MORITA EQUIVALENCES OF CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(r,p,n)$

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Abstract. We prove a Morita reduction theorem for the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}(q,\mathbf{Q})$ of type $G(r,p,n)$. As a consequence, we show that computing the decomposition numbers of $\mathscr{H}_{r,p,n}(\mathbf{Q})$ reduces to computing the p'-*splittable decomposition numbers* (see Definition 1.1) of the cyclotomic Hecke algebras $\mathscr{H}_{r',p',n'}(\mathbf{Q}')$, where $1 \le r' \le r$, $1 \le n' \le n$, $p' \mid p$ and where the parameters $\mathbf{Q}'$ are contained in a single (ϵ,q)-orbit and ϵ is a primitive p'th root of unity.

1. Introduction

Motivated by "generic features" of the representation theory of finite reductive groups Broué and Malle [**3**] attached a *cyclotomic Hecke algebra* to each complex reflection group. These algebras have many good properties and, conjecturally, they arise as the endomorphism algebras of Deligne–Lusztig representations.

This paper is concerned with the cyclotomic Hecke algebras of type $G(r,p,n)$ with $p > 1$. These algebras were first considered by Broué and Malle [**3**] and by Ariki [**1**] in the semisimple case. These algebras have been studied extensively in the non–semisimple case, notably by the first author [**9–12**] and by Genet and Jacon [**19**]. In particular, the simple modules of these algebras have been classified over any field of characteristic coprime to p.

In the case $p = 1$ the cyclotomic Hecke algebras of type $G(r,1,n)$ are known as the Ariki–Koike algebras. These algebras are much better understood; see [**17**] and the references therein. The highlight of this theory is Ariki's celebrated theorem which says that the decomposition numbers of these algebras in characteristic zero can be computed using the canonical bases of the higher level Fock spaces of the quantized affine special linear groups. Another fundamental result for the Ariki–Koike algebras is the Morita equivalence theorem of Dipper and the second author [**6**] which says that, up to Morita equivalence, these algebras are determined by the q–orbits of their parameters.

The first main result in this paper gives an analogue of the Dipper–Mathas Morita equivalence theorem for the Hecke algebras of type $G(r,p,n)$. To state this result explicitly, fix positive integers r, p and n with $r = pt$, for some integer t, and let K be an algebraically closed field of characteristic coprime to p. Fix parameters $q, Q_1, \dots, Q_t \in K^\times$ and let $\mathbf{Q} = (Q_1, \dots, Q_t)$. Let $\mathscr{H}_{r,n}(\mathbf{Q})$ be the Ariki–Koike algebra and let $\mathscr{H}_{r,p,n}(\mathbf{Q})$ be the Hecke algebra of type $G(r,p,n)$ with parameters q and $\mathbf{Q}$. The algebra $\mathscr{H}_{r,n}(\mathbf{Q})$ is equipped with an automorphism σ of order p and $\mathscr{H}_{r,p,n}(\mathbf{Q})$ is the fixed point subalgebra of $\mathscr{H}_{r,n}(\mathbf{Q})$ under σ. There is a second automorphism τ on $\mathscr{H}_{r,n}$ which fixes $\mathscr{H}_{r,p,n}$ setwise. For the precise definitions see section 2.

Fix a primitive pth root of unity ε in K and say that Q_i and Q_j are in the **same** (ε,q)–**orbit** if $Q_i = \varepsilon^a q^b Q_j$, for some integers $a, b \in \mathbb{Z}$. Given two ordered

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2000 *Mathematics Subject Classification.* 20C08, 20C30.

Key words and phrases. Cyclotomic Hecke algebras, Morita equivalence.

tuples $\mathbf{X} = (X_1, \dots, X_s)$ and $\mathbf{Y} = (Y_1, \dots, Y_l)$ of elements of $K^\times$ set $\mathbf{X} \vee \mathbf{Y} = (X_1, \dots, X_s, Y_1, \dots, Y_l)$.

Suppose that A is an algebra and that $\mathbb{Z}_p$ is a group which acts on A as a group of algebra automorphisms. Let $A \rtimes \mathbb{Z}_p$ be the algebra with elements $\{\, ag \mid a \in A \text{ and } g \in \mathbb{Z}_p \,\}$ and with multiplication

$$ag \cdot bh = ab^g \cdot gh, \qquad \text{for } a, b \in A \text{ and } g, h \in \mathbb{Z}_p.$$

The first main result of this paper is the following.

Theorem A. *Suppose that $\mathbf{Q} = \mathbf{Q}_1 \vee \dots \vee \mathbf{Q}_\kappa$, where $Q_i \in \mathbf{Q}_\alpha$ and $Q_j \in \mathbf{Q}_\beta$ are in the same (ε, q)–orbit only if $\alpha = \beta$. Let $t_\alpha = |\mathbf{Q}_\alpha|$, for $1 \le \alpha \le \kappa$. Then the following two algebras are Morita equivalent:*

a) $\mathscr{H}_{r,p,n}(\mathbf{Q})$,

b) $\displaystyle\bigoplus_{\substack{b_1,\cdots,b_\kappa \ge 0\\ b_1+\cdots+b_\kappa = n}} \Big(\mathscr{H}_{pt_1,b_1}(\mathbf{Q}_1) \otimes \dots \otimes \mathscr{H}_{pt_\kappa,b_\kappa}(\mathbf{Q}_\kappa)\Big) \rtimes \mathbb{Z}_p$,

In part (b), each of the algebras $\mathscr{H}_{pt_\alpha,b_\alpha}(\mathbf{Q}_\alpha)$ has an automorphism σ_α of order p and, in the direct sum, the automorphism $\sigma_1 \otimes \dots \otimes \sigma_\kappa$ acts diagonally on the algebra $\mathscr{H}_{pt_1,b_1}(\mathbf{Q}_1) \otimes \dots \otimes \mathscr{H}_{pt_\kappa,b_\kappa}(\mathbf{Q}_\kappa)$. Observe that $\langle \sigma_1 \otimes \dots \otimes \sigma_\kappa \rangle \cong \mathbb{Z}_p$.

The second result of this paper uses Theorem A to prove a reduction theorem for computing the decomposition numbers of $\mathscr{H}_{r,p,n}$. In order to state this result fix a modular system $(F, \mathcal{O}, K)$ "with parameters". That is, we fix an algebraically closed field F of characteristic zero, a discrete valuation ring $\mathcal{O}$ with maximal ideal π and residue field $K \cong \mathcal{O}/\pi$, together with parameters $\hat{q}, \hat{Q}_1, \dots, \hat{Q}_t \in \mathcal{O}^\times$ such that $q = \hat{q} + \pi$ and $Q_i = \hat{Q}_i + \pi$ for each i. Let $\mathscr{H}^F_{r,p,n} = \mathscr{H}^F_{r,p,n}(\hat{q}, \hat{\mathbf{Q}})$ be the Hecke algebra of type $G(r,p,n)$ over F with parameters $\hat{q}$ and $\hat{\mathbf{Q}} = (\hat{Q}_1, \dots, \hat{Q}_t)$ and similarly let $\mathscr{H}^{\mathcal{O}}_{r,p,n} = \mathscr{H}_{r,p,n}(\hat{q}, \hat{\mathbf{Q}})$ and write $\mathscr{H}^K_{r,p,n} = \mathscr{H}_{r,p,n}$. We assume that $\mathscr{H}^F_{r,p,n}$ is semisimple. By freeness we have that $\mathscr{H}^F_{r,p,n} \cong \mathscr{H}^{\mathcal{O}}_{r,p,n} \otimes_{\mathcal{O}} F$ and $\mathscr{H}^K_{r,p,n} \cong \mathscr{H}^{\mathcal{O}}_{r,p,n} \otimes_{\mathcal{O}} K$. Thus, by choosing $\mathcal{O}$–lattices we can talk of modular reduction from $\mathscr{H}^F_{r,p,n}$–Mod to $\mathscr{H}^K_{r,p,n}$–Mod.

Using the definitions below, it is straightforward to check that the automorphisms σ and τ commute with modular reduction. Thus, we have compatible automorphisms σ and τ on $\mathscr{H}^F_{r,n}$ and on $\mathscr{H}^K_{r,n}$.

Let $R \in \{F, K\}$ and let M be an $\mathscr{H}^R_{r,p,n}$–module. Then we define a new $\mathscr{H}^R_{r,p,n}$–module M^τ by "twisting" the action of $\mathscr{H}^R_{r,p,n}$using the automorphism τ. Explicitly, $M^\tau = M$ as a vector space and the $\mathscr{H}^R_{r,p,n}$–action on M^τ is defined by

$$m \cdot h = m\tau(h), \qquad \text{for all } m \in M \text{ and } h \in \mathscr{H}^R_{r,p,n}.$$

Since τ^p is an inner automorphism of the algebra $\mathscr{H}^R_{r,p,n}$, it follows that $M \cong M^{\tau^p}$ for any $\mathscr{H}^R_{r,p,n}$–module M. Therefore, there is a natural action of the cyclic group $\mathbb{Z}_p$ on the set of isomorphism classes of $\mathscr{H}^R_{r,p,n}$–modules. We define the **inertia group** of M to be $G_M = \{\, k \mid 1 \le k \le p, M \cong M^{\tau^k} \,\} \le \mathbb{Z}_p$.

If A is any algebra let $\operatorname{Irr}(A)$ be the complete set of isomorphism classes of irreducible A–modules. We are interested in the inertia group G_S and G_D, for $S \in \operatorname{Irr}(\mathscr{H}^F_{r,p,n})$ and $D \in \operatorname{Irr}(\mathscr{H}^K_{r,p,n})$.

1.1. **Definition.** *Suppose that $S \in \operatorname{Irr}(\mathscr{H}^F_{r,p,n})$ and $D \in \operatorname{Irr}(\mathscr{H}^K_{r,p,n})$. The decomposition number $[S : D]$ is a p-**splittable** decomposition number of $\mathscr{H}_{r,p,n}$ if $G_S = \{0\} = G_D$.*

The second main result of this paper is the following:

Theorem B. *Suppose that $q \neq 1$. Then the decomposition numbers of the cyclotomic Hecke algebras of type $G(r,p,n)$ are completely determined by the p'-splittable decomposition numbers of certain cyclotomic Hecke algebras $\mathscr{H}_{r',p',n'}(\mathbf{Q}')$, where p' divides p, $1 \le r' \le r$, $1 \le n' \le n$ and where the parameters $\mathbf{Q}'$ are contained in a single (ε,q)–orbit.*

The proof of Theorem B explicitly describes the algebras $\mathscr{H}_{r',p',n'}(\mathbf{Q}')$ and the parameters $\mathbf{Q}'$ which appear in this reduction. Thus, once the p'-splittable decomposition numbers are known this result gives an algorithm for computing the decomposition matrices of the cyclotomic Hecke algebras of type $G(r,p,n)$.

This paper is organized as follows. In the next section we define the cyclotomic Hecke algebras of type $G(r,p,n)$ and prove the Morita equivalence result for the Hecke algebras of type $G(r,1,n)$ which underpins all of the results in this paper. In the third section we apply the results for the algebras of type $G(r,1,n)$ to prove Theorem A. The fourth section of the paper uses Clifford theory to show that if an algebra can be written as a semidirect product then its decomposition numbers are determined by a suitable family of p'-splittable decomposition numbers. This result is then applied in section 5 to prove Theorem B.

2. Morita equivalence theorems for Hecke algebras of type $G(r,1,n)$

In this section we define the cyclotomic Hecke algebras and set our notation. We then recall and generalize the Morita equivalence results that we need for the cyclotomic algebras of type $G(r,1,n)$.

Throughout this paper we fix positive integers r, p and n such that $r = pt$ for some integer t. Let K be an algebraically closed field which contains a primitive pth root of unity ε. In particular, the characteristic of K is coprime to p. Fix parameters $q, Q_1, \cdots, Q_t \in K^\times$ and, as in the introduction, let $\mathbf{Q} := (Q_1, \cdots, Q_t)$ and write $|\mathbf{Q}| = t$.

Let $\mathscr{H}_{r,n}(\mathbf{Q})$ be the cyclotomic Hecke algebra of type $G(r,1,n)$. As a K-algebra $\mathscr{H}_{r,n}(\mathbf{Q})$ is generated by $T_0, T_1, \cdots, T_{n-1}$ subject to the relations:

$$\begin{aligned}
&(T_0^p - Q_1^p)(T_0^p - Q_2^p)\cdots(T_0^p - Q_t^p) = 0,\\
&T_0T_1T_0T_1 = T_1T_0T_1T_0,\\
&(T_i + 1)(T_i - q) = 0, \quad 1 \le i \le n-1,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad 1 \le i \le n-2,\\
&T_iT_j = T_jT_i, \quad 0 \le i < j-1 \le n-2.
\end{aligned}$$

That is, $\mathscr{H}_{r,n}(\mathbf{Q})$ is the cyclotomic Hecke algebra of type $G(r,1,n)$ with parameters $\{Q'_1, \dots, Q'_r\}$, where $Q'_{i+pj} = \varepsilon^i Q_{j+1}$ for $0 \le i < p$ and $0 \le j < t$.

2.1. **Definition.** *The* ***cyclotomic Hecke algebra of type*** $G(r,p,n)$ *is the subalgebra $\mathscr{H}_{r,p,n}(\mathbf{Q})$ of $\mathscr{H}_{r,n}(\mathbf{Q})$ which is generated by the elements $T_0^p, T_u = T_0^{-1}T_1T_0$ and $T_1, T_2, \cdots, T_{n-1}$.*

When the choice of parameters $q, \mathbf{Q}$ is clear we write $\mathscr{H}_{r,p,n} = \mathscr{H}_{r,p,n}(\mathbf{Q})$ and $\mathscr{H}_{r,n} = \mathscr{H}_{r,n}(\mathbf{Q})$. When we want to emphasize the coefficient ring we write $\mathscr{H}^K_{r,p,n} = \mathscr{H}^K_{r,p,n}(\mathbf{Q})$ and $\mathscr{H}^K_{r,n} = \mathscr{H}^K_{r,n}(\mathbf{Q})$, respectively.

Let $\mathfrak{S}_n$ be the symmetric group of degree n. As the type A braid relations hold in $\mathscr{H}_{r,n}$ for each $w \in \mathfrak{S}_n$ there is a well–defined element $T_w \in \mathscr{H}_{r,n}$, where $T_w = T_{i_1} \dots T_{i_k}$ whenever k is minimal such that $w = (i_1, i_1+1)\dots(i_k, i_k+1)$. Set $L_1 = T_0$ and $L_{k+1} = q^{-1}T_kL_kT_k$, for $k = 1, \dots, n-1$. Then Ariki and Koike [**2**, Theorem 3.10] showed that

$$\left\{ L_1^{c_1} \dots L_n^{c_n} T_w \,\middle|\, w \in \mathfrak{S}_n \text{ and } 0 \le c_i < r \right\}$$

is a basis of $\mathscr{H}_{r,n}$.

All of the Morita equivalences in Theorem A are a consequence of the following result for the cyclotomic Hecke algebras $\mathscr{H}_{r,n}$.

