MORITA EQUIVALENCES OF CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(r, p, n)$ II: THE (ε, q)-SEPARATED CASE

JUN HU AND ANDREW MATHAS

Abstract. The paper studies the modular representation theory of the cyclotomic Hecke algebras of type $G(r, p, n)$ with (ε, q)-separated parameters. We show that the decomposition numbers of these algebras are completely determined by the decomposition matrices of related cyclotomic Hecke algebras of type $G(s, 1, m)$, where $1 \leq s \leq r$ and $1 \leq m \leq n$. Furthermore, the proof gives an explicit algorithm for computing these decomposition numbers meaning that the decomposition matrices of these algebras are now known in principle.

In proving these results, we develop a Specht module theory for these algebras, explicitly construct their simple modules and introduce and study analogues of the cyclotomic Schur algebras of type $G(r, p, n)$ when the parameters are (ε, q)-separated.

The main results of the paper rest upon two Morita equivalences: the first reduces the calculation of all decomposition numbers to the case of the l-*splittable decomposition numbers* and the second Morita equivalence allows us to compute these decomposition numbers using an analogue of the cyclotomic Schur algebras for the Hecke algebras of type $G(r, p, n)$.

Contents

2000 *Mathematics Subject Classification.* 20C08, 20C30.
Key words and phrases. Cyclotomic Hecke algebras, Morita equivalence.

1. Introduction

The cyclotomic Hecke algebras [**5**] are an important class of algebras which arise in the representation theory of finite reductive groups. These algebras can be defined using generators and relations and they are deformations of the group algebras of the complex reflection groups. The cyclotomic Hecke algebras can also be constructed using the monodromy representation of the associated braid groups [**6**] and, in characteristic zero, they are closely connected with category $\mathcal{O}$ of the rational Cherednik algebras by the Knizhnik-Zamolodchikov functor [**17**].

This paper is concerned with the representation theory of the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}$ of type $G(r,p,n)$, where $r = pd$, $p > 1$ and $n \geq 3$. Throughout we work over a field K which contains a primitive pth root of unity ε. The algebra $\mathscr{H}_{r,p,n}$ depends upon ε and parameters $q \in K$ and $\mathbf{Q} = (Q_1, \dots, Q_d) \in K^d$ (see Definition 2.1). The parameters $\mathbf{Q}$ are (ε, q)**-separated** over K if

$$\prod_{1\le i,j\le d}\ \prod_{-n<k<n}\ \prod_{1\le t<p} \big(Q_i - \varepsilon^t q^k Q_j\big) \neq 0$$

in K. In general, $\mathscr{H}_{r,p,n}$ is not semisimple if $\mathbf{Q}$ is (ε, q)-separated over K.

Our main result is the following.

Theorem A. *Suppose that K is a field of characteristic zero and that $\mathbf{Q}$ is (ε, q)-separated over K. Then the decomposition matrix of $\mathscr{H}_{r,p,n}$ is determined by the decomposition matrices of the cyclotomic Hecke algebras of type $G(s,1,m)$, where $1 \leq s \leq r$ and $1 \leq m \leq n$.*

To prove Theorem A we explicitly compute the l–splittable decomposition numbers (Definition 1.2) of $\mathscr{H}_{r,p,n}$, where l divides p. Theorem D at the end of this introduction gives our closed formula for the l-splittable decomposition numbers. This formula depends on the decomposition numbers of the Hecke algebras $\mathscr{H}_{s,m}$ of type $G(s,1,m)$, ε and certain scalars $\mathfrak{g}_{\boldsymbol{\lambda}}$ which are roots of certain quotients of the (known) Schur elements of these algebras. This result implies Theorem A because in earlier work [**23**, Theorem B and Theorem 5.7] we showed that every decomposition number of $\mathscr{H}_{r,p,n}$ is a sum of l-splittable decomposition numbers for certain Hecke algebras $\mathscr{H}_{s,l,m}$, where $1 \leq s \leq r$ and $1 \leq m \leq n$. The proof of Theorem A also gives detailed information about the decomposition numbers of $\mathscr{H}_{r,p,n}$ in positive characteristic.

Our proof of Theorem A, when combined with the results of [**23**], gives an explicit algorithm for computing the decomposition numbers of $\mathscr{H}_{r,p,n}$ when $\mathbf{Q}$ is (ε, q)-separated. Ariki [**2**] determined the decomposition numbers of the Hecke algebras $\mathscr{H}_{r,n}$ of type $G(r,1,n)$ when he, famously, proved and generalised the LLT conjecture. Hence, combining [**2**] and Theorem A implies the following.

Corollary. *Suppose that K is a field of characteristic zero and that $\mathbf{Q}$ is (ε, q)-separated over K. Then the decomposition matrix of $\mathscr{H}_{r,p,n}$ is, in principle, known.*

We note that Theorem A and its corollary have both been obtained by the first author in the special case of the Hecke algebras of type D, when $r = p = 2$ [**22**].

All of the results in this paper are geared towards computing the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ and this requires a considerable amount of new representation theory.

This story begins with the Morita equivalence theorem of Dipper and the second author [**10**] which shows, modulo some technical assumptions on the parameters $\mathbf{Q}$, that there is a Morita equivalence

$$\mathscr{H}_{r,n} \xrightarrow[\text{Morita}]{\simeq} \bigoplus_{\mathbf{b}\in\mathscr{C}_{p,n}} \mathscr{H}_{d,\mathbf{b}}, \tag{1.1}$$

where $\mathscr{C}_{p,n}$ is the set of compositions of n into p parts and if $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ then $\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,b_1} \otimes \cdots \otimes \mathscr{H}_{d,b_p}$. This result is proved by constructing an explicit $(\mathscr{H}_{d,\mathbf{b}}, \mathscr{H}_{r,n})$-bimodule $V_\mathbf{b} = v_\mathbf{b}\mathscr{H}_{r,n}$ (Definition 2.6), and showing that $V_\mathbf{b}$ is projective as an $\mathscr{H}_{r,n}$-module and that $\mathscr{H}_{d,\mathbf{b}} \cong \operatorname{End}_{\mathscr{H}_{r,n}}(V_\mathbf{b})$.

