PRO-p GROUPS OF POSITIVE DEFICIENCY

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Abstract. Let Γ be a finitely presentable pro-p group with a nontrivial finitely generated closed normal subgroup N of infinite index. Then $\mathrm{def}(\Gamma) \leq 1$, and if $\mathrm{def}(\Gamma) = 1$ then Γ is a pro-p duality group of dimension 2, N is a free pro-p group and Γ/N is virtually free. In particular, if the centre of Γ is nontrivial and $\mathrm{def}(\Gamma) \geq 1$, then $\mathrm{def}(\Gamma) = 1$, $cd\ G \leq 2$ and Γ is virtually a direct product $F \times \mathbb{Z}_p$, with F a finitely generated free pro-p group.

If π is a finitely presentable group with $\mathrm{def}(\pi) \geq 1$ and $\beta_1^{(2)}(\pi) = 0$ then $\mathrm{def}(\pi) = 1$ and $cd\ \pi = 2$ or $\pi \cong \mathbb{Z}$. (See Theorem 2.5 of [Hi1].) The L^2-Betti number condition holds if π has a finitely generated infinite normal subgroup of infinite index, or if π has an infinite amenable normal subgroup. (See Chapter 7 of [Lue].)

We are interested in finding analogous results for pro-p groups. There is at present no good p-adic analogue of the von Neumann algebra, providing an invariant with the formal properties of the L^2-Betti numbers in the discrete case. Nevertheless, we shall show that if a finitely presentable pro-p group Γ with $\mathrm{def}(\Gamma) \geq 1$ has a nontrivial finitely generated closed normal subgroup N of infinite index then $\mathrm{def}(\Gamma) = 1$, Γ is a pro-p duality group of dimension 2, N is a free pro-p group and Γ/N is virtually free. In particular, if a finitely presentable pro-p group has a nontrivial finite normal subgroup then it has deficiency ≤ 0. (We do not know whether the analogue of the latter result holds in the discrete case.) We derive two corollaries from the main result. Firstly, if the centre of Γ is nontrivial and $\mathrm{def}(\Gamma) \geq 1$, then $\mathrm{def}(\Gamma) = 1$, $cd\ \Gamma \leq 2$ and Γ is virtually a direct product $F \times \mathbb{Z}_p$, with F a finitely generated free pro-p group. Secondly, if Γ has a finitely generated abelian closed normal subgroup A and $\mathrm{def}(\Gamma) \geq 1$, then $\mathrm{def}(\Gamma) = 1$, $cd\ \Gamma \leq 2$ and $A \cong \mathbb{Z}_p$ or A has finite index in Γ and $A \cong \mathbb{Z}_p^2$.

If G is a pro-p group let $H^i(G) = H^i(G; \mathbb{F}_p)$ and $h^i(G) = \dim_{\mathbb{F}_p} H^i(G)$, where $\mathbb{F}_p$ is the trivial G-module of order p. The following proposition is well-known. (See, for example, Proposition 3.9.4 of [NSW^2].)

2000 *Mathematics Subject Classification.* 20E18.

Key words and phrases. deficiency, normal subgroup, pro-p group.

Proposition 1. *Let p be a prime number and let G be a pro-p group. Then G is finitely presentable if and only if $h^1(G) < \infty$ and $h^2(G) < \infty$, and in this case* $\operatorname{def}(G) = h^1(G) - h^2(G)$. ☐

If the numbers $h^i(G)$ are finite for $i = 0, \dots, n$ we may define the nth partial Euler-Poincaré characteristic by $\chi_n(G) = \sum_{i=0}^{n}(-1)^i h^i(G)$. On applying Lemma 3.3.15 of [NSW^2] (first ed. 3.3.12) and Shapiro's Lemma to the (co)induced module $A = \mathbb{F}_p[G/U] \cong Hom_U(\mathbb{F}_p[G], \mathbb{F}_p)$, we obtain the inequality

$$\chi_n(U)(-1)^n \leq (G : U)\chi_n(G)(-1)^n$$

for every open subgroup $U \subseteq G$. In particular, $\chi_2(G) = 1 - \operatorname{def}(G)$ is submultiplicative: if U is an open subgroup of G then $\chi_2(U) \leq \chi_2(G)[G : U]$.

