^{-1}h)$ for $g, h \in G$ defines an isotropic random walk on V_P. Since the k-th convolution power $\varphi^{(*k)}(g)$ is simply $p^{(k)}(o, go)$, Theorem 2.16 may immediately be interpreted as a local limit theorem for bi-K-invariant probability measures on G (the assumption that $a_\lambda > 0$ for some $\lambda \neq 0$ simply means that φ is not the indicator function on K).

As an important modification, suppose now that G is a group of type preserving simplicial complex automorphisms acting *strongly transitively* on $\mathscr{X}$, meaning that G acts transitively on pairs $(\mathcal{A}, c)$ of apartments $\mathcal{A}$ and chambers $c \subset \mathcal{A}$. Fix an apartment $\mathcal{A}_0$ and a chamber $c_0 \subset \mathcal{A}_0$. The subgroups $B = \mathrm{stab}_G(c_0)$ and $N = \mathrm{stab}_G(\mathcal{A}_0)$ form a BN-pair in G with associated Weyl group $N/(B \cap N)$ isomorphic to W [14, Theorem 5.2].

Indeed the set of left cosets $\{gB \mid g \in G\}$ defines an affine building (as a chamber system) isomorphic to $\mathscr{X}$, where $gB \sim_i hB$ if and only if $g^{-1}h \in B\langle s_i \rangle B$ (where wB means nB for any $n \in N$ with image $w \in W$). Let o be the type 0 vertex of c_0. The subgroup $K = \mathrm{stab}_G(o)$ of G equals $BW_0B = \bigcup_{w \in W_0} BwB$ (see [14, Theorem 5.4(iii)]), and since G acts strongly transitively and $B \cap N$ is transitive on the chambers of $\mathcal{A}_0$, it follows that K is transitive on each set $V_\lambda(o)$, $\lambda \in Q \cap P^+$.

Let φ be the density function of a bi-K-invariant probability measure on G. To study convolution powers $\varphi^{(*k)}(g)$, $g \in G$, it is natural to study an associated random walk on $V_Q = \{x \in V_P \mid \tau(x) = 0\} \subseteq V_P$, where we define $p(go, ho) = \varphi(g^{-1}h)$ for $g, h \in G$. To apply our local limit theorem we consider these random walks as reducible isotropic random walks on V_P by setting $p(x,y) = \varphi(g^{-1}h)$ if $y \in V_\lambda(x)$ and $go \in V_\lambda(ho)$ for some $\lambda \in P^+$ (necessarily $\lambda \in Q \cap P^+$), and $p(x,y) = 0$ otherwise. These random walks have the property that $p(x,y) = 0$ if $\tau(x) \neq \tau(y)$, and it is simple to see that they are indeed isotropic. Theorem 2.16 is now applicable, and in particular, by taking $\mathscr{X}$ to be the Bruhat-Tits building of a group of p-adic type (see Remark 1.6(ii)) we have a local limit theorem for these groups.

Finally we remark that the methods here can be extended to deal with groups acting (in a type rotating fashion) on subsets V_L of V_P. Here L is a lattice in E with $Q \subseteq L \subseteq P$, and $V_L = \{x \in V_P \mid \tau(x) \in I_L\}$, where $I_L = \{\tau(\lambda) \mid \lambda \in L\} \subseteq I_P$. Thus $V_Q \subseteq V_L \subseteq V_P$, and our discussion above deals with the extreme cases of $L = P$ and $L = Q$.

3. The Rate of Escape Theorem

Let X be any set, and let $P = (p(x,y))_{x,y \in X}$ be a transition probability matrix. Let $X = \bigcup_{i \in I} X_i$ be a partition of X. We call P *factorisable over* I if for each $i, j \in I$, the sum

$$\sum_{y \in X_j} p(x,y)$$

has the same value for all $x \in X_i$. In this case we write $\overline{p}(i,j)$ for this value, and let $\overline{P} = (\overline{p}(i,j))_{i,j \in I}$. Clearly $\overline{p}(i,j) \geq 0$ for all $i, j \in I$, and for each $i \in I$,

$$\sum_{j \in I} \overline{p}(i,j) = \sum_{j \in I} \sum_{y \in X_j} p(x,y) = \sum_{y \in X} p(x,y) = 1$$

where $x \in X_i$. Thus $\overline{P}$ is a transition probability matrix (on I). Furthermore, if $(Z_k)_{k \in \mathbb{N}}$ is a Markov chain on X with transition probability matrix P, then $(\overline{Z}_k)_{k \in \mathbb{N}}$, where $\overline{Z}_k = i$ if $Z_k \in X_i$, defines a Markov chain on I with transition probability matrix $\overline{P}$.