2.2. **Theorem** (Dipper–Mathas [**6**, Theorem 1.1]). *Suppose that* $\mathbf{Q}=\mathbf{Q}_1\vee\dots\vee\mathbf{Q}_\kappa$, *such that* $\alpha=\beta$ *whenever* $Q_i\in\mathbf{Q}_\alpha$, $Q_j\in\mathbf{Q}_\beta$ *and* $Q_i=q^aQ_j$ *for some* $a\in\mathbb{Z}$. *Let* $t_\alpha=|\mathbf{Q}_\alpha|$. *Then* $\mathscr{H}_{r,n}(\mathbf{Q})$ *is Morita equivalent to the algebra*

$$\bigoplus_{\substack{b_1,\dots,b_\kappa\ge0\\ b_1+\dots+b_\kappa=n}}\mathscr{H}_{pt_1,b_1}(\mathbf{Q}_1)\otimes\dots\otimes\mathscr{H}_{pt_\kappa,b_\kappa}(\mathbf{Q}_\kappa).$$

As noted in [**6**] the proof of Theorem 2.2 quickly reduces to the case $\kappa=2$, so only this case is considered in [**6**]. Unfortunately, to prove Theorem A we need detailed information about the bimodule which induces the Morita equivalence of Theorem 2.2 for arbitrary $\kappa\ge1$. Consequently, we need to generalize the results of [**6**] and construct the bimodule which induces the Morita equivalence of Theorem 2.2 (in the special case when $\mathbf{Q}$ is partitioned into a disjoint union of (ε,q)–orbits). In constructing this bimodule we refer the reader back to [**6**] whenever the details are not substantially different from the case $\kappa=2$.

First, fix non-negative integers a and b with $a+b\le n$ and an integer s with $1\le s\le t$. Define

$$v_{a,b}(s)=\prod_{k=1}^{s}(L_1^p-Q_k^p)\dots(L_a^p-Q_k^p)\cdot T_{w_{a,b}}\cdot\prod_{k=s+1}^{t}(L_1^p-Q_k^p)\dots(L_b^p-Q_k^p),$$

where $w_{a,b}=(s_{a+b-1}\dots s_1)^b$. (So $v_{n-b,b}(s)$ is the element v_b of [**6**, Definition 3.3].) It may help the reader to observe that if we write $w_{a,b}\in\mathfrak{S}_{a+b}$ as a permutation in two-line notation then

$$w_{a,b}=\begin{pmatrix}1&\cdots&a&a+1&\cdots&a+b\\ b+1&\cdots&a+b&1&\cdots&b\end{pmatrix}.$$

We will use the following notation extensively.

Notation. *Given any sequence* $\mathbf{a}=(a_1,\dots,a_k)$ *and integers* $1\le i\le j\le k$ *we set* $a_{i,j}=a_i+\dots+a_j$. *If* $i<j$ *then let* $a_{j,i}=0$.

Until further notice we fix a partition $\mathbf{Q}=\mathbf{Q}_1\vee\dots\vee\mathbf{Q}_\kappa$ of $\mathbf{Q}$ such that $Q_i\in\mathbf{Q}_\alpha$ and $Q_j\in\mathbf{Q}_\beta$ are in the same (ε,q)-orbit only if $\alpha=\beta$. Set $\mathbf{t}=(t_1,\dots,t_\kappa)$ where $t_\alpha=|\mathbf{Q}_\alpha|$, for $1\le\alpha\le\kappa$. Without loss of generality we assume that $\mathbf{Q}_\alpha=(Q_{t_{1,\alpha-1}+1},\dots,Q_{t_{1,\alpha}})$, for $\alpha=1,\dots,\kappa$ (set $t_0=0$).

Let $\Lambda(n,\kappa)=\{\,\mathbf{b}=(b_1,\dots,b_\kappa)\mid b_{1,\kappa}=n\text{ and }b_\alpha\ge0\text{ for }1\le\alpha\le\kappa\,\}$ be the set of compositions of n into κ parts. If $\mathbf{b}\in\Lambda(n,\kappa)$ then, for convenience, we set $b_{\kappa+1}=0$.

2.3. **Definition.** *Suppose that* $\mathbf{b}\in\Lambda(n,\kappa)$. *Define*

$$v_{\mathbf{b}}=v_{b_\kappa,b_{1,\kappa-1}}(t_{1,\kappa-1})v_{b_{\kappa-1},b_{1,\kappa-2}}(t_{1,\kappa-2})\dots v_{b_2,b_{1,1}}(t_{1,1})$$

and let $w_{\mathbf{b}}=w_{b_\kappa,b_{1,\kappa-1}}w_{b_{\kappa-1},b_{1,\kappa-2}}\dots w_{b_2,b_{1,1}}$. *Define* $V^{\mathbf{b}}=v_{\mathbf{b}}\mathscr{H}_{r,n}$.

Note that $v_{\mathbf{b}}$ depends crucially on our fixed partition $\mathbf{Q}=\mathbf{Q}_1\vee\dots\vee\mathbf{Q}_\kappa$ of $\mathbf{Q}$, so we should really write $v_{\mathbf{b}}=v_{\mathbf{b}}(\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa)$. Our first goal is to understand $V^{\mathbf{b}}$.

The key property of the elements $v_{\mathbf{b}}$ is the following.

2.4. **Proposition.** *Suppose that* $\mathbf{b}\in\Lambda(n,\kappa)$. *Then*

a) $T_iv_{\mathbf{b}}=v_{\mathbf{b}}T_{w_{\mathbf{b}}(i)}$, *whenever* $1\le i<n$ *and* $i\ne b_{\alpha,\kappa}$ *for* $\alpha=1,\dots,\kappa$.

b) $L_kv_{\mathbf{b}}=v_{\mathbf{b}}L_{w_{\mathbf{b}}(k)}$, *whenever* $1\le k\le n$.

Proof. After translating notation, [**6**, Prop. 3.4] says that if $1 \le s \le t$ and a and b are non–negative integers with $a + b \le n$ then

$$T_i v_{a,b}(s) = v_{a,b}(s) T_{w_{a,b}(i)} \quad \text{and} \quad L_k v_{a,b}(s) = v_{a,b}(s) L_{w_{a,b}(k)} \tag{2.5}$$

whenever $1 \le i < a + b$, $i \ne a$, and $1 \le k \le a + b$. This is precisely the special case of the Proposition when $\kappa = 2$. The general case follows from this result together with the observation that $T_i v_{a,b}(s) = v_{a,b}(s) T_i$ and $L_k v_{a,b}(s) = v_{a,b}(s) L_k$ whenever $a + b < i < n$ and $a + b < k \le n$ for non–negative integers a and b. □

Observe that $v_{\mathbf{b}} T_j = T_{w_{\mathbf{b}}^{-1}(j)} v_{\mathbf{b}}$ and $v_{\mathbf{b}} L_m = L_{w_{\mathbf{b}}^{-1}(m)} v_{\mathbf{b}}$ by Proposition 2.4, for $1 \le j < n$, $1 \le m \le n$ with $j \ne b_{1,\alpha}$ for $\alpha = 1, \dots, \kappa$.

2.6. **Lemma.** *Suppose that* $\mathbf{b} \in \Lambda(n, \kappa)$, $1 \le \alpha \le \kappa$ *and* $b_\alpha \ne 0$. *Then*

$$\prod_{Q_i \in \mathbf{Q}_\alpha} (L^p_{1+b_{\alpha+1,\kappa}} - Q^p_i) \cdot v_{\mathbf{b}} = 0 = v_{\mathbf{b}} \cdot \prod_{Q_i \in \mathbf{Q}_\alpha} (L^p_{b_{1,\alpha-1}+1} - Q^p_i)$$

Proof. Recall that $\prod_{i=1}^t (L^p_1 - Q^p_i) = 0$ since $L_1 = T_0$. Therefore, it follows from the definitions that if $b_\kappa \ne 0$ then

$$\prod_{Q_i \in \mathbf{Q}_\kappa} (L^p_1 - Q^p_i) \cdot v_{b_\kappa, b_{1,\kappa-1}}(t_{1,\kappa-1}) = 0.$$

Hence, $\prod_{Q_i \in \mathbf{Q}_\kappa} (L^p_1 - Q^p_i) \cdot v_{\mathbf{b}} = 0$. Similarly, if $b_1 \ne 0$, then

$$v_{\mathbf{b}} \cdot \prod_{Q_j \in \mathbf{Q}_1} (L^p_1 - Q^p_j) = 0.$$

Now suppose that $1 \le \alpha < \kappa$, $b_\alpha \ne 0$, and set $L(\alpha) = \prod_{Q_i \in \mathbf{Q}_\alpha} (L^p_{1+b_{\alpha+1,\kappa}} - Q^p_i)$. Then, using (2.5),

$$L(\alpha) v_{\mathbf{b}} = \prod_{\beta=\alpha}^{\kappa-1} v_{b_{\beta+1}, b_{1,\beta}}(t_{1,\beta}) \cdot \prod_{Q_i \in \mathbf{Q}_\alpha} (L^p_1 - Q^p_i) \cdot \prod_{\beta=1}^{\alpha-1} v_{b_{\beta+1}, b_{1,\beta}}(t_{1,\beta}),$$

where the terms in the two outside products appear in order with β decreasing from left to right. Using the definitions, $\prod_{k=t_{1,\alpha}+1}^{t} (L^p_1 - Q^p_k)$ is a right divisor of $v_{b_{\alpha+1}, b_{1,\alpha}}(t_{1,\alpha})$, whereas $\prod_{k=1}^{t_{1,\alpha-1}} (L^p_1 - Q^p_k)$ is a left divisor of $v_{b_\alpha, b_{1,\alpha-1}}(t_{1,\alpha-1})$. Hence, $L(\alpha) v_{\mathbf{b}} = 0$ by combining the relation $\prod_{i=1}^t (L^p_1 - Q^p_i) = 0$ with the last displayed equation. The second statement

$$v_{\mathbf{b}} \cdot \prod_{Q_i \in \mathbf{Q}_\alpha} (L^p_{b_{1,\alpha-1}+1} - Q^p_i) = 0,$$

for $\alpha = 2, \dots, \kappa$, is equivalent to what we have just proved because $v_{\mathbf{b}} L_{b_{1,\alpha-1}+1} = L_{w_{\mathbf{b}}^{-1}(b_{1,\alpha-1}+1)} v_{\mathbf{b}} = L_{1+b_{\alpha+1,\kappa}} v_{\mathbf{b}}$ by Proposition 2.4. □

To proceed we recall the cellular basis of the algebras $\mathscr{H}_{r,n}$, and the associated combinatorics, introduced in [**5**]. A **multipartition of** n is an ordered r–tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ such that $|\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. Let Λ^+_n be the set of multipartitions of n. Then Λ^+_n is a poset under the **dominance order**, where $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if

$$\sum_{a=1}^{s-1} |\lambda^{(a)}| + \sum_{j=1}^{i} \lambda^{(s)}_j \ge \sum_{a=1}^{s-1} |\mu^{(a)}| + \sum_{j=1}^{i} \mu^{(s)}_j,$$

for all $1 \le s \le r$ and all $i \ge 1$.

The **diagram** of $\boldsymbol{\lambda}$ is the set $[\boldsymbol{\lambda}] = \{\, (i, j, s) \mid 1 \le j \le \lambda^{(s)}_i \text{ for } 1 \le s \le r \,\}$. A $\boldsymbol{\lambda}$**-tableau** is a bijection $\mathfrak{t}\colon [\boldsymbol{\lambda}] \longrightarrow \{1, 2, \dots, n\}$. The $\boldsymbol{\lambda}$–tableau $\mathfrak{t}$ is **standard** if $\mathfrak{t}(i, j, s) < \mathfrak{t}(i', j', s)$ whenever $i \le i'$, $j \le j'$ and (i, j, s) and (i', j', s) are distinct elements of $[\boldsymbol{\lambda}]$. Let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$–tableaux. Observe that $\mathfrak{S}_n$

acts from the right on the set of $\boldsymbol{\lambda}$–tableaux. In particular, if $w \in \mathfrak{S}_n$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\mathfrak{t}w$ is a $\boldsymbol{\lambda}$–tableau, however, it is not necessarily standard.

If $\boldsymbol{\lambda} \in \Lambda_n^+$ let $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ be the corresponding Young (or parabolic) subgroup of $\mathfrak{S}_n$. We set

$$x_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w \quad \text{and} \quad u_{\boldsymbol{\lambda}}^+ = \prod_{s=2}^{r} \prod_{k=1}^{|\lambda^{(1)}|+\cdots+|\lambda^{(s-1)}|} (L_k - Q'_s).$$

Then $x_{\boldsymbol{\lambda}}$ and $u_{\boldsymbol{\lambda}}^+$ are commuting elements of $\mathscr{H}_{r,n}$. Next, if $\mathfrak{s}$ is a standard $\boldsymbol{\lambda}$-tableau let $d(\mathfrak{s})$ be the corresponding distinguished right coset representative of $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$. Finally, given a pair $(\mathfrak{s}, \mathfrak{t})$ of standard $\boldsymbol{\lambda}$–tableaux define $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} u^+_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})}$, where $*$ is the unique anti–isomorphism of $\mathscr{H}_{r,n}$ which fixes $T_0, \dots, T_{n-1}$. Then

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \Lambda_n^+ \,\}$$

is a cellular basis of $\mathscr{H}_{r,n}$ by [**5**, Theorem 3.26].

We can relate $V^{\mathbf{b}}$ to the combinatorics of the cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ by defining $\omega_{\mathbf{b}} = (\omega_{\mathbf{b}}^{(1)}, \dots, \omega_{\mathbf{b}}^{(r)})$ to be the multipartition with

$$\omega_{\mathbf{b}}^{(s)} = \begin{cases} (1^{b_\alpha}), & \text{if } s = pt_{1,\alpha} \text{ for some } \alpha, \\ (0), & \text{otherwise.} \end{cases}$$

From the definitions, $u^+_{\omega_{\mathbf{b}}} = \prod_{\alpha=1}^{\kappa-1} \prod_{Q_i \in \mathbf{Q}_{\alpha+1} \vee \cdots \vee \mathbf{Q}_\kappa} (L_1^p - Q_i^p) \dots (L^p_{b_{1,\alpha}} - Q_i^p)$. Hence, using (2.5) we obtain the following.

2.7. **Lemma.** *Suppose that* $\mathbf{b} \in \Lambda(n, \kappa)$. *Then* $v_{\mathbf{b}} = v^-_{\mathbf{b}} u^+_{\omega_{\mathbf{b}}}$, *where*

$$v^-_{\mathbf{b}} = \prod_{\alpha=2}^{\kappa} \prod_{i=1}^{t_{1,\alpha-1}} (L_1^p - Q_i^p) \dots (L^p_{b_\alpha} - Q_i^p) \cdot T_{w_{b_\alpha, b_{1,\alpha-1}}}$$

where in the product α *decreases in order from left to right.*

Following [**5**] define $M^{\omega_{\mathbf{b}}} = u^+_{\omega_{\mathbf{b}}} \mathscr{H}_{r,n}$. By the Lemma, there is a surjective $\mathscr{H}_{r,n}$–module homomorphism $\theta_{\mathbf{b}} \colon M^{\omega_{\mathbf{b}}} \longrightarrow V^{\mathbf{b}}$ given by $\theta_{\mathbf{b}}(h) = v^-_{\mathbf{b}} h$, for all $h \in M^{\omega_{\mathbf{b}}}$.

