GEOMETRIES AND INFRASOLVMANIFOLDS IN DIMENSION 4

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Abstract. We show that every torsion-free virtually poly-Z group of Hirsch length 4 is the fundamental group of a closed 4-manifold with a geometry of solvable Lie type.

Geometric 4-manifolds of solvable Lie type are infrasolvmanifolds, and their fundamental groups are torsion-free virtually poly-Z group π of Hirsch length 4. We shall show that every such group is realized geometrically. In general, every such group is the fundamental group of a closed infrasolvmanifold [1], and closed infrasolvmanifolds are diffeomorphic if and only if their fundamental groups are isomorphic [2]. However it is not *a priori* obvious that infrasolvmanifolds admit geometries in the sense of Thurston. It is well-known that torsion-free virtually abelian groups are realized by flat manifolds (see [3]), while Dekimpe has dealt with the virtually nilpotent cases [5]. If π is of Seifert type (i.e, is an extension of a flat 2-orbifold group by Z^2 or $Z \rtimes_{-1} Z$) then this is due to Ue in the orientable case and Kemp in general [9, 10, 11]. In the remaining cases we shall give explicit representations of the possible fundamental groups as lattices in the appropriate isometry groups. (Some calculations are deferred to an appendix.) We also give an *ad hoc* low-dimensional argument to show that the group determines the manifold up to diffeomorphism (with the exception of a handful of groups, for which we must appeal to [2].)

1. Notation

If G is a group let G', ζG and $\sqrt{G}$ denote the commutator subgroup, centre and Hirsch-Plotkin radical of G. Let $I(G) = \{g \in G \mid \exists n > 0, g^n \in G'\}$ and let $C_G(H)$ be the centralizer of the subgroup H.

Let Γ_q be the nilpotent group with presentation

$$\langle x, yz \mid xz = zx, yz = zy, xy = z^q yx \rangle.$$

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Let $\mathbb{R}^n$ be the space of column vectors $\phi = (\phi_1, \ldots, \phi_n)^{tr}$ of length n, and let $e_1, \ldots, e_n$ be the standard basis vectors. Let $E(r) = Isom(\mathbb{E}^r)$ be the group of euclidean isometries of R^n, and let I_n be the $n \times n$ identity matrix.

2. Seifert fibrations and geometries

An n-dimensional orbifold B has an open covering by subspaces of the form D^n/G, where G is a finite subgroup of $O(n)$. The orbifold B is *good* if $B = \Gamma\backslash M$, where Γ is a discrete group acting properly discontinuously on a manifold M; otherwise it is *bad*. A good orbifold B is *aspherical* if $B = \Gamma\backslash M$ with M aspherical, and is *closed* if Γ acts cocompactly. An *orbifold bundle with general fibre F over B* is a map $f : M \to B$ which is locally equivalent to a projection $G\backslash(F \times D^n) \to G\backslash D^n$, where G acts freely on F and effectively and orthogonally on D^n.

A 4-manifold S is *Seifert fibred* if it is the total space of an orbifold bundle with general fibre a torus or Klein bottle over a 2-orbifold. (In [10, 11, 13, 14] it is required that the general fibre be a torus. This is always so if the manifold is orientable.) It is easily seen that $\chi(S) = 0$ and that if the base is aspherical $\pi_1(S)$ has Z^2 as a normal subgroup. Seifert fibred 4-manifolds over aspherical bases are determined up to diffeomorphism by their fundamental groups. This is due to Zieschang for the cases with base a hyperbolic orbifold with no reflector curves and general fibre a torus [13], and the general result is due to Vogt [12].

Theorem [Vogt] *Let M and M' be two closed 4-manifolds which are Seifert fibred over euclidean or hyperbolic orbifolds. Then the Seifert fibrations are isomorphic if and only if the corresponding fundamental group sequences are isomorphic.* $\square$

In particular, if the bases are hyperbolic the fundamental group of the general fibre is the unique solvable normal subgroup with quotient a discrete cocompact subgroup of $Isom(\mathbb{H}^2)$, and so such 4-manifolds have esentially unique Seifert fibrations. The fibration is also unique if π is not virtually nilpotent of class at most 2.

If $\mathbb{X}$ is one of the geometries $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol^3 \times \mathbb{E}^1$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{F}^4$ its model space X has a canonical foliation with leaves diffeomorphic to R^2 and which is preserved by isometries. (For the Lie groups $Nil^3 \times R$, Nil^4 and $Sol^3 \times R$ we may take the foliations by cosets of the normal subgroups $\zeta Nil^3 \times R$, $Nil^{4'}$ and $Sol^{3'}$.) These foliations induce Seifert fibrations on quotients by lattices. All $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds are also Seifert fibred. Case-by-case inspection of the 74 flat 4-manifold groups shows that all but three have Z^2 as a normal subgroup, and the representations given in [3] may be used to show that the corresponding manifolds are Seifert fibred. The exceptions are certain semidirect products $G_6 \rtimes_\theta Z$ where G_6 is the Hantzsche-Wendt flat 3-manifold group. No other closed geometric 4-manifolds are Seifert fibred, as they either have nonzero Euler characteristic or their fundamental groups do not admit abelian normal subgroups of rank 2.

Conversely, Ue has shown that an orientable 4-manifold which is Seifert fibred over an aspherical 2-orbifold is diffeomorphic to a geometric 4-manifold [10, 11]. This has been extended to the nonorientable case by Kemp (using similar arguments) [9]. These results for the euclidean base cases follow also from our constructions below, together with Vogt's theorem. (A new treatment of the cases with hyperbolic bases is given in Chapter 11 of the revised version of [8].)

3. $\mathbb{S}ol^4_{m,n}$- AND $\mathbb{S}ol^4_0$-MANIFOLDS

Let π be a torsion-free virtually poly-Z group of Hirsch length 4. If π is virtually nilpotent then it is realized geometrically [5]. Otherwise $\sqrt{\pi} \cong Z^3$ or Γ_q, for some $q \geq 1$, by Theorem 1.5 of [8]. Hence $\pi/\sqrt{\pi}$ is an extension of Z or $D_\infty = (Z/2Z) * (Z/2Z)$ by a finite normal subgroup. Thus π has a characteristic subgroup ν such that $[\nu : \sqrt{\pi}] < \infty$ and $\pi/\nu \cong Z$ or D_∞. We shall consider the groups with $\sqrt{\pi} \cong Z^3$ here and those with $\sqrt{\pi} \cong \Gamma_q$ in the next section.

If π is the fundamental group of a closed $\mathbb{S}ol^4_{m,n}$-manifold (with $m \neq n$) or $\mathbb{S}ol^4_0$-manifold then $\pi/\nu \cong Z$, by Corollary 8.4.1 of [8].

Theorem 1. *Let π be a torsion-free group with $\sqrt{\pi} \cong Z^3$ and such that $\pi/\sqrt{\pi}$ maps onto Z with finite kernel. Then π is the fundamental group of a closed $\mathbb{S}ol^4_{m,n}$- or $\mathbb{S}ol^4_0$-manifold.*

Proof. Let ν be the characteristic subgroup of π containing $\sqrt{\pi}$ and such that $\pi/\nu \cong Z$, and let $t \in \pi$ represent a generator of π/ν. Then $\pi \cong \nu \rtimes_\tau Z$, where τ is the automorphism of ν determined by conjugation by t. Let $M = \tau|_{\sqrt{\pi}}$. If the eigenvalues κ, λ, μ of M were all roots of unity, of order dividing k, say, the subgroup generated by $\sqrt{\pi}$ and t^k would be nilpotent, and of finite index in π. Therefore we may assume that κ, λ and μ are distinct and that neither κ nor λ is a root of unity. If $\nu \neq \sqrt{\pi}$ then $[\nu : \sqrt{\pi}] = 2$ and $\mu = \pm 1$, by Theorem 8.3 of [8].

Suppose first that the eigenvalues are all real. Then the eigenvalues of M^2 are all strictly positive. Since $\sqrt{\pi} \cong Z^3$ there is a monomorphism $f : \sqrt{\pi} \to \mathbb{R}^3$ such that $fM = Tf$, where $T = diag[\kappa, \lambda, \mu] \in GL(3, \mathbb{R})$.