In our setting, consider the partition (for fixed $o \in V_P$ and $\omega \in \Omega$) $V_P = \bigcup_{\lambda \in P} V_\lambda$, where

$$V_\lambda = \{x \in V_P \mid h(o, x; \omega) = \lambda\}.$$

Proposition 3.1. *The matrices (operators) $A_\lambda = (p_\lambda(x,y))_{x,y \in V_P}$, $\lambda \in P^+$, are factorisable over P. Moreover, $\overline{p}_\lambda(\mu,\nu)$ does not depend on o or ω, and $\overline{p}_\lambda(\mu,\nu) = \overline{p}_\lambda(0, \nu - \mu)$.*

Proof. Let $\mu,\nu\in P$ and $x\in V_\mu$. By the cocycle relations we have $h(o,y;\omega)=h(x,y;\omega)+\mu$ for all $y\in V_P$, and so

$$\begin{aligned}\sum_{y\in V_\nu}p_\lambda(x,y)&=\frac{1}{N_\lambda}|\{y\in V_\lambda(x)\mid h(o,y;\omega)=\nu\}|\\&=\frac{1}{N_\lambda}|\{y\in V_\lambda(x)\mid h(x,y;\omega)=\nu-\mu\}|.\end{aligned}\tag{3.1}$$

It follows from [13, Lemma 3.19] that A_λ is factorisable, and the final claims are clear. □

Corollary 3.2. *Let $A=(p(x,y))_{x,y\in V_P}$ be as in (0.1). Then A is factorisable over P. Moreover, for each $\mu,\nu\in P$ we have $\overline{p}(\mu,\nu)=\overline{p}(0,\nu-\mu)$, and this value does not depend on o and ω. Finally, if $(Z_k)_{k\in\mathbb{N}}$ is a Markov chain with transition probability matrix A, then $\overline{Z}_k=h(o,Z_k;\omega)$, so that $\overline{p}(\mu,\nu)=\mathbb{P}(h(o,Z_{k+1};\omega)=\nu\mid h(o,Z_k;\omega)=\mu)$.*

Proof. The first statements follow easily from Proposition 3.1 and the elementary fact that a (finite or infinite) convex combination of factorisable transition matrices is again factorisable. The final claim is immediate from the definition of $\overline{Z}_k$. □

Remark 3.3. We note that the random walk $(\overline{Z}_k)_{k\in\mathbb{N}}$ on P from Corollary 3.2 can be explicitly studied using classical methods, since $P\cong\mathbb{Z}^n$. In the notation of (A.1), by (3.1) and [13, Lemma 3.19 and Theorem 6.2] we have $\overline{p}_\lambda(0,\mu)=r^{-\mu}a_{\lambda,\mu}$. Assuming that $\sum_{\mu\in P}|\mu|\,\overline{p}(0,\mu)<\infty$, a simple calculation using (A.1) and (3.2) shows that the mean $\mathfrak{m}=\sum_{\mu\in P}\mu\,\overline{p}(0,\mu)$ of the random walk $(\overline{Z}_k)_{k\in\mathbb{N}}$ is $\mathfrak{m}=\sum_{j=1}^n\gamma_{j^*}\lambda_j$, where γ_j is as in Theorem 3.9, and j^* is as described before Corollary 3.5. A similar calculation shows that the characteristic function for the walk is

$$\sum_{\mu\in P}\overline{p}(0,\mu)e^{i\langle\mu,\theta\rangle}=\widehat{A}(r^{-1}e^{i\theta}).$$

By Corollary 3.2 this walk is transitionally invariant, and so the usual Fourier inversion (as in [16, §II.6, Proposition 3]) gives

$$\overline{p}^{(k)}(0,\mu)=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\big(\widehat{A}(r^{-1}e^{i\theta})\big)^k e^{-i\langle\mu,\theta\rangle}\,d\theta_1\cdots d\theta_n.$$

The asymptotic behaviour may now be extracted using the methods in [17, §III.13] and the calculations in Lemma 4.4.

Let $\{T_j\}_{j\in J}$ be a partition of R_2 according to root length (so $|J|=1$ or 2). For $j\in J$, let $T_j^+=R_2^+\cap T_j$, and $B_j=B\cap T_j$ (so $B=\bigcup_{j\in J}B_j$, as $B\subset R_2$). For each $j\in J$, let

$$\rho_j=\frac{1}{2}\sum_{\alpha\in T_j^+}\alpha.$$

Finally, for each $j\in J$ fix some $\beta_j\in T_j^+$.

Proposition 3.4. *With the above definitions:*

(i) *For $i, j \in J$, if $\alpha \in B_i$, then $\langle \alpha^\vee, \rho_j\rangle = \delta_{i,j}$. Thus, for each $j \in J$, $\rho_j \in \overline{\mathcal{S}}_0$.*

(ii) *Let $\lambda \in P$. Then*

$$r^\lambda = \prod_{j\in J} (\tau_{\beta_j}\tau_{2\beta_j}^2)^{\langle\lambda,\rho_j\rangle}.$$

(iii) *$r^{w_0\lambda} = r^{-\lambda}$ for all $\lambda \in P$ (here w_0 is the longest element of W_0).*

(iv) *If $\lambda \in P$ and $\mu \preceq \lambda$, then $r^\mu \le r^\lambda$, with equality if and only if $\mu = \lambda$.*

(v) *For $w \in W_0$, we have $c(wr) = \delta_{w,1}W_0(q^{-1})$.*

Proof. (i) Since $\alpha \in B$, s_α permutes $R_2^+\backslash\{\alpha\}$, and since the sets T_j, $j \in J$, are W_0-invariant, we have $s_\alpha(T_j^+) = T_j^+$ if $j \in J\backslash\{i\}$. Thus for any $j \in J$ we have

$$s_\alpha(\rho_j) = \rho_j - \delta_{i,j}\alpha,$$

and so $\langle\alpha^\vee, \rho_j\rangle = \delta_{i,j}$. Then $\langle\alpha^\vee, \rho_j\rangle \ge 0$ for all $\alpha \in B$, and so $\rho_j \in \overline{\mathcal{S}}_0$ for all $j \in J$.