For each $\mathbf{b} \in \mathscr{C}_{p,n}$, we show in Theorem 2.26 that there exists an invertible central element $z_\mathbf{b}$ in the centre of $\mathscr{H}_{d,\mathbf{b}}$ such that $e_\mathbf{b} = z_\mathbf{b}^{-1} \cdot v_\mathbf{b} T_\mathbf{b}$ is the idempotent in $\mathscr{H}_{r,n}$ which generates $V_\mathbf{b}$, where $T_\mathbf{b} = T_{w_\mathbf{b}}$ for a certain permutation $w_\mathbf{b}$. As a byproduct we construct a *parabolic subalgebra* of $\mathscr{H}_{r,n}$ which is isomorphic to $\mathscr{H}_{d,\mathbf{b}}$ and we show that the Morita equivalence (1.1) corresponds to induction from these subalgebras.

More importantly, however, the element $z_\mathbf{b}$ allows us to decompose certain $\mathscr{H}_{r,n}$-modules when we restrict them to $\mathscr{H}_{r,p,n}$. To describe this, recall from [**9**] that $\mathscr{H}_{r,n}$ is a cellular algebra with cell modules the **Specht modules** $S(\boldsymbol{\lambda})$, which are indexed by the r-multipartitions $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \dots, \boldsymbol{\lambda}^{(r)})$ of n. If $\mathscr{H}_{r,n}$ is semisimple then the Specht modules gives a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{r,n}$-modules. More generally, define $D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda})/\operatorname{rad} S(\boldsymbol{\lambda})$, where $\operatorname{rad} S(\boldsymbol{\lambda})$ is the radical of the bilinear form on $S(\boldsymbol{\lambda})$. Then the non-zero $D(\boldsymbol{\lambda})$ give a complete set of pairwise non-isomorphic $\mathscr{H}_{r,n}$-modules.

For each $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$, we write $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{[1]}, \cdots, \boldsymbol{\lambda}^{[p]})$, where

$$\boldsymbol{\lambda}^{[t]} = (\boldsymbol{\lambda}^{(dt-d+1)}, \boldsymbol{\lambda}^{(dt-d+2)}, \dots, \boldsymbol{\lambda}^{(dt)}), \quad \text{for } 1 \le t \le p.$$

Let $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ and set $\mathscr{P}_{d,\mathbf{b}} = \{\, \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \mid |\boldsymbol{\lambda}^{[t]}| = b_t \text{ for } 1 \le t \le p \,\}$. We want to describe the Specht modules of $\mathscr{H}_{d,\mathbf{b}}$ for each $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. By [**9**] the algebra $\mathscr{H}_{d,\mathbf{b}}$ is a cellular algebra with cell modules $S_\mathbf{b}(\boldsymbol{\lambda}) \cong S(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes S(\boldsymbol{\lambda}^{[p]})$, for $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, and $D_\mathbf{b}(\boldsymbol{\lambda}) = S_\mathbf{b}(\boldsymbol{\lambda})/\operatorname{rad} S_\mathbf{b}(\boldsymbol{\lambda})$ is either absolutely irreducible or zero.

Let $\mathcal{F} = \mathbb{Q}(\dot\varepsilon, \dot q, \dot{\mathbf{Q}}, A(\dot\varepsilon, \dot q, \dot{\mathbf{Q}})$, where $\dot\varepsilon \in \mathbb{C}$ is a fixed primitive pth root of unity in $\mathbb{C}$, $\dot q$ and $\dot{\mathbf{Q}}$ are indeterminates and $A(\dot\varepsilon, \dot q, \dot{\mathbf{Q}})$ is a certain polynomial which ensures that $\dot{\mathbf{Q}}$ is $(\dot\varepsilon, \dot q)$-separated over $\mathcal{F}$; see Definition 2.17. Then the cyclotomic Hecke algebras $\mathscr{H}_{r,n}^{\mathcal{F}}$ and $\mathscr{H}_{d,\mathbf{b}}^{\mathcal{F}}$ over $\mathcal{F}$ are semisimple and they come equipped with non-degenerate trace forms Tr and $\operatorname{Tr}_\mathbf{b}$, respectively. Define the **Schur elements** $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}$ and $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}$ in $\mathcal{F}$ by

$$\operatorname{Tr} = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{r,n}} \frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}} \chi^{\boldsymbol{\lambda}} \quad \text{and} \quad \operatorname{Tr}_\mathbf{b} = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}} \frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}} \chi_\mathbf{b}^{\boldsymbol{\lambda}},$$

where $\chi^{\boldsymbol{\lambda}}$ and $\chi_\mathbf{b}^{\boldsymbol{\lambda}}$ are the characters of $S(\boldsymbol{\lambda})$ and $S_\mathbf{b}(\boldsymbol{\lambda})$, respectively. The Schur elements $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}$ and $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}$ are explicitly known [**27**].

Given an integer $k \in \mathbb{Z}$ and a sequence $\mathbf{a} = (a_1, a_2, \dots, a_m)$ define $\mathbf{a}\langle k\rangle = (a_{k+1}, a_{k+2}, \dots, a_{k+m})$, where we set $a_{i+jm} = a_i$ whenever $j \in \mathbb{Z}$ and $1 \le i \le m$. Now define $\mathsf{o}_m(\mathbf{a}) = \min\{\, k \ge 1 \mid \mathbf{a}\langle k\rangle = \mathbf{a} \,\}$. In particular, for $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{[1]}, \dots, \boldsymbol{\lambda}^{[p]}) \in \mathscr{P}_{d,\mathbf{b}}$ we define

$$\mathsf{o}_{\boldsymbol{\lambda}} = \mathsf{o}_p(\boldsymbol{\lambda}) \quad \text{and} \quad p_{\boldsymbol{\lambda}} = p/\mathsf{o}_{\boldsymbol{\lambda}}.$$

Note that $\mathsf{o}_{\boldsymbol{\lambda}}$ divides p so that $p_{\boldsymbol{\lambda}}$ is an integer, for all $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$.

Theorem B. *Suppose that $\mathbf{Q}$ is (ε, q)-separated over K and that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Then there exists a nonzero scalar $\mathfrak{f}_{\boldsymbol{\lambda}} \in K$ such that $z_\mathbf{b} \cdot v = \mathfrak{f}_{\boldsymbol{\lambda}} v$, for all $v \in S(\boldsymbol{\lambda})$.*

Moreover,

$$\mathfrak{f}_{\boldsymbol{\lambda}} = (\mathfrak{s}_{\boldsymbol{\lambda}}/\mathfrak{s}^{\mathbf{b}}_{\boldsymbol{\lambda}})\,\mathrm{Tr}(v_{\mathbf{b}}T_{\mathbf{b}}) = \varepsilon^{\frac{1}{2}do_{\boldsymbol{\lambda}}n(1-p_{\boldsymbol{\lambda}})}\mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}},$$

where $\mathfrak{g}_{\boldsymbol{\lambda}} \in K$ *and where* $(\mathfrak{s}_{\boldsymbol{\lambda}}/\mathfrak{s}^{\mathbf{b}}_{\boldsymbol{\lambda}})(\varepsilon, q, \mathbf{Q}) = (\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}/\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}})(\varepsilon, q, \mathbf{Q})$ *is the specialization of the rational function* $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}/\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}$ *at* $(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) = (\varepsilon, q, \mathbf{Q})$ (*which is well-defined and non-zero*).