Suppose that $\boldsymbol{\lambda}$ is a multipartition and that $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau. For each integer k, with $1 \le k \le n$, define $\operatorname{comp}_{\mathfrak{t}}(k) = s$ if (i, j, s) is the unique node in $[\boldsymbol{\lambda}]$ such that $\mathfrak{t}(i, j, s) = k$.

2.8. **Definition.** *Suppose that* $\boldsymbol{\lambda}$ *is a multipartition of* n. *Define*

$$\operatorname{Std}_{\mathbf{b}}(\boldsymbol{\lambda}) = \{\, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid \operatorname{comp}_{\mathfrak{t}}(k) \le pt_{1,\alpha} \textit{ if } 1 \le k \le b_{1,\alpha} \,\}$$

and

$$\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda}) = \{\, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid pt_{1,\alpha-1} < \operatorname{comp}_{\mathfrak{t}}(k) \le pt_{1,\alpha} \textit{ if } b_{1,\alpha-1} < k \le b_{1,\alpha} \,\}$$

Then $\operatorname{Std}_{\mathbf{b}}(\boldsymbol{\lambda}) \ne \emptyset$ if only if $\sum_{s=1}^{pt_{1,\alpha}} |\lambda^{(s)}| \ge b_{1,\alpha}$ for $1 \le \alpha \le \kappa$ and $\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda}) \ne 0$ if and only if $\sum_{s=pt_{1,\alpha-1}+1}^{pt_{1,\alpha}} |\lambda^{(s)}| = b_\alpha$, for $1 \le \alpha \le \kappa$. Note that $\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda}) \subseteq \operatorname{Std}_{\mathbf{b}}(\boldsymbol{\lambda})$.

2.9. **Lemma.** *Suppose that* $\mathbf{b} \in \Lambda(n, \kappa)$.

a) $M^{\omega_{\mathbf{b}}}$ *has basis* $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \in \operatorname{Std}_{\mathbf{b}}(\boldsymbol{\lambda}), \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ *for some* $\boldsymbol{\lambda} \in \Lambda_n^+ \,\}$.

b) *Suppose that* $\mathfrak{s} \in \operatorname{Std}_{\mathbf{b}}(\boldsymbol{\lambda}) \setminus \operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$ *and* $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. *Then* $\theta_{\mathbf{b}}(m_{\mathfrak{s}\mathfrak{t}}) = 0$.

Proof. Part (a) is a translation of [**5**, Theorem 4.14] into the current notation. See the proof of [**6**, Lemma 3.9] for more details.

For part (b) we follow the proof of [**6**, Lemma 3.10]. Let $\mathbf{c} = (c_1, \dots, c_\kappa)$, where $c_\alpha = |\lambda^{(pt_{1,\alpha-1}+1)}| + \cdots + |\lambda^{(pt_{1,\alpha})}|$, for $1 \le \alpha \le \kappa$. Then $c_{1,\alpha} \ge b_{1,\alpha}$, for $1 \le \alpha \le \kappa$, since $\mathfrak{s} \in \operatorname{Std}_{\mathbf{b}}(\boldsymbol{\lambda})$ and $\mathbf{c} \ne \mathbf{b}$ since $\mathfrak{s} \notin \operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$. Choose β to be minimal such that $c_\beta > b_\beta$. Then $1 \le \beta < \kappa$ and if $1 \le \alpha \le \beta$ then

$pt_{1,\alpha-1} < \operatorname{comp}_{\mathfrak s}(k) \le pt_{1,\alpha}$ if $b_{1,\alpha-1} < k \le b_{1,\alpha}$ since $c_1 = b_1, \dots, c_{\beta-1} = b_{\beta-1}$ and $\mathfrak s \in \operatorname{Std}_{\mathbf b}(\boldsymbol\lambda)$. Choose $\gamma \ge \beta$ to be minimal such that $b_\gamma \ne 0$. Let w be a permutation of $\{b_{1,\gamma}+1, b_{1,\gamma}+2, \dots, n\}$ of minimal length such that $\mathfrak s' = \mathfrak s w$ is a standard $\boldsymbol\lambda$–tableau with $pt_{1,\alpha-1} < \operatorname{comp}_{\mathfrak s}(k) \le pt_{1,\alpha}$ if $c_{1,\alpha-1} < k \le c_{1,\alpha}$ for $\gamma+1 \le \alpha \le \kappa$. (Such a permutation exists because we can first swap integers k, with $c_{1,\kappa-1} < k \le n$ and $\operatorname{comp}_{\mathfrak s}(k) \le pt_{1,\kappa-1}$, with the integers l, where $b_{1,\kappa-1} < l \le n$ and $pt_{1,\kappa-1} < \operatorname{comp}_{\mathfrak s}(l) \le pt_{1,\kappa}$; let $\mathfrak s_1$ be the resulting λ–tableau. Then we swap integers k, with $c_{1,\kappa-2} < k \le c_{1,\kappa-1}$ and $\operatorname{comp}_{\mathfrak s_1}(k) \le pt_{1,\kappa-2}$, with the integers l, where $b_{1,\kappa-2} < l \le n$ and $pt_{1,\kappa-2} < \operatorname{comp}_{\mathfrak s_1}(l) \le pt_{1,\kappa-1}$; $\cdots$, and so on; compare [**6**, Lemma 3.10].) As a result, $\operatorname{comp}_{\mathfrak s}(k) \le pt_{1,\gamma}$ if $k \le c_{1,\gamma}$. Then $d(\mathfrak s) = d(\mathfrak s')w$, with the lengths adding, so that $m_{\mathfrak{st}} = T^*_w m_{\mathfrak s'\mathfrak t}$. Furthermore, by construction, there is a composition $\mathbf c' \in \Lambda(n,\kappa)$ such that $\mathfrak s' \in \operatorname{Std}_{\mathbf c'}(\boldsymbol\lambda)$, $c'_\alpha = c_\alpha$, for $1 \le \alpha < \beta$ or $\gamma \le \alpha \le \kappa$, and $c'_{1,\alpha} \ge b_{1,\alpha}$, for $1 \le \alpha \le \kappa$. Hence, $m_{\mathfrak s'\mathfrak t} \in M^{\omega_{\mathbf c'}}$ by part (a), so that $m_{\mathfrak s'\mathfrak t} = u^+_{\omega_{\mathbf c'}} h$ for some $h \in \mathscr H_{r,n}$. Therefore, $\theta_{\mathbf b}(m_{\mathfrak{st}}) = v^-_{\mathbf b} T^*_w m_{\mathfrak s'\mathfrak t} = v^-_{\mathbf b} T^*_w v^+_{\omega_{\mathbf c'}} h$.

To simplify the notation, for the remainder of the proof set $w(\alpha) = w_{b_\alpha, b_{1,\alpha-1}}$ and $u^-(m,s) = \prod_{i=1}^{s}(L_1^p - Q_i^p)\dots(L_m^p - Q_i^p)$, for $1 \le \alpha \le \kappa$, $1 \le m \le n$ and $1 \le s \le t$. Similarly, set $u^+(m,s) = \prod_{i=s}^{t}(L_1^p - Q_i^p)\dots(L_m^p - Q_i^p)$. Then

$$\theta_{\mathbf b}(m_{\mathfrak{st}}) = v^-_{\mathbf b} T^*_w u^+_{\omega_{\mathbf c'}} h = \prod_{\alpha=2}^{\kappa} u^-(b_\alpha, t_{1,\alpha-1}) T_{w(\alpha)} \cdot T^*_w u^+_{\omega_{\mathbf c'}} h,$$

where the product is taken in order with α decreasing from left to right. Now, $u^\pm(m,s)$ commutes with T_i if $i \ne m$. Therefore, since w is a permutation of $\{b_{1,\gamma}+1, b_{1,\gamma}+2, \dots, n\}$, we have

$$\theta_{\mathbf b}(m_{\mathfrak{st}}) = \prod_{\alpha=\gamma+1}^{\kappa} u^-(b_\alpha, t_{1,\alpha-1}) T_{w(\alpha)} \cdot T^*_w \cdot \prod_{\alpha=2}^{\gamma} u^-(b_\alpha, t_{1,\alpha-1}) T_{w(\alpha)} \cdot u^+_{\omega_{\mathbf c'}} h,$$

for some $w' \in \mathfrak S_n$, where again both products are ordered with α decreasing from left to right. By definition, $u^+_{\omega_{\mathbf c'}} = \prod_{\alpha=1}^{\kappa-1} u^+(c'_{1,\alpha}, t_{1,\alpha}+1)$, where this product can be taken in any order. So,

$$\prod_{\alpha=2}^{\gamma} u^-(b_\alpha, t_{1,\alpha-1}) T_{w(\alpha)} \cdot u^+_{\omega_{\mathbf c'}} = \prod_{\alpha=2}^{\gamma} u^-(b_\alpha, t_{1,\alpha-1}) T_{w(\alpha)} \cdot \prod_{\alpha=1}^{\kappa-1} u^+(c'_{1,\alpha}, t_{1,\alpha}+1).$$

Now, $w(\alpha) = w_{b_\alpha, b_{1,\alpha-1}} \in \mathfrak S_{b_{1,\alpha}}$. So if $1 \le \alpha < \gamma$ then $T_{w(\alpha)}$ commutes with $u^+(c'_{1,\gamma}, t_{1,\gamma}+1)$ since $c'_{1,\gamma} = c_{1,\gamma} > b_{1,\alpha}$. Consequently, the last displayed equation contains $u^-(b_\gamma, t_{1,\gamma-1}) T_{w(\gamma)} u^+(c_{1,\gamma}, t_{1,\gamma}+1)$ as a factor. Note that $c_{1,\gamma} > b_{1,\gamma-1}$. Looking at the definitions, this element is equal to

$$v_{b_\gamma, b_{1,\gamma-1}}(t_{1,\gamma-1}) \prod_{s=t_{1,\gamma}+1}^{t} (L^p_{b_{1,\gamma-1}+1} - Q^p_s) \dots (L^p_{c_{1,\gamma}} - Q^p_s) = 0,$$

where the last equality comes from applying the right hand equation of Lemma 2.6 in the special case when $\kappa = 2$. Putting all of these equations together, we have shown that $\theta_{\mathbf b}(m_{\mathfrak{st}}) = 0$, as required. ☐

Suppose that $\mathfrak t$ is a standard $\boldsymbol\lambda$–tableau and that $1 \le k \le n$. Let $\operatorname{Shape}_k(\mathfrak t)$ be the multipartition with diagram $\mathfrak t^{-1}(\{1,\dots,k\})$; that is, $\operatorname{Shape}_k(\mathfrak t)$ is the multipartition given by the positions of $\{1,\dots,k\}$ in $\mathfrak t$. If $\mathfrak t \in \operatorname{Std}(\boldsymbol\lambda)$ and $\mathfrak v \in \operatorname{Std}(\boldsymbol\mu)$ then we write $\mathfrak t \trianglerighteq \mathfrak v$ if $\boldsymbol\lambda \triangleright \boldsymbol\mu$ or if $\boldsymbol\lambda = \boldsymbol\mu$ and $\operatorname{Shape}_k(\mathfrak t) \trianglerighteq \operatorname{Shape}_k(\mathfrak v)$ for $1 \le k \le n$. We extend this partial order to pairs of standard tableaux in the obvious way.

2.10. **Lemma.** *Suppose that* $\boldsymbol{\lambda}$ *is a multipartition of* n *and that* $\mathfrak{s}\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$ *and* $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. *Let* $\mathfrak{s}'=\mathfrak{s}w_{\mathbf{b}}^{-1}$. *Then there exists an invertible element* $u\in R$ *such that*

$$\theta_{\mathbf{b}}(m_{\mathfrak{s}\mathfrak{t}})=um_{\mathfrak{s}'\mathfrak{t}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s}',\mathfrak{t})}r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}}\in R$.

Proof. By [**13**, Prop. 3.7], if $1\le k\le n$ and $p(a-1)<\operatorname{comp}_{\mathfrak{t}}(k)\le pa$ then

$$m_{\mathfrak{s}\mathfrak{t}}L_k^p=q^{p(j-i)}Q_a^pm_{\mathfrak{s}\mathfrak{t}}+\sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{s},\mathfrak{t})}r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}}\in R$. Using this formula we can compute $\theta(m_{\mathfrak{s}\mathfrak{t}})=v_{\mathbf{b}}^-m_{\mathfrak{s}\mathfrak{t}}$ directly, which shows that $m_{\mathfrak{u}\mathfrak{v}}$ appears with non-zero coefficient in $\theta_{\mathbf{b}}(m_{\mathfrak{s}\mathfrak{t}})$ only if $(\mathfrak{u},\mathfrak{v})\trianglerighteq(\mathfrak{s}',\mathfrak{t})$. Finally, $m_{\mathfrak{s}'\mathfrak{t}}$ appears with non-zero coefficient in $\theta_{\mathbf{b}}(m_{\mathfrak{s}\mathfrak{t}})$ because $\mathbf{Q}=\mathbf{Q}_1\vee\cdots\vee\mathbf{Q}_\kappa$ is a partition of $\mathbf{Q}$ into (ε,q)-orbits. (Compare with the proof of [**6**, Lemma 3.11].) □

Suppose that $\boldsymbol{\lambda}$ is a multipartition of n. Let $\mathscr{H}^{\boldsymbol{\lambda}}_{r,n}$ be the module with basis $\{m_{\mathfrak{u}\mathfrak{v}}\}$ where $\mathfrak{u}$ and $\mathfrak{v}$ range over the standard $\boldsymbol{\mu}$–tableaux with $\boldsymbol{\mu}\rhd\boldsymbol{\lambda}$. It follows from the general theory of cellular algebras [**16**, Lemma 2.3] that $\mathscr{H}^{\boldsymbol{\lambda}}_{r,n}$ is a two–sided ideal of $\mathscr{H}_{r,n}$.

As a vector space, the **Specht module** (or cell module) $S(\boldsymbol{\lambda})$ is the module with basis $\{\,m_{\mathfrak{s}\mathfrak{t}}+\mathscr{H}^{\boldsymbol{\lambda}}_{r,n}\mid\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$. The theory of cellular algebras [**16**, 2.4] shows that $S(\boldsymbol{\lambda})$ is an $\mathscr{H}_{r,n}$-module and that, up to isomorphism, $S(\boldsymbol{\lambda})$ does not depend on the choice of $\mathfrak{s}$.