(ii) By (1.4) and the fact that $R_1^+\backslash R_3^+ = 2(R_2^+\backslash R_3^+)$ we calculate

$$r^\lambda = \Big(\prod_{\alpha\in R_3^+} \tau_\alpha^{\frac{1}{2}\langle\lambda,\alpha\rangle}\Big)\Big(\prod_{\beta\in R_2^+\backslash R_3^+} (\tau_\beta\tau_{2\beta}^2)^{\frac{1}{2}\langle\lambda,\beta\rangle}\Big).$$

Since $\tau_{2\alpha} = 1$ if $\alpha \in R_3^+$, and since $\tau_\beta = \tau_{\beta_j}$ if $\beta \in T_j$, it follows that

$$r^\lambda = \prod_{\beta\in R_2^+} (\tau_\beta\tau_{2\beta}^2)^{\frac{1}{2}\langle\lambda,\beta\rangle} = \prod_{j\in J}(\tau_{\beta_j}\tau_{2\beta_j}^2)^{\langle\lambda,\rho_j\rangle}.$$

(iii) Since $w_0\rho_j = -\rho_j$ for $j \in J$, by (ii) we have $r^{w_0\lambda} = r^{-\lambda}$ for all $\lambda \in P$.

(iv) Observe that $\tau_\alpha\tau_{2\alpha}^2 = q_\alpha$ if $\alpha \in R_3$, and $\tau_\alpha\tau_{2\alpha}^2 = q_0q_\alpha$ if $\alpha \in R_2\backslash R_3$. Thus, by thickness, $\tau_\alpha\tau_{2\alpha}^2 > 1$ for all $\alpha \in R_2$. Since $\rho_j \in \overline{\mathcal{S}}_0$ for $j \in J$, and since $\mu \preceq \lambda$ implies that $\lambda - \mu \in Q^+$, it follows from (ii) that $r^{\lambda-\mu} \ge 1$, with equality if and only if $\mu = \lambda$ (for if $\mu \ne \lambda$, then $\langle\lambda - \mu, \rho_j\rangle > 0$ for at least one $j \in J$).

(v) Observe that if $w \ne 1$, then $wR_2^+ \cap (-B) \ne \emptyset$. To see this, if $\alpha \in R_2^+$, and if $-\alpha \notin wR_2^+$, then $-w^{-1}\alpha \notin R_2^+$, and so $w^{-1}\alpha \in R_2^+$. It follows that if $wR_2^+ \cap (-B) = \emptyset$, then $w^{-1}B \subset R_2^+$, and so $w^{-1}R_2^+ = R_2^+$. Thus $w = 1$ (for by [1, VI, §1, No.6, Corollary 2] we have $\ell(w) = |\{\alpha \in R_2^+ \mid w\alpha \in R_2^-\}|$ for all $w \in W_0$).

Suppose that $w \ne 1$, and take $\alpha \in R_2^+$ such that $w\alpha = -\beta \in -B$. Then by (i) and (ii),

$$r^{-w\alpha^\vee} = r^{\beta^\vee} = \tau_\beta\tau_{2\beta}^2 = \tau_\alpha\tau_{2\alpha}^2,$$

and so $1 - \tau_{2\alpha}^{-1}\tau_\alpha^{-1/2}r^{-w\alpha^\vee/2} = 0$. Thus $c(wr) = 0$ by (2.10).

Since $h_u : \mathscr{A} \to \mathbb{C}$ is an algebra homomorphism we have

$$1 = h_u(A_0) = \frac{1}{W_0(q^{-1})}\sum_{w\in W_0} c(wu)$$

for all nonsingular $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, and so $c(r) = W_0(q^{-1})$. $\square$

For each $j \in I_0$, let $j^* \in I_0$ be defined by $-w_0\alpha_j = \alpha_{j^*}$. Note that $(j^*)^* = j$.

Corollary 3.5. *Let $x \in V_P$, $\lambda \in P^+$, and $y \in V_\lambda(x)$.*

(i) *We have*
$$P_\lambda(re^{i\theta}) = \int_\Omega e^{i\langle h(x,y;\omega), w_0\theta\rangle} d\nu_x(\omega)$$
where w_0 is the longest element of W_0.