Roughly the first half of this paper is devoted to proving Theorem B, but the payoff is considerable as the scalars $\mathfrak{g}_{\boldsymbol{\lambda}}$ play a role in everything that follows. To show that $\varepsilon^{\frac{1}{2}do_{\boldsymbol{\lambda}}n(p_{\boldsymbol{\lambda}}-1)}\mathfrak{f}_{\boldsymbol{\lambda}}$ has a $p_{\boldsymbol{\lambda}}$th root we use seminormals forms for the Specht modules over $\mathcal{F}$ and an integral independence result from commutative algebra. Further, we explicitly compute $\mathrm{Tr}(v_{\mathbf{b}}T_{\mathbf{b}})$ in Corollary 2.33. This yields an explicit closed formula for $\mathfrak{f}_{\boldsymbol{\lambda}}$.

As $\mathscr{H}_{r,n}$ is a cellular algebra, $\{\, D(\boldsymbol{\lambda}) \mid D(\boldsymbol{\lambda} \neq 0\,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{r,n}$-modules. Let $\mathscr{K}_{r,n} = \{\boldsymbol{\lambda} \in \mathscr{P}_{r,n} | D(\boldsymbol{\lambda}) \neq 0\}$. Then $\mathscr{K}_{r,n}$ is the set of **Kleshchev multipartitions** (for $\mathbf{Q}^{\vee\varepsilon}$). Define an equivalence relation $\sim_\sigma$ on $\mathscr{P}_{r,n}$ by $\boldsymbol{\lambda} \sim_\sigma \boldsymbol{\mu}$ if $\boldsymbol{\lambda} = \boldsymbol{\mu}\langle k\rangle$, for some $k \in \mathbb{Z}$ and $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{r,n}$. If $\mathbf{Q}$ is (ε, q)-separated over K, then $\sim_\sigma$ induces an equivalence relation on $\mathscr{K}_{r,n}$ (cf. Lemma 3.3). Let $\mathscr{P}^{\sigma}_{r,n}$ and $\mathscr{K}^{\sigma}_{r,n}$ be the sets of $\sim_\sigma$-equivalence classes in $\mathscr{P}_{r,n}$ and $\mathscr{K}_{r,n}$, respectively.

As a first application of Theorem B we develop a *Specht module theory* for $\mathscr{H}_{r,p,n}$. More precisely, if $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and $1 \le t \le p_{\boldsymbol{\lambda}}$ define

$$S^{\boldsymbol{\lambda}}_t = \{\, x \in S(\boldsymbol{\lambda}) \mid \theta_{\boldsymbol{\lambda}}(x) = \varepsilon^{to_{\boldsymbol{\lambda}}}\mathfrak{g}_{\boldsymbol{\lambda}}x \,\},$$

where $\theta_{\boldsymbol{\lambda}}$ is an $\mathscr{H}_{r,p,n}$-module endomorphism of $S(\boldsymbol{\lambda})$ which depends on the central element $z_{\mathbf{b}} \in \mathscr{H}_{d,\mathbf{b}}$; see Definition 4.30. Then $S^{\boldsymbol{\lambda}}_t$ is an $\mathscr{H}_{r,p,n}$-module. Let $D^{\boldsymbol{\lambda}}_t$ be the head of $S^{\boldsymbol{\lambda}}_t$. Then we have the following explicit construction of the simple $\mathscr{H}_{r,p,n}$-modules.

Theorem C. *Suppose that* K *is a field and that* $\mathbf{Q}$ *is* (ε, q)*-separated over* K*. Then:*

a) *If* $\boldsymbol{\mu} \in \mathscr{K}_{r,n}$ *then* $D^{\boldsymbol{\mu}}_t$ *is an absolutely irreducible* $\mathscr{H}_{r,p,n}$*-module, for* $1 \le i \le p_{\boldsymbol{\mu}}$.
b) $\{\, D^{\boldsymbol{\mu}}_t \mid \boldsymbol{\mu} \in \mathscr{K}^{\sigma}_{r,n}$ *and* $1 \le i \le p_{\boldsymbol{\mu}}\,\}$ *is a complete set of pairwise non-isomorphic absolutely irreducible* $\mathscr{H}_{r,p,n}$*-modules. Hence,* K *is a splitting field for* $\mathscr{H}_{r,p,n}$.
c) *The decomposition matrix of* $\mathscr{H}_{r,p,n}$ *is unitriangular.*

We are able to say quite a bit more about the structure of the Specht modules $S^{\boldsymbol{\lambda}}_t$ and the simple modules $D^{\boldsymbol{\lambda}}_t$ for $\mathscr{H}_{r,p,n}$; see Theorem 4.33 and Theorem 4.35 for details.

We are finally able to define the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$. Suppose that A is a K-algebra and suppose that M is an A-module and D is an irreducible A-module. Let $[M : D]$ be the composition multiplicity of D as a composition factor of M.

1.2. **Definition.** *Suppose that* l *divides* p, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ *and that* $1 \le i \le p_{\boldsymbol{\lambda}}$ *and* $1 \le j \le p_{\boldsymbol{\mu}}$. *The decomposition number* $[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j]$ *is* l***-splittable*** *if* $p_{\boldsymbol{\lambda}} = l = p_{\boldsymbol{\mu}}$.

By the results in section 4, and the general theory developed in [**23**], the decomposition number $[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j]$ is p-splittable if and only if $S^{\boldsymbol{\lambda}}_j$ and $D^{\boldsymbol{\mu}}_j$ both have trivial *inertia groups* in the usual sense of Clifford theory.