Finally, we need the classification of the blocks for $\mathscr{H}_{r,n}$. For each $\boldsymbol{\lambda}\in\Lambda^+_n$ define a "content function" $c_{\boldsymbol{\lambda}}\colon R\longrightarrow\mathbb{N}$ by

$$c_{\boldsymbol{\lambda}}(x)=\#\{\,(i,j,a+pb)\in[\boldsymbol{\lambda}]\mid 0\le a<p\text{ and }x=q^{j-i}\varepsilon^aQ_b\,\},$$

for $x\in R$. Then the Specht modules $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ are in the same block only if $c_{\boldsymbol{\lambda}}(x)=c_{\boldsymbol{\mu}}(x)$, for all $x\in R$, by [**8**, Prop. 5.9(ii)]. (Although we will not need this we note that the converse is also true by [**14**, Theorem A].)

With the results that we have now proved we can complete the proof of Theorem 2.2 with only minor modifications of the arguments of [**6**]. Consequently, we sketch the rest of the proof and give references to [**6**] for those readers who require more detail.

2.11. **Definition.** *Suppose that* $\mathbf{b}\in\Lambda(n,\kappa)$ *and that* $\mathfrak{s}\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$, $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ *for some* $\boldsymbol{\lambda}\in\Lambda^+_n$.

a) *Set* $v_{\mathfrak{s}\mathfrak{t}}=\theta_{\mathbf{b}}(m_{\mathfrak{s}\mathfrak{t}})\in V^{\mathbf{b}}$.
b) *If further* $\mathfrak{t}\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$, *then let* $\theta_{\mathfrak{s}\mathfrak{t}}\in\operatorname{End}_R(V^{\mathbf{b}})$ *be the endomorphism* $\theta_{\mathfrak{s}\mathfrak{t}}(v_{\mathbf{b}}h)=v_{\mathfrak{s}\mathfrak{t}}h$, *for all* $h\in\mathscr{H}_{r,n}$.

We remark that it is not clear from the definition that the maps $\theta_{\mathfrak{s}\mathfrak{t}}$ are well–defined.

2.12. **Proposition.** *Suppose that* $\mathbf{b}\in\Lambda(n,\kappa)$. *Then:*

a) $V^{\mathbf{b}}$ *has basis*

$$\{\,v_{\mathfrak{s}\mathfrak{t}}\mid\mathfrak{s},\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda}),\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\textit{ for some }\boldsymbol{\lambda}\in\Lambda^+_n\,\}.$$

b) *If* $\mathbf{b}\ne\mathbf{c}\in\Lambda(n,\kappa)$ *then* $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}},V^{\mathbf{c}})=0$.
c) $\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}})$ *is a vector space with basis*

$$\{\,\theta_{\mathfrak{s}\mathfrak{t}}\mid\mathfrak{s},\mathfrak{t}\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})\textit{ for some }\boldsymbol{\lambda}\in\Lambda^+_n\,\}.$$

Proof. (a) This follows directly from Lemma 2.9(b) and Lemma 2.10.

(b) As in [**6**, Theorem 3.16] it follows from part (a) and the construction of the Specht modules that $V^{\mathbf{b}}$ has a filtration $V^{\mathbf{b}} = V_1 \supset V_2 \cdots \supset V_k = 0$ such that (1) $V_i/V_{i+1} \cong S(\boldsymbol{\lambda}_i)$, for some $\boldsymbol{\lambda}_i \in \Lambda_n^+$, and (2) if $\boldsymbol{\mu} \in \Lambda_n^+$ then

$$\#\operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\mu}) = \#\{\, 1 \le i < k \mid V_i/V_{i+1} \cong S(\boldsymbol{\mu}) \,\}.$$

Now, if $\mathbf{b} \ne \mathbf{c}$ and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two multipartitions such that $\operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}) \ne \emptyset$ and $\operatorname{Std}_{\mathbf{c}}^+(\boldsymbol{\mu}) \ne \emptyset$ then it is easy to see (cf. the proof of [**6**, Cor. 3.17]) that $c_{\boldsymbol{\lambda}} \ne \mathbf{c}_{\boldsymbol{\mu}}$. Consequently, by the remarks before the Theorem, the Specht modules $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ are in different blocks. Therefore, all of the composition factors of $V^{\mathbf{b}}$ and $V^{\mathbf{c}}$ belong to different blocks, so $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}}, V^{\mathbf{c}}) = 0$.

(c) The proof is identical to that of [**6**, Theorem 3.19]. In outline, the argument is as follows. By [**5**, Theorem 6.16] and Lemma 2.9(a), $\operatorname{End}_{\mathscr{H}_{r,n}}(M^{\omega_{\mathbf{b}}})$ has basis $\{\, \varphi_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \Lambda_n^+ \,\}$, where $\varphi_{\mathfrak{st}}(u_{\omega_{\mathbf{b}}}^+ h) = m_{\mathfrak{st}} h$ for all $h \in \mathscr{H}_{r,n}$. As in the proof of part (b), the filtration $0 \subseteq \ker\theta_{\mathbf{b}} \subset M^{\omega_{\mathbf{b}}}$ can be extended to a Specht filtration which is compatible with the Specht filtration of $V^{\mathbf{b}} \cong M^{\omega_{\mathbf{b}}}/\ker\theta_{\mathbf{b}}$. Using this Specht filtration and the classification of the blocks of $\mathscr{H}_{r,n}$ given above it follows that all of the irreducible constituents of $V_{\mathbf{b}}$ and $\ker\theta_{\mathbf{b}}$ belong to different blocks. Therefore, the map $\theta_{\mathbf{b}}\colon M^{\omega_{\mathbf{b}}} \longrightarrow V^{\mathbf{b}}$ splits. Let $\theta_{\mathbf{b}}^{-1}$ be a *right* inverse to $\theta_{\mathbf{b}}$. Then a straightforward calculation shows that

$$\theta_{\mathbf{b}}\varphi_{\mathfrak{st}}\theta_{\mathbf{b}}^{-1} = \begin{cases} \theta_{\mathfrak{st}}, & \text{if } \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}), \\ 0, & \text{otherwise.} \end{cases}$$

As the maps $\{\, \theta_{\mathbf{b}}\varphi_{\mathfrak{st}}\theta_{\mathbf{b}}^{-1} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}) \,\}$ span $\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}})$, this proves part (c). □

Let $\mathscr{H}_{\mathbf{b}} = \mathscr{H}_{pt_1,b_1}(\mathbf{Q}_1) \otimes \cdots \otimes \mathscr{H}_{pt_\kappa,b_\kappa}(\mathbf{Q}_\kappa)$. Let $T_i^{(\alpha)} = 1 \otimes \cdots \otimes T_i \otimes \cdots \otimes 1$ be a generator of $\mathscr{H}_{\mathbf{b}}$, where the T_i occurs in the α^{th} tensor factor, for $1 \le \alpha \le \kappa$. Then, as an algebra, $\mathscr{H}_{\mathbf{b}}$ is generated by the elements $\{\, T_i^{(\alpha)} \mid 1 \le \alpha \le \kappa \text{ and } 0 \le i < b_\alpha \,\}$.

We need some combinatorial machinery to describe the $\mathscr{H}_{\mathbf{b}}$–modules. Let

$$\Lambda_{\mathbf{b}}^+ = \{\, \boldsymbol{\lambda} \in \Lambda_n^+ \mid b_\alpha = |\lambda^{(pt_{1,\alpha-1}+1)}| + \cdots + |\lambda^{(pt_{1,\alpha})}|, \text{ for } \alpha = 1, \dots, \kappa \,\}.$$

Note that $\operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}) \ne \emptyset$ if and only if $\boldsymbol{\lambda} \in \Lambda_{\mathbf{b}}^+$.

For $\boldsymbol{\lambda} \in \Lambda_{\mathbf{b}}^+$ let $\boldsymbol{\lambda}_{\mathbf{b}} = (\boldsymbol{\lambda}_{\mathbf{b}}^{(1)}, \dots, \boldsymbol{\lambda}_{\mathbf{b}}^{(\kappa)})$, where $\boldsymbol{\lambda}_{\mathbf{b}}^{(\alpha)} = (\lambda^{(pt_{1,\alpha-1}+1)}, \dots, \lambda^{(pt_{1,\alpha})})$. Then the Specht modules of $\mathscr{H}_{\mathbf{b}}$ are all of the form $S(\boldsymbol{\lambda}_{\mathbf{b}}^{(1)}) \otimes \cdots \otimes S(\boldsymbol{\lambda}_{\mathbf{b}}^{(\kappa)})$, for $\boldsymbol{\lambda} \in \Lambda_{\mathbf{b}}^+$, and there is a natural bijection $\operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}) \cong \operatorname{Std}(\boldsymbol{\lambda}_{\mathbf{b}}^{(1)}) \times \cdots \times \operatorname{Std}(\boldsymbol{\lambda}_{\mathbf{b}}^{(\kappa)})$.

Let $\mathfrak{S}_{\mathbf{b}} = \mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_\kappa}$, which we consider as a subgroup of $\mathfrak{S}_n$ via the natural embedding. Let $\mathcal{D}_{\mathbf{b}}$ be the set of distinguished (minimal length) right coset representatives for $\mathfrak{S}_{\mathbf{b}}$ in $\mathfrak{S}_n$. Observe that if $\boldsymbol{\lambda} \in \Lambda_{\mathbf{b}}^+$ then

$$\operatorname{Std}(\boldsymbol{\lambda}) = \coprod_{d \in \mathcal{D}_{\mathbf{b}}} \operatorname{Std}_{\mathbf{b}}^+(\boldsymbol{\lambda}) d. \tag{2.13}$$

Recall that a **progenerator**, or projective generator, for an algebra A is a projective A–module V which contains every projective indecomposable A–module as a direct summand. The algebras A and $\operatorname{End}_A(V)$ are Morita equivalent and, moreover, every Morita equivalence arises in this way.

2.14. **Proposition.**

a) *Let* $V = \bigoplus_{\mathbf{b} \in \Lambda(n,\kappa)} V^{\mathbf{b}}$. *Then* V *is a progenerator for* $\mathscr{H}_{r,n}$.

b) *Suppose that* $\mathbf{b} \in \Lambda(n,\kappa)$. *Then:*
   i) $V^{\mathbf{b}}$ *is a projective* $\mathscr{H}_{r,n}$*–module;*
   ii) $\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}}) \cong \mathscr{H}_{\mathbf{b}}$*; and*

iii) *as left* $\mathscr{H}_{\mathbf{b}}$*–modules,* $\mathscr{H}_{\mathbf{b}} \cong V_0^{\mathbf{b}}$ *and* $V^{\mathbf{b}} = \bigoplus_{d\in\mathfrak{D}_b} V_0^{\mathbf{b}} T_d$*, where* $V_0^{\mathbf{b}}$ *is the subspace of* $V^{\mathbf{b}}$ *with basis* $\{\, v_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t}\in \operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$ *for some* $\boldsymbol{\lambda}\in\Lambda^+_{\mathbf{b}}\,\}$.

Sketch of proof. Using Proposition 2.12 and (2.13) it is straightforward to show that $\mathscr{H}_{r,n} \cong \bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} \bigoplus_{d\in\mathcal{D}_{\mathbf{b}}} T_d^* V^{\mathbf{b}}$ (see the proof of [**6**, Theorem 3.20]). Hence, $V^{\mathbf{b}}$ is a projective $\mathscr{H}_{r,n}$–module, proving b(i). Part (a) now follows because $V^{\mathbf{b}} \cong T_d^* V^{\mathbf{b}}$, for all $d\in\mathcal{D}_{\mathbf{b}}$.

Now consider the remaining parts of (b). By Proposition 2.4 and Lemma 2.6 there is an action of $\mathscr{H}_{\mathbf{b}}$ on $V^{\mathbf{b}}$ which is uniquely determined by letting the generator $T_i^{(\alpha)}$ of $\mathscr{H}_{\mathbf{b}}$ act as left multiplication by $L_{1+b_{\alpha+1,\kappa}}$ if $i=0$ and by $T_{i+1+b_{\alpha+1,\kappa}}$ if $1\le i<b_\alpha$. Thus, there is a map from $\mathscr{H}_{\mathbf{b}}$ into $\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}})$. The argument used to prove [**6**, Theorem 4.7] now shows that if $\boldsymbol{\lambda}\in\Lambda^+_{\mathbf{b}}$ and $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$ then the map $\theta_{\mathfrak{s}\mathfrak{t}}\in\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}})$ corresponds to left multiplication by the corresponding Murphy basis element of $\mathscr{H}_{\mathbf{b}}$, where we use the bijection $\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda}) \cong \operatorname{Std}(\boldsymbol{\lambda}^{(1)}_{\mathbf{b}}) \times \cdots \times \operatorname{Std}(\boldsymbol{\lambda}^{(\kappa)}_{\mathbf{b}})$. That is if $h\in\mathscr{H}_{r,p,n}$ then $\theta_{\mathfrak{s}\mathfrak{t}}(v_{\mathbf{b}}h) = (m_{\mathfrak{s}^{(1)}\mathfrak{t}^{(1)}}\otimes\cdots\otimes m_{\mathfrak{s}^{(\kappa)}\mathfrak{t}^{(\kappa)}})v_{\mathbf{b}}h$, where $\mathfrak{u}\in\operatorname{Std}^+_{\mathbf{b}}(\boldsymbol{\lambda})$ maps to $(\mathfrak{u}^{(1)},\dots,u^{(\kappa)})$ under the bijection above; see the proof of [**6**, Lemma 4.6]. This shows that $\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}}) \cong \mathscr{H}_{\mathbf{b}}$. Finally, part b(iii) follows from b(ii) and the observation that $V^{\mathbf{b}} = \bigoplus_{d\in\mathcal{D}_{\mathbf{b}}} V_0^{\mathbf{b}} T_d$, as a left $\mathscr{H}_{\mathbf{b}}$–module. □

Notice, in particular, that Theorem 2.2 is an immediate Corollary of the Proposition. We will also need the following result which follows directly from Proposition 2.14b(iii); compare [**6**, Remark 3.15].

2.15. **Corollary.** *Suppose that* $\mathbf{b}\in\Lambda(n,\kappa)$*. Then*

$$\{\, v_{\mathbf{b}} L_1^{c_1}\cdots L_n^{c_n} T_w \mid w\in\mathfrak{S}_n \text{ and } 0\le c_i< pt_\alpha \text{ whenever } b_{1,\alpha-1}<i\le b_{1,\alpha}\,\}$$

is a basis of $v_{\mathbf{b}}\mathscr{H}_{r,n,}$.

We now have the information that we need to start proving Theorem A from the introduction.

3. Morita equivalence theorems for algebras of type $G(r,p,n)$

In this section we prove Theorem A, the Morita reduction theorem for the Hecke algebras of type $G(r,p,n)$, by analyzing the structure of $V^{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ as an $\mathscr{H}_{r,p,n}$–module. We maintain our notation from the previous section. In particular, we fix a partitioning $\mathbf{Q} = \mathbf{Q}_1\vee\cdots\vee\mathbf{Q}_\kappa$ of $\mathbf{Q}$ such that Q_i and Q_j are in different (ε,q)–orbits whenever $Q_i\in\mathbf{Q}_\alpha$, $Q_j\in\mathbf{Q}_\beta$ and $\alpha\ne\beta$.