(ii) *For each $j \in I_0$ the integral*
$$\gamma_j^{(\lambda)} = \int_\Omega h_{j^*}(y,x;\omega) d\nu_x(\omega)$$
is independent of the particular pair $x, y \in V_P$ with $y \in V_\lambda(x)$ (this index reversal makes the statements of the main theorems simpler).

Proof. By Proposition 3.4(iii) (and the fact that $w_0^{-1} = w_0$) we have
$$P_\lambda(re^{i\theta}) = P_\lambda(w_0(re^{i\theta})) = P_\lambda(r^{-1}w_0(e^{i\theta})) = \int_\Omega e^{i\langle h(x,y;\omega), w_0\theta\rangle} d\nu_x(\omega),$$
proving (i).

Since $w_0\theta = -\sum_{j=1}^n \theta_j\alpha_{j^*}$, by (i) we have
$$\frac{\partial}{\partial\theta_j} P_\lambda(re^{i\theta})\big|_{\theta=0} = i\gamma_j^{(\lambda)}, \tag{3.2}$$
proving (ii). $\square$

The following proposition gives a symmetry property of the numbers $\gamma_j^{(\lambda)}$ generalising [4, Proposition 3.5(iii)]. We will not use this result in this paper.

Proposition 3.6. *Let $j \in I_0$ and $\lambda \in P^+$. We have $\gamma_j^{(\lambda^*)} = \gamma_{j^*}^{(\lambda)}$.*

Proof. Observe that for $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ and $\lambda \in P^+$, $P_{\lambda^*}(u) = P_\lambda(u^{-1})$. To prove this note that for $\lambda \in P^+$, using Proposition 3.4 and the fact that $\lambda^* = -w_0\lambda$ we have $r^{\lambda^*} = r^{-w_0\lambda} = r^\lambda$. Furthermore, if $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ is nonsingular then
$$c(w_0u) = \prod_{\alpha\in R^+} \frac{1 - \tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}u^{-w_0\alpha^\vee}}{1 - \tau_{\alpha/2}^{-1/2}u^{-w_0\alpha^\vee}} = \prod_{\alpha\in R^+} \frac{1 - \tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}u^{\alpha^\vee}}{1 - \tau_{\alpha/2}^{-\alpha/2}u^{\alpha^\vee}} = c(u^{-1}),$$
where we have used the facts that $w_0R^+ = R^-$ and $\tau_\alpha = \tau_\beta$ if $\beta \in W_0\alpha$. Thus
$$P_{\lambda^*}(u) = \frac{r^{-\lambda^*}}{W_0(q^{-1})} \sum_{w\in W_0} c(wu)u^{w\lambda^*} = \frac{r^{-\lambda}}{W_0(q^{-1})} \sum_{w\in W_0} c(ww_0u)u^{ww_0\lambda^*} = P_\lambda(u^{-1}),$$
and so by (3.2) we have
$$\gamma_j^{(\lambda^*)} = -i\frac{\partial}{\partial\theta_j} P_\lambda(r^{-1}e^{-i\theta})\big|_{\theta=0} = -i\frac{\partial}{\partial\theta_j} P_\lambda(w_0(re^{-iw_0\theta}))\big|_{\theta=0} = -i\frac{\partial}{\partial\varphi_{j^*}} P_\lambda(re^{i\varphi})\big|_{\varphi=0},$$
and so $\gamma_j^{(\lambda^*)} = \gamma_{j^*}^{(\lambda)}$. $\square$

Lemma 3.7. *Let $\lambda \in P^+$ and $j \in I_0$. Then*
$$\gamma_j^{(\lambda)} = \langle \lambda, \alpha_j \rangle + \mathcal{O}(1).$$

Proof. Let us temporarily write ρ_θ in place of $re^{i\theta}$. Then for $w \in W_0$,
$$\frac{\partial}{\partial \theta_j} c(w\rho_\theta)\rho_\theta^{w\lambda} = i\langle w\lambda, \alpha_j\rangle c(w\rho_\theta)\rho_\theta^{w\lambda} + \rho_\theta^{w\lambda}\frac{\partial}{\partial \theta_j} c(w\rho_\theta).$$
It follows from Proposition 3.4 that for $w \in W_0$,
$$\frac{r^{-\lambda}}{W_0(q^{-1})}\frac{\partial}{\partial \theta_j} c(w\rho_\theta)\rho_\theta^{w\lambda}\bigg|_{\theta=0} = \begin{cases} i\langle \lambda, \alpha_j\rangle + \mathcal{O}(1) & \text{if } w = 1 \\ \mathcal{O}(1) & \text{if } w \neq 1. \end{cases}$$
The result follows from (3.2). □

Lemma 3.8. *If $(Z_k)_{k\in\mathbb{N}}$ is a Markov chain in V_P with $Z_0 = x$ and transition operator A_λ, then for any $\omega \in \Omega$, $\mathbb{E}(h_j(Z_1, x; \omega)) = \gamma_{j^*}^{(\lambda)}$.*

Proof. Since $Z_1 \in V_\lambda(x)$ with probability 1, we have $\int_\Omega h_j(Z_1, x; \omega) d\nu_x(\omega) = \gamma_{j^*}^{(\lambda)}$. As in [4, Proposition 3.5(ii)] we see that we may take expectations under the integral sign, and so $\gamma_{j^*}^{(\lambda)} = \int_\Omega \mathbb{E}(h_j(Z_1, x; \omega)) d\nu_x(\omega)$. By Corollary 3.2, the distribution of $h_j(Z_1, x; \omega)$, and hence $\mathbb{E}(h_j(Z_1, x; \omega))$, is independent of $\omega \in \Omega$. The result follows. □

We now prove our rate of escape theorem.