Now suppose that l divides p and let $m = p/l$. To describe the l-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$ let $V(l)$ be the $l \times l$ Vandermonde matrix

$$V(l) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \varepsilon^{m} & \varepsilon^{2m} & \dots & \varepsilon^{lm} \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon^{(l-1)m} & \varepsilon^{2(l-1)m} & \dots & \varepsilon^{l(l-1)m} \end{pmatrix}.$$

For $1 \le i \le p$ define $V_i(l)$ to be the matrix obtained from $V(l)$ by replacing its ith column with the column vector

$$\begin{pmatrix} d^{l}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m} \\ \left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^{1} d^{l_1}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m} \\ \vdots \\ \left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^{l-1} d^{l_{l-1}}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m} \end{pmatrix}$$

where $d_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m} = [S(\boldsymbol{\lambda}^{[1]}, \dots, \boldsymbol{\lambda}^{[m]}) : D(\boldsymbol{\mu}^{[1]}, \dots, \boldsymbol{\mu}^{[m]})]$ and $l_t = \gcd(l, t)$.

Theorem D. *Suppose that K is a field, that $\mathbf{Q}$ is (ε, q)-separated over K and that the decomposition number $[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j]$ is l-splittable so that $p_{\boldsymbol{\lambda}} = l = p_{\boldsymbol{\mu}}$, for some l dividing p. Then*

$$[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j] \equiv \frac{\det V_{j-i}(l)}{\det V(l)} \pmod{\operatorname{char} K},$$

for $1 \le i, j \le l = p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}}$.

The main idea underpinning Theorem D is the introduction of a new algebra $\mathscr{S}_{r,p,n}$, which is an analogue of the cyclotomic Schur algebra [**9**] for $\mathscr{H}_{r,p,n}$. We introduce Weyl modules and simple modules for $\mathscr{S}_{r,p,n}$ and then compute the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ using the **twining characters** of $\mathscr{S}_{r,p,n}$. These characters are a generalization of the formal characters of a quasi-hereditary algebra and they compute the trace of a certain element $\vartheta_{\boldsymbol{\lambda}} \in \mathscr{S}_{r,p,n}$ on certain weight spaces of $\mathscr{S}_{r,p,n}$-modules. The map $\vartheta_{\boldsymbol{\lambda}}$ is constructed from the action of $z_{\mathbf{b}}$ upon certain $\mathscr{S}_{r,p,n}$-modules. Finally, Theorem D is proved using a considerable amount of Clifford theory and some natural functors

$$\mathscr{S}_{r,p,n} = \bigoplus_{\mathbf{b}\in\mathscr{C}^{\sigma}_{p,n}} \mathscr{S}_{r,p,n}(\mathbf{b}) \xrightarrow{\oplus \mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}} \mathscr{E}_d = \bigoplus_{\mathbf{b}\in\mathscr{C}^{\sigma}_{p,n}} \mathscr{E}_{d,\mathbf{b}} \xrightarrow[\text{Morita}]{\simeq} \mathscr{H}_{r,p,n}.$$

where the first functor is an analogue of the Schur functor and the second functor lifts the Morita equivalence of (1.1) up to $\mathscr{H}_{r,p,n}$.

Very briefly, the outline of this paper is as follows. Section 2 studies the right ideals $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ in long and technical detail. The main results are Lemma 2.21 which shows the existence of the central element $z_{\mathbf{b}}$, Theorem 2.26 which produces a subalgebra of $\mathscr{H}_{r,n}$ isomorphic to $\mathscr{H}_{d,\mathbf{b}}$, and Theorem 2.31 which is a comparison theorem for the natural trace forms on $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$. In Section 3 these results are used to compute the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$, which describe the action of $z_{\mathbf{b}}$ on the Specht modules $S(\boldsymbol{\lambda})$ of $\mathscr{H}_{r,n}$ and proves the first half of Theorem B. Section 4 marks the first direct appearance of the algebras $\mathscr{H}_{r,p,n}$. Using seminormal forms we factorize the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$, completing the proof of Theorem B. We then use the roots of the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ to decompose the Specht modules, culminating in Theorem 4.33 and Theorem 4.35 which describe the Specht modules and simple modules of $\mathscr{H}_{r,p,n}$, respectively. Section 4 concludes by lifting the Morita equivalence (1.1) to a new Morita equivalence between $\mathscr{H}_{r,p,n}$ and a new algebra E_d in Corollary 4.41. Section 5 begins by introducing analogues of the cyclotomic

Schur algebras for $\mathscr{H}_{r,p,n}$. Theorem 5.29 computes the l-splittable decomposition numbers of these algebras using twining characters, using Schur functors and the algebras $\mathscr{E}_d$ we then prove Theorem D and hence complete the proof of Theorem A.