Theorem 2. *Let Γ be a finitely presentable pro-p group with a nontrivial finite normal subgroup N. Then* $\operatorname{def}(\Gamma) \leq 0$.

Proof. Let $a \in N$ be an element of order p and set $A = \langle a \rangle \subset N$. Let $U = C_\Gamma(A)$ be the centralizer of A. Then U is a closed subgroup of finite index in Γ, and hence is also open. Since $A \cong \mathbb{F}_p$ is central, the group extension $0 \to A \to U \to U/A \to 1$ is classified by an element $\alpha \in H^2(U/A)$. Now $H^2(U/A) = \varinjlim H^2(U/U')$, where the limit is taken over open subgroups $U' \leq U$ which contain A, and so α is in the image of $H^2(U/U')$ for some such U'. Clearly the restriction of α to $H^2(U'/A)$ vanishes. Then $U' \cong A \times U'/A$. As the Hochschild-Serre spectral sequence associated to a direct product degenerates at E^2, see [Ja] or Theorem 2.4.6 of [NSW^2], we obtain $h^1(U') = h^1(U'/A) + 1$ and $h^2(U') = h^2(U'/A) + 1 + h^1(U'/A)$. This implies $\chi_2(U') > 0$, hence $\chi_2(\Gamma) > 0$ and $\operatorname{def}(\Gamma) \leq 0$. ☐

In the discrete case there is no general result asserting that torsion cohomology classes restrict to 0 on suitable finite-index subgroups. (This is however true if the group is a surface group. See [Hi2].)

Theorem 3. *Let Γ be a finitely presentable pro-p group with a nontrivial finitely generated closed normal subgroup N of infinite index. Then* $\operatorname{def}(\Gamma) \leq 1$.

Proof. After passing to an open subgroup U we may assume that U/N acts trivially on the finite group $N/N^p[N,N]$ and that the corresponding extension splits: $U/N^p[N,N] \cong (U/N) \times N/N^p[N,N]$. Hence the transgression from $H^i(U/N; H^1(N))$ to $H^{i+2}(U/N)$ in the Hochschild-Serre spectral sequence for U as an extension of U/N by N is trivial. (See Theorem 2.4.4 of [NSW^2], first ed. 2.1.8.) It follows that

$h^1(U) = h^1(U/N) + h^1(N)$ and $h^2(U) \geq h^2(U/N) + h^1(U/N)h^1(N)$. Hence

$$\begin{aligned} \chi_2(U) &\geq 1 - h^1(U/N) - h^1(N) + h^1(U/N)h^1(N) \\ &= (h^1(U/N) - 1)(h^1(N) - 1). \end{aligned}$$

Since $h^1(U/N) \geq 1$ and $h^1(N) \geq 1$ it follows that $\chi_2(U) \geq 0$ and so $\chi_2(\Gamma) \geq 0$. Therefore $\mathrm{def}(\Gamma) \leq 1$. □

A slight sharpening of the estimate for $h^2(U)$ gives a stronger result.

Theorem 4. *Let Γ be a finitely presentable pro-p group with $\mathrm{def}(\Gamma) = 1$ and with a nontrivial finitely generated closed normal subgroup N of infinite index. Then Γ is a pro-p duality group of dimension 2, N is a free pro-p group and Γ/N is virtually free. Moreover, either $N \cong \mathbb{Z}_p$ or Γ/N is virtually $\mathbb{Z}_p$.*

Proof. We note first that N must be infinite, by Theorem 2. If U is an open subgroup of Γ then $U \cap N$ has finite index in N, and thus is a nontrivial finitely generated closed normal subgroup of infinite index in U. Hence $\mathrm{def}(U) \leq 1$, by Theorem 3. Thus $0 \leq \chi_2(U) \leq \chi_2(\Gamma)[\Gamma : U] = 0$, by the submultiplicativity of χ_2 and the hypothesis that $\mathrm{def}(\Gamma) = 1$. Therefore $\chi_2(U) = 0$ for all such subgroups U, and so $cd\ \Gamma \leq 2$, by Theorem 3.3.16 of [NSW^2] (first ed. 3.3.13).