Let $F(g) = \left(\begin{smallmatrix} I_3 & f(g) \\ 0 & 1 \end{smallmatrix}\right)$, for $g \in \sqrt{\pi}$. We shall extend F to ν below, and let $F(t) = \left(\begin{smallmatrix} T & \xi \\ 0 & 1 \end{smallmatrix}\right)$, where $\xi \in \mathbb{R}^3$ is to be chosen so that $F(t)F(g)F(t)^{-1} = F(\tau(g))$, for $g \in \nu$. (In this theorem there is usually an obvious simplest choice, but we shall need greater flexibility in Theorem 6 below.) This condition needs checking only on a set of coset representatives for $\nu/\sqrt{\pi}$, as it clearly holds for $g \in \sqrt{\pi}$.

If $\nu = \sqrt{\pi}$ we may choose ξ arbitrarily. In this case F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol^4_{m,n})$, where $m = trace(M^2)$ and $n = trace(M^{-2})$. (If one of the eigenvalues of M is ± 1 then $m = n$ and the geometry is $\mathbb{S}ol^3 \times \mathbb{E}^1$. See Chapter 7.§3 of [8].)

If $\nu \neq \sqrt{\pi}$ then $\nu \cong Z^2 \rtimes_\theta Z$, where $\theta = -I_2$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, by Theorem 8.3 of [8]. If $\nu \cong G_2 = Z^2 \rtimes_\theta Z$ where $\theta = -I_2$ then ν has a presentation

$$\langle x, y, z \mid xy = yx,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1}\rangle,$$

and $\{x, y, z^2\}$ is a basis for $\sqrt{\pi}$. The subgroup $I(\nu)$ generated by $\{x, y\}$ is characteristic in ν and therefore normal in π. The image of z generates $\nu/I(\nu)$, and z^2 generates $\zeta\nu$. Hence $tzt^{-1} = z^\epsilon k$ for some $\epsilon = \pm 1$ and $k \in I(\nu)$, and so $M(z^2) = z^{2\epsilon}$. Therefore $\epsilon = \mu$. Since $\mathbb{R}^3$ is generated by the images of $I(\nu)$ and $\zeta\nu$ and τ preserves each of these subgroups we may assume that $f(I(\nu))$ is in the span of $\{e_1, e_2\}$ and $f(z^2) = e_3$. We extend F to π by setting $F(z) = \left(\begin{smallmatrix} J & e_3 \\ 0 & 1 \end{smallmatrix}\right)$, where $J = diag[-1, -1, 1]$, and choosing ξ so that $f(k) + 2\xi \in \mathbb{R}e_3$.

If $\nu \cong B_1 = Z^2 \rtimes_\theta Z$ where $\theta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then ν has a presentation

$$\langle x, y, z \mid xy = yx,\ xz = xz,\ yzy^{-1} = z^{-1}\rangle,$$

and $\{x, y^2, z\}$ represents a basis for $\sqrt{\pi}$. In this case $\{x, y^2\}$ generates $\zeta\nu$ and z generates $I(\nu)$. Since $\mathbb{R}^3$ is generated by the images of $\zeta\nu$ and $I(\nu)$ and τ preserves each of these subgroups we may assume that $f(\zeta\nu)$ is in the span of $\{e_1, e_2\}$ and $f(z) = e_3$. Suppose that $tyt^{-1} = mz^r y$, where $m \in \zeta\nu$ and $r \in \mathbb{Z}$. Then $ty^2t^{-1} = m^2y^2$, and so $Tf(y^2) = 2f(m) + f(y^2)$. We extend F to π by setting $F(y) = \left(\begin{smallmatrix} -J & \beta \\ 0 & 1 \end{smallmatrix}\right)$, where $\beta = \frac{1}{2}f(y^2) + he_3$, and $\xi_3 = \frac{1}{2}(h(1-\mu) + r)$. (Here h, ξ_1, ξ_2 may be chosen freely.)

If $\nu \cong B_2 = Z^2 \rtimes_\theta Z$ where $\theta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then ν has a presentation

$$\langle x, y, z \mid xyx^{-1} = yz,\ xz = zx,\ yzy^{-1} = z^{-1}\rangle$$

and $\{x, y^2, z\}$ represents a basis for $\sqrt{\pi}$. In this case $\{y^2, x^2z\}$ represent a basis for $\zeta\nu$. Since $\mathbb{R}^3$ is generated by the images of $\zeta\nu$ and z and τ preserves each of these subgroups we may assume that $f(\zeta\nu)$ is in the span of $\{e_1, e_2\}$ and $f(z) = e_3$. Suppose that $tyt^{-1} = nx^sy$, where $n \in \zeta\nu$ and $s \in \mathbb{Z}$. Then $ty^2t^{-1} = n^2(x^2z)^sy^2$. Let $\gamma = \frac{1}{2}f(y^2) + he_3$

and $\xi = \frac{1}{2}(2h(1-\mu) - s)e_3$. (Here h may be chosen freely.) Then we may extend F to π by setting $F(y) = \left(\begin{smallmatrix} -J & \gamma \\ 0 & 1 \end{smallmatrix}\right)$ and $F(t) = \left(\begin{smallmatrix} T & \xi \\ 0 & 1 \end{smallmatrix}\right)$.

In each case, F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$.

If the eigenvalues are not all real we may assume that $\lambda = \bar{\kappa}$ and $\mu \neq \pm 1$. In this case we again have $\nu = \sqrt{\pi}$, by Theorem 8.3 of [8]. Let $R_\phi \in SO(2)$ be rotation of $\mathbb{R}^2$ through the angle $\phi = Arg(\kappa)$. There is a monomorphism $f : \sqrt{\pi} \to \mathbb{R}^3$ such that $fM = Tf$ where $T = \left(\begin{smallmatrix} |\kappa| R_\phi & 0 \\ 0 & \mu \end{smallmatrix}\right)$. Let $F(n) = \left(\begin{smallmatrix} I_3 & f(n) \\ 0 & 1 \end{smallmatrix}\right)$, for $n \in \sqrt{\pi}$, and let $F(t) = \left(\begin{smallmatrix} T & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol_0^4)$. □

It remains to be shown that if ν is virtually Z^3 and $\pi/\nu \cong D_\infty$ then π is the fundamental group of a closed $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold. (As recalled in Theorem 1, $\mathbb{S}ol^4_{m,m} = \mathbb{S}ol^3 \times \mathbb{E}^1$, for any m.)

Theorem 2. *Let π be a torsion-free group with $\sqrt{\pi} \cong Z^3$ and such that $\pi/\sqrt{\pi}$ maps onto D_∞ with finite kernel. Then π is the fundamental group of a closed $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold.*

Proof. Let ν be the characteristic subgroup of π containing $\sqrt{\pi}$ and such that $\pi/\nu \cong D_\infty$, and let $t \in \pi$ represent a generator of $\sqrt{\pi/\nu} \cong Z$. Let $\tilde{\pi}$ be the subgroup of index 2 in π generated by ν and t, and let $f : \sqrt{\pi} \to \mathbb{R}^3$ and $F : \tilde{\pi} \to Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$ be the embeddings given in Theorem 1. Let $u \in \pi$ represent an element of order 2 in π/ν. Then $v = u^2$ and $w = (ut)^2$ are elements of ν, and are nontrivial since π is torsion-free. The subgroups $\langle \nu, u \rangle$ and $\langle \nu, ut \rangle$ are flat 3-manifold groups, with holonomy of order dividing 4. Since B_2 is not a subgroup of index 2 in any such group, we may assume that $\nu \cong Z^3$, G_2 or B_1. However each of these groups is such a subgroup of several different flat 3-manifold groups.

If $\nu \neq \sqrt{\pi}$ let q represent the nontrivial coset of $\sqrt{\pi}$ in ν. Then $c = uqu^{-1}q^{-1}$ is in $\sqrt{\pi}$. We must define $F(u)$ so that

$$F(ugu^{-1}) = F(u)F(g)F(u)^{-1} \quad \forall g \in \sqrt{\pi},$$

$$F(u)F(q) = F(c)F(q)F(u),$$

$$F(u)^2 = F(v) \quad \text{and}$$

$$(F(u)F(t))^2 = F(w).$$

Let $U \in GL(3, \mathbb{R})$ be the matrix such that $f(ugu^{-1}) = Uf(g)$ for $g \in \sqrt{\pi}$.