Index of notation

$\sim_\sigma$	Equivalence relation $\mathbf{b} \sim \mathbf{b}\langle t\rangle$
$\sim_{\mathbf{b}}$	Equivalence relation $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}\langle \mathsf{o}_{\mathbf{b}}\rangle$
$\uparrow_B^A, \downarrow_B^A$	Induction/restriction functors
$A(\varepsilon, q, \mathbf{Q})$	$\prod_{i,j,\lvert k\rvert<n,1\le t<p}(Q_i - \varepsilon^t q^k Q_j)$
$\mathcal{A}$	$\mathbb{Z}[\dot\varepsilon, \dot q^{\pm1}, \dot Q_1^{\pm1}, \dots, \dot Q_d^{\pm1}, \frac{1}{A(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})}]$
$\mathbf{a} \vee \mathbf{b}$	The concatenation of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{b}\langle z\rangle$	The shift of the sequence $\mathbf{b}$ by z
$\mathscr{C}_{p,n}$	Compositions of n of length p
$\operatorname{ch} M$	$\sum_{\boldsymbol{\mu}}(\dim M_{\boldsymbol{\mu}})e^{\boldsymbol{\mu}}$
$\operatorname{ch}_{\mathrm{t}}^{\mathrm{l}} M$	$\sum_\gamma \operatorname{Tr}(\vartheta_{\boldsymbol{\lambda}}^t, M_{\gamma^{l_t}})e^\gamma$
$e_{\mathbf{b}}$	The idempotent $z_{\mathbf{b}}^{-1} \cdot v_{\mathbf{b}} T_{\mathbf{b}}$
d	r/p
$D(\boldsymbol{\lambda})$	Simple module for $\mathscr{H}_{r,n}$
$D_{\mathbf{b}}(\boldsymbol{\lambda})$	Simple module for $\mathscr{H}_{d,\mathbf{b}}$
$D_{i,p}^{\boldsymbol{\lambda}}$	Simple module for $\mathscr{E}_{d,\mathbf{b}}$
$D_i^{\boldsymbol{\lambda}}$	Simple module for $\mathscr{H}_{r,p,n}$
$\Delta(\boldsymbol{\lambda})$	Weyl module for $\mathscr{S}_{r,n}$
$\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$	Weyl module for $\mathscr{S}_{d,\mathbf{b}}$
$\Delta_{i,p}^{\boldsymbol{\lambda}}$	Weyl module for $\mathscr{S}_{r,p,n}$
ε	Primitive pth root of unity in K
$\dot\varepsilon$	Primitive pth root of unity in $\mathbb{C}$
$\mathscr{E}_{d,\mathbf{b}}$	$= \operatorname{End}_{\mathscr{H}_{r,p,n}}(V_{\mathbf{b}})$
$\mathcal{F}$	The field of fractions of $\mathcal{A}$
$\mathfrak{f}_{\boldsymbol{\lambda}}$	The scalar: $z_{\mathbf{b}}\downarrow_{S(\boldsymbol{\lambda})} = \mathfrak{f}_{\boldsymbol{\lambda}} \operatorname{id}_{S(\boldsymbol{\lambda})}$
$\mathfrak{f}_{\boldsymbol{\lambda}}^{(t)}$	$= \mathfrak{f}_{\boldsymbol{\lambda}}^{(t:m)}$, a factor of $\mathfrak{f}_{\boldsymbol{\lambda}}$
$\mathfrak{g}_{\boldsymbol{\lambda}}$	$\mathfrak{f}_{\boldsymbol{\lambda}} = \varepsilon^{\frac12 d\mathsf{o}_{\boldsymbol{\lambda}} n(1-p_{\boldsymbol{\lambda}})}\mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}$
$\mathtt{H}_{\mathbf{b}}$	A functor Mod-$\mathscr{H}_{d,\mathbf{b}} \to$ Mod-$\mathscr{H}_{r,n}$
θ_t'	The map $h \mapsto Y_t h$
$\theta_{t,m}'$	The map $h \mapsto Y_{t,m} h$
$\theta_{t,m}$	$= \sigma^m \circ \theta_{t,m}'$
$\theta_{\mathbf{b}}$	$= \theta_{0,\mathsf{o}_p(\mathbf{b})}$ restricted to $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$
$\theta_{\boldsymbol{\lambda}}$	$= \theta_{0,\mathsf{o}_p(\boldsymbol{\lambda})}$ restricted to $S(\boldsymbol{\lambda})$
$\vartheta_{\mathbf{b}}$	$\theta_{\mathbf{b}}$ restricted to $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$
$\vartheta_{\boldsymbol{\lambda}}$	$\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$
$\widehat\Theta_{\mathbf{b}}, \Theta_{\mathbf{b}}$	Two maps $\mathscr{H}_{d,\mathbf{b}} \to \mathscr{H}_{r,n}$
$\mathscr{H}_{r,n}^R$	Hecke algebra of type $G(r,1,n)$
$\mathscr{H}_{r,p,n}^R$	Hecke algebra of type $G(r,p,n)$
$\mathscr{H}_{d,\mathbf{b}}$	$= \mathscr{H}_{d,b_1} \otimes \dots \otimes \mathscr{H}_{d,b_p}$
$\widehat{\mathscr{H}_{d,\mathbf{b}}}$	$= \mathscr{H}_{d,\mathbf{b}} \cdot e_{\mathbf{b}} \cong \mathscr{H}_{d,\mathbf{b}}$
$\mathscr{K}_{r,n}$	Kleshchev multipartitions in $\mathscr{P}_{r,n}$
$\mathscr{K}_{d,\mathbf{b}}$	Kleshchev multipartitions in $\mathscr{P}_{d,\mathbf{b}}$

