COMMENSURATORS AND DEFICIENCY

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Abstract. We show that if G is a finitely generated group the kernel of the natural homomorphism from G to its abstract commensurator $Comm(G)$ is locally nilpotent by locally finite, and is finite if $\mathrm{def}(G) > 1$. We also give a simple proof that $Comm_{GL(n,\mathbb{R})}(SL(n,\mathbb{Z}) = \mathbb{R}^\times GL(n,\mathbb{Q})$.

Two groups are *commensurable* if they have subgroups of finite index which are isomorphic. If $G \leq \pi$ the *commensurator of* G *in* π is

$$Comm_\pi(G) = \{\alpha \in \pi \mid [G : G \cap \alpha G\alpha^{-1}] < \infty,\ [G : G \cap \alpha^{-1}G\alpha] < \infty\}.$$

The *abstract commensurator* of a group G is $Comm(G)$, the group of equivalence classes of isomorphisms $\alpha : H \cong J$ between subgroups of finite index in G, where α and α' are equivalent if they agree on some common subgroup of finite index, and the product of the equivalence classes of α and β is represented by the partially defined composite $\alpha \circ \beta$. (Thus $Comm_\pi(G) = \{\alpha \in \pi \mid c_\alpha \in Comm(G)\}$, where c_α is conjugation by α). It is easily seen that if H is a subgroup of finite index in G there is a natural homomorphism from $Comm(H)$ to $Comm(G)$, and that this is an isomorphism. Therefore if G_1 and G_2 are commensurable $Comm(G_1) \cong Comm(G_2)$. If G has only finitely many subgroups of finite index then on letting H be their intersection we find that $Comm_\pi(G) = N_\pi(H)$ and $Comm(G) = Aut(H)$. (Commensurability is sometimes defined as the equivalence relation determined by homomorphisms with finite kernel and cokernel. It is not clear whether the issues considered here behave as well under this broader definition.)

We shall show that if G is finitely generated the kernel of the natural homomorphism from G to $Comm(G)$ is is locally nilpotent by locally finite, and is finite if $\mathrm{def}(G) > 1$. We attempt to examine the structure of $Comm(G)$ more closely, through the subgroup $VA(G)$ generated by the equivalence classes of automorphisms of subgroups of finite index in G. This subgroup is normal in $Comm(G)$, and we observe that if every partial isomorphism $\alpha : H \cong J$ is equivalent to a virtual automorphism $\beta : K \cong K$ then G satisfies the volume condition. We conclude by showing that (as is no doubt well-known) $Comm(Z^n) \cong GL(n,\mathbb{Q})$ and $Comm_{GL(n,\mathbb{R})}(GL(n,\mathbb{Z})) = \mathbb{R}^\times GL(n,\mathbb{Q})$, and that taking determinants gives an isomorphism $Comm(Z^n)/VA(Z^n) \cong \mathbb{Q}^\times/\{\pm 1\}$.

1. The kernel

Let $F(r)$ be the free group of finite rank r. If S is a subset of a group G let $\langle\langle S\rangle\rangle_G$ denote the normal closure of S in G. Let G', ζG and $\sqrt{G}$ be the commutator subgroup, centre and Hirsch-Plotkin radical (respectively) of G. Let $Hol(G) = G \rtimes Aut(G)$, with the natural action of $Aut(G)$ on G.

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If $G \leq \pi$ let $C_\pi(G)$ be the centralizer of G in π, $\gamma_\pi(G) : Comm_\pi(G) \to Comm(G)$ be the natural homomorphism and $K_\pi(G) = \mathrm{Ker}(\gamma_\pi(G))$. Let K_G and $K_{aut}(G) = Aut(G) \cap K_{Hol(G)}(G)$ be the kernels of the natural homomorphisms from G and $Aut(G)$ (respectively) to $Comm(G)$. Clearly $K_\pi(H) = H \cap K_\pi(G)$ if H is a subgroup of finite index in G.