Suppose first that $v, w \in \sqrt{\pi}$. Then $U^2 = I_3$ and $UTU = T^{-1}$, so $\{\kappa, \lambda, \mu\} = \{\kappa^{-1}, \lambda^{-1}, \mu^{-1}\}$. Since $\kappa^2, \lambda^2 \neq 1$ it follows that $\kappa\lambda = 1$ and $\mu^2 = 1$, and since $T = diag[\kappa, \lambda, \mu]$ we then see that

$$U = \begin{pmatrix} 0 & \eta & 0 \\ \eta^{-1} & 0 & 0 \\ 0 & 0 & e \end{pmatrix}$$

for some $\eta \neq 0$ and $e = \pm 1$. (In particular, $U \neq I_3$ and $UT \neq I_3$.) We shall assume that $F(t) = \left(\begin{smallmatrix} T & \xi \\ 0 & 1 \end{smallmatrix}\right)$ and $F(u) = \left(\begin{smallmatrix} U & \rho \\ 0 & 1 \end{smallmatrix}\right)$, for some $\xi, \rho \in \mathbb{R}^3$. Then $F(ugu^{-1}) = F(u)F(g)F(u)^{-1}$ for all $g \in \sqrt{\pi}$.

If $\nu = \sqrt{\pi}$ the remaining consistency conditions $F(u)^2 = F(v)$ and $F(ut)^2 = F(w)$ reduce to the two equations

$$(U + I_3)\rho = f(v) \quad \text{and}$$
$$(UT + I_3)(U\xi + \rho) = f(w).$$

We must have $f(v)_1 = \eta f(v)_2$, $f(v)_3 = ef(v)_3$, $f(w)_1 = \eta\kappa f(w)_2$ and $f(w)_3 = e\mu f(w)_3$, and we obtain four linear equations in six unknowns: $\rho_1 + \eta\rho_2 = f(v)_1$, $(e+1)\rho_3 = f(v)_3$, $\rho_1 + \eta\kappa\rho_2 = f(w)_1 - \kappa\xi_1 - \eta\xi_2$ and $(e\mu + 1)(e\xi_3 + \rho_3) = f(w)_3$. Thus ρ_1 and ρ_2 are determined uniquely by ξ_1 and ξ_2. If $e = \mu = +1$ then we must have $\rho_3 = \frac{1}{2}f(v)_3$ and $\xi_3 = \frac{1}{2}(f(w)_3 - f(v)_3)$. If $e = -1$ then $f(v)_3 = 0$, since $uvu^{-1} = v$, while if $e\mu = -1$ then $f(w)_3 = 0$, since $utw(ut)^{-1} = w$. In either case these linear equations impose no further constraint on ξ and may be solved for ρ. In this case any solution ξ, ρ determines a discrete cocompact embedding $F : \pi \to Isom(\mathbb{S}ol_0^4)$.

If $\nu \cong G_2$ we set $q = z$. Let $uzu^{-1} = mz^e$ and $v = v_0 z^{2r}$, where $m, v_0 \in I(\nu)$ and $e, r \in \mathbb{Z}$. Then $c = mz^{e-1}$, $u^2zu^{-2} = umu^{-1}mz$ and also $u^2zu^{-2} = vzv^{-1} = v_0zv_0^{-1} = v_0^2z = v^2z^{1-4r}$. Since $v = uvu^{-1} = uv_0u^{-1}z^{2er}$ we see that if $e = -1$ then $r = 0$. We also have $tzt^{-1} = z^\mu k$, where $\mu = \pm 1$ and $k \in I(\nu)$, and so $utz(ut)^{-1} = uz^\mu u^{-1}uku^{-1} = muk^{-1}u^{-1}z^{e\mu}$. Similarly, if $w = w_0z^{2s}$, where $w_0 \in I(\nu)$ and $s \in \mathbb{Z}$, then $utw(ut)^{-1} = w$, $w^2z^{1-4s} = ut(muk^{-1}u^{-1})(ut)^{-1}muk^{-1}u^{-1}z$, and $s = 0$ if $e\mu = -1$.

In this case $F(z) = \left(\begin{smallmatrix} J & e_3 \\ 0 & 1 \end{smallmatrix}\right)$ and $f(k) + 2\xi \in \mathbb{R}e_3$. The consistency conditions reduce to the equations

$$(J - I_3)\rho = (U - I_3)e_3 - f(c) = -f(m),$$
$$(U + I_3)\rho = f(v) \quad \text{and}$$
$$(UT + I_3)(U\xi + \rho) = f(w).$$

The two calculations of u^2zu^{-2} in the previous paragraph show that $(U + I_3)f(m) = f(v^2z^{-4r}) = (I_3 - J)f(v)$, while the calculations of $(ut)^2z(ut)^{-2}$ show that $(UT + I_3)(f(m) - Uf(k)) = f(w_0^2)$. Thus the consistency conditions reduce further to the four equations $\rho_1 = \frac{1}{2}f(c)_1$, $\rho_2 = \frac{1}{2}f(c)_2$, $(1+e)\rho_3 = f(v)_3 = 2r$ and $(1+e\mu)(\xi_3+\rho_3) = f(w)_3 = 2s$. If $e = \mu = +1$ we must have $\rho_3 = r$ and $\xi_3 = s-r$. If $e = -1$ or $e\mu = -1$ then $r = 0$ or $s = 0$ (respectively), and there is a one-parameter family of solutions.

If $\nu \cong B_1$ we set $q = y$. Suppose that $tyt^{-1} = mz^r y$ and $uyu^{-1} = nz^s y$, where $m, n \in \zeta\nu$ and $r, s \in \mathbb{Z}$. Then $uty(ut)^{-1} = umu^{-1}nz^{er+s}y$, $c = nz^s$ and $uy^2u^{-1} = n^2y^2$. Hence $y^2uy^{-2}u^{-1}c^2 = z^{2s}$. Let $v = v_0z^a$ and $w = w_0z^b$, where $v_0, w_0 \in \zeta\nu$ and $a, b \in \mathbb{Z}$. Calculating $vyv^{-1} = u^2yu^{-2} = ucyu^{-1}$ in two ways shows that $2a = (1+e)s$. Similarly, $2b = (1+e\mu)(er+s)$.

In this case $F(y) = \left(\begin{smallmatrix} -J & \beta \\ 0 & 1 \end{smallmatrix}\right)$, where $\beta = \frac{1}{2}f(y^2) + he_3$, and $\xi_3 = \frac{1}{2}(h(1-\mu)+r)$. (Here h, ξ_1, ξ_2 may be chosen freely.) The consistency conditions are

$$(J + I_3)\rho = (I_3 - U)\beta + f(c),$$
$$(U + I_3)\rho = f(v) \quad \text{and}$$
$$(UT + I_3)(U\xi + \rho) = f(w).$$

The equation $y^2uy^{-2}u^{-1}c^2 = z^{2s}$ implies that $(I_3 - U)f(y^2) + 2f(c) = 2se_3$. Hence the first two conditions reduce to the equations

$$2\rho_3 = (h(1-e)+s),$$
$$\rho_1 + \eta\rho_2 = f(v)_1$$
$$\eta^{-1}\rho_1 + \rho_2 = f(v)_2 \quad \text{and}$$
$$(e+1)\rho_3 = f(v)_3.$$

If $e = 1$ then $\rho_3 = \frac{1}{2}f(v)_3 = \frac{1}{2}s$, while if $e = -1$ then $f(v)_3 = 0$ and $\rho_3 = h + \frac{1}{2}s$. Thus the first and fourth equations have an unique solution for ρ_3. The third equation is equivalent to the second since $Uf(v) = f(uvu^{-1}) = f(v)$, and so $\eta f(v)_2 = f(v)_1$.

The third condition reduces to the equations

$$\lambda\xi_1 + \eta\xi_2 + \rho_1 + \eta\lambda\rho_2 = f(w)_1$$
$$\eta^{-1}\xi_1 + \kappa\xi_2 + \eta^{-1}\kappa\rho_1 + \rho_2 = f(w)_2 \quad \text{and}$$
$$(e\mu + 1)(e\xi_3 + \rho_3) = f(w)_3.$$

The second equation is equivalent to the first, since $\kappa\lambda = 1$ and $UTf(w) = f(w)$, so $\kappa\eta f(w)_2 = f(w)_1$. If $e\mu = 1$ then $f(w)_3 = er + s$, while $f(w)_3 = 0$ if $e\mu = -1$. Hence the third equation holds since $e(h(1-\mu)+r) + h(1-e) + s) = er + s$ if $e\mu = 1$, while both sides are 0 if $e\mu = -1$. Thus our consistency conditions are satisfied.