$L(\boldsymbol{\lambda})$	Simple module for $\mathscr{S}_{r,n}$
$L_{\mathbf{b}}(\boldsymbol{\lambda})$	Simple module for $\mathscr{S}_{d,\mathbf{b}}$
$L_{i,p}^{\boldsymbol{\lambda}}$	Simple module for $\mathscr{S}_{r,p,n}$
$M_{\mathbf{b}}^{\boldsymbol{\lambda}}$	Permutation module in $V_{\mathbf{b}}$
$\mathsf{o}_p(\mathbf{b})$	$= \min\{\, z > 0 \mid \mathbf{b}\langle z\rangle = \mathbf{b}\,\}$
$\mathsf{o}_{\mathbf{b}}$	$= \mathsf{o}_p(\mathbf{b})$
$\mathsf{o}_{\boldsymbol{\lambda}}$	$= \mathsf{o}_p(\boldsymbol{\lambda})$
$p_{\boldsymbol{\lambda}}$	$p/\mathsf{o}_p(\boldsymbol{\lambda}) = p/\mathsf{o}_{\boldsymbol{\lambda}}$
$p_{\mathbf{b}/\boldsymbol{\lambda}}$	$p_{\mathbf{b}}/p_{\boldsymbol{\lambda}} = \mathsf{o}_{\boldsymbol{\lambda}}/\mathsf{o}_{\mathbf{b}}$
$p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}$	$p_{\boldsymbol{\mu}}/p_{\boldsymbol{\lambda}} = \mathsf{o}_{\boldsymbol{\lambda}}/\mathsf{o}_{\boldsymbol{\mu}}$
$\mathscr{P}_{r,n}$	The set of r–partitions of n
$\mathscr{P}_{d,\mathbf{b}}$	$\{\, \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \mid b_i = \lvert\boldsymbol{\lambda}^{[i]}\rvert, 1 \le i \le d\,\}$
$\boldsymbol{\lambda}^{[i]}$	$(\lambda^{(d(i-1)+1)}, \dots, \lambda^{(di)})$
$\mathcal{L}_k^{(s)}$	$\prod_{i=1}^d (L_k - \varepsilon^s Q_i)$
$\mathcal{L}_{l,m}^{(i,j)}$	$\prod_{l\le k\le m}\prod_{s\in I_{ij}} \mathcal{L}_k^{(s)}$
$\mathbf{Q}$	$(Q_1, \dots, Q_d)$
$\mathbf{Q}^{\vee\varepsilon}$	$\varepsilon\mathbf{Q} \vee \varepsilon^2\mathbf{Q} \vee \dots \vee \varepsilon^p\mathbf{Q}$
σ	$\mathscr{H}_{r,n} \to \mathscr{H}_{r,n}; T_i \mapsto \varepsilon^{\delta_{i0}} T_i$
$\hat\sigma$	An automorphism of $\mathscr{S}_{r,p,n}$
s_i	$(i, i+1) \in \mathfrak{S}_n$
$S(\boldsymbol{\lambda})$	Specht module for $\mathscr{H}_{r,n}$
$S_{\mathbf{b}}(\boldsymbol{\lambda})$	Specht module for $\mathscr{H}_{d,\mathbf{b}}$
$S_{i,p}^{\boldsymbol{\lambda}}$	Specht module for $\mathscr{E}_{d,\mathbf{b}}$
$S_t^{\boldsymbol{\lambda}}$	Specht module for $\mathscr{H}_{r,p,n}$
$\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}$	Schur element of $S(\boldsymbol{\lambda})$
$\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}^{\mathbf{b}}$	Schur element of $S_{\mathbf{b}}(\boldsymbol{\lambda})$
$\mathscr{S}_{r,n}$	Cyclotomic Schur algebra for $\mathscr{H}_{r,n}$
$\mathscr{S}_{d,\mathbf{b}}$	Cyclotomic Schur algebra for $\mathscr{H}_{d,\mathbf{b}}$
$\mathscr{S}_{r,p,n}$	Cyclotomic Schur algebra for $\mathscr{H}_{r,p,n}$
$T_{a,b}$	$T_{w_{a,b}}$
$T_{a,b}^{\langle k\rangle}$	T_w, where $w = w_{a,b}^{\langle k\rangle}$
τ	$h \mapsto T_0^{-1} h T_0$
$v_{\mathbf{b}}$	$v_{\mathbf{b}}(\mathbf{Q}) = v_{\mathbf{b}}^+ u_{\mathbf{b}}^+ = u_{\mathbf{b}}^- v_{\mathbf{b}}^-$
$v_{\mathbf{b}}^{(t)}$	$v_{\mathbf{b}}(\varepsilon^t\mathbf{Q})$
$V_{\mathbf{b}}$	The ideal $v_{\mathbf{b}}\mathscr{H}_{r,n}$
$V_{\mathbf{b}}^{(t)}$	$= v_{\mathbf{b}}^{(t)}\mathscr{H}_{r,n}$
$\{v_{\mathfrak{s}}^{(tm)}\}$	Seminormal basis of $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$
$w_{a,b}^{\langle k\rangle}$	$(s_{a+b+k-1} \dots s_{k+1})^b$
$w_{\mathbf{b}}$	$w_{b_{p-1},\mathbf{b}_p^p}^{\langle \mathbf{b}_1^{p-2}\rangle} \dots w_{b_2,\mathbf{b}_3^p}^{\langle \mathbf{b}_1^1\rangle} w_{b_1,\mathbf{b}_2^p}$
Y_t	$\mathcal{L}_{1,b_t}^{(t+1,t+p-1)} T_{b_t,n-b_t}$
$Y_{t,m}$	$Y_{tm} Y_{tm-1} \dots Y_{t(m-1)+1}$
$z_{\mathbf{b}}$	Central element of $\mathscr{H}_{d,\mathbf{b}}$

2. Morita equivalences for the Hecke algebras of type $G(r,1,n)$

This section introduces and studies some very useful right ideals $V_{\mathbf{b}}$ of the cyclotomic Hecke algebras of type $G(r,1,n)$. In the next section we use these ideals to construct 'shifting homomorphisms' linking certain Specht modules. These maps, which will turn out to be multiplication by a scalar, are the key to the main results of this paper. We start by recalling the definition of the cyclotomic Hecke algebras.

Throughout this paper we fix positive integers n, r, p and d such that $n \geq 3$, $p > 1$ and $r = pd$. Let R be a commutative ring which contains a primitive pth root of unity ε and suppose that $q, Q_1, \dots, Q_d$ are invertible elements of R. Let $\mathbf{Q} = (Q_1, \dots, Q_d)$ and $\mathbf{Q}^{\vee\varepsilon} = \varepsilon\mathbf{Q} \vee \varepsilon^2\mathbf{Q} \vee \cdots \vee \varepsilon^p\mathbf{Q}$.

2.1. **Cyclotomic Hecke algebras.** The **Ariki-Koike algebra** $\mathscr{H}^R_{r,n} = \mathscr{H}^R_{r,n}(q, \mathbf{Q})$ with parameters q and $\mathbf{Q}^{\vee\varepsilon}$ is the unital associative R-algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_0^p - Q_1^p) \dots (T_0^p - Q_d^p) &= 0, \\
(T_i - q)(T_i + 1) &= 0, && \text{for } 1 \leq i \leq n-1, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i, && \text{for } 1 \leq i \leq n-2, \\
T_iT_j &= T_jT_i, && \text{for } 0 \leq i < j-1 \leq n-2.
\end{aligned}$$

When the ring R and the parameters $\mathbf{Q}$ are understood we write $\mathscr{H}_{r,n} = \mathscr{H}^R_{r,n}(\mathbf{Q})$.

2.1. **Definition.** *The **cyclotomic Hecke algebra of type** $G(r, p, n)$ is the subalgebra $\mathscr{H}_{r,p,n}(\mathbf{Q})$ of $\mathscr{H}_{r,n}(\mathbf{Q})$ which is generated by the elements $T_0^p, T_u = T_0^{-1}T_1T_0$ and $T_1, T_2, \cdots, T_{n-1}$.*

In this paper we are interested in understanding the decomposition matrices of the algebra $\mathscr{H}_{r,p,n}$. We have chosen the ordering of the parameters $\varepsilon\mathbf{Q} \vee \varepsilon^2\mathbf{Q} \vee \cdots \vee \varepsilon^p\mathbf{Q}$ so that we can extend the Morita equivalences developed in [**10**, **23**] to prove a new Morita reduction theorem for $\mathscr{H}_{r,p,n}$; see Corollary 4.41.

Let $\mathfrak{S}_n$ be the symmetric group on n letters and let $s_i = (i, i+1) \in \mathfrak{S}_n$ be a simple transposition, for $1 \leq i < n$. Then $\{s_1, \dots, s_{n-1}\}$ are the standard Coxeter generators of the symmetric group $\mathfrak{S}_n$. Let $\ell : \mathfrak{S}_n \longrightarrow \mathbb{N}$ be the length function on $\mathfrak{S}_n$, so that $\ell(w) = k$ if k is minimal such that $w = s_{i_1} \dots s_{i_k}$, where $1 \leq i_1, \dots, i_k < n$. As the type A braid relations hold in $\mathscr{H}_{r,n}$ for each $w \in \mathfrak{S}_n$ there is a well–defined element $T_w \in \mathscr{H}_{r,n}$, where $T_w = T_{i_1} \dots T_{i_k}$ whenever $w = s_{i_1} \dots s_{i_k}$ and $k = \ell(w)$.