Lemma 1. *Let $G \leq \pi$ be a pair of groups, with G finitely generated. Then $K_\pi(G) = \cup C_\pi(H)$ and $K_G = \cup C_G(H)$, where the unions are taken over all normal subgroups H of finite index in G.*

Proof. This is clear, since normal subgroups are cofinal among all subgroups of finite index in a finitely generated group. □

Theorem 2. *Let G be a finitely generated group. Then*

(1) K_G is locally nilpotent by locally finite and ${K_G}'$ is locally finite;
(2) if G has no nontrivial locally-finite normal subgroup then K_G is abelian;
(3) if $C_G(H)$ is finite for all normal subgroups H of finite index in G then K_G and $K_{aut}(G)$ are locally finite;
(4) if G has no infinite elementary amenable normal subgroup then K_G is the maximal finite normal subgroup of G.

Proof. (1) We have $[C_G(H) : \zeta C_G(H)] \leq [C_G(H) : \zeta H] \leq [G : H]$, since $\zeta H \leq \zeta C_G(H)$. Therefore $C_G(H)$ has centre of finite index, if $[G : H] < \infty$. Hence $C_G(H)'$ is finite, by Schur's Theorem (Proposition 10.1.4 of [7]). Let H_n be a cofinal descending sequence of normal subgroups of finite index in G, and let $K_n = C_G(H_n)$. Then K_n is a normal subgroup of K_{n+1} for all $n \geq 1$ and $K_G = \cup K_n$. Hence ${K_G}' = \cup {K_n}'$ is locally finite. Since $\zeta K_n \leq \sqrt{K_n} \leq \sqrt{K_{n+1}}$ the union $N = \cup\sqrt{K_n}$ is locally nilpotent, with locally finite quotient. (In particular, K_G is elementary amenable.)

(2) This follows from (1), since K_G' is a normal subgroup of G.

(3) If $C_G(H)$ is finite for all normal subgroups H of finite index in G then $K_G = \cup C_G(H)$ is locally finite. Let F be a finitely generated subgroup of $K_{aut}(G)$. Then there is a subgroup H of finite index in G such that $\alpha(h) = h$ for all $\alpha \in F$ and $h \in H$. We may assume without loss of generality that H is normal in G. Let $f_\alpha(g) = g\alpha(g^{-1})$ for all $g \in G$ and $\alpha \in F$. Then $f_\alpha(gh) = f_\alpha(g)$ and $hf_\alpha(g)h^{-1} = f_\alpha(hg) = f_\alpha(gg^{-1}hg) = f_\alpha(g)$ for all $g \in G$ and $h \in H$. Therefore f_α factors through the finite group G/H, and takes values in $C_G(H)$. Since $C_G(H)$ is finite and $f_\alpha = f_\beta$ if and only if $\alpha = \beta$ it follows that F is finite. Thus $K_{aut}(G)$ is locally finite.

(4) If G has no infinite elementary amenable normal subgroup then K_G is finite. Moreover if F is the maximal locally finite normal subgroup of G it is finite, and acts trivially on $C_G(F)$, which has finite index in G. It follows easily that $K_G = F$. □

In particular, if $\sqrt{G}$ is locally finite then K_G is locally finite. If moreover $G \leq GL(n, \mathbb{Q})$ then K_G must be finite, since locally finite subgroups of such linear groups are finite. In case (4) if $H = C_G(K_G)$ then $[G : H] < \infty$ and $K_H = \zeta H$ is a finite abelian subgroup.

2. Groups of positive deficiency

In this section we shall show that if G is a finitely presentable group with def(G) ≥ 1 then K_G is usually finite, the exceptions being groups commensurable with $Z \times F(2)$ or Z^2.

Theorem 3. *Let G be a group with a presentation of deficiency ≥ 1. Then either*

(1) K_G is the maximal finite normal subgroup of G; or

(2) def(G) $= 1$, $c.d.G \leq 2$ and $K_G = \sqrt{G} \cong Z$ or Z^2.

In the latter case either $G \cong Z^2$ or $Z\tilde{\times}Z$ or G is commensurable with $Z \times F(2)$.