Theorem 3.9. *Let A be as in (0.1), and suppose that $\sum_{\lambda\in P^+} |\lambda| a_\lambda < \infty$. Let $(Z_k)_{k\in\mathbb{N}}$ be the corresponding Markov chain, and for each $k \in \mathbb{N}$ let $\nu_k \in P^+$ be such that $Z_k \in V_{\nu_k}(x)$, where $x = Z_0$. Then for each $j \in I_0$, with probability 1*
$$\frac{1}{k}\langle \nu_k, \alpha_j\rangle \to \gamma_j \quad \text{as } k \to \infty,$$
where $\gamma_j = \sum_{\lambda\in P^+} a_\lambda \gamma_j^{(\lambda)}$.

Proof. Observe first that $\gamma_j < \infty$ by Lemma 2.5 and the finite first moment assumption. By Lemma 3.7 we have $\frac{1}{k}\langle \nu_k, \alpha_j\rangle = \frac{1}{k}\gamma_j^{(\nu_k)} + \mathcal{O}(k^{-1})$, and so it suffices to prove that
$$\int_\Omega \frac{h_{j^*}(Z_k, x; \omega)}{k} d\nu_x(\omega) \to \gamma_j$$
with probability 1.

By Corollary 3.2 we see that for each fixed $\omega \in \Omega$, $h_{j^*}(Z_k, x; \omega)$ is a random variable distributed like a sum of k independent real random variables, each with the distribution of $h_{j^*}(Z_1, x; \omega)$. Now $\mathbb{E}(h_{j^*}(Z_1, x; \omega)) = \gamma_j$, and so by the classical law of large numbers we have
$$\frac{h_{j^*}(Z_k, x; \omega)}{k} \to \gamma_j$$
with probability 1.

By Remark 2.6 and the second part of [17, Proposition 8.8(a)] we see that $h_{j^*}(Z_k,x;\omega)/k$ is bounded with probability 1. Thus by the Bounded Convergence Theorem we have

$$\lim_{k\to\infty}\int_\Omega \frac{h_{j^*}(Z_k,x;\omega)}{k}d\nu_x(\omega)=\int_\Omega \lim_{k\to\infty}\frac{h_{j^*}(Z_k,x;\omega)}{k}d\nu_x(\omega)=\gamma_j$$

with probability 1, completing the proof. □

4. The Central Limit Theorem

Lemma 4.1. *Let $\lambda\in P^+$. There exists a constant C, independent of θ and λ, such that*

$$\left|h_{re^{i\theta}}(A_\lambda)-e^{i\langle\lambda,\theta\rangle}\right|\leq C|\theta|.$$

Proof. Recall that $r^{w\lambda}\leq r^\lambda$ and $c(wr)=\delta_{w,1}W_0(q^{-1})$ for all $w\in W_0$ (see Proposition 3.4). Thus

$$\left|h_{re^{i\theta}}(A_\lambda)-e^{i\langle\lambda,\theta\rangle}\right|\leq\frac{1}{W_0(q^{-1})}\sum_{w\in W_0}\left|c(w(re^{i\theta}))-c(wr)\right|.$$

The result follows since each $c(w(re^{i\theta}))$ is a smooth function in $\theta_1,\ldots,\theta_n$. □

For $1\leq j,k\leq n$ and $\lambda\in P^+$, let

$$\gamma_{j,k}^{(\lambda)}=\int_\Omega h_{j^*}(y,x;\omega)h_{k^*}(y,x;\omega)d\nu_x(\omega) \tag{4.1}$$

where $x,y\in V_P$ are any vertices with $y\in V_\lambda(x)$ (as in Corollary 3.5(ii) this is easily seen to be independent of the particular pair $x,y\in V_P$ with $y\in V_\lambda(x)$ chosen).

If we suppose a finite second moment assumption:

$$\sum_{\lambda\in P^+}|\lambda|^2a_\lambda<\infty, \tag{4.2}$$

then for all $1\leq j,k\leq n$ we have $\sum_{\lambda\in P^+}a_\lambda\gamma_{j,k}^{(\lambda)}<\infty$, and we denote this value by $\gamma_{j,k}$.

Lemma 4.2. *(See Theorem A.3) The homomorphisms $h_{re^{i\theta}}:\mathscr{A}\to\mathbb{C}$, $\theta\in E$, are bounded.*

Proof. For each $\lambda\in P^+$ we have $|h_{re^{i\theta}}(A_\lambda)|\leq 1$ by Corollary 3.5(i). □

Let $x\in V_P$. The *spherical function (with respect to x)* associated to h_u is the function $F_u^x:V_P\to\mathbb{C}$ which for each $\lambda\in P^+$ takes the constant value $h_u(A_\lambda)$ on the set $V_\lambda(x)$.