Inspecting the relations, there is a unique anti-isomorphism $*$ of $\mathscr{H}_{r,n}$ which fixes $T_0, T_1, \dots, T_{n-1}$. We have $T_w^* = T_{w^{-1}}$.

2.2. **Jucys-Murphy elements.** For non-negative integers a, b with $0 < a + b \leq n$ we set $w_{a,b} = (s_{a+b-1} \dots s_1)^b$. (In particular, $w_{a,0} = 1 = w_{0,b}$.) If we write $w_{a,b} \in \mathfrak{S}_{a+b}$ as a permutation in two-line notation then

$$w_{a,b} = \begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & a+b \\ b+1 & \cdots & a+b & 1 & \cdots & b \end{pmatrix}.$$

For simplicity, we write $T_{a,b} = T_{w_{a,b}}$. Similarly, if k is a non-negative integer such that $0 < a + b + k \leq n$ then we set $w^{\langle k \rangle}_{a,b} = (s_{a+b+k-1} \dots s_{k+1})^b$. Then $w_{a,b} = w^{\langle 0 \rangle}_{a,b}$ and, abusing notation slightly, we write $T^{\langle k \rangle}_{a,b} = T_{w^{\langle k \rangle}_{a,b}}$.

The following result is easily checked.

2.2. **Lemma.** *Suppose that a, b and c are non-negative integers such that $a+b+c \leq n$. Then $w_{a,b+c} = w_{a,b}w^{\langle b \rangle}_{a,c}$ and $w_{a+b,c} = w^{\langle a \rangle}_{b,c}w_{a,c}$, with the lengths adding. Consequently, $T_{a,b+c} = T_{a,b}T^{\langle b \rangle}_{a,c}$ and $T_{a+b,c} = T^{\langle a \rangle}_{b,c}T_{a,c}$. Moreover, $T_iT^{\langle c \rangle}_{a,b} = T^{\langle c \rangle}_{a,b}T_{(i)w^{\langle c \rangle}_{a,b}}$ if $1 \leq i < n$ and $i \neq a + c$.*

Set $L_1 = T_0$ and $L_{k+1} = q^{-1}T_kL_kT_k$, for $k = 1, \dots, n-1$. These elements L_i are the Jucys–Murphy elements of $\mathscr{H}_{r,n}$ and they generate a commutative subalgebra of $\mathscr{H}_{r,n}$. We will use the following well-known properties of the Jucys-Murphy elements without mention.

2.3. **Lemma** (cf. [**3**, Lemma 3.3]). *Suppose that* $1 \le i < n$ *and* $1 \le k \le n$. *Then*

a) T_i *and* L_k *commute if* $i \ne k, k-1$.
b) T_k *commutes with* L_kL_{k+1} *and* $L_k + L_{k+1}$.
c) $T_kL_k = L_{k+1}(T_k - q + 1)$ *and* $T_kL_{k+1} = L_kT_k + (q-1)L_{k+1}$.

For integers k and s, with $1 \le k \le n$ and $1 \le s \le p$, set

$$\mathcal{L}_k^{(s)} = \prod_{i=1}^{d}(L_k - \varepsilon^s Q_i).$$

More generally, if $1 \le l \le m \le n$ and $1 \le i, j \le p$ then set

$$\mathcal{L}_{l,m}^{(i,j)} = \prod_{\substack{l\le k\le m\\ s\in I_{ij}}} \mathcal{L}_k^{(s)} = \prod_{\substack{l\le k\le m\\ s\in I_{ij}}} \prod_{t=1}^{d}(L_k - \varepsilon^s Q_t),$$

where $I_{ij} = \{i, i+1, \dots, j\}$, if $i \le j$, and $I_{ij} = \{1, 2 \dots, j, i, i+1, \dots, p\}$ if $i > j$.

A key property of the Jucys-Murphy elements of $\mathscr{H}_{r,n}$ is that T_i commutes with any polynomial in $L_1, \dots, L_n$ which is symmetric with respect to L_i and L_{i+1}. In particular, any symmetric polynomial in $L_1, \dots, L_n$ is central in $\mathscr{H}_{r,n}$. Hence, we have the following.

2.4. **Lemma.** *Suppose that* $1 \le l < m \le n$ *and* $1 \le t \le p$. *Then*

$$T_i\mathcal{L}_{l,m}^{(t)} = \mathcal{L}_{l,m}^{(t)}T_i \qquad \text{and} \qquad L_j\mathcal{L}_{l,m}^{(t)} = \mathcal{L}_{l,m}^{(t)}L_j,$$

for all i, j *such that* $1 \le i < n$, $1 \le j \le n$ *and* $i \ne l-1, m$.

2.3. **The elements $v_{\mathbf{b}}$ and $v_{\mathbf{b}}^{(t)}$.** As remarked after Theorem D in the introduction, the main results of this paper rely on a Morita equivalence between $\mathscr{H}_{r,p,n}$ and a direct sum of certain algebras $\mathscr{E}_{d,\mathbf{b}}$. This equivalence builds upon previous work [**8**, **10**, **23**] which gave similar Morita equivalences for the algebras $\mathscr{H}_{r,n}$ and $\mathscr{H}_{r,p,n}$. The starting point for all of this work is a generalization of a fundamental lemma of Dipper and James [**8**, Lemma 3.10].

2.5. **Lemma** ([**10**, Proposition 3.4]). *Suppose that* a, b, s *and* t *are positive integers with* $1 \le a + b < n$ *and* $1 \le s \le t \le p$. *Let* $v_{a,b}^{(s,t)} = \mathcal{L}_{1,a}^{(s,t)}T_{a,b}\mathcal{L}_{1,b}^{(t+1,s-1)}$. *Then*

$$T_iv_{a,b}^{(s,t)} = v_{a,b}^{(s,t)}T_{(i)w_{a,b}} \qquad \text{and} \qquad L_jv_{a,b}^{(s,t)} = v_{a,b}^{(s,t)}L_{(j)w_{a,b}},$$

for all i, j *such that* $1 \le i, j \le a + b$ *and* $i \ne a, a + b$.