Proof. If K_G is not the maximal finite normal subgroup of G then G has an infinite amenable normal subgroup, by Theorem 2. Hence $\beta_1^{(2)}(G) = 0$ [2], and so def(G) $=$ 1 and $c.d.G \leq 2$, by Corollary 2.4.1 of [4]. In this case K_G is nontrivial and torsion free. Let k be a nontrivial element of K_G. Then k centralizes some subgroup H of finite index in G. We may assume that H is normal in G. Then $k^{[G:H]}$ is a nontrivial element of ζH, so $\sqrt{G} \neq 1$. Either $\sqrt{G} \cong Z$ or G' is abelian, by Theorem 2.7 of [4].

If $\sqrt{G} \cong Z$ then $G/\sqrt{G}$ is virtually free, by Theorem 8.4 of [1], and $[G : C_G(\sqrt{G}] \leq 2$. (In particular, the preimage in G of a free subgroup F of finite index in $C_G(\sqrt{G}/\sqrt{G}$ is isomorphic to $Z \times F$.) Hence $\sqrt{G} \leq K_G$. Since $K_G/\sqrt{G}$ is a torsion group, and is a normal subgroup of $G/\sqrt{G}$ it is finite. Since K_G is torsion free it follows that $K_G \cong Z$ and hence that $K_G = \sqrt{G}$.

If G' is abelian and $\sqrt{G}$ is not infinite cyclic then $G \cong Z*_m$, the ascending HNN extension with presentation $\langle a, t \mid tat^{-1} = a^m \rangle$, for some $m \neq 0$, by Corollary 2.6 of [4]. But it is easy to see that K_G must then be trivial unless $m = \pm 1$, in which case $G \cong Z^2$ or $Z\tilde{\times}Z$ and $K_G = \sqrt{G} \cong Z^2$. □

If def(G) ≥ 1 and K_G is finite must it be trivial?

3. Hopficity, the volume condition and commensurability

A group G is *hopfian* if surjective endomorphisms of G are automorphisms, and is *cohopfian* if injective endomorphisms are automorphisms. Residually finite groups are hopfian. It is easy to see that residual finiteness is a property of commensurability classes. To what extent is this true for hopficity or cohopficity and related notions?

Let G be a finitely generated group and $\phi : G \to G$ be an epimorphism. If $H \leq G$ then $\phi(\phi^{-1}(H)) = H$ and $[G : \phi^{-1}(H)] = [G : H]$. Hence ϕ maps the set of subgroups of G of index $[G : H]$ onto itself. In particular, if $[G : H] < \infty$ this set is finite and so $\phi^N(H) = H$ for some $N \geq 1$. Therefore if H is also hopfian $\phi^N|_H$ is an isomorphism, and so ϕ^N is itself an isomorphism. Hence G is hopfian.

The converse is not so clear. Suppose that G is hopfian. After replacing H by the intersection of its images under automorphisms of G we may assume that H is characteristic in G. But in order to use hopficity of G we need to be able to extend a surjective endomorphism of H to a surjective endomorphism of G.

The *volume condition* holds for G if whenever H_1 and H_2 are isomorphic subgroups of finite index then $[G : H_1] = [G : H_2]$. This holds if def$(G) > 1$ [6]. More generally, this is the case if $\beta_1^{(2)}(G) \neq 0$, since L^2 Betti numbers are multiplicative in finite extensions. It also holds if G is of type FP and $\chi(G) \neq 0$. Thus finitely generated nonabelian free groups and hyperbolic surface groups satisfy the volume condition. If G is a PD_n-group and satisfies the volume condition then G is cohopfian, since subgroups of infinite index in PD_n-groups have cohomological dimension strictly less than n [8]. On the other hand, nontrivial free abelian groups do not satisfy the volume condition and nontrivial free groups are not cohopfian.