Following Ariki [**1**], for each integer m with $1\le m\le n$, define:

$$S_m = \begin{cases} T_0^p, & \text{if } m=1;\\ T_0^{-1}L_m, & \text{if } 2\le m\le n.\end{cases}$$

The elements $S_1, S_2, \cdots, S_n$ are the **Murphy operators** of $\mathscr{H}_{r,p,n}$. We need these elements to prove the following fundamental fact.

3.1. **Lemma.** *The algebra* $\mathscr{H}_{r,p,n}$ *has basis*

$$\{\, L_1^{c_1}\dots L_n^{c_n} T_w \mid w\in\mathfrak{S}_n, 0\le c_i<r \text{ and } c_1+\cdots+c_n\equiv 0 \bmod p\,\}.$$

Proof. Ariki [**1**, Prop. 1.6] showed that $\mathscr{H}_{r,p,n}$ is the subspace of $\mathscr{H}_{r,n}$ with basis

$$\left\{ S_1^{c_1}\cdots S_n^{c_n} T_w \,\middle|\, \begin{matrix} w\in\mathfrak{S}_n, 0\le c_i<r \text{ for } 2\le i\le n,\\ \text{and } 0\le pc_1-c_2-\cdots-c_n<r \end{matrix} \right\}.$$

Applying the definitions $S_1^{c_1}\cdots S_n^{c_n}T_w = L_1^{pc_1-c_2-\cdots-c_n} L_2^{c_2}\dots L_n^{c_n} T_w$. Hence, the Lemma is just a reformulation of Ariki's result. □

Recall from the introduction that there are two algebra automorphisms σ and τ of $\mathscr{H}_{r,n}$. The automorphism τ is the K-algebra automorphism of $\mathscr{H}_{r,n}$ which is defined on generators of $\mathscr{H}_{r,n}$ by

$$\tau(T_1) = T_0^{-1} T_1 T_0 \qquad \text{and} \qquad \tau(T_i) = T_i, \qquad \text{for } i \neq 1.$$

The map σ is the K-algebra automorphism of $\mathscr{H}_{r,n}$ which is determined by

$$\sigma(T_0) = \varepsilon T_0 \qquad \text{and} \qquad \sigma(T_i) = T_i, \qquad \text{for } i = 1, \dots, n-1.$$

The reader can check that $\mathscr{H}_{r,p,n}$ is the fixed point subalgebra of $\mathscr{H}_{r,n}$ under σ.

Suppose that A is an algebra with an automorphism θ of order p. Define $A \rtimes_\theta \mathbb{Z}_p$ to be the K-algebra with elements

$$\{\, a\theta^k \mid a \in A \text{ and } 0 \le k < p \,\}$$

and with multiplication $a\theta^k \cdot b\theta^l = a\theta^k(b)\theta^{k+l}$, for $a, b \in A$ and $0 \le k, l < p$. As in the introduction, if M is an A–module then we can define a new A–module M^θ which is isomorphic to M as a vector space but with the A–action twisted by θ. Formally, it is convenient to think of M^θ as the set of elements $\{\, m\theta \mid m \in M \,\}$ with A-action $m\theta \cdot a = \big(m\theta(a)\big)\theta$, for $m \in M$ and $a \in A$.

3.2. **Lemma.** *Suppose that $0 \le b \le n$. Then $\sigma(v_{\mathbf{b}}) = v_{\mathbf{b}}$ and $\tau(v_{\mathbf{b}}) \in v_{\mathbf{b}}\mathscr{H}_{r,p,n}$. Consequently, $\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big)^\sigma = v_{\mathbf{b}}\mathscr{H}_{r,n}$ and $\big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big)^\tau = v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ as $\mathscr{H}_{r,p,n}$-modules.*

Proof. Since $\sigma(T_i) = T_i$, for $1 \le i < n$, and $\sigma(L_k^p) = L_k^p$, for $1 \le k \le n$, we see that $\sigma(v_{\mathbf{b}}) = v_{\mathbf{b}}$. Furthermore, $T_0 v_{\mathbf{b}} = v_{\mathbf{b}} L_{b_{1,\kappa-1}+1}$ and $v_{\mathbf{b}} T_0 = L_{b_{2,\kappa}+1} v_{\mathbf{b}}$, by Proposition 2.4, so $\tau(v_{\mathbf{b}}) = T_0^{-1} v_{\mathbf{b}} T_0 = v_{\mathbf{b}} L_{b_{1,\kappa-1}+1}^{-1} L_1 = v_{\mathbf{b}} S_{b_{1,\kappa-1}+1}^{-1} \in v_{\mathbf{b}}\mathscr{H}_{r,p,n}$. ¿From what we have proved, $\sigma(v_{\mathbf{b}}\mathscr{H}_{r,n}) = v_{\mathbf{b}}\mathscr{H}_{r,n}$. Consequently, the map $v_{\mathbf{b}}h \mapsto \sigma(v_{\mathbf{b}}h) = \sigma(v_{\mathbf{b}})\sigma(h)$ defines a module isomorphism $v_{\mathbf{b}}\mathscr{H}_{r,n} \cong \big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big)^\sigma$, for $h \in \mathscr{H}_{r,n}$. Similarly, $\big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big)^\tau \cong v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ as $\mathscr{H}_{r,p,n}$-modules. $\square$

3.3. **Proposition.** *Suppose that $0 \le b \le n$. Then*

$$\left\{ v_{\mathbf{b}} L_1^{c_1} \cdots L_n^{c_n} T_w \;\middle|\; \begin{array}{c} w \in \mathfrak{S}_n \text{ and } 0 \le c_i < pt_\alpha \text{ whenever } b_{1,\alpha-1} < i \le b_{1,\alpha} \\ \text{and } c_1 + \cdots + c_n \equiv 0 \pmod p \end{array} \right\}$$

is a basis of $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$. In particular, $\dim v_{\mathbf{b}}\mathscr{H}_{r,p,n} = \frac{1}{p}\dim v_{\mathbf{b}}\mathscr{H}_{r,n}$.

Proof. First, observe that $\dim v_{\mathbf{b}}\mathscr{H}_{r,p,n} \ge \frac{1}{p}\dim v_{\mathbf{b}}\mathscr{H}_{r,n}$ since $\mathscr{H}_{r,n}$ is a free $\mathscr{H}_{r,p,n}$–module of rank p. By Corollary 2.15 the number of the elements given in the statement of the Proposition is exactly $\frac{1}{p}\dim v_{\mathbf{b}}\mathscr{H}_{r,n}$. Therefore, it suffices to show that the elements in the statement in the Proposition span $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$.

By Lemma 3.1 the module $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ is spanned by the elements

$$\left\{ v_{\mathbf{b}} L_1^{c_1} \cdots L_n^{c_n} T_w \;\middle|\; \begin{array}{c} w \in \mathfrak{S}_n, 0 \le c_i < r \text{ for } 1 \le i \le n \\ c_1 + \cdots + c_n \equiv 0 \pmod p \end{array} \right\}.$$

The elements $L_1, \dots, L_n$ commute by [**2**, Lemma 3.3]. Therefore, to prove the Proposition it is enough to show for $\alpha = 1, \dots, \kappa$ that if $b_{1,\alpha-1} < i \le b_{1,\alpha}$ then $v_{\mathbf{b}} L_{i-1}^c$ is a linear combination of terms of the form $v_{\mathbf{b}} L_{b_{1,\alpha-1}+1}^{a_{b_{1,\alpha-1}+1}} \cdots L_{i-1}^{a_{i-1}} T_w$, where $0 \le a_j < pt_\alpha$ for all j and w is an element of the symmetric group on the letters $\{b_{1,\alpha-1}+1, \dots, i-1\}$. We prove this by induction on i.

Fix α such that $b_\alpha \neq 0$ and $1 \le \alpha \le \kappa$. Suppose first that $i = i_0 \le b_{1,\alpha}$, where $i_0 = b_{1,\alpha-1} + 1$. Recall from Lemma 2.6 that

$$v_{\mathbf{b}} \cdot \prod_{Q_i \in \mathbf{Q}_\alpha} (L_{i_0}^p - Q_i^p) = 0.$$

Therefore, $v_{\mathbf{b}}L_{i_0}^{pt_\alpha}$ can be written as a linear combination of the elements $v_{\mathbf{b}}L_{i_0}^{pk}$, for $0 \le k < t_\alpha$. Note that, modulo p, we have not changed the exponent of L_{i_0}. Hence, we may assume that $0 \le c_i < pt_\alpha$ when $i = i_0$. Now suppose that $i_0 < i \le b_{1,\alpha}$. Arguing by induction (see [**2**, Lemma 3.3]), it follows easily that

$$L_i^c = q^{-1}T_{i-1}L_{i-1}^cT_{i-1} + (1-q^{-1})\sum_{d=1}^{c-1} L_{i-1}^{c-d}L_i^dT_{i-1}. \tag{3.4}$$

Therefore, using Proposition 2.5,

$$\begin{aligned} v_{\mathbf{b}}L_i^c &= q^{-1}v_{\mathbf{b}}T_{i-1}L_{i-1}^cT_{i-1} + (1-q^{-1})\sum_{d=1}^{c-1} v_{\mathbf{b}}L_{i-1}^{c-d}L_i^dT_{i-1}\\ &= q^{-1}T_{w_{\mathbf{b}}^{-1}(i-1)}v_{\mathbf{b}}L_{i-1}^cT_{i-1} + (1-q^{-1})\sum_{d=1}^{c-1} v_{\mathbf{b}}L_{i-1}^{c-d}L_i^dT_{i-1}. \end{aligned}$$

If $c \ge pt_\alpha$ then, by induction on i, we can rewrite $v_{\mathbf{b}}L_{i-1}^c$ as a linear combination of terms of the form $v_{\mathbf{b}}L_{i_0}^{a_{i_0}}\dots L_{i-1}^{a_{i-1}}T_w$, where $0 \le a_j < pt_\alpha$ for all j and w is an element of the symmetric group on the letters $\{i_0, \dots, i-1\}$. Now, $L_1, \dots, L_n$ commute with each other, and T_{i-1} commutes with L_j if $j \ne i-1, i$, so

$$\begin{aligned} T_{w_{\mathbf{b}}^{-1}(i-1)}v_{\mathbf{b}}L_{i_0}^{a_{i_0}}\dots L_{i-1}^{a_{i-1}}T_wT_{i-1} &= v_{\mathbf{b}}T_{i-1}L_{i_0}^{a_{i_0}}\dots L_{i-1}^{a_{i-1}}T_wT_{i-1}\\ &= v_{\mathbf{b}}L_{i_0}^{a_{i_0}}\dots L_{i-2}^{a_{i-2}}T_{i-1}L_{i-1}^{a_{i-1}}T_wT_{i-1} \end{aligned}$$

Hence, using (3.4) once again, we can rewrite $v_{\mathbf{b}}L_i^c$ as a linear combination of terms of the form $v_{\mathbf{b}}L_{i_0}^{a_{i_0}}\dots L_i^{a_i}T_w$, where $0 \le a_j < pt_\alpha$ for all j and w is an element of the symmetric group on the letters $\{i_0, \dots, i\}$. This proves our claim. Moreover, this completes the proof of the Proposition because, modulo p, the sums of the exponents of $L_1, \dots, L_n$ are unchanged in all of the formulae above. ☐

If M is an $\mathscr{H}_{r,p,n}$–module let $M\!\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} = M \otimes_{\mathscr{H}_{r,p,n}} \mathscr{H}_{r,n}$ be the corresponding **induced** $\mathscr{H}_{r,n}$–module. Similarly, if N is an $\mathscr{H}_{r,n}$–module let $N\!\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}$ be the **restriction** of N to $\mathscr{H}_{r,p,n}$. Since $\mathscr{H}_{r,n}$ is free as an $\mathscr{H}_{r,p,n}$–module both induction and restriction are exact functors.

3.5. **Corollary.** a) $v_{\mathbf{b}}\mathscr{H}_{r,n} \cong \big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big)\!\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}$,

b) $\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big)\!\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong \big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big)^{\oplus p}$,

c) $\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big)\!\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}} \cong \big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big)^{\oplus p}$.

Proof. Since $\mathscr{H}_{r,n} = \bigoplus_{k=0}^{p-1} T_0^k\mathscr{H}_{r,p,n}$, there is a surjective homomorphism from $\big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big)\!\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}$ onto $v_{\mathbf{b}}\mathscr{H}_{r,n}$. By Corollary 2.15 and Proposition 3.3 both modules have the same dimension so this map must be an isomorphism, proving (a). Part (b) now follows from Proposition 2.14b(iii); alternatively, use part (a) and Lemma 3.2. Part (c) follows from parts (a) and (b). ☐

3.6. **Corollary.** *Suppose that* $0 \le b \le n$ *as above. Then*

$$\frac{1}{p}\dim\operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big) = \dim\operatorname{End}_{\mathscr{H}_{r,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big) = p\dim\operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big).$$

Proof. The left and right hand equalities follow using Corollary 3.5 and Frobenius reciprocity. ☐

We can now prove Theorem A from the introduction.

3.7. **Theorem.** *Suppose that* $\mathbf{Q} = \mathbf{Q}_1 \vee \cdots \vee \mathbf{Q}_\kappa$*, where* $Q_i \in \mathbf{Q}_\alpha$ *and* $Q_j \in \mathbf{Q}_\beta$ *are in the same* (ε, q)*–orbit only if* $\alpha = \beta$*. Then* $\mathscr{H}_{r,p,n}$ *is Morita equivalent to the algebra*

$$\bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} \mathscr{H}_{\mathbf{b}} \rtimes \mathbb{Z}_p$$

Proof. By Proposition 2.14, $\bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} V^{\mathbf{b}}$ is a progenerator for $\mathscr{H}_{r,n}$. Hence, by restriction, it is also a progenerator for $\mathscr{H}_{r,p,n}$. By Proposition 2.12(b), Frobenius reciprocity and Corollary 3.5(c) if $\mathbf{b} \neq \mathbf{c}$ then $\operatorname{Hom}_{\mathscr{H}_{r,p,n}}(V^{\mathbf{b}}, V^{\mathbf{c}}) = 0$. Therefore, $\mathscr{H}_{r,p,n}$ is Morita equivalent to

$$\operatorname{End}_{\mathscr{H}_{r,p,n}}\Big(\bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} V^{\mathbf{b}}\Big) = \bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(V^{\mathbf{b}}\big).$$

Hence, to prove the Proposition it suffices to show that

$$\operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big) \cong \mathscr{H}_{\mathbf{b}} \rtimes \mathbb{Z}_p.$$

for all $\mathbf{b} \in \Lambda(n, \kappa)$.