Lemma 4.3. *Let $Z_0=x$, and suppose that $u\in\mathrm{Hom}(P,\mathbb{C}^\times)$ is such that $h_u:\mathscr{A}\to\mathbb{C}$ is bounded. Then $\mathbb{E}(F_u^x(Z_k))=(\widehat{A}(u))^k$.*

Proof. We have $A^k=\sum_{\lambda\in P^+}a_\lambda^{(k)}A_\lambda$ where $a_\lambda^{(k)}=\mathbb{P}(Z_k\in V_\lambda(x))$. Since $F_u^x(Z_k)=h_u(A_\lambda)$ if $Z_k\in V_\lambda(x)$, we have

$$\mathbb{E}(F_u^x(Z_k))=\sum_{\lambda\in P^+}a_\lambda^{(k)}h_u(A_\lambda)=h_u(A^k)=(\widehat{A}(u))^k,$$

where we have used the continuity of h_u on the closure of $\mathscr{A}$ in the space of bounded linear operators on $\ell^1(V_P)$ to justify the last two equalities. $\square$

Lemma 4.4. *Suppose that (4.2) holds. Then with γ_j, $1 \le j \le n$ as defined in Theorem 3.9, and $\gamma_{j,k}$, $1 \le j,k \le n$ as defined above,*

$$\widehat{A}(re^{i\theta}) = 1 + i\sum_{j=1}^{n} \gamma_j\theta_j - \frac{1}{2}\sum_{j,k=1}^{n} \gamma_{j,k}\theta_j\theta_k + o(|\theta|^2).$$

Furthermore, if $\theta \neq 0$ then $\left(\sum_{j=1}^{n}\gamma_j\theta_j\right)^2 < \sum_{j,k=1}^{n}\gamma_{j,k}\theta_j\theta_k$.

Proof. Consider the case $A = A_\lambda$, $\lambda \neq 0$. Using Corollary 3.5(i), the elementary result $e^{i\varphi} = 1 + i\varphi - \frac{1}{2}\varphi^2 + o(\varphi^2)$ implies that

$$\widehat{A}_\lambda(re^{i\theta}) = 1 + i\sum_{j=1}^{n} \gamma_j^{(\lambda)}\theta_j - \frac{1}{2}\sum_{j,k=1}^{n} \gamma_{j,k}^{(\lambda)}\theta_j\theta_k + o(|\lambda||\theta|),$$

where we have used Lemma 2.5. The first claim follows.

To deduce the final claim, let $B_\lambda = \sum_{j=1}^{n}\gamma_j^{(\lambda)}\theta_j$ and $C_\lambda = \sum_{j,k=1}^{n}\gamma_{j,k}^{(\lambda)}\theta_j\theta_k$. Then

$$B_\lambda^2 = \left(\int_\Omega \sum_{j=1}^{n} h_{j^*}(y,x;\omega)d\nu_x(\omega)\right)^2 \le \int_\Omega \left(\sum_{j=1}^{n} h_{j^*}(y,x;\omega)\right)^2 d\nu_x(\omega) = C_\lambda,$$

and

$$\left(\sum_{j=1}^{n}\gamma_j\theta_j\right)^2 = \left(\sum_{\lambda\in P^+} a_\lambda B_\lambda\right)^2 \le \sum_{\lambda\in P^+} a_\lambda B_\lambda^2 \le \sum_{\lambda\in P^+} a_\lambda C_\lambda = \sum_{j,k=1}^{n}\gamma_{j,k}\theta_j\theta_k.$$

To see that the inequality is strict if $\theta \neq 0$, recall that by hypothesis there exists $\lambda \neq 0$ such that $a_\lambda > 0$. If equality holds in the inequality $B_\lambda^2 \le C_\lambda$, then for $y \in V_\lambda(x)$,

$$\langle h(x,y;\omega), w_0\theta\rangle = \sum_{j=1}^{n} h_{j^*}(y,x;\omega)\theta_j$$

is independent of $\omega \in \Omega$, and thus by Corollary 3.5(ii) this quantity is independent of the particular pair $x, y \in V_P$ with $y \in V_\lambda(x)$ too. Choosing $z \in V_\lambda(x) \cap V_{\tilde{\alpha}^\vee}(y)$ as in Lemma 2.1(i), we have

$$\langle h(y,z;\omega), w_0\theta\rangle = \langle h(x,z;\omega), w_0\theta\rangle - \langle h(x,y;\omega), w_0\theta\rangle = 0.$$

By modifying the proof of Lemma 2.1(ii), it is easy to see that for each $w \in W_0$ there exists $\omega_w \in \Omega$ such that $h(y,z;\omega_w) = w\tilde{\alpha}^\vee$, and thus by the above $\langle w\tilde{\alpha}^\vee, w_0\theta\rangle = 0$ for all $w \in W_0$. Thus $\theta = 0$, since $W_0\tilde{\alpha}^\vee$ spans E [7, Lemma 10.4.B]. $\square$

Let $\Gamma_1(\theta) = \sum_{j=1}^{n}\gamma_j\theta_j$ and $\Gamma_2(\theta) = \sum_{j,k=1}^{n}\gamma_{j,k}\theta_j\theta_k$, and write

$$\Gamma(\theta) = \Gamma_2(\theta) - \Gamma_1^2(\theta) = \sum_{j,k=1}^{n} g_{j,k}\theta_j\theta_k.$$

By Lemma 4.4, $\Gamma = (g_{j,k})_{j,k=1}^n$ is a positive definite matrix.