Theorem 4. *The volume condition is a property of commensurability classes.*

Proof. Suppose that H has finite index n in G. If G satisfies the volume condition then clearly so does H. Conversely, suppose that H satisfies the volume condition. If H_1 and H_2 are isomorphic subgroups of finite index in G then $\widetilde{H_1}^n \cong \widetilde{H_2}^n$, and each is contained in H. Therefore

$$[G:H_1]=[G:H][H:\widetilde{H_1}^n]/[H_1:\widetilde{H_1}^n]=[G:H][H:\widetilde{H_2}^n]/[H_2:\widetilde{H_2}^n]=[G:H_2].$$

The theorem follows easily. □

4. Virtual automorphisms

We may give a somewhat more explicit description of $Comm(G)$ when G is finitely generated. Define inductively a descending sequence of subgroups $\widetilde{G}^n$ of finite index in G by setting $\widetilde{G}^1 = G$ and $\widetilde{G}^{n+1}$ to be the intersection of all subgroups of index $\leq n+1$ in $\widetilde{G}^n$, for all $n \geq 1$. Then $\widetilde{G}^{n+1}$ is characteristic in $\widetilde{G}^n$, and the sequence $\{\widetilde{G}^n\}_{n\geq 1}$ is cofinal among all subgroups of finite index in G. Let

$$X_n = \{\phi \in Hom(\widetilde{G}^n, G) \mid [G:\phi(\widetilde{G}^n)] < \infty,\ \phi \textit{ is mono}\}$$

and let $i_n : \widetilde{G}^n \to G$ be the natural inclusion. Composition defines a right action of $A_n = Aut(\widetilde{G}^n)$ on X_n, with trivial point stabilizers, by $(\phi,\alpha) \mapsto \phi\circ\alpha$, for $\phi \in X_n$ and $\alpha \in A_n$.

Restriction defines a function $\rho_n : X_n \to X_{n+1}$ such that $\rho_n(i_n) = i_{n+1}$. Since $\widetilde{G}^{n+1}$ is characteristic in $\widetilde{G}^n$ automorphisms of $\widetilde{G}^n$ restrict to automorphisms of $\widetilde{G}^{n+1}$ and $\rho_n(\phi\circ\alpha) = \phi\circ(\alpha|_{\widetilde{G}^{n+1}})$. If the subgroups $\widetilde{G}^n$ have trivial centralizers in G then $\rho_n|A_n$ is a monomorphism for all $n \geq 1$. Are the functions ρ_n injective?

The natural map from the direct limit $\varinjlim X_n$ to $Comm(G)$ which sends ϕ to the equivalence class of the corresponding isomorphism $\widetilde{G}^n \cong \phi(\widetilde{G}^n)$ is a bijection, for if $\alpha : H \cong J$ is an isomorphism of subgroups of finite index and $[G:H] = n$ then $\widetilde{G}^n \leq H$ and $\alpha|_{\widetilde{G}^n} \in X_n$. The direct limit $A_\infty = \varinjlim A_n$ acts on $\varinjlim X_n$.

A *virtual automorphism* of G is an element of $Comm(G)$ determined by an automorphism of a subgroup of finite index. In general the set of virtual automorphisms is not a subgroup. Let $A = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$. Then A and BAB^{-1} represent virtual automorphisms of Z^2 but $ABAB^{-1} = \begin{pmatrix}2&0\\0&\frac{1}{2}\end{pmatrix}$ does not preserve any lattice, and so does not represent a virtual automorphism of Z^2. Let $VA(G) \leq Comm(G)$ be the subgroup generated by virtual automorphisms. If H is a subgroup of finite index in G the natural homomorphism from $VA(H)$ to $VA(G)$ is an isomorphism.

Theorem 5. *Let G be a finitely generated group. Then*

(1) $VA(G)$ is a normal subgroup of $Comm(G)$;

(2) if every element of $VA(G)$ is represented by a virtual automorphism then $Comm(G)/VA(G)$ is torsion-free.

(3) $A_\infty = \varinjlim Aut(\widetilde{G}^n) \leq VA(G)$

Proof. Suppose that H, J, K are subgroups of finite index in G and $\alpha : H \to H$ and $\beta : J \to K$ are isomorphisms. Let $m = [H : H \cap J]$. Then $\widetilde{H}^m \leq J$ and α restricts to an automorphism of $\widetilde{H}^m$. Hence $\beta\alpha\beta^{-1}$ is an automorphism of $\beta(\widetilde{H}^m)$. It follows that $VA(G)$ is a normal subgroup of $Comm(G)$.