Recall that each of the algebras $\mathscr{H}_{pt_\alpha,b_\alpha}(\mathbf{Q}_\alpha)$, for $1 \le \alpha \le \kappa$, has an automorphism σ_α of order p. The automorphism $\sigma_1 \otimes \cdots \otimes \sigma_\kappa$ acts diagonally on the algebra $\mathscr{H}_{\mathbf{b}}$. Note that $\langle \sigma_1 \otimes \cdots \otimes \sigma_\kappa\rangle \cong \mathbb{Z}_p$. By Proposition 2.14, we have that

$$\mathscr{H}_{\mathbf{b}} \cong \operatorname{End}_{\mathscr{H}_{r,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big) \hookrightarrow \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big).$$

On the other hand, $\sigma(v_{\mathbf{b}}) = v_{\mathbf{b}}$ by Lemma 3.2. So, σ induces an automorphism of $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ which is given by $v_{\mathbf{b}}h \mapsto \sigma(v_{\mathbf{b}}h) = v_{\mathbf{b}}\sigma(h)$, for all $h \in \mathscr{H}_{r,p,n}$. Hence, we have an injective map $\mathbb{Z}_p \hookrightarrow \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big)$. Since σ is an outer automorphism of $\mathscr{H}_{r,p,n}$ it follows that we have an embedding

$$\mathscr{H}_{\mathbf{b}} \rtimes \mathbb{Z}_p \hookrightarrow \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big).$$

Using Corollary 3.6 to compare the dimensions on both sides of this equation, we conclude that $\operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,n}\big) \cong \mathscr{H}_{\mathbf{b}} \rtimes \mathbb{Z}_p$. □

If instead of $V^{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ we consider the $\mathscr{H}_{r,p,n}$–module $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ then we obtain a second Morita reduction theorem for $\mathscr{H}_{r,p,n}$. To state this result, if $\mathbf{b} \in \Lambda(n, \kappa)$ set $\mathscr{H}_{p,\mathbf{b}} = \mathscr{H}_{pt_1,p,b_1}(\mathbf{Q}_1) \otimes \cdots \otimes \mathscr{H}_{pt_\kappa,p,b_\kappa}(\mathbf{Q}_\kappa)$. Observe that $\mathscr{H}_{p,\mathbf{b}}$ is a subalgebra of $\mathscr{H}_{\mathbf{b}}$ and that $\dim \mathscr{H}_{\mathbf{b}} = p^\kappa \dim \mathscr{H}_{p,\mathbf{b}}$. Next, let $\mathscr{H}'_{p,\mathbf{b}}$ be the subalgebra of $\mathscr{H}_{\mathbf{b}}$ generated by $\mathscr{H}_{p,\mathbf{b}}$ and the elements $\{\, T_0^{(1)}\big(T_0^{(\alpha)}\big)^{-1} \mid 1 < \alpha \le \kappa\,\}$.

3.8. **Proposition.** *Suppose that* $\mathbf{Q} = \mathbf{Q}_1 \vee \cdots \vee \mathbf{Q}_\kappa$*, where* Q_i *and* Q_j *are in different* (ε, q)*–orbits whenever* $Q_i \in \mathbf{Q}_\alpha$ *and* $Q_j \in \mathbf{Q}_\beta$ *for some* $\alpha \neq \beta$*. Let* $\mathbf{b} \in \Lambda(n, \kappa)$*. Then* $\mathscr{H}_{r,p,n}$ *is Morita equivalent to the algebra*

$$\bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} \mathscr{H}'_{p,\mathbf{b}}.$$

Proof. By Proposition 2.14(a), $\bigoplus_{\mathbf{b}} v_{\mathbf{b}}\mathscr{H}_{r,n}$ is a progenerator for $\mathscr{H}_{r,n}$. Therefore, $\bigoplus_{\mathbf{b}} v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ is a progenerator for $\mathscr{H}_{r,p,n}$ by Corollary 3.5(b). Furthermore, if $\mathbf{b} \neq \mathbf{c} \in \Lambda(n, \kappa)$ then $\operatorname{Hom}_{\mathscr{H}_{r,n}}(v_{\mathbf{b}}\mathscr{H}_{r,n}, v_{\mathbf{c}}\mathscr{H}_{r,n}) = 0$ by Proposition 2.12(b). By Corollary 3.5 and Frobenius reciprocity, $\operatorname{Hom}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}, v_{\mathbf{c}}\mathscr{H}_{r,p,n}\big) = 0$. Combining these results we see that $\mathscr{H}_{r,p,n}$ and

$$\operatorname{End}_{\mathscr{H}_{r,p,n}}\big(\bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big) = \bigoplus_{\mathbf{b}\in\Lambda(n,\kappa)} \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(v_{\mathbf{b}}\mathscr{H}_{r,p,n}\big)$$

are Morita equivalent.

Fix $\mathbf{b} \in \Lambda(n, \kappa)$ and let $E_{\mathbf{b}} = \operatorname{End}_{\mathscr{H}_{r,p,n}}(v_{\mathbf{b}}\mathscr{H}_{r,p,n})$. To complete the proof it is enough to show that $E_{\mathbf{b}} \cong \mathscr{H}'_{p,\mathbf{b}}$. As a left $\mathscr{H}_{\mathbf{b}}$–module, $v_{\mathbf{b}}\mathscr{H}_{r,n}$ is isomorphic to a direct sum of $[\mathfrak{S}_n{:}\mathfrak{S}_{\mathbf{b}}]$ copies of the regular representation of $\mathscr{H}_{\mathbf{b}}$ by Corollary 2.15. By Proposition 2.14b(iii), $\mathscr{H}_{p,\mathbf{b}}$ acts faithfully on $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ by restriction and this action commutes with the action of $\mathscr{H}_{r,p,n}$ from the right. Hence, we can identify $\mathscr{H}_{p,\mathbf{b}}$ with a subalgebra of $E_{\mathbf{b}}$.

By Lemma 3.2 and Proposition 2.14b(iii), $\tau(v_{\mathbf{b}}) = L_1^{-1}L_{b_{2,\kappa}+1}v_{\mathbf{b}} \in v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ acts on $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ in the same way that $T_0^{(1)}\big(T_0^{(\kappa)}\big)^{-1} = T_0 \otimes 1 \otimes \dots \otimes 1 \otimes T_0^{-1} \in \mathscr{H}_{\mathbf{b}}$ acts on $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$. More generally, for $\alpha = 1, \dots, \kappa-1$ let ρ_α be the automorphism of $v_{\mathbf{b}}\mathscr{H}_{r,p,n}$ given by left multiplication by $L_{b_{2,\kappa}+1}L^{-1}_{b_{\alpha+1,\kappa}+1} \in \mathscr{H}_{\mathbf{b}}$. By Proposition 2.4,

$$L_{b_{2,\kappa}+1}L^{-1}_{b_{\alpha+1,\kappa}+1}v_{\mathbf{b}} = v_{\mathbf{b}}L_1L^{-1}_{b_{1,\alpha-1}+1} = v_{\mathbf{b}}S^{-1}_{b_{1,\alpha-1}+1} \in v_{\mathbf{b}}\mathscr{H}_{r,p,n}.$$

Therefore, $\rho_\alpha \in E_{\mathbf{b}}$, for $2 \le \alpha \le \kappa$, since ρ_α commutes with the action of $\mathscr{H}_{r,p,n}$. Thus, ρ_α coincides with the action of $T_0^{(1)}\big(T_0^{(\alpha)}\big)^{-1} \in \mathscr{H}_{\mathbf{b}}$ on $V^{\mathbf{b}}$ in Proposition 2.14. Consequently, $\rho_\alpha \notin \mathscr{H}_{p,\mathbf{b}}$ and the automorphisms $\rho_2, \dots, \rho_\kappa$ commute. By definition, $\mathscr{H}'_{p,\mathbf{b}}$ is isomorphic to the subalgebra of $E_{\mathbf{b}}$ generated by $\mathscr{H}_{p,\mathbf{b}}$ and $\rho_2, \dots, \rho_\kappa$. Hence, $\mathscr{H}'_{p,\mathbf{b}}$ is a subalgebra of $E_{\mathbf{b}}$.

For each α, the map ρ_α^p acts as left multiplication by $\big(T_0^{(1)}\big)^p\big(T_0^{(\alpha)}\big)^{-p} \in \mathscr{H}_{\mathbf{b}}$, so $\rho_\alpha^p \in \mathscr{H}_{p,\mathbf{b}}$. Note that the extensions of the endomorphisms $\rho_\alpha, \dots, \rho_\alpha^{p-1}$ to $\operatorname{End}_{\mathscr{H}_{r,n}}(v_{\mathbf{b}}\mathscr{H}_{r,n})$, for $2 \le \alpha \le \kappa$, are all linearly independent since $\operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}}) \cong \mathscr{H}_b$ by Proposition 2.14. As these maps act on different components of $\mathscr{H}_{p,\mathbf{b}}$ it follows that $\dim \mathscr{H}'_{p,\mathbf{b}} = p^{\kappa-1}\dim \mathscr{H}_{p,\mathbf{b}}$. However, by Lemma 3.6 and Proposition 2.14b(ii),

$$\dim E_{\mathbf{b}} = \frac{1}{p}\dim \operatorname{End}_{\mathscr{H}_{r,n}}(V^{\mathbf{b}}) = \frac{1}{p}\dim \mathscr{H}_{\mathbf{b}} = p^{\kappa-1}\dim \mathscr{H}_{p,\mathbf{b}}.$$

Therefore, $\dim \mathscr{H}'_{p,\mathbf{b}} = p^{\kappa-1}\dim \mathscr{H}_{p,\mathbf{b}} = \dim E_{\mathbf{b}}$. So $E_{\mathbf{b}} \cong \mathscr{H}'_{p,\mathbf{b}}$, as required. □

4. Splittable decomposition numbers

In this section we use Clifford theory to show that if an algebra can be written be a semidirect product then its decomposition numbers are determined by the corresponding"p'-splittable" decomposition numbers of a related family of algebras. First, we recall some general results about the representation theory of semidirect product algebras. The basic references for this topic are [**4**, **15**, **18**].

Let A be a finite dimensional algebra over an algebraically closed field K and suppose that θ is an algebra automorphism of A of order p. We identify $\mathbb{Z}_p$ with the group generated by θ and consider the algebra $A \rtimes \mathbb{Z}_p$. Then, as a set,

$$A \rtimes \mathbb{Z}_p = \{\, a\theta^k \mid a \in A \text{ and } 0 \le k \le p \,\}$$

and the multiplication in $A \rtimes \mathbb{Z}_p$ is defined by

$$(a\theta^k) \cdot (b\theta^m) = a\theta^k(b) \cdot \theta^{k+m},$$

for $a, b \in A$ and $0 \le k, m < p$. If H is a subgroup of $\mathbb{Z}_p$ we identify $A \rtimes H$ with a subalgebra of $A \rtimes \mathbb{Z}_p$ in the natural way. In particular, by taking $H = 1$ we can view A as a subalgebra of $A \rtimes \mathbb{Z}_p$. Moreover, there are natural induction and restriction functors between the module categories of all of these algebras.

Suppose that L is an A–module. Then we can twist L by θ to get a new simple A-module L^θ. As a vector space we set $L^\theta = L$, and we define the action of A on L^θ by

$$v \cdot a := v\theta(a), \quad \text{for all } v \in L \text{ and } a \in A.$$

It is straightforward to check that L is irreducible if and only if L^θ is irreducible.

The **inertia group** of L is the group

$$G_L := \{\, \theta^k \in \mathbb{Z}_p \mid L \cong L^{\theta^k} \,\}.$$

Then G_L is a subgroup of $\mathbb{Z}_p$ and, in particular, it is cyclic. Let $l = |G_L|$. Then $p = lk$ and G_L is generated by θ^k. Recall that we have fixed a primitive pth root of unity $\varepsilon \in K$. Since K is algebraically closed we can choose an A-module isomorphism $\phi\colon L \longrightarrow L^{\theta^k}$ such that $\phi^l = 1_L$, the identity map on L. For each integer $i \in \mathbb{Z}$ define the $(A \rtimes G_L)$-module $L_{l,i}$ as follows: as a vector space $L_{l,i} := L$ and the action of $(A \rtimes G_L)$ on $L_{l,i}$ is given by:

$$v \cdot \big(a\theta^{mk}\big) := \varepsilon^{mki}\phi^m(va), \quad \text{for all } m \in \mathbb{Z}, v \in L_{l,i} \text{ and } a \in A.$$

It is easy to check that $L_{l,i}$ is an $(A \rtimes G_L)$-module and, by definition, $L_{l,i+l} \cong L_{l,i}$ for all $i \in \mathbb{Z}$.

Recall that $\operatorname{Irr}(A)$ is the complete set of isomorphism classes of simple A-modules. Let $L \in \operatorname{Irr}(A)$. Then, since $L_{l,i}\downarrow_A \cong L$, it follows that $L_{l,i}$ is a simple $(A \rtimes G_L)$-module. In fact, it is shown in [**4**, **15**, **18**] that

$$\Big\{ L_{l,1}\!\uparrow_{A\rtimes G_L}^{A\rtimes \mathbb{Z}_p}, \cdots, L_{l,l}\!\uparrow_{A\rtimes G_L}^{A\rtimes \mathbb{Z}_p} \;\Big|\; L \in \operatorname{Irr}(A), l = |G_L| \Big\}$$

is a complete set of pairwise non–isomorphic simple $(A \rtimes \mathbb{Z}_p)$-modules.

4.1. **Lemma.** *Suppose that L is a simple A–module and that H is a subgroup of G_L. Let $l = |G_L|$, $h = |H|$ and fix i with $1 \le i \le l$. Then $L_{l,i}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}$ is an irreducible $(A \rtimes H)$-module and*

$$L_{l,i}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}\uparrow_{A\rtimes H}^{A\times G_L} \cong \bigoplus_{j=1}^{[G_L:H]} L_{l,i+hj}.$$

Proof. As remarked above, we can view A and $A \rtimes H$ as subalgebras of $A \rtimes G_L$. Since $L_{l,i}\!\downarrow_{A}^{A\rtimes G_L} \cong L$ is irreducible we see that $L_{l,i}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}$ is irreducible. Next, observe that $\theta^{p/h}$ is a generator of H. Hence, by the argument e if $1 \le j \le l = |G_L|$ then $L_{l,i+hj}\!\downarrow_{A\rtimes H}^{A\rtimes G_L} \cong L_{l,i}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}$ is irreducible. Therefore, by Frobenius reciprocity and Schur's Lemma,

$$\begin{aligned}\operatorname{Hom}_{A\rtimes G_L}\big(L_{l,i+hj}, L_{l,i}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}\uparrow_{A\rtimes H}^{A\rtimes G_L}\big) &\cong \operatorname{Hom}_{A\rtimes H}\big(L_{l,i+hj}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}, L_{l,i}\!\downarrow_{A\rtimes H}^{A\rtimes G_L}\big)\\ &\cong K.\end{aligned}$$

The lemma now follows by comparing dimensions. □

Now suppose that we have a modular system $(F, \mathcal{O}, K)$ such that $A = A_K$ has an $\mathcal{O}$-lattice $A_{\mathcal{O}}$ which is an $\mathcal{O}$-algebra, θ can be lifted to an automorphism of $A_{\mathcal{O}}$ of order p, F is an algebraically closed field of characteristic zero, and $A_F := A_{\mathcal{O}} \otimes_{\mathcal{O}} F$ is a (split) semisimple F-algebra. We abuse notation and write θ for the corresponding automorphism of A_F. Note that if H is a subgroup of $\mathbb{Z}_p$ then $A_K \rtimes H \cong (A_{\mathcal{O}} \rtimes H) \otimes_{\mathcal{O}} K$, so that we also have a modular reduction system for the algebras $A_F \rtimes H$ and $A_K \rtimes H$.