Suppose now that v is not in $\sqrt{\pi}$. On considering the possible holonomy groups we see that we may assume that $\langle\nu, u\rangle \cong G_4$, $\nu \cong G_2$

and $u^2 = z$. Since $\mathbb{R}^3$ is generated by the images of $I(\nu)$ and $\zeta\nu$ and conjugation by u preserves the subgroup $I(\nu)$ and fixes $\zeta\nu$ we must have $U = \left(\begin{smallmatrix} V & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $V^2 = -I_2$. Moreover $\rho = \frac{1}{2}e_3$, since $u^2 = z$, and $f(k) + 2\xi \in \mathbb{R}e_3$. Let $F(w) = \left(\begin{smallmatrix} W & \psi \\ 0 & 1 \end{smallmatrix}\right)$. Then we must have $W = (UT)^2$. Since $c = 1$ our consistency conditions are now

$$(W + I_3)\psi = f(w^2) \quad \text{and}$$

$$(UT + I_3)(U\xi + \frac{1}{2}e_3) = \psi.$$

Let $w = w_0 z^s$, where $w_0 \in I(\nu)$ and $s \in \mathbb{Z}$. Then $s = 0$ if $\mu = -1$, since $utw(ut)^{-1} = w$. If $w \in \sqrt{\pi}$ then s is even, and $W = I_3$ and $\psi = f(w)$. Since the equations $wzw^{-1} = w_0^2 z$ and $wzw^{-1} = ut(uz^\mu ku^{-1})(ut)^{-1} = (ut)uk^{-1}u^{-1}(ut)^{-1}uk^{-1}u^{-1}z$ imply that $(UT + I_3)Uf(k) + 2f(w) = 2se_3$, and since $s = 0$ if $\mu = -1$, our consistency conditions are satisfied.

If $w \notin \sqrt{\pi}$ then s is odd. Hence $w^2 = z^{2s}$, $W^2 = I_3$ and $f(w^2) = 2se_3$. In this case our conditions our satisfied if we set $\psi = se_3$ and $\xi = \xi_3 e_3$, where $(\mu + 1)(2\xi_3 + 1) = 2s$. (Once again we observe that $s = 0$ if $\mu = -1$.) □

Theorems 1 and 2, with the work of De Kimpe [5] provide another approach to the following result of Kemp.

Theorem 3. [9] *Let π be a torsion-free group with a normal subgroup K such that K and π/K are each virtually Z^2. Then π is the fundamental group of a Seifert fibred geometric 4-manifold of solvable Lie type.*

Proof. If $K \cong \pi_1(Kb)$ then $\sqrt{K} \cong Z^2$ and $Out(K)$ is finite. Hence $\sqrt{K}$ is centralized by a subgroup of finite index, and so $\sqrt{K} \cong Z^2 \leq \zeta\sqrt{\pi}$. Hence π is virtually abelian or virtually 2-step nilpotent, and so is the fundamental group of a flat 4-manifold or a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold, respectively [5].

If π is virtually nilpotent but $Z^2 \not\leq \zeta\sqrt{\pi}$ then π is the fundamental group of a $\mathbb{N}il^4$-manifold [5].

If π is not virtually nilpotent $h(\sqrt{\pi}) = 3$ and $K \cong Z^2$. Clearly $K < \sqrt{\pi}$, since $\sqrt{\pi}$ is the unique maximal locally-nilpotent normal subgroup. We may assume that K is maximal among abelian normal subgroups of rank 2, and so π/K has no nontrivial finite normal subgroup. The image of π/K in $Aut(K)$ is virtually cyclic, since π/K is solvable and $Aut(K) \cong GL(2, \mathbb{Z})$ is virtually free. It follows easily that $\sqrt{\pi} = C_\pi(K) \cong Z^3$.

If $\widetilde{K} \neq K$ were a second normal subgroup of π which is maximal abelian of rank 2 then $\widetilde{K} < \sqrt{\pi}$ also. But then π would have a chain of normal subgroups $K \cap \widetilde{K} < K < K\widetilde{K}$ of strictly increasing Hirsch length, and so would be virtually nilpotent. Thus K is the unique maximal normal subgroup of π isomorphic to Z^2.

Thus π has a characteristic subgroup ν such that $[\nu : \sqrt{\pi}] < \infty$ and $\pi/\nu \cong Z$ or D_∞. The result now follows from Theorems 1 and 2. $\square$

The observation that if the general fibre is Kb then the geometry must be $\mathbb{E}^4$ or $\mathbb{N}il^3 \times \mathbb{E}^1$ is due to Kemp [9]. His proof of this theorem follows Ue in using the fact that (if K is maximal) π/K is the fundamental group of a flat 2-orbifold. Hence π has a presentation lifting a standard presentation for π/K, and one may construct matrices realizing the standard generators and respecting the relations. (He treats the virtually nilpotent cases explicitly, as well as the $\mathbb{S}ol^3 \times \mathbb{E}^1$ cases.)

4. $\mathbb{S}ol_1^4$-manifolds

In this section we shall show that every torsion-free virtually poly-Z group π of Hirsch length 4 with $\sqrt{\pi} \cong \Gamma_q$ (for some $q \geq 1$) is the fundamental group of a closed $\mathbb{S}ol_1^4$-manifold. The model space for this geometry is the linear group

$$Sol_1^4 = \{\begin{pmatrix} 1 & y & z \\ 0 & t & x \\ 0 & 0 & 1 \end{pmatrix} \mid t > 0,\ x,\ y,\ z,\ \in \mathbb{R}\}.$$

The subgroup of such matrices with $t = 1$ is the group Nil^3. Let $\mathfrak{G}$ be the subgroup of $GL(3,\mathbb{R})$ generated by Sol_1^4 and the diagonal matrices $diag[-1,1,1]$ and $diag[1,1,-1]$, and let $P \in GL(3,\mathbb{R})$ be the permutation matrix which reverses the order of the standard basis of $\mathbb{R}^3$. Let $\Omega(g) = P(g^{tr})^{-1}P$, for all $g \in \mathfrak{G}$. Then $Isom(\mathbb{S}ol_1^4) \cong \mathfrak{G} \rtimes Z/2Z$, where the multiplication is given by $(g,i)(h,j) = (g\Omega^i(h), i+j)$ for $g, h \in \mathfrak{G}$ and $i, j \in Z/2Z$. (Note that while Sol_1^4 acts on itself by left multiplication the diagonal matrices act by conjugation.) For simplicity of notation we shall identify each $g \in \mathfrak{G}$ with $(g, 0) \in Isom(\mathbb{S}ol_1^4)$.

If M is a $\mathbb{S}ol_1^4$-manifold with fundamental group π then it is orientable if and only if $\pi/\sqrt{\pi}$ maps onto Z, by Corollary 8.7.1 of [8].

Theorem 4. *Let π be a torsion-free group with $\sqrt{\pi} \cong \Gamma_q$ and such that $\pi/\sqrt{\pi}$ maps onto Z with finite kernel. Then π is the fundamental group of an orientable closed $\mathbb{S}ol_1^4$-manifold.*

Proof. Let ν be the characteristic subgroup of π containing $\sqrt{\pi}$ and such that $\pi/\nu \cong Z$, and let $t \in \pi$ represent a generator of π/ν. Then $\pi \cong \nu \rtimes_\tau Z$, where τ is the automorphism of ν determined by conjugation by t. Let A be the induced automorphism of $\sqrt{\pi}/\zeta\sqrt{\pi} \cong Z^2$. Since

π is not virtually nilpotent the eigenvalues α,β of A are distinct and not ± 1. Moreover $\nu=\sqrt{\pi}$ or $[\nu:\sqrt{\pi}]=2$ and $\nu/\zeta\sqrt{\pi}\cong Z^2\rtimes_{-I}(Z/2Z)$. (See Chapter 8.§7 of [8].)