Suppose that every element of $VA(G)$ is represented by a virtual automorphism. If $\alpha : J \cong K$ represents an element of order n in $Comm(G)/VA(G)$ there is a subgroup H of finite index and an automorphism $\theta : H \cong H$ such that $\alpha^n(h) = \theta(h)$ for all $h \in H_0 \leq H$. Let $J_i \leq J$ be the domain of α^i, for $1 \leq i \leq n$. Then $J_{i+1} \leq J_i$ for $i < n$ and $H_0 \leq J_n$. Let $m = [H : H_0]$. On replacing H with $\widetilde{H}^m$ we may assume that $H \leq J_n$ and that $\alpha^n(H) = H$. In particular, $H \leq J_i$, for all $1 \leq i \leq n$, and α restricts to an automorphism of $\cap_{1 \leq k \leq n}\alpha^i(H)$. Thus α represents a virtual automorphism and so $Comm(G)/VA(G)$ is torsion-free.

The final assertion is clear. □

Similarly, if $G < \pi$ and we let $VN_\pi(G)$ be the subgroup of π generated by elements which normalize a subgroup of finite index in G then $VN_\pi(G)$ is normal in $Comm_\pi(G)$ and $\gamma_\pi(G)(VN_\pi(G)) \leq VA(G)$.

To explore the quotient $Comm(G)/VA(G)$ it may be useful to consider the following generalization. Let Γ be a countable lattice which is a disjoint union of finite subsets Γ_n, for $n \geq 0$, with $|\Gamma_0| = 1$ and such that if $\gamma \in \Gamma_k$, $\delta \in \Gamma_l$ and $\gamma < \delta$ then $k > l$. (Thus elements of the same subset Γ_k are incomparable.) Let L_γ be the sublattice with elements $\{\delta \in \Gamma \mid \delta \leq \gamma\}$. A *partial permutation* of Γ is a lattice isomorphism $f : L_\gamma \to L_{\gamma'}$. Two such partial permutations are equivalent if they agree on L_δ for some $\delta \leq \gamma, \gamma'$. Let $PP(\Gamma)$ be the set of equivalence classes, with the group multiplication determined by partial composition. A partial permutation f is a *virtual automorphism* if it fixes a vertex γ. The set of equivalence classes containing virtual automorphisms generates a subgroup $VA(\Gamma) \leq PP(\Gamma)$.

If G satisfies the volume condition $[G : \phi(G_n)] = [G : G_n]$ for all $\phi \in X_n$ and so the set of orbits X_n/A_n is finite. In particular, if $G \cong F(r)$ or is an orientable hyperbolic surface group then subgroups of finite index in G are hopfian, and two such subgroups are isomorphic if and only if they have the same index. Moreover the centralizers of such subgroups are trivial, since G is torsion-free and $\sqrt{G} = 1$. Therefore A_∞ embeds as a subgroup of $Comm(G)$.

Let Γ be the lattice with Γ_n the set of subgroups of index n in G, partially ordered by inclusion. Then $Comm(G)$ is the subgroup of $PP(\Gamma)$ represented by partial permutations which preserve the index and $VA(G) = VA(\Gamma)$.

Lemma 6. *If G is finitely generated and every element of $Comm(G)$ is represented by a virtual automorphism then $VA(G) = Comm(G)$ and G satisfies the volume condition.*

Proof. The first assertion is clear. Suppose J and K are subgroups of finite index and $\alpha : J \to K$ is an isomorphism. If $\alpha|_L = \beta|_L$ for some automorphism $\beta : H \to H$ of a finite index subgroup H and some $L \leq J \cap H$ of finite index in G then we map assume L is characteristic in H, so $\beta(L) = L$. Hence

$$[G : J] = [G : L]/[J : L] = [G : L]/[\alpha(J) : \alpha(L)] = [G : L]/[K : L] = [G : K]$$

and so G satisfies the volume condition. □

Does $VA(G) = Comm(G)$ already imply that G satisfies the volume condition?