As in Theorem 3 there is an open subgroup U containing N such that $U/N^p[N,N] \cong (U/N) \times N/N^p[N,N]$. Let $d_3 : H^0(U/N; H^2(N)) \to H^3(U/N)$ be the d_3^{02} differential of the Hochschild-Serre spectral sequence, and let $c = \dim_{\mathbb{F}_p}\mathrm{Ker}(d_3)$. Then $h^1(U) = h^1(U/N) + h^1(N)$ and $h^2(U) = h^2(U/N) + h^1(U/N)h^1(N) + c$. Since $\chi_2(U) = 0$ it follows that

$$(h^1(N) - 1)(h^1(U/N) - 1) + h^2(U/N) + c = 0.$$

Since N and U/N are each nontrivial $h^1(N) \geq 1$ and $h^1(U/N) \geq 1$. Thus the three summands are all non-negative and so must be 0. Since $h^2(U/N) = 0$ the quotient U/N is a free pro-p group. In particular, Γ/N is virtually free. Since $H^3(U/N) = 0$ and $c = 0$, it follows that $H^0(U/N; H^2(N)) = 0$. Since U/N is a pro-p group and $H^2(N)$ is a discrete p-torsion module, it follows that $H^2(N) = 0$. (See Corollary 1.6.13 of [NSW^2], first ed. 1.7.4.) Thus N is also a free pro-p group. (The fact that N is free if $h^1(U/N) = 1$ follows also from [Ko], since $\mathrm{def}(U) = 1$ and N is finitely generated.)

In particular, U is an extension of finitely generated free pro-p groups. Hence U is a pro-p duality group of dimension 2, by [Ple] Theorem 3.9. Since U is an open subgroup of Γ and $cd\ \Gamma \leq 2$, the group Γ is also a pro-p duality group of dimension 2, by [Ple] Theorem 3.8.

If $h^1(N) = 1$ then $N \cong \mathbb{Z}_p$; otherwise $h^1(U/N) = 1$ and so Γ/N is virtually $\mathbb{Z}_p$. $\square$

Corollary 1. *Let Γ be a finitely presentable pro-p group with nontrivial centre $\zeta\Gamma$. Then $\mathrm{def}(\Gamma) \leq 1$. If $\mathrm{def}(\Gamma) = 1$ then either $\Gamma \cong \mathbb{Z}_p^2$ or $\zeta\Gamma \cong \mathbb{Z}_p$. Moreover, Γ is virtually a direct product $F \times \mathbb{Z}_p$, with F a finitely generated free pro-p group.*

Proof. We may clearly assume that $\mathrm{def}(\Gamma) \geq 1$, since there is nothing to prove if $\mathrm{def}(\Gamma) < 1$.

If $[\Gamma : \zeta\Gamma]$ is finite the commutator subgroup $\Gamma' = [\Gamma, \Gamma]$ is finite, by a lemma of Schur. (See Proposition 10.1.4 of [Rob]. The argument given there for the discrete case extends without material change to the pro-p case.) Hence $\Gamma' = 1$, by Theorem 2, and so Γ is abelian. Moreover, Γ is torsion-free, by Theorem 2 again. Therefore $\Gamma \cong \mathbb{Z}_p^2$ or $\mathbb{Z}_p$, since $\mathrm{def}(\Gamma) \geq 1$. In particular, $\Gamma \cong F \times \mathbb{Z}_p$ with F free of rank 1 or 0.

Suppose now that $[\Gamma : \zeta\Gamma] = \infty$. Let C be a nontrivial finitely generated closed subgroup of $\zeta\Gamma$. Then C is a closed normal subgroup of infinite index in Γ, and is infinite, by Theorem 2. Hence $\mathrm{def}(\Gamma) = 1$, C is free and Γ/C is virtually free, by Theorem 3. But then $C \cong \mathbb{Z}_p$ and $\zeta\Gamma/C$ is finite, since it is a central closed subgroup of infinite index in a virtually free pro-p group. Thus $\zeta\Gamma$ is also finitely generated and so $\zeta\Gamma \cong \mathbb{Z}_p$, by the same argument.