Suppose first that $\nu=\sqrt{\pi}$. Then $\tau(x)=x^ay^bz^m$ and $\tau(y)=x^cy^dz^n$, for some $a,\dots,n\in\mathbb{Z}$, and $\tau(z)=z^{\det A}=z^{\alpha\beta}$. Let $e,f\in R^2$ be the eigenvectors of $A=\left(\begin{smallmatrix}a&c\\b&d\end{smallmatrix}\right)$ corresponding to α and β, respectively. Let $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)=x_1e+x_2f$, $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)=y_1e+y_2f$, and let $X=\left(\begin{smallmatrix}x_1&y_1\\x_2&y_2\end{smallmatrix}\right)$ be the change-of-basis matrix. Then $XA=\left(\begin{smallmatrix}\alpha&0\\0&\beta\end{smallmatrix}\right)X$. In particular, $\alpha x_1=ax_1+by_1$, $\beta x_2=ax_2+by_2$, $\alpha y_1=cx_1+dy_1$ and $\beta y_2=cx_2+dy_2$. Let $[x,y]=\frac{1}{q}(x_2y_1-x_1y_2)$. If $v=v_1e+v_2f$ let

$$vN=\begin{pmatrix}0&v_2&0\\0&0&v_1\\0&0&0\end{pmatrix}\quad\text{and}\quad N^2=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

Then $(vN)^2=v_1v_2N^2$ and $(vN)^3=0$. Hence the exponential series gives $e^{vN}=I_3+vN+\frac{1}{2}(vN)^2$. An easy calculation shows that $(e^{vN})^n=e^{nvN}$ for all $n\in\mathbb{Z}$. We may define a homomorphism $F:\nu\to Nil^3$ by setting $F(x)=e^{xN}$, $F(y)=e^{yN}$ and $F(z)=e^{[x,y]N^2}$. Let $F(t)=\frac{\alpha}{|\alpha|}T$, where

$$T=\begin{pmatrix}\alpha\beta&t_2&t_3\\0&\alpha&t_1\\0&0&1\end{pmatrix}.$$

(The sign term $\frac{\alpha}{|\alpha|}$ here is to ensure that $F(t)\in\mathfrak{G}$, and plays no role in the computations. Similarly in Theorem 6.) Then $F(t)F(z)=F(z)^{\alpha\beta}F(t)$ and so F extends to a homomorphism from π to $Isom(\mathbb{S}ol_1^4)$ provided that

$$F(t)F(x)=F(x)^aF(y)^bF(z)^mF(t)\quad\text{and}$$

$$F(t)F(y)=F(x)^cF(y)^dF(z)^nF(t).$$

These equations have unique solutions for t_1 and t_2:

$$t_1=\frac{1}{2q\beta}((2n+cdq)x_1-(2m+abq)y_1)\quad\text{and}$$

$$t_2=\frac{1}{2q}((2n+cdq)x_2-(2m+abq)y_2).$$

(The entry t_3 may be chosen freely.) It is easily seen that the corresponding homomorphism F is a discrete cocompact embedding.

If $\nu\neq\sqrt{\pi}$ let $w\in\nu$ represent a generator of $\nu/\sqrt{\pi}$. Sine $\nu/\zeta\sqrt{\pi}\cong Z^2\rtimes_{-I}(Z/2Z)$ we may assume that $wxw^{-1}=x^{-1}z^i$ and $wyw^{-1}=$

$y^{-1}z^j$ for some $i,j\in\mathbb{Z}$, and so $w^2\in\zeta\sqrt{\pi}$. Hence $w^2=z^k$, for some odd $k\in\mathbb{Z}$, since ν is torsion-free. Thus ν has a presentation

$$\langle w,x,y,z\mid wxw^{-1}=x^{-1}z^i, wyw^{-1}=y^{-1}z^j, xy=z^qyx, w^2=z^k\rangle.$$

Then $\tau(x)=x^ay^bz^m$, $\tau(y)=x^cy^dz^n$ and $\tau(w)=x^ry^sz^pw$, for some $a,\dots,s\in\mathbb{Z}$. These formulae define an automorphism of ν if and only if

$$2m=q(rb-as-ab)-ai-bj+i\det A,$$
$$2n=q(rd-cs-cd)-ci-dj+j\det A\quad\text{and}$$
$$qrs+2p+ri+sj=k(\det A-1).$$

Define $F(x)$, $F(y)$ and $F(t)$ as before, and let

$$F(w)=\begin{pmatrix}-1&w_2&w_3\\0&1&w_1\\0&0&-1\end{pmatrix},$$

where $w_1=\frac{1}{q}(jx_1-iy_1)$, $w_2=\frac{1}{q}(-jx_2+iy_2)$ and $w_3=\frac{1}{2}(w_1w_2-k[x,y])$. Then $F(w)F(x)=F(x)^{-1}F(z)^iF(w)$, $F(w)F(y)=F(y)^{-1}F(z)^jF(w)$ and $F(w)^2=F(z)^k$. The condition

$$F(t)F(w)=F(x)^rF(y)^sF(z)^pF(w)F(t)$$

reduces to the equations

$$2t_1=rx_1+sy_1+(\alpha-1)w_1,$$
$$2t_2=\alpha(rx_2+sy_2+(1-\beta)w_2)\quad\text{and}$$
$$(\alpha\beta-1)w_3=t_1w_2-t_2w_1+(t_1+w_1-\frac{rx_1+sy_1}{2})(rx_2+sy_2)-\frac{qrs+2p}{2}[x,y].$$

On solving $\alpha x_1=ax_1+by_1$ for y_1 in terms of x_1 and remembering that $\alpha+\beta=a+d$ and $\alpha\beta=\det A$ we see that the first equation is satisfied. Similarly for the second equation. The third equation follows from the first since $qrs+2p+ri+sj=k(\det A-1)$. Thus F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol_1^4)$. □

We could arrange that $i=j=0$ and $k=1$ in Theorem 4 by replacing x, y and w by xz^{-i}, yz^{-j} and $wz^{\frac{1-k}{2}}$, respectively. However the version given is more convenient for use in Theorem 6 below, which treats the case when $\pi/\sqrt{\pi}$ is a finite extension of D_∞. Here the verification that our constructions work needs more effort, and we shall defer the details of our calculations to an appendix.

Lemma 5. *Let ν be a torsion-free group with $\sqrt{\nu}\cong\Gamma_q$ and suppose that ν is generated by $\sqrt{\nu}$ and an element u such that $u^2\in\sqrt{\nu}$ and $ugu^{-1}=g^{-1}$ for $g\in\zeta\sqrt{\nu}$. Then q is even.*

Proof. Let z generate $\zeta\sqrt{\nu}$, and let U be the automorphism of $\sqrt{\nu}/\zeta\sqrt{\nu}$ induced by conjugation by u. Then $\det(U) = -1$, since $uzu^{-1} = z^{-1}$, and $u^2 \notin \zeta\nu$, since $u(u^2)u^{-1} = u^2$. Therefore U has eigenvalues $\{1, -1\}$, and so may be diagonalized (over $\mathbb{Z}$). Thus we may assume $\sqrt{\nu}$ has generators x, y and z such that $uxu^{-1} = xz^i$ and $uyu^{-1} = y^{-1}z^j$. Let $u^2 = x^p y^q z^r$. Then $q = 0$, since $u(u^2)u^{-1} = u^2$. On replacing u by ux^{-P} we see that we may vary p by any even number. In particular, we may assume that $p = 1$, since $u^2 \notin \zeta\nu$. We may now replace x by u^2z^{-r}, so that $u^2 = x$. Hence $xyx^{-1} = u(y^{-1}z^j)u^{-1} = yz^{-2j}$, and so q must be even. □

Theorem 6. *Let π be a torsion-free group with $\sqrt{\pi} \cong \Gamma_q$ and such that $\pi/\sqrt{\pi}$ maps onto D_∞ with finite kernel. Then π is the fundamental group of a nonorientable closed $\mathbb{S}ol_1^4$-manifold.*

Proof. Let ν be the characteristic subgroup of π containing $\sqrt{\pi}$ and such that $\pi/\nu \cong D_\infty$, and let $u, v \in \pi$ represent a pair of generators of order 2 for π/ν. Then u^2 and v^2 are elements of ν, and are non-trivial since π is torsion-free. Let $\tilde{\pi}$ be the subgroup of index 2 in π generated by ν and $t = uv$. Let U, V and T be the automorphisms of $\sqrt{\pi}/\zeta\sqrt{\pi} \cong Z^2$ induced by conjugation by u, v and t, respectively. Since π is not virtually nilpotent $T = UV$ has infinite order and distinct real eigenvalues α and β which are not ± 1.

Suppose first that $\nu = \sqrt{\pi}$. Then $U^2 = V^2 = I_2$. Since $T = UV$ has infinite order neither of U or V is $\pm I_2$. Therefore $\det U = \det V = -1$, and so $\alpha\beta = \det T = +1$. As in the lemma, we may assume the generators of ν so chosen that $u^2 = x$ and $uyu^{-1} = y^{-1}z^{-\frac{q}{2}}$. We shall assume also that $txt^{-1} = x^a y^b z^m$ and $tyt^{-1} = x^c y^d z^n$, for some $a, \dots, n \in \mathbb{Z}$, as in Theorem 4. (Note that $abcd \neq 0$, since π is not virtually nilpotent.)