5. Examples

The results in this section are surely known, but we include simple expositions for convenience.

We consider first finitely generated free abelian groups.

Theorem 7. *$Comm(Z^n) \cong GL(n,\mathbb{Q})$, $VA(Z^n)$ contains $SL(n,\mathbb{Q})$ as a subgroup of index 2 and $Comm(Z^n)/VA(Z^n) \cong \mathbb{Q}^\times/\{\pm 1\} \cong Z^\infty$.*

Proof. Let $\alpha : H \cong J$ be an isomorphism between subgroups of finite index in Z^n, and let $m = [Z^n : H]$. Then $\alpha m = \alpha|_{mZ^n} \circ m$ determines an injective endomorphism of Z^n. Let $M(\alpha m)$ be the matrix of this endomorphism with respect to the standard basis of Z^n. Then $f([\alpha]) = \frac{1}{m}M(\alpha m)$ gives a well-defined isomomorphism from $Comm(Z^n)$ to $GL(n,\mathbb{Q})$. If $N \in GL(n,\mathbb{Q})$ and d is a common denominator for the entries of N let $H = dZ^n$, $J = dN(Z^n)$ and $\beta(h) = dN(d^{-1}h)$ for all $h \in H$. Then $f([\beta]) = N$ and so f is onto.

If $A : L \to L$ is an automorphism of a lattice $L \leq Z^n$ then $L = P(Z^n)$ for some change of basis matrix $P \in GL(n,\mathbb{Q})$ and so $P^{-1}AP \in GL(n,\mathbb{Z})$. Therefore $VA(Z^n) = \langle\langle GL(n,\mathbb{Z})\rangle\rangle_{GL(n,\mathbb{Q})}$. Now $VA(Z^n) \cap SL(n,\mathbb{Q})$ is a normal subgroup which properly contains $\zeta SL(n,\mathbb{Q})$. Hence $VA(Z^n) \cap SL(n,\mathbb{Q}) = SL(n,\mathbb{Q})$, by Proposition 3.2.8 of [7]. Since $VA(Z^n)$ certainly has elements represented by automorphisms of determinant -1 the homomomorphism from $Comm(Z^n)/VA(Z^n)$ to $\mathbb{Q}^\times/\{\pm 1\}$ induced by taking determinants is an isomorphism. $\square$

These results extend immediately to finitely generated, virtually abelian groups. Note however that the natural homomorphism from $Aut(D_\infty)$ to $VA(D_\infty)$ has infinite kernel.

The simplest infinite nonabelian groups are perhaps the free groups $F(r)$ of rank $r > 1$. All subgroups of index k in $F(2)$ are free of rank $k+1$, and so all such groups are commensurable with $F(2)$. We may give a "lower bound" for $Comm(F(2))$ by first finding $Comm_{\mathfrak{P}}(F(2))$, for a suitable embedding of $F(2)$ into $\mathfrak{P} = PSL(2,\mathbb{R})$. By a deep theorem of Margulis, a lattice H in $\mathfrak{P}$ has infinite index in $Comm_{\mathfrak{P}}(H)$ if and only if H is arithmetic [5]. The most familiar arithmetic embedding is given by the isomorphisms $F(2) \cong SL(2,\mathbb{Z})' \cong PSL(2,\mathbb{Z})'$. (These commutator subgroups have finite index in $SL(2,\mathbb{Z})$ and $PSL(2,\mathbb{Z})$, respectively.) Our argument applies more generally, to $n \times n$ matrix groups. Let $\mathfrak{G}(n) = GL(n,\mathbb{R})$ and $\mathfrak{P}(n) = PSL(n,\mathbb{R})$

Theorem 8. *$\mathbb{R}^\times GL(n,\mathbb{Q}) = Comm_{\mathfrak{G}(n)}(SL(n,\mathbb{Z}))$.*