Theorem 4.5. *Let A be as in (0.1) and suppose that (4.2) holds. As in Theorem 3.9, for each $k \in \mathbb{N}$ let $\nu_k \in P^+$ be such that $Z_k \in V_{\nu_k}(x)$, where $x = Z_0$. Then*

$$\left(\frac{\langle \nu_k, \alpha_1\rangle - \gamma_1 k}{\sqrt{k}}, \ldots, \frac{\langle \nu_k, \alpha_n\rangle - \gamma_n k}{\sqrt{k}}\right)$$

converges in distribution to the normal distribution $N(0, \Gamma)$, with Γ as above.

Proof. Following the proof of the classical Central Limit Theorem (see [16, Proposition II.8] for example), it suffices to show that

$$\lim_{k\to\infty} \mathbb{E}(e^{i(\langle \nu_k, \theta\rangle - k\Gamma_1(\theta))/\sqrt{k}}) = e^{-\frac{1}{2}\Gamma(\theta)}. \tag{4.3}$$

By Lemma 4.1 we have

$$e^{i\langle \nu_k, \theta\rangle/\sqrt{k}} = P_{\nu_k}(re^{i\theta/\sqrt{k}}) + o(k^{-1/2}) = F^x_{re^{i\theta/\sqrt{k}}}(Z_k) + o(k^{-1/2}),$$

and so by Lemmas 4.2 and 4.3 we have

$$\mathbb{E}(e^{i\langle \nu_k, \theta\rangle/\sqrt{k}}) = (\widehat{A}(re^{i\theta/\sqrt{k}}))^k + o(k^{-1/2}).$$

Thus using Lemma 4.4 we compute

$$\mathbb{E}(e^{i(\langle \nu_k, \theta\rangle - k\Gamma_1(\theta))/\sqrt{k}}) = \left(1 - \frac{1}{2k}\Gamma(\theta) + o(k^{-1})\right)^k + o(k^{-1/2}) = e^{-\frac{1}{2}\Gamma(\theta)} + o(k^{-1/2}).$$

Thus (4.3) holds, completing the proof. □

Appendix A. Bounded Spherical Functions

It is easy to see that each $A \in \mathscr{A}$ maps $\ell^1(V_P)$ into itself. Let $\mathscr{A}_1$ denote the closure of $\mathscr{A}$ in the space $\mathscr{L}(\ell^1(V_P))$ of bounded linear operators on $\ell^1(V_P)$. Thus $\mathscr{A}_1$ is a commutative unital Banach $*$-algebra. The algebra homomorphisms $h : \mathscr{A}_1 \to \mathbb{C}$ are precisely the extensions of those algebra homomorphisms $h_u : \mathscr{A} \to \mathbb{C}$ which are bounded. In this appendix we determine the $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$ for which this holds.

In the notation of Remark 1.6(ii), it is shown in [9, Theorem 4.7.1] that $h_u : \mathscr{A}_Q \to \mathbb{C}$ is bounded if and only $|u^{w\lambda}| \leq r^\lambda$ for all $\lambda \in Q \cap P^+$ and all $w \in W_0$. The proof given in [9] requires some knowledge of the *singular cases* (when the denominator of a $c(wu)$ function vanishes). While it should be possible to generalise the proof in [9] to cover the more general setting of homomorphisms $h_u : \mathscr{A} \to \mathbb{C}$, we will provide a different proof which does not require any specific details of the singular cases (instead our proof uses the Plancherel measure).

We restrict our attention to the standard case (where $\tau_\alpha \geq 1$ for all $\alpha \in R$). In the exceptional case (where $\tau_\alpha < 1$ for some $\alpha \in R$) we have $R = BC_n$ for some $n \geq 1$ and so $Q = P$ and $\mathscr{A}_Q = \mathscr{A}$. Thus Macdonald's analysis in [9] covers this specific case.

Remark A.1. In fact in [9, Theorem 4.7.1] Macdonald proves a geometric analog of the result stated above. For $u \in \mathrm{Hom}(Q, \mathbb{C}^\times)$, identify $\log|u| \in \mathrm{Hom}(Q, \mathbb{R})$ with the unique element $x_u \in E$ which satisfies $\langle \lambda, x_u \rangle = \log|u^\lambda|$ for all $\lambda \in Q$. Let $D = \{x_{wr} \mid w \in W_0\}$. Then [9, Theorem 4.7.1] says that $h_u : \mathscr{A}_Q \to \mathbb{C}$ is bounded if and only if $x_u \in \mathrm{conv}(D)$.