Let $e \in \mathbb{R}^2 = \mathbb{R} \otimes_{\mathbb{Z}} (\sqrt{\pi}/\zeta\sqrt{\pi})$ be an eigenvector of T corresponding to α. Then $TUe = \alpha^{-1}TUTe = \beta Ue$. Thus $f = Ue$ is an eigenvector of T corresponding to β. Let $[x] = x_1e + x_2f$ and $[y] = y_1e + y_2f$ be the images of x and y in $\mathbb{R}^2$. Since $U[x] = [x]$ and $U[y] = -[y]$ are eigenvectors of U, we have $x_2 = x_1$ and $y_2 = -y_1$. Hence all these terms are nonzero.

Let $F : \tilde{\pi} \to Isom(\mathbb{S}ol_1^4)$ be the embedding given in Theorem 4, and let $F(u) = (K, \Omega)$, where

$$K = \begin{pmatrix} -1 & \frac{1}{2}x_1 & 0 \\ 0 & 1 & -\frac{1}{2}x_1 \\ 0 & 0 & -1 \end{pmatrix}.$$

It follows easily from the equations $x_2 = x_1$ and $y_2 = -y_1$ that $x_1y_2 = -x_2y_1 = -\frac{q}{2}[x,y]$, and hence that $F(u)^2 = F(x) = F(u^2)$, $F(u)F(y)F(u)^{-1} = F(uyu^{-1})$ and $F(u)F(z)F(u)^{-1} = F(uzu^{-1})$. Since $t = uv$ we must set $F(v) = F(u)^{-1}F(t) = F(x)^{-1}F(u)F(t)$.

Let $v^2 = x^g y^k z^p$. Since $v = u^{-1}t$ we have $vxv^{-1} = x^a y^{-b} z^{\frac{bq}{2}-m}$, $vyv^{-1} = x^c y^{-d} z^{\frac{dq}{2}-n}$ and $vzv^{-1} = z^{-1}$. On expanding out the left-hand side of the equation $vx^g y^k z^p v^{-1} = x^g y^k z^p$ we see that

$$(a-1)g + ck = bg + (d+1)k = 0 \quad \text{and}$$

$$2p = \frac{q}{2}(g(g-1)ab + k(k-1)cd + 2bcgk + bg + dk) - gm - kn.$$

The latter equation simplifies further to

$$p = \frac{qk}{4}(a^2 + a - ac - 2g - 1) - \frac{1}{2}(gm + kn).$$

The equation $v(vxv^{-1})v^{-1} = v^2xv^{-2} = y^kxy^{-k} = xz^{-qk}$ gives

$$k = \frac{1}{2}(ab(a-c-1) + b) + \frac{1}{q}((a-1)m - bn).$$

We see also that $a = d$ and $a^2 - 1 = bc$. Similarly, the equation $v(vyv^{-1})v^{-1} = x^g yx^{-g} = yz^{qg}$ gives

$$g = \frac{1}{2}(ac(a-b+1) - a - 1) + \frac{1}{q}(-cm + (a+1)n).$$

(The last two equations are equivalent, since $bc = (a+1)(a-1)$ and $ck = (1-a)g$.) In the appendix we shall verify that

$$F(v)^2 = F(v^2) = F(x)^g F(y)^k F(z)^p.$$

Then F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol_1^4)$.

If $\nu \neq \sqrt{\pi}$ then ν is generated by x, y and w, with $wxw^{-1} = x^{-1}z^i$, $wyw^{-1} = y^{-1}z^j$ and $w^2 = z^k$, for some $i, j, k \in \mathbb{Z}$ with k odd. Suppose first that $u^2 \in \sqrt{\pi}$. Then we may assume that $u^2 = x$ and $uyu^{-1} = y^{-1}z^{-\frac{q}{2}}$, by the first part of this theorem (applied to the subgroup of index 2 generated by $\sqrt{\pi}$, u and t). Since $uwu^{-1}w^{-1} \in \sqrt{\pi}$ we have $uwu^{-1}w^{-1} = x^\phi y^\psi z^\rho$, for some $\phi, \psi, \rho \in \mathbb{Z}$. Squaring both sides of the equation $wuw^{-1} = z^{-\rho}y^{-\psi}x^{-\phi}u$ gives

$$\phi = 1.$$

Similarly, squaring both sides of the equation $uwu^{-1} = xy^\psi z^\rho w$ gives

$$i + (j+q)\psi + 2\rho = -2.$$

Conjugating x by $uwu^{-1}w^{-1}$ gives $y^\psi xy^{-\psi} = xz^{2i}$, and so

$$2i = -q\psi.$$

(Conjugating y and z by $uwu^{-1}w^{-1}$ gives no further constraints.)

We again let $F : \tilde{\pi} \to Isom(\mathbb{S}ol_1^4)$ be the embedding given in Theorem 4, and let $F(u) = (K, \Omega)$, where K is as above. In the appendix we shall verify that

$$F(u)F(w) = F(x)F(y)^{\psi}F(z)^{\rho}F(w)F(u).$$

Then F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol_1^4)$.

If $u^2 \notin \sqrt{\pi}$ then $\langle \nu, u\rangle/\sqrt{\pi} \cong Z/4Z$, and we may assume that $w = u^2$, $uxu^{-1} = y$ and $uyu^{-1} = x^{-1}z^i$, for some $i \in \mathbb{Z}$. Hence $u^2xu^{-2} = x^{-1}z^i$ and $u^2yu^{-2} = y^{-1}z^i$, so $U^2 = -I_2$. We may also assume that $u^4 = z^k$ and $tu^2t^{-1} = x^ry^sz^pu^2$, for some $k, p, r, s \in \mathbb{Z}$ with k odd. Since $C = utut$ is in ν we have $C = x^{\phi}y^{\psi}z^{\rho}u^{2\eta}$, for some $\phi, \psi, \rho \in \mathbb{Z}$ and $\eta = 0$ or 1. Hence $UTUT = (-I_2)^{\eta}$. On computing the left-hand side and recalling that $abcd \neq 0$, we see that $b = c$ and $ad - b^2 = (-1)^{\eta+1}$. If $Te = \alpha e$ then $TUe = (-1)^{\eta+1}UT^{-1}e = \beta Ue$, and we again choose the eigenvectors e, f so that $f = Ue$. It then follows that $y_1 = -x_2$ and $y_2 = x_1$. The exponents $a, b, d, i, m, n, p, r, s$ satisfy the constraints of Theorem 4 for $w = u^2$. Conjugation of x and y by C gives equations $\phi = \frac{1}{2}(-dr + (b-1)s)$ and $\psi = \frac{1}{2}((b+1)r - as)$. We shall defer consideration of other constraints on the exponents to the appendix.

Let $F : \tilde{\pi} \to Isom(\mathbb{S}ol_1^4)$ be the embedding given in Theorem 4 and let $F(u) = (L, \Omega)$, where

$$L = \begin{pmatrix} -1 & \frac{i}{q}x_1 & \frac{1}{4q^2}((i^2 - kq)x_1^2 - (i^2 + kq)x_2^2) \\ 0 & 1 & -\frac{i}{q}x_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $F(u)^2 = F(u^2)$ and $F(u)^4 = F(z)^k$. It is also easily seen that $F(y)F(u) = F(u)F(x)$. (This uses the fact that $y_1 = -x_2$ and $y_2 = x_1$.) Since $\langle \nu, u\rangle$ is generated by u, x and z it follows that $F(ugu^{-1}) = F(u)F(g)F(u)^{-1}$ for all $g \in \nu$. In the appendix we shall verify that

$$F(u)F(t)F(u)F(t) = F(utut) = F(x)^{\phi}F(y)^{\psi}F(z)^{\rho}F(u)^{2\eta}.$$

Then F determines a discrete cocompact embedding of π in $Isom(\mathbb{S}ol_1^4)$. □

5. Orbifold bundles and diffeomorphisms

An *infrasolvmanifold* is a quotient $M = \Gamma\backslash S$ where S is a 1-connected solvable Lie group and Γ is a closed torsion free subgroup of the semidirect product $Aff(S) = S \rtimes_{\alpha} Aut(S)$ such that Γ_o (the component of the identity of Γ) is contained in the *nilradical* of S (the maximal connected nilpotent normal subgroup of S), $\Gamma/\Gamma \cap S$ has compact closure in $Aut(S)$ and M is compact. The pair (S, Γ) is called a presentation for M, and is discrete if Γ is a discrete subgroup of $Aff(S)$, in which case $\pi_1(M) = \Gamma$. Every infrasolvmanifold has a presentation such that $\Gamma/\Gamma \cap S$ is finite [6], but Γ need not be discrete, and S is not determined by π. (For example, Z^3 is a lattice in both R^3 and $\widetilde{E(2)}^+ = \mathbb{C} \rtimes_{\tilde{\alpha}} \mathbb{R}$, where $\tilde{\alpha}(t)(z) = e^{2\pi i t} z$ for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$. However if M is an infranilmanifold, with $S = N$ solvable and Γ discrete then N is determined by Γ.)