By definition, $\operatorname{Irr}(A_F)$ is the complete set of isomorphism classes of simple A_F-modules — the "semisimple" A_F–modules. As the automorphism θ lifts to A_F for each simple A_F–module $S \in \operatorname{Irr}(A_F)$ we have an inertia group $G_S \le \mathbb{Z}_p$ and, as above, we can define $(A_F \rtimes G_S)$-modules $S_{s,j}$, for $j \in \mathbb{Z}$ where $s = |G_S|$. Consequently,

$$\Big\{ S_{s,1}\!\uparrow_{A_F\rtimes G_S}^{A_F\rtimes \mathbb{Z}_p}, \cdots, S_{s,s}\!\uparrow_{A_F\rtimes G_S}^{A_F\rtimes \mathbb{Z}_p} \;\Big|\; S \in \operatorname{Irr}(A_F) \text{ and } s = |G_S| \Big\}$$

is a complete set of pairwise non-isomorphic simple $(A_F \rtimes \mathbb{Z}_p)$-modules.

Suppose that $S \in \operatorname{Irr}(A_F)$ and that $D \in \operatorname{Irr}(A_K)$ and let $s = |G_S|$ and $d = |G_D|$. Given i and j with $1 \le i \le s$ and $1 \le j \le d$, we want to determine the decomposition numbers

$$[S_{s,i}\!\uparrow_{A_F\rtimes G_S}^{A_F\rtimes \mathbb{Z}_p} : D_{d,j}\!\uparrow_{A\rtimes G_D}^{A\rtimes \mathbb{Z}_p}],$$

which gives the multiplicity of $D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}$ as an irreducible composition factor of a modular reduction of $S_{s,i}\uparrow_{A\rtimes G_S}^{A\rtimes\mathbb{Z}_p}$.

4.2. **Definition.** *Suppose that* $S\in\operatorname{Irr}(A_F)$ *and* $D\in\operatorname{Irr}(A_K)$ *and set* $s=|G_S|$ *and* $d=|G_D|$. *Then the pair* (S,D) *has* ***cyclic decomposition numbers*** *if*

$$[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]=[S_{s,i+1}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+1}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}],$$

for all $i,j\in\mathbb{Z}$.

In the next section we show that all pairs of irreducible $\mathscr{H}_{r,n}$-modules have cyclic decomposition numbers.

4.3. **Proposition.** *Let* $S\in\operatorname{Irr}(A_F)$ *and* $D\in\operatorname{Irr}(A_K)$ *and suppose that* (S,D) *have cyclic decomposition numbers. Set* $s=|G_S|$, $d=|G_D|$ *and let* $d_0=\gcd(s,d)$.

Then

$$[S_{s,i+d_0l}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+d_0l'}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]=[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]$$

for all $i,i'j,j',l,l'\in\mathbb{Z}$.

Proof. Since the groups G_S, G_D and $G_0=G_S\cap G_D$ are all cyclic subgroups of $\mathbb{Z}_p$, we have that $|G_0|=\gcd(|G_S|,|G_D|)=\gcd(s,d)=d_0$. Therefore, there exist integers u and v such that $d_0=us+vd$. Consequently, if $1\le i\le s$, $1\le j\le d$ and $l\in\mathbb{Z}$ then

$$\begin{aligned}[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+d_0l}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]&=[S_{s,i+us}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+d_0l+us}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]\\&=[S_{s,i+us}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+d_0l+d_0-vd}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]\\&=[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+d_0(l+1)}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}],\end{aligned}$$

where the first equality follows since,by assumption, (S,D) has cyclic decomposition numbers. Similarly, we have that

$$[S_{s,i+d_0l}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]=[S_{s,i+d_0(l+1)}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}],$$

whenever $1\le i\le s$, $1\le j\le d$ and $l\in\mathbb{Z}$. Together these two identities imply that if $1\le i,i'\le s$, $1\le j,j'\le d$ and $l,l'\in\mathbb{Z}$ then

$$[S_{s,i+d_0l}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j+d_0l'}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]=[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}\colon D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}].$$

This completes the proof. ☐

4.4. **Definition.** *Suppose that* $S\in\operatorname{Irr}(A_F)$ *and* $D\in\operatorname{Irr}(A_K)$ *and let* $s=|G_S|$ *and* $d=|G_D|$ *as above. Then the decomposition number* $[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}:D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]$ *is* ***p-splittable*** *if* $G_S=\mathbb{Z}_p=G_D$.

We will see in Corollary 5.6 below that this definition of p–splittable decomposition number agrees with Definition 1.1 when applied to the algebras $\mathscr{H}_{r,p,n}$.

4.5. **Corollary.** *Suppose that* $S\in\operatorname{Irr}(A_F)$ *and* $D\in\operatorname{Irr}(A_K)$ *have cyclic decomposition numbers and let* $G_0=G_S\cap G_D$, $s=|G_S|$ *and* $d=|G_D|$. *Then*

$$[S_{s,i}\uparrow_{A_F\rtimes G_S}^{A_F\rtimes\mathbb{Z}_p}:D_{d,j}\uparrow_{A\rtimes G_D}^{A\rtimes\mathbb{Z}_p}]=[S_{s,i}\downarrow_{A_F\rtimes G_0}^{A_F\rtimes G_S}:D_{d,j}\downarrow_{A\rtimes G_0}^{A\rtimes G_D}]$$

for all $1\le i\le s$ *and all* $1\le j\le d$. *In particular, these decomposition numbers are* p'*-splittable in the sense of Definition 4.4, where* $p'=|G_0|$.

Proof. Observe that if L and L' are non–isomorphic simple $(A\rtimes G_0)$-modules then the induced modules $L\uparrow_{A\rtimes G_0}^{A\rtimes\mathbb{Z}_p}$ and $L'\uparrow_{A\rtimes G_0}^{A\rtimes\mathbb{Z}_p}$ have no common irreducible composition factors. Next observe that if $1\le i\le s$ then $S_{s,i}\downarrow_{A_F\rtimes G_0}^{A_F\rtimes G_S}$ is an irreducible $(A\rtimes G_0)$-module by Lemma 4.1. Similarly, if $1\le j\le d$ then $D_{d,j}\downarrow_{A\rtimes G_0}^{A\rtimes G_S}$ is an irreducible

$(A \rtimes G_0)$-module. The first claim now follows by combining this observation with Lemma 4.1 and Proposition 4.3. Finally, these decomposition numbers are p'-splittable because the inertia groups of the modules $S_{s,i}$ and $D_{d,j}$ inside G_0 is exactly G_0. ☐

Corollary 4.5 reduces the calculation of decomposition numbers of the algebra $A \rtimes \mathbb{Z}_p$ to determining the p'-splittable decomposition numbers of the subalgebras $A \rtimes \mathbb{Z}_{p'}$, where p' divides p.

We now apply Corollary 4.5 in the special case where A has a tensor product decomposition. That is, we suppose that $A = A^{(1)} \otimes \cdots \otimes A^{(\kappa)}$, for some finite dimensional K–algebras $A^{(1)}, \dots, A^{(\kappa)}$ which are equipped with automorphisms $\theta_1, \dots, \theta_\kappa$, respectively, of order p. Set $\theta = \theta_1 \otimes \cdots \otimes \theta_\kappa$. We assume that the tensor product decomposition of A is compatible with the modular system $(F, \mathcal{O}, K)$ so that $A_F = A_F^{(1)} \otimes \cdots \otimes A_F^{(\kappa)}$.

Fix an integer $f \in \{1, \dots, \kappa\}$ and let $S^{(f)} \in \operatorname{Irr}(A_F^{(f)})$ and $D^{(f)} \in \operatorname{Irr}(A_f)$ be irreducible modules. Then $S = S^{(1)} \otimes \cdots \otimes S^{(\kappa)} \in \operatorname{Irr}(A_F)$ and $D = D^{(1)} \otimes \cdots \otimes D^{(\kappa)} \in \operatorname{Irr}(A_K)$. Further, $G_S = G_{S^{(1)}} \cap \cdots \cap G_{S^{(\kappa)}}$, and $G_D = G_{D^{(1)}} \cap \cdots \cap G_{D^{(\kappa)}}$. Set $s = |G_S|$ and $d = |G_D|$. Then $s = |G_S| = \gcd(|G_{S^{(1)}}|, \dots, |G_{S^{(\kappa)}}|)$ and $d = |G_D| = \gcd(|G_{D^{(1)}}|, \dots, |G_{D^{(\kappa)}}|)$.

Suppose that (S, D) has cyclic decomposition numbers and let $G_0 = G_S \cap G_D$. Then, by Corollary 4.5, the decomposition numbers of $A \rtimes \mathbb{Z}_p$ are completely determined by the decomposition numbers of the form $[S_{s,i}{\downarrow}^{A_F \rtimes G_S}_{A_F \rtimes G_0} : D_{d,j}{\downarrow}^{A_K \rtimes G_D}_{A \rtimes G_0}]$, for $1 \le i \le s$ and $1 \le j \le d$.

4.6. **Theorem.** *Suppose that* $S = S^{(1)} \otimes \cdots \otimes S^{(\kappa)} \in \operatorname{Irr}(A_F)$ *and* $D = D^{(1)} \otimes \cdots \otimes D^{(\kappa)} \in \operatorname{Irr}(A_K)$. *Let* $s = |G_S|$, $d = |G_D|$ *and* $G_0 = G_S \cap G_D$ *and set* $d_0 = |G_0|$. *Then*

$$[S_{s,i}{\downarrow}^{A_F \rtimes G_S}_{A_F \rtimes G_0} : D_{d,j}{\downarrow}^{A_K \rtimes G_D}_{A_K \rtimes G_0}] = \sum_{\substack{0 \le j_1, \dots, j_\kappa < d_0 \\ j_1 + \cdots + j_\kappa \equiv (\kappa - 1)i + j \pmod{d_0}}} \prod_{\alpha=1}^{\kappa} [S^{(\alpha)}_{d_0,i} : D^{(\alpha)}_{d_0,j_\alpha}]$$

for all i *and* j *with* $1 \le i \le s$ *and* $1 \le j \le d$.

Proof. Suppose that $R \in \{F, K\}$. Let $k = \frac{p}{d_0} = |\mathbb{Z}_p / G_0|$, so that $G_0 = \langle \theta^k \rangle$. Consider the algebra

$$\widehat{A}_R = RG_0 \otimes (A_R^{(1)} \rtimes \langle \theta_1^k \rangle) \otimes \cdots \otimes (A_R^{(\kappa)} \rtimes \langle \theta_\kappa^k \rangle).$$

Then it is straightforward to check that there is an embedding of algebras $A_R \rtimes G_0 \hookrightarrow \widehat{A}_R$ given by

$$(a_1 \otimes \cdots \otimes a_\kappa)\theta^{mk} \mapsto \theta^{mk} \otimes \big(a_1 \theta_1^{mk}\big) \otimes \cdots \otimes \big(a_\kappa \theta_\kappa^{mk}\big),$$

for $a_1, \cdots, a_\kappa \in A_R$ and $m \in \mathbb{Z}$.

Let $L = L^{(1)} \otimes \cdots \otimes L^{(\kappa)}$ be a simple A_R-module, where $L = S$ if $R = F$, or $L = D$ if $R = K$. Let $l = |G_L|$ and suppose that $1 \le i \le l$. Then $L_{l,i}$ is a simple $(A_R \rtimes G_L)$-module. Recall that the modules $L_{l,i}$ are defined using a fixed isomorphism $\phi : L \longrightarrow L^{\theta^{p/l}}$ satisfying $\phi^l = \mathrm{id}$. We may assume that ϕ is compatible with the tensor decomposition of L; that is, $\phi = \widetilde{\phi}_1 \otimes \cdots \otimes \widetilde{\phi}_\kappa$, where for $f = 1, \dots, \kappa$, the order of the inertia group $G_{L^{(f)}}$ of $L^{(f)}$ in $\mathbb{Z}_p$ is l_f, $\widetilde{\phi}_f = (\phi_f)^{l_f/l}$, $\phi_f : L^{(f)} \longrightarrow \big(L^{(f)}\big)^{\theta_f^{p/l_f}}$ is an isomorphism defining the module $L^{(f)}_{l_f,i}$. Note that

$$L^{(f)}_{l_f,i}{\downarrow}^{A_R^{(f)} \rtimes G_{L^{(f)}}}_{A_R^{(f)} \rtimes \langle \theta_f^k \rangle} \cong L^{(f)}_{d_0,i}.$$

Applying the definitions, given any integers $i, i_0, i_1, \dots, i_\kappa$ with $i \equiv i_1 + \dots + i_\kappa - i_0 \pmod{d_0}$ there is a natural isomorphism of $(A_R \rtimes G_0)$-modules

$$(\dagger) \qquad L_{l,i}\downarrow^{A_R \rtimes G_L}_{A_R \rtimes G_0} \cong \big(\varepsilon^{-i_0} \otimes L^{(1)}_{d_0,i_1} \otimes \dots \otimes L^{(\kappa)}_{d_0,i_\kappa}\big)\downarrow^{\widehat{A}_R}_{A_R \rtimes G_0},$$

where ε^{-i_0} is the one dimensional representation of $G_0 = \langle \theta^k \rangle$ upon which θ^k acts as multiplication by ε^{-i_0}.