Proof. The diagonal matrix $D(m) = \left(\begin{smallmatrix} m & 0 \\ 0 & I_{n-1}\end{smallmatrix}\right)$ with m a nonzero integer conjugates the subgroup of matrices $A = \left(\begin{smallmatrix} a & \beta \\ \gamma & *\end{smallmatrix}\right) \in SL(n,\mathbb{Z})$ with $\gamma \equiv 0 \ mod\ (m)$ onto the subgroup with $\beta \equiv 0\ mod\ (m)$. These subgroups have finite index in $SL(n,\mathbb{Z})$, and so $D(m)$ is in $Comm_{\mathfrak{G}(n)}(SL(n,\mathbb{Z}))$. Since $GL(n,\mathbb{Q})$ is generated by $SL(n,\mathbb{Z})$ and these diagonal matrices it follows that $GL(n,\mathbb{Q}) \leq Comm_{\mathfrak{G}(n)}(SL(n,\mathbb{Z}))$.

Let H be a subgroup of finite index m in $SL(n,\mathbb{Z})$ and for each $1 \leq p \neq q \leq n$ let $E_{p,q}$ be the $n \times n$ matrix with (p,q)-entry 1 and 0s elsewheres. Then $I_n + mE_{p,q} = (I_n + E_{p,q})^m$ is in H for all such p,q. Therefore if $M \in \mathfrak{G}(n)$ conjugates H into $SL(n,\mathbb{Z})$ we must have $M(I_n + mE_{p,q}) = A_{p,q}M$ for some $A_{p,q} \in SL(n,\mathbb{Z})$. The set of matrices in $\mathbb{M}_{n\times n}(\mathbb{R})$ satisfying such an equation for given m, p, q and $A_{p,q}$ is a nontrivial rationally defined linear subspace. If M' is any matrix in this subspace then $M' = MN$, where $N = M^{-1}M'$ commutes with $E_{p,q}$. Thus the p^{th} column and q^{th} row of N are zero, except for the (p,p)- and (q,q)-entries, which are equal. As we vary p,q we see that the intersection of all such subspaces $\{MN\}$ is $\{M(rI_n) \mid r \in \mathbb{R}\}$. Therefore M lies on a rational line through the origin, and so is proportional to a matrix with rational entries. Thus $Comm_{\mathfrak{G}(n)}(SL(n,\mathbb{Z})) = \mathbb{R}^{\times}GL(n,\mathbb{Q})$. □

Corollary 9. *$VN_{\mathfrak{P}(n)}(PSL(n,\mathbb{Z})) = Comm_{\mathfrak{P}(n)}(PSL(n,\mathbb{Z})) = PGL(n,\mathbb{Q})$ and $\gamma_{\mathfrak{P}(n)}(PSL(n,\mathbb{Z}))$ is a monomorphism.*

Proof. The equality $Comm_{\mathfrak{P}(n)}(PSL(n,\mathbb{Z})) = PGL(n,\mathbb{Q})$ is immediate from the theorem, while $VN_{\mathfrak{P}(n)}(PSL(n,\mathbb{Z})) = Comm_{\mathfrak{P}}(n)(PSL(n,\mathbb{Z}))$ then follows from the fact that $PSL(n,\mathbb{Q})$ is simple.

If H has finite index in $PSL(n,\mathbb{Z})$ then $C_{\mathfrak{P}(n)}(H) = 1$, since there are matrices in H with distinct sets of eigenvalues. Hence $K_{\mathfrak{P}(n)}(PSL(n,\mathbb{Z})) = 1$. □

The homomorphism from $Aut(F(n))$ to $Comm(F(2))$ corresponding to an inclusion of $F(n)$ as a subgroup of index $n-1$ in $F(2)$ is injective, since noncyclic subgroups of free groups have trivial centralizers. The groups $Aut(F(n))$ are not linear if $n \geq 3$ [3]. Hence $Comm(F(2))$ is not linear and $\gamma_{\mathfrak{P}}(PSL(2,\mathbb{Z}))$ is not an isomorphism.

Is $VA(F(2)) = Comm(F(2))$? Is $Comm(F(2))$ simple?

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