Geometric 4-manifolds of solvable Lie type are infrasolvmanifolds, and infrasolvmanifolds are the total spaces of orbifold bundles with infranilmanifold fibre and flat base. (See Chapter 7 of [8].) Baues showed that in all dimensions infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups [2]. In dimensions ≤ 3 this follows from standard results of low dimensional topology. We shall show that related arguments also cover most 4-dimensional orbifold bundle spaces. The following theorem extends the main result of [4] (in which it was assumed that π is not virtually nilpotent).

Theorem 7. *Let M and M' be 4-manifolds which are total spaces of orbifold bundles $p : M \to B$ and $p' : M' \to B'$ with fibres infranilmanifolds F and F' (respectively) and bases flat orbifolds, and suppose that $\pi_1(M) \cong \pi_1(M') \cong \pi$. If π is virtually abelian and $\beta_1(\pi) = 1$ assume that π is orientable. Then M and M' are diffeomorphic.*

Proof. We may assume that $d = dim(B) \leq d' = dim(B')$. Suppose first that π is not virtually abelian or virtually nilpotent of class 2. Then all subgroups of finite index in π have $\beta_1 \leq 2$, and so $1 \leq d \leq d' \leq 2$. Moreover π has a characteristic nilpotent subgroup $\tilde{\nu}$ such that $h(\pi/\tilde{\nu}) = 1$, by Theorems 1.5 and 1.6 of [8]. Let ν be the preimage in π of the maximal finite normal subgroup of $\pi/\tilde{\nu}$. Then ν is a characteristic virtually nilpotent subgroup (with $\sqrt{\nu} = \tilde{\nu}$) and $\pi/\nu \cong Z$ or D. If $d = 1$ then $\pi_1(F) = \nu$ and $p : M \to B$ induces this isomorphism. If $d = 2$ the image of ν in $\pi_1^{orb}(B)$ is normal. Hence there is an orbifold map q from B to the circle S^1 or the reflector interval $\mathbb{I}$ such that qp is an orbifold bundle projection. A similar analysis applies to M'. In either case, M and M' are canonically mapping tori or unions of two twisted I-bundles, and the theorem follows via standard 3-manifold theory.

If π is virtually nilpotent it is realized by an infranilmanifold M_0 [5]. Hence we may assume that $M' = M_0$, $d' = 4$, $h(\sqrt{\pi}) = 4$ and $\sqrt{\pi}' \cong Z$ or 1. If $d = 0$ or 4 then M is also an infranilmanifold and the result is clear. If there is an orbifold bundle projection from B to S^1 or $\mathbb{I}$ then M is a mapping torus or a union of twisted I-bundles, and π is a semidirect product $\kappa \rtimes Z$ or a generalized free product with amalgamation $G *_J H$ where $[G : J] = [H : J] = 2$. The model M_0 then has a corresponding structure as a mapping torus or a union of twisted I-bundles, and we may argue as before.

If $\beta_1(\pi) + d > 4$ then $\pi_1^{orb}(B)$ maps onto Z, and so B is an orbifold bundle over S^1. Hence if $d = 1$ or $\beta_1(\pi) + d > 4$ the above argument applies.

If there is no such orbifold bundle projection we may assume that $d = 2$ or 3 and that $\beta_1(\pi) \leq 4 - d$. (If moreover $\beta_1(\pi) = 4-d$ and there is no such projection then $\pi' \cap \pi_1(F) = 1$ and so π is virtually abelian.) If $d = 2$ then M is Seifert fibred. Since M' is an infranilmanifold (and π cannot be one of the three exceptional flat 4-manifold groups $G_6 \rtimes_\theta Z$ with $\theta = j$, cej or $abcej$) it is also Seifert fibred, and so M and M' are diffeomorphic, by [12].

If $d = 3$ then $\pi_1(F) \cong Z$. The group π has a normal subgroup K such that $\pi/K \cong Z$ or D, by Lemma 3.14. If $\pi_1(F) < K$ then $\pi_1^{orb}(B)$ maps onto Z or D and we may argue as before. Otherwise $\pi_1(F) \cap K = 1$, since Z and D have no nontrivial finite normal subgroups, and so π is virtually abelian. If $\beta_1(\pi) = 1$ then $\pi_1(F) \cap \pi' = 1$ (since π/K does not map onto Z) and so $\pi_1(F)$ is central in π. It follows that p is the orbit map of an S^1-action on M. Once again, the model M_0 has an S^1-action inducing the same orbifold fundamental group sequence. Orientable 4-manifolds with S^1-action are determined up to diffeomorphism by the orbifold data and an Euler class corresponding to the central extension of $\pi_1^{orb}(B)$ by Z [7]. Thus M and M' are diffeomorphic. It is not difficult to determine the maximal infinite cyclic normal subgroups of the flat 4-manifold groups π with $\beta_1(\pi) = 0$, and to verify that in each case the quotient maps onto D. □

It is highly probable that the arguments of Fintushel can be extended to all 4-manifolds which admit smooth S^1-actions, and Theorem 7 is surely true without any restrictions on π. (Note that the algebraic argument of the final sentence of this theorem does not work for nine of the 30 nonorientable flat 4-manifold groups π with $\beta_1(\pi) = 1$.)

Since all such groups are realized geometrically, every smooth 4-manifold admitting such an orbifold fibration is diffeomorphic to a geometric 4-manifold of solvable Lie type.

Theorem 8. *Let M be a closed 4-dimensional infrasolvmanifold. Then M is diffeomorphic to a geometric 4-manifold of solvable Lie type.*

Proof. Let $\pi = \pi_1(M)$. Then π is a torsion-free virtually poly-Z group of Hirsch length 4. If π is virtually nilpotent then it is the fundamental group of a $\mathbb{E}^4$-, $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{N}il^4$-manifold [5]. Otherwise π is the fundamental group of a $\mathbb{S}ol^4_{m,n}$-, $\mathbb{S}ol^4_1$-manifold, by Theorems 1,2, 4 and 6. Since all such manifolds are infrasolvmanifolds (cf. Chapter 7 of [8]) the result follows from Theorem 7 (unless π is virtually abelian, $\beta_1(\pi) = 1$ and $w_1(\pi) \neq 0$) or from [2] (in general). □

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Appendix: the calculations for Theorem 6

Here we shall outline some of the calculations which support the claims that the functions defined on generators of π in Theorem 6 extend to homomorphisms from π to $Isom(\mathbb{S}ol_1^4)$.

5.1. **The case $\nu = \sqrt{\pi}$:** We assume that $txt^{-1} = x^a y^b z^m$, $tyt^{-1} = x^c y^d z^n$ and $v^2 = x^g y^k z^p$. The matrices of U and T with respect to the basis $\{[x],[y]\}$ of $\sqrt{\pi}/\zeta\sqrt{\pi}$ are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, respectively. Here $a = d$ and $a^2 - 1 = bc$. Since e and f are eigenvectors of T we have also $cx_1 = (\alpha - a)y_1 = (a - \beta)y_1$ and $cx_2 = (\beta - a)y_2 = (a - \alpha)y_2$. As observed in the proof of Theorem 6, $x_1 = x_2$ and $y_1 = -y_2$, so all of these terms are nonzero. Moreover,