Let F be a free pro-p subgroup of finite index in $\Gamma/\zeta\Gamma$. Since $\Gamma/\zeta\Gamma$ is virtually free Γ has an open subgroup U containing $\zeta\Gamma$ and such that $F = U/\zeta\Gamma$ is a finitely generated free pro-p group. Since F is free and acts trivially on $\zeta\Gamma$ the extension splits, and so $U \cong F \times \mathbb{Z}_p$. $\square$

Such direct products $F \times \mathbb{Z}_p$ with F a finitely generated free pro-p group of rank > 1 clearly have deficiency 1 and centre $\mathbb{Z}_p$.

Corollary 2. *Let Γ be a finitely presentable pro-p group with a nontrivial finitely generated closed abelian normal subgroup A. Then we have $\mathrm{def}(\Gamma) \leq 1$. If $\mathrm{def}(\Gamma) = 1$, then $cd\,\Gamma \leq 2$. Moreover, $A \cong \mathbb{Z}_p$ or A has finite index and $A \cong \mathbb{Z}_p^2$.*

Proof. If A has a nontrivial torsion subgroup T, then T is finite, and so $\mathrm{def}(\Gamma) < 1$, by Theorem 2. So we may assume that $A \cong \mathbb{Z}_p^n$ for some $n \geq 1$. If A has infinite index in Γ, then $\mathrm{def}(\Gamma) \leq 1$ by Theorem 3. So assume that $(\Gamma : A) < \infty$. Then $0 \leq \chi_2(A) \leq (\Gamma : A)\chi_2(\Gamma)$, implying $\chi_2(\Gamma) \geq 0$, hence $\mathrm{def}(\Gamma) \leq 1$. The same argument shows $\mathrm{def}(\Gamma) < 1$ if $n \geq 3$.

Finally, assume that $\mathrm{def}(\Gamma) = 1$. If A has finite index in Γ, then $A \cong \mathbb{Z}_p$ or $A \cong \mathbb{Z}_p^2$ by the argument above. The open subgroups U of

A are cofinal among the open subgroups of Γ and we have $\chi_2(U) = 0$. Since $\chi_2(\Gamma) = 1 - \mathrm{def}(\Gamma) = 0$, we obtain $cd\ \Gamma \leq 2$ by Theorem 3.3.16 of [NSW^2] (first ed. 3.3.13).

If A has infinite index, it is free by Theorem 4. Hence $A \cong \mathbb{Z}_p$ in this case. Furthermore $cd\ \Gamma = 2$, again by Theorem 4. □

The questions considered here can be traced back to Murasugi and Gottlieb. Murasugi [Mu] conjectured that if a finitely presentable (discrete) group π has nontrivial centre then $\mathrm{def}(\pi) \leq 1$, with equality only if $\pi \cong \mathbb{Z}^2$ or $\zeta\pi \cong \mathbb{Z}$, and verified this for one-relator groups and classical link groups, while Gottlieb [Go] showed that if the fundamental group of an aspherical finite complex X has nontrivial centre then $\chi(X) = 0$.

Nakamura [Na] has given a direct analogue of Gottlieb's Theorem for pro-p groups: if G is a pro-p group with $\zeta G \neq 1$, $cd_p G < \infty$ and $\beta_i(G;\mathbb{F}_p) < \infty$ for all i then $\chi(G;\mathbb{F}_p) = 0$. (He uses a pro-p version of Stallings' argument involving universal trace functions.) This result and the above Corollary 1 overlap, but neither implies the other.

Amenable groups have no nonabelian free subgroups. The latter notion extends naturally to the pro-p case. If a finitely presentable discrete group has deficiency > 1 then it contains a nonabelian free group [Rom]. If a finitely presentable pro-p group Γ with $h^1(\Gamma) = d > 1$ has no nonabelian free subgroup then $h^2(\Gamma) \geq \frac{d^2}{4}$ [Ze]. Thus either $d = 2$ and Γ is a one-relator pro-p group or $\mathrm{def}(\Gamma) \leq 0$. Do either of these arguments extend to (discrete or pro-p) groups with infinite normal subgroups having no nonabelian free subgroup?

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