For $\lambda \in P^+$ and $u \in \mathrm{Hom}(P, \mathbb{C}^\times)$, define the *monomial symmetric function* $m_\lambda(u)$ by

$$m_\lambda(u) = \sum_{\mu \in W_0\lambda} u^\mu.$$

By [13, (6.1)] there are numbers $a_{\lambda,\mu}$ such that

$$P_\lambda(u) = \sum_{\mu \in P} a_{\lambda,\mu} u^\mu = \sum_{\mu \preceq \lambda} a_{\lambda,\mu} m_\mu(u) \tag{A.1}$$

(where the second sum is over $\mu \in P^+$), and it follows (using [12, Theorem 6.11]) that for $\lambda, \mu \in P^+$ there are numbers $b_{\lambda,\mu}$ such that

$$m_\lambda(u) = \sum_{\mu \preceq \lambda} b_{\lambda,\mu} P_\mu(u). \tag{A.2}$$

Lemma A.2. *Let $\lambda, \mu \in P^+$ and $\mu \preceq \lambda$. In the standard case there exists a constant $K > 0$ independent of λ and μ such that $|b_{\lambda,\mu}| \le K r^\mu$. Thus there is a constant $C > 0$ independent of $\lambda \in P^+$ such that*

$$\sum_{\mu \preceq \lambda} |b_{\lambda,\mu}| \le C|\Pi_\lambda| r^\lambda.$$

Proof. Since we assume that we are in the standard case, by [13, Lemma 6.1] we have

$$b_{\lambda,\mu} = N_\mu \int_{\mathbb{U}} m_\lambda(u)\overline{P_\mu(u)} d\pi(u).$$

Using (1.5) and the techniques used to derive [13, (5.2)] we see that

$$\begin{aligned} b_{\lambda,\mu} &= \frac{W_0(q^{-1})}{W_{0\mu}(q^{-1})|W_0|} r^\mu \int_{\mathbb{U}} \sum_{w \in W_0} \frac{m_\lambda(wu) u^{-w\mu} \overline{c(wu)}}{|c(wu)|^2} du \\ &= \frac{W_0(q^{-1})}{W_{0\mu}(q^{-1})} r^\mu \int_{\mathbb{U}} m_\lambda(u) \frac{u^{-\mu}}{c(u)} du, \end{aligned}$$

and the first claim easily follows. The final claim follows from Proposition 3.4(iv). $\square$

Let $\Upsilon = \{u \in \mathrm{Hom}(P, \mathbb{C}^\times) : |u^{w\lambda}| \le r^\lambda \text{ for all } \lambda \in P^+ \text{ and all } w \in W_0\}$.

Theorem A.3. *The algebra homomorphism $h_u : \mathscr{A} \to \mathbb{C}$ is bounded if and only if $u \in \Upsilon$.*

Proof. In the exceptional case this follows from [9, Theorem 4.7.1], as remarked at the beginning of this appendix. Suppose we are in the standard case. If $u \in \Upsilon$ is *nonsingular*, then by (1.6) we have

$$|h_u(A_\lambda)| \leq \frac{1}{W_0(q^{-1})} \sum_{w \in W_0} |c(wu)| \quad \text{for all } \lambda \in P^+.$$

Thus $h_u : \mathscr{A} \to \mathbb{C}$ is bounded, and so by [5, Theorem I.2.5], $|h_u(A_\lambda)| \leq 1$ for all $\lambda \in P^+$.

If $u \in \Upsilon$ is *singular*, it is clear that there exists a sequence $(u_{(k)})_{k \in \mathbb{N}}$ in Υ such that each $u_{(k)}$ is nonsingular and $u_{(k)} \to u$. By the above we have $|h_{u_{(k)}}(A_\lambda)| \leq 1$ for all $\lambda \in P^+$, and since each $h_{u_{(k)}}$ is a Laurent polynomial (in the variables $\{u_{(k)}^{\lambda_i}\}_{i \in I_0}$) it follows that $|h_u(A_\lambda)| \leq 1$ for all $\lambda \in P^+$. Thus h_u is bounded for all $u \in \Upsilon$.

Suppose now that $h_u : \mathscr{A} \to \mathbb{C}$ is bounded (so $|h_u(A_\lambda)| \leq 1$ for all $\lambda \in P^+$). Then for all $\lambda \in P^+$, by (A.2) and Lemma A.2,

$$|m_\lambda(u)| \leq \sum_{\mu \preceq \lambda} |b_{\lambda,\mu}| \leq C|\Pi_\lambda| r^\lambda.$$

Thus, fixing λ, for $k \in \mathbb{N}$,

$$\left| \sum_{\mu \in W_0\lambda} (r^{-\lambda} u^\mu)^k \right| \leq p(k),$$

where $p(k)$ is a polynomial. It is elementary that this implies that $|r^{-\lambda} u^\mu| \leq 1$ for all $\mu \in W_0\lambda$, hence the result. □

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