$$g = \frac{1}{2}(ac(a-b+1) - a - 1) + \frac{1}{q}(-cm + (a+1)n),$$

$$k = \frac{1}{2}(ab(a-c-1) + b) + \frac{1}{q}((a-1)m - bn) \quad \text{and}$$

$$p = \frac{qk}{4}(a^2 + a - ac - 2g - 1) - \frac{1}{2}(gm + kn).$$

Let $F(v) = F(u)^{-1}F(t) = F(x)^{-1}F(u)F(t) = (P, \Omega)$, where

$$P = \begin{pmatrix} 1 & -x_2 & \frac{1}{2}x_1x_2 \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & \frac{1}{2}x_1 & 0 \\ 0 & 1 & -\frac{1}{2}x_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha^{-1}t_1 & \frac{t_1t_2 - t_3}{\alpha} \\ 0 & \alpha^{-1} & -\alpha^{-1}t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \alpha^{-1}(t_1 - \frac{1}{2}x_1) & D \\ 0 & (\alpha & \frac{1}{2}x_1 - \alpha^{-1}t_2 \\ 0 & 0 & -1 \end{pmatrix},$$

where $D = t_3 + \frac{1}{2}x_1x_2 + \alpha^{-1}(\frac{1}{2}x_1t_2 - t_1t_2)$. Therefore $P\Omega(P) =$

$$\begin{pmatrix} -1 & \alpha^{-1}(t_1 - \frac{1}{2}x_1) & D \\ 0 & \alpha^{-1} & \frac{1}{2}x_1 - \alpha^{-1}t_2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & \frac{1}{2}\alpha x_1 - t_2 & D^* \\ 0 & \alpha & t_1 - \frac{1}{2}x_1 \\ 0 & 0 & -1 \end{pmatrix},$$

where $D^* = \alpha^{-1}(t_1 - \frac{1}{2}x_1)(\frac{\alpha}{2}x_1 - t_2) - D$. Hence $P\Omega(P) =$

$$\begin{pmatrix} 1 & t_2 + t_1 - \frac{1}{2}(\alpha+1)x_1 & \alpha^{-1}(t_1 - \frac{1}{2}x_1)(t_1 + t_2 - \frac{\alpha+1}{2}x_1) \\ 0 & 1 & \alpha^{-1}(t_2 + t_1 - \frac{1}{2}(\alpha+1)x_1) \\ 0 & 0 & 1 \end{pmatrix}.$$

It remains to check that $P\Omega(P) = F(x)^g F(y)^k F(z)^p$. This condition reduces to three equations

$$gx_2 + ky_2 = t_2 + t_1 - \frac{1}{2}(\alpha+1)x_1,$$

$$gx_1 + ky_1 = \alpha^{-1}(t_2 + t_1 - \frac{1}{2}(\alpha+1)x_1) \quad \text{and}$$

$$\frac{1}{2}(a^2x_1x_2+2abx_2y_1+b^2y_1y_2)+p[x,y] = \alpha^{-1}(t_1-\frac{1}{2}x_1)(t_1+t_2-\frac{\alpha+1}{2}x_1).$$

On multiplying the first of these equations through by $2q$, expressing t_1 and t_2 in terms of $x_1, \dots, y_2$ and using the facts that $\alpha\beta = 1$, $\alpha+\beta = 2a$, $x_1 = x_2$ and $y_1 = -y_2$ we obtain the equation

$$2q(gx_2 + ky_2) = (1+\alpha)(2n + cdq - q)x_2 + (\alpha - 1)(2m + abq)y_2).$$

Multiply through by c and write $cx_2 = (a - \alpha)y_2$. Now use the fact that $ck = (1 - a)g$, and divide by y_2, and this reduces to the above equation for g.

The second equation is equivalent to the first, since $[v^2] = g[x]+k[y]$ is fixed by UT, and so $\alpha(gx_1 + ky_1) = gx_2 + ky_2$. The third equation follows after similar substitutions and reductions.

5.2. **The case $\nu \neq \sqrt{\pi}$ but $u^2 \in \sqrt{\pi}$:** A similar calculation shows that the condition $F(u)F(w) = F(x)F(y)^{\psi}F(z)^{\rho}F(w)F(u)$ reduces to the equations

$$\frac{1}{2}x_1 - w_1 = -\frac{1}{2}x_1 + (w_2 + x_2 + \psi y_2),$$

$$\frac{1}{2}x_1 + w_2 = x_1 + \psi y_1 - \frac{1}{2}x_1 - w_1 \quad \text{and}$$

$$2w_3 - w_1w_2 + x_1w_2 = \rho[x,y] + \frac{1}{2}(x_1x_2 + 2\psi x_2y_1 + \psi^2 y_1y_2)$$
$$-(x_2 + \psi y_2)(\frac{1}{2}x_1 + w_1).$$

We see that these equations hold on setting $w_1 = \frac{1}{q}(jx_1 - iy_1)$, $w_2 = \frac{1}{q}(-jx_2 + iy_2)$, $w_3 = \frac{1}{2}(w_1w_2 - [x,y])$, $x_2 = x_1$ and $y_2 = -y_1$, and using the conditions on the exponents $a, \dots, s, \psi, \rho$ given in Theorem 4.

5.3. **The case $u^2 \notin \sqrt{\pi}$:** This is the most computationally tedious case. We may assume that $uxu^{-1} = y$, $uyu^{-1} = x^{-1}z^i$ and $utut = x^{\phi}y^{\psi}z^{\rho}u^{2\eta}$, for some $i, \phi, \psi, \rho \in \mathbb{Z}$ and $\eta = 0$ or 1, and hence that $u^4 = z^k$, for some odd $k \in \mathbb{Z}$. We may also assume that $tu^2t^{-1} = x^r y^s z^p u^2$, for some $p, r, s \in \mathbb{Z}$. The constraints of Theorem 4 for the exponents $a, b, d, i, m, n, p, r, s$ become

$$2m = q(rb - as - ab) + i(2\eta - a - b - 1),$$

$$2n = q(rd - bs - bd) + i(2\eta - b - d - 1) \quad \text{and}$$

$$qrs + 2p + i(r + s) = 2k(\eta - 1),$$

since $c=b$ and $j=i$. The action of $C=utut$ by conjugation on x and y gives

$$\phi=\frac{1}{2}(-dr+(b-1)s) \quad \text{and}$$
$$\psi=\frac{1}{2}((b+1)r-as).$$

The equation $utC=Cut$ gives the further equation: $\rho=$

$$\det A\rho+\phi(m+bi)+\psi(n+di)+q(ab\binom{\phi}{2}+cd\binom{\psi}{2}+ab\phi^2+ad\phi\psi+bd\psi^2).$$

If $\det A=-1$ this determines ρ in terms of the other exponents; otherwise it gives no constraint on ρ. Note also that $\det A=ad-b^2=(-1)^{\eta+1}=2\eta-1$.

We have $F(u)F(t)F(u)F(t)=L\Omega(F(t))\Omega(L)F(t)=$

$$\begin{pmatrix} -\alpha\beta & t_1-t_2+\frac{i}{q}(x_1-\alpha x_2) & \Delta \\ 0 & 1 & \alpha^{-1}(t_1+\frac{i}{q}x_1)+\beta(t_2+\frac{i}{q}\alpha x_2) \\ 0 & 0 & -\alpha\beta \end{pmatrix},$$

where $\Delta=$

$$\alpha^{-1}(t_1+\frac{i}{q}x_1)^2-2\eta(t_3+\frac{i^2-kq}{4q^2}x_1^2-\frac{i^2+kq}{4q^2}x_2^2)+(\beta t_2-\frac{i}{q}x_2)(t_1+\frac{i}{q}x_1).$$

The condition $F(u)F(t)F(u)F(t)=F(x)^\phi F(y)^\psi F(z)^\rho F(u)^{2\eta}$ reduces to the equations

$$q(t_1-t_2)+i(x_1-\alpha x_2)=q(\phi x_2+\psi y_2)+\eta i(x_1-x_2)$$
$$\alpha^{-1}(qt_1+ix_1)+\beta(qt_2+i\alpha x_2)=q(\phi x_1+\psi y_1)+\eta i(x_1+x_2) \quad \text{and}$$
$$\eta(w_3+w_1(\phi x_2+\psi y_2))+(-1)^\eta(\rho[x,y]+\phi\psi x_2y_1+\frac{1}{2}(\phi^2x_1x_2+\psi^2y_1y_2))=\Delta,$$

where $w_1=\frac{i}{q}(x_1+x_2)$, $w_2=\frac{i}{q}(x_1-x_2)$ and $w_3=\frac{1}{2}(w_1w_2-k[x,y])$. The first two equations may be verified on using the equations $y_1=-x_2$ and $y_2=x_1$ and the above constraints on the exponents. (Note also that $\alpha x_1=ax_1-bx_2$, etc.) If $\eta=1$ then we may choose t_3 to satisfy the third equation. If $\eta=0$ then we may substitute for ρ and then show that everything cancels.

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