By (†), if $1 \le i \le s$ then $S_{s,i}\downarrow^{A_F \rtimes G_S}_{A_F \rtimes G_0} \cong \big(\varepsilon^{-(\kappa-1)i} \otimes S^{(1)}_{d_0,i} \otimes \dots \otimes S^{(\kappa)}_{d_0,i}\big)\downarrow^{\widehat{A}_F}_{A_F \rtimes G_0}$. Therefore, we can find the $(A_F \rtimes G_0)$-module composition factors of $S_{s,i}\downarrow^{A_F \rtimes G_S}_{A_F \rtimes G_0}$ by first finding the $\widehat{A}_K$-module composition factors of $\varepsilon^{-(\kappa-1)i} \otimes S^{(1)}_{d_0,i} \otimes \dots \otimes S^{(\kappa)}_{d_0,i}$ and then restricting to $A_K \rtimes G_0$. The $\widehat{A}_K$-module composition factors of $\varepsilon^{-(\kappa-1)i} \otimes S^{(1)}_{d_0,i} \otimes \dots \otimes S^{(\kappa)}_{d_0,i}$ are all of the form $\varepsilon^{-(\kappa-1)i} \otimes D^{(1)}_{d_0,j_1} \otimes \dots \otimes D^{(\kappa)}_{d_0,j_\kappa}$, where $0 \le j_1, \dots, j_\kappa < d_0$. Further, by (†), if $j \equiv j_1 + \dots + j_\kappa - (\kappa-1)i \pmod{d_0}$ then

$$D_{d,j}\downarrow^{A_K \rtimes G_D}_{A_K \rtimes G_0} \cong \big(\varepsilon^{-(\kappa-1)i} \otimes D^{(1)}_{d_0,j_1} \otimes \dots \otimes D^{(\kappa)}_{d_0,j_\kappa}\big)\downarrow^{\widehat{A}_K}_{A_K \rtimes G_0}$$

The theorem now follows. □

4.7. **Corollary.** *With the same notation as in Theorem 4.6, and suppose that* $S \in \operatorname{Irr}(A_F)$ *and* $D \in \operatorname{Irr}(A_K)$ *have cyclic decomposition numbers, then we have*

$$[S_{s,i}\uparrow^{A_F \rtimes \mathbb{Z}_p}_{A_F \rtimes G_S} : D_{d,j}\uparrow^{A \rtimes \mathbb{Z}_p}_{A \rtimes G_D}] = \sum_{\substack{0 \le j_1, \dots, j_\kappa < d_0 \\ j_1 + \dots + j_\kappa \equiv (\kappa-1)i + j \pmod{d_0}}} \prod_{\alpha=1}^{\kappa} [S^{(\alpha)}_{d_0,i} : D^{(\alpha)}_{d_0,j_\alpha}]$$

for all i *and* j *with* $1 \le i \le s$ *and* $1 \le j \le d$.

Proof. This follows from Corollary 4.5 and Theorem 4.6. □

5. Reduction theorems for the decomposition numbers of $\mathscr{H}_{r,p,n}$

We now combine the results of last two sections to show the decomposition numbers of the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}$ are completely determined by the p'-splittable decomposition numbers of an explicitly determined family of cyclotomic Hecke algebras. Throughout this section we assume that $q \ne 1$. We first show that the simple $\mathscr{H}_{r,n}$-modules always have cyclic decomposition numbers.

5.1. **Lemma.** *The algebras* $\mathscr{H}_{r,p,n}$ *and* $\mathscr{H}_{r,n} \rtimes \mathbb{Z}_p$ *are Morita equivalent.*

Proof. As a right $\mathscr{H}_{r,p,n}$–module $\mathscr{H}_{r,n} = \bigoplus_{k=0}^{p-1} T_0^k \mathscr{H}_{r,p,n}$ by Lemma 3.1. Consequently, $\mathscr{H}_{r,p}$ is a progenerator for $\mathscr{H}_{r,p,n}$, so $\mathscr{H}_{r,p,n}$ is Morita equivalent to $\operatorname{End}_{\mathscr{H}_{r,p,n}}(\mathscr{H}_{r,p})$. Observe that $\sigma \in \operatorname{End}_{\mathscr{H}_{r,p,n}}(\mathscr{H}_{r,n})$ since σ is trivial on $\mathscr{H}_{r,p,n}$. Furthermore, as vector spaces,

$$\begin{aligned}
\operatorname{End}_{\mathscr{H}_{r,p,n}}(\mathscr{H}_{r,n}) &\cong \operatorname{Hom}_{\mathscr{H}_{r,p,n}}(\mathscr{H}_{r,n}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}, \mathscr{H}_{r,p,n}^{\oplus p}) \\
&\cong \operatorname{Hom}_{\mathscr{H}_{r,p,n}}(\mathscr{H}_{r,n}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}, \mathscr{H}_{r,p,n})^{\oplus p} \\
&\cong \operatorname{Hom}_{\mathscr{H}_{r,n}}(\mathscr{H}_{r,n}, \mathscr{H}_{r,p,n}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}})^{\oplus p} \\
&\cong \operatorname{Hom}_{\mathscr{H}_{r,n}}(\mathscr{H}_{r,n}, \mathscr{H}_{r,n})^{\oplus p} \cong \mathscr{H}_{r,n}^{\oplus p}.
\end{aligned}$$

where the third isomorphism comes from Frobenius reciprocity. Hence, by counting dimensions, $\operatorname{End}_{\mathscr{H}_{r,p,n}}(\mathscr{H}_{r,n}) \cong \mathscr{H}_{r,n} \rtimes \mathbb{Z}_p$. (Alternatively, apply Theorem A with $\kappa = 1$.) □

The proof of this Lemma gives a Morita equivalence from the category of (finite dimensional right) $\mathscr{H}^R_{r,n} \rtimes \mathbb{Z}_p$–modules to the category of $\mathscr{H}^R_{r,p,n}$–modules. This functor sends the finite dimensional $(\mathscr{H}_{r,n} \rtimes \mathbb{Z}_p)$–module M to the $\mathscr{H}_{r,p,n}$–module

$$F(M) = M \otimes_{\mathscr{H}^R_{r,n} \rtimes \mathbb{Z}_p} \mathscr{H}^R_{r,n}.$$

5.2. **Lemma.** *Suppose that L is a simple $\mathscr{H}^R_{r,n}$–module. Then*

$$L{\downarrow}^{\mathscr{H}^R_{r,n}}_{\mathscr{H}^R_{r,p,n}} \cong F\big(L{\uparrow}^{\mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^R_{r,n}}\big).$$

Proof. Applying the definitions and standard properties of tensor products,

$$F\big(L{\uparrow}^{\mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^R_{r,n}}\big) = \big(L\otimes_{\mathscr{H}^R_{r,n}} \mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p\big)\otimes_{\mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p}\mathscr{H}^R_{r,n} \cong L = L{\downarrow}^{\mathscr{H}^R_{r,n}}_{\mathscr{H}^R_{r,p,n}}.$$

□

5.3. **Proposition** ([**7**, (2.2)], [**19**, (2.2)] and [**9**, (5.4), (5.5), (5.6)])**.** *Suppose that L is a simple $\mathscr{H}^R_{r,n}$-module and let $k>0$ be minimal such that $L\cong L^{\sigma^k}$. Then $1\le k\le p$ and $l:=\frac{p}{k}$ is the smallest positive integer such that $L'\cong L'^{\tau^l}$ whenever L' is a simple $\mathscr{H}_{r,p,n}$-submodule of L.*

Now fix an isomorphism $\phi\colon L\longrightarrow L^{\sigma^k}$ such that $\phi^l=1$ and for $i\in\mathbb{Z}$ define

$$L_i := \big\{v\in L \mid \phi(v)=\varepsilon^{-ik}v\big\}.$$

Then $L{\downarrow}^{\mathscr{H}^R_{r,n}}_{\mathscr{H}^R_{r,p,n}} = L_0\oplus\dots\oplus L_{l-1}$. Moreover, $L_i=L_{i+l}$ and $L_{i+1}\cong L_i^\tau$, for any $i\in\mathbb{Z}$. Consequently, $L{\downarrow}^{\mathscr{H}^R_{r,n}}_{\mathscr{H}^R_{r,p,n}} \cong L_0\oplus {L_0}^\tau\oplus\dots\oplus {L_0}^{\tau^{l-1}}$.

For each simple $\mathscr{H}^R_{r,n}$-module L we henceforth fix an isomorphism $\phi\colon L\longrightarrow L^{\sigma^k}$ such that $\phi^l=1$, where k and $l=\frac{p}{k}$ as in the Lemma. Observe that $l=|G_L|$, where G_L is the inertia group of L. For each integer i we have defined $\mathscr{H}_{r,p,n}$–modules L_i and $L_{l,i}$. The next result gives the connection between these two modules.

5.4. **Lemma.** *Suppose that L is a simple $\mathscr{H}^R_{r,n}$-module with inertia group G_L and let $l=|G_L|$. Then, for each $i\in\mathbb{Z}$, we have*

$$L_i \cong F\big(L_{l,i}{\uparrow}^{\mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^R_{r,n}\rtimes G_L}\big).$$

Proof. As in the proof of Lemma 5.2, $F\big(L_{l,i}{\uparrow}^{\mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^R_{r,n}\rtimes G_L}\big) \cong L_{l,i}\otimes_{\mathscr{H}^R_{r,n}\rtimes G_L}\mathscr{H}^R_{r,n}$. Therefore, there is a natural map $\psi\colon F\big(L_{l,i}{\uparrow}^{\mathscr{H}^R_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^R_{r,n}\rtimes G_L}\big)\longrightarrow L$ given by $\psi(v\otimes_{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_d} h)=vh$, for $v\in L_{l,i}$ and $h\in\mathscr{H}^R_{r,n}$. Clearly, $\psi\ne 0$ so it suffices to show that the image of ψ is contained in L_i. Now, if $v\in L_{l,i}$ and $h\in\mathscr{H}^R_{r,n}$ as above then, using the definition of $L_{l,i}$, we see that

$$\varepsilon^{ik}\phi(vh) = v\cdot(h\sigma^k) = \psi\big(v\cdot(h\sigma^k)\otimes_{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_d}1\big) = \psi\big(v\otimes_{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_d}h\big) = vh.$$

Hence, $\psi(vh)\in L_i$ as required. □

5.5. **Corollary.** *Suppose that $S\in\operatorname{Irr}(\mathscr{H}_{r,n})$ and $D\in\operatorname{Irr}(\mathscr{H}^K_{r,n})$ and let $s=|G_S|$ and $d=|G_D|$. Then*

$$[S_{s,i}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_S}\colon D_{d,j}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_D}] = [S_{s,i+1}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_S}\colon D_{d,j+1}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_D}],$$

for all $i,j\in\mathbb{Z}$. That is, (S,D) has cyclic decomposition numbers.

Proof. Using Lemma 5.4 and Proposition 5.3 we have

$$\begin{aligned}[S_{s,i+1}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_S}\colon D_{d,j+1}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_D}] &= [S_{i+1}:D_{j+1}] = [S_i^\tau:D_j^\tau] = [S_i:D_j]\\ &= [S_{s,i}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_S}\colon D_{d,j}{\uparrow}^{\mathscr{H}_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}_{r,n}\rtimes G_D}],\end{aligned}$$

as required. □

Note that Proposition 5.3 and Lemma 5.4 also imply that the two notions of p–splittable decomposition numbers for the algebras $\mathscr{H}_{r,p,n}$ coincide.

5.6. **Corollary.** *Suppose that $S \in \operatorname{Irr}(\mathscr{H}^F_{r,n})$ and $D \in \operatorname{Irr}(\mathscr{H}^K_{r,n})$ and let $s = |G_S|$ and $d = |G_D|$. Then the decomposition number*

$$[S_{s,i}\uparrow^{\mathscr{H}^F_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^F_{r,n}\rtimes G_S} : D_{d,j}\uparrow^{\mathscr{H}^K_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^K_{r,n}\rtimes G_D}]$$

is p–splittable in the sense of Definition 4.4 *if and only if the decomposition number*

$$[F(S_{s,i}\uparrow^{\mathscr{H}^F_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^F_{r,n}\rtimes G_S}) : F(D_{d,j}\uparrow^{\mathscr{H}^K_{r,n}\rtimes\mathbb{Z}_p}_{\mathscr{H}^K_{r,n}\rtimes G_D})]$$

is p–splittable in the sense of Definition 1.1*.*

5.7. **Theorem.** *Suppose that $q \neq 1$. Then every decomposition number of $\mathscr{H}_{r,p,n}(\mathbf{Q})$ is equal to a p'–splittable decomposition number of some cyclotomic Hecke algebra $\mathscr{H}_{r,p',n}(q, \mathbf{Q}')$, where $p = kp'$ and*

$$\mathbf{Q}' = (Q_1, \varepsilon Q_1, \dots, \varepsilon^{k-1}Q_1, Q_2, \dots, \varepsilon^{k-1}Q_2, \dots, Q_t, \dots, \varepsilon^{k-1}Q_t).$$

Proof. By Proposition 5.3 every irreducible $\mathscr{H}^F_{r,p,n}$-module is equal to S_i for some $S \in \operatorname{Irr}(\mathscr{H}^F_{r,n})$ and with $1 \le i \le s = |G_S|$. Similarly, every irreducible $\mathscr{H}^K_{r,p,n}$-module is equal to D_j for some $D \in \operatorname{Irr}(\mathscr{H}^K_{r,n})$ and with $1 \le j \le s = |G_D|$. Therefore, by Lemma 5.4, Corollary 5.5 and Corollary 4.5,

$$[S_i : D_j] = [S_{s,i}\downarrow^{\mathscr{H}^F_{r,n}\rtimes G_S}_{\mathscr{H}^F_{r,n}\rtimes G_0} : D_{d,j}\downarrow^{\mathscr{H}^F_{r,n}\rtimes G_D}_{\mathscr{H}^K_{r,n}\rtimes G_0}],$$

where $G_0 = G_S \cap G_D$. Suppose that $G_0 = \langle\sigma^k\rangle$ and write $p = kp'$. Then we have shown that $[S_i : D_j]$ is a p'–splittable decomposition number of $\mathscr{H}_{r,n} \rtimes \mathbb{Z}_{p'}$.

As at the beginning of section 2, write $r = pt$. Then $r = p'kt$ and in $\mathscr{H}_{r,p,n}(\mathbf{Q})$ the 'order relation' for T_0 is

$$0 = \prod_{b=1}^{t}\big(T_0^p - Q_b^p\big) = \prod_{b=1}^{t}\prod_{a=0}^{k-1}\big(T_0^{p'} - (\varepsilon^a Q_b)^{p'}\big).$$

Observe that the right hand side is the 'order relation' for $T_0^{p'}$ in $\mathscr{H}_{r,p',n}(\mathbf{Q}')$. It now follows using Lemma 5.1 that $\mathscr{H}_{r,n} \rtimes \mathbb{Z}_{p'}$ is Morita equivalent to $\mathscr{H}_{r,p',n}(q, \mathbf{Q}')$, where the parameters $\mathbf{Q}'$ are as given in the statement of the theorem. This completes the proof of the theorem. □

Proof of Theorem B. This follows from an recursive application of Theorem A, Corollary 4.7, Corollary 5.5 and Theorem 5.7. □

Theorem B gives a recursive algorithm for computing all of the decomposition numbers of a cyclotomic Hecke algebra $\mathscr{H}_{r,p,n}(\mathbf{Q})$ in terms of the p'-splittable decomposition numbers of a family of "smaller" cyclotomic Hecke algebras $\mathscr{H}_{r',p',n'}(\mathbf{Q}')$, where $1 \le r' \le r$, $1 \le n' \le n$, $1 \le p' \mid p$, and the parameters $\mathbf{Q}'$ are contained in a single (ε, q)-orbit of $\mathbf{Q}$. Therefore, the p'-splittable decomposition numbers of the cyclotomic Hecke algebras of type $G(r', p', n')$ completely determine the decomposition numbers of all cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}(\mathbf{Q})$.

Acknowledgments

The first author was supported partly by the National Natural Science Foundation of China (Project 10401005) and the Program NCET. He wishes to thank the School of Mathematics and Statistics at University of Sydney for their hospitality during his visit in 2006. Both authors were supported by an Australian Research Council discovery grant.

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