Recall from the introduction that $\mathscr{C}_{p,n}$ is the set of compositions of n into p parts. Thus, $\mathbf{b} \in \mathscr{C}_{p,n}$ if and only if $\mathbf{b} = (b_1, \dots, b_p)$, $b_1 + \dots + b_p = n$ and $b_i \ge 0$, for all i. Finally, if $\mathbf{b} \in \mathscr{C}_{p,n}$ and i and j are integers then we set $\mathbf{b}_i^j = b_i + \dots + b_j$ if $i \le j$ and $b_i^j = 0$ if $i > j$.

The following elements of $\mathscr{H}_{r,n}$, which generalize the elements $v_{a,b}^{(s,t)}$ and were introduced in [**23**, Definition 2.4], play an important role throughout this paper.

2.6. **Definition.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$. *Let*

$$v_{\mathbf{b}}(\mathbf{Q}) = \mathcal{L}_{1,b_p}^{(1,p-1)}T_{b_p,\mathbf{b}_1^{p-1}}\mathcal{L}_{1,b_{p-1}}^{(1,p-2)}T_{b_{p-1},\mathbf{b}_1^{p-2}} \dots \mathcal{L}_{1,b_2}^{(1,1)}T_{b_2,\mathbf{b}_1^1}\mathcal{L}_{1,\mathbf{b}_1^1}^{(2)}\mathcal{L}_{1,\mathbf{b}_1^2}^{(3)} \dots \mathcal{L}_{1,\mathbf{b}_1^{p-1}}^{(p)}$$

We write $v_{\mathbf{b}} = v_{\mathbf{b}}(\mathbf{Q})$ *and for* $t \in \mathbb{Z}$ *we set* $v_{\mathbf{b}}^{(t)} = v_{\mathbf{b}}(\varepsilon^t\mathbf{Q})$.

Set $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ *and, more generally, let* $V_{\mathbf{b}}^{(t)} = v_{\mathbf{b}}^{(t)}\mathscr{H}_{r,n}$.

We start by showing that $v_{\mathbf{b}}$ can be written in many different (and useful) ways. This requires several long and involved calculations. On the first reading the reader might prefer to skip these calculations and start reading from Section 3.

Recall from the introduction that if $\mathbf{b} \in \mathscr{C}_{p,n}$ and $k \in \mathbb{Z}$ then $\mathbf{b}\langle k\rangle = (b_{k+1}, b_{k+2}, \dots, b_{k+p})$, where we set $b_{i+p} = b_i$ for $1 \le i \le p$.

2.7. **Lemma.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$. *Then*

a) $v_{\mathbf{b}} = \mathcal{L}^{(2,p-1)}_{1,b_p} T_{b_p,\mathbf{b}_2^{p-1}} \dots \mathcal{L}^{(2,2)}_{1,b_3} T_{b_3,\mathbf{b}_2^2} \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(p)}_{1,\mathbf{b}_2^{p-1}} \mathcal{L}^{(1)}_{1,\mathbf{b}_2^p} T_{\mathbf{b}_2^p,b_1} \mathcal{L}^{(2,p)}_{1,b_1}$.

b) $v_{\mathbf{b}} \in \mathcal{L}^{(1)}_{1,\mathbf{b}_2^p} \mathscr{H}_{r,n}$.

Proof. To prove the Lemma, if $1 \le s < p$ then define

$$L(s) = \mathcal{L}^{(1,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_1^s} \dots \mathcal{L}^{(1,1)}_{1,b_2} T_{b_2,\mathbf{b}_1^1} \mathcal{L}^{(2)}_{1,\mathbf{b}_1^1} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_1^{s-1}} \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}_1^s},$$

$$R(s) = \mathcal{L}^{(2,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_2^s} \dots \mathcal{L}^{(2,2)}_{1,\mathbf{b}_3} T_{b_3,\mathbf{b}_2^2} \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_2^{s-1}} \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}_2^s} \mathcal{L}^{(1)}_{1,\mathbf{b}_2^{s+1}} T_{\mathbf{b}_2^{s+1},b_1} \mathcal{L}^{(2,p)}_{1,b_1}.$$

Part (a) of the Lemma is the claim that $L(p-1) = R(p-1)$. To prove this we show by induction on s that $L(s) = R(s)$, for $1 \le s < p$. If $s = 1$ then

$$L(1) = \mathcal{L}^{(1)}_{1,b_2} T_{b_2,b_1} \mathcal{L}^{(2,p)}_{1,b_1} = R(1),$$

and there is nothing to prove. Assume by induction that $L(s) = R(s)$. Then

$$\begin{aligned}
L(s+1) &= \mathcal{L}^{(1,s+1)}_{1,b_{s+2}} T_{b_{s+2},\mathbf{b}_1^{s+1}} \cdot L(s) \cdot \mathcal{L}^{(s+2,p)}_{\mathbf{b}_1^s+1,\mathbf{b}_1^{s+1}} \\
&= \mathcal{L}^{(1,s+1)}_{1,b_{s+2}} T_{b_{s+2},\mathbf{b}_1^{s+1}} \cdot R(s) \cdot \mathcal{L}^{(s+2,p)}_{\mathbf{b}_1^s+1,\mathbf{b}_1^{s+1}} \\
&= \mathcal{L}^{(1,s+1)}_{1,b_{s+2}} T_{b_{s+2},\mathbf{b}_1^{s+1}} \cdot \mathcal{L}^{(2,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_2^s} \dots \mathcal{L}^{(2,2)}_{1,\mathbf{b}_3} T_{b_3,\mathbf{b}_2^2} \\
&\quad \times \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_2^{s-1}} \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}_2^s} \mathcal{L}^{(1)}_{1,\mathbf{b}_2^{s+1}} T_{\mathbf{b}_2^{s+1},b_1} \mathcal{L}^{(2,p)}_{1,b_1} \cdot \mathcal{L}^{(s+2,p)}_{\mathbf{b}_1^s+1,\mathbf{b}_1^{s+1}} \\
&= \mathcal{L}^{(1,s+1)}_{1,b_{s+2}} \cdot T_{b_{s+2},\mathbf{b}_2^{s+1}} T^{\langle \mathbf{b}_2^{s+1}\rangle}_{b_{s+2},b_1} \cdot \mathcal{L}^{(2,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_2^s} \dots \mathcal{L}^{(2,2)}_{1,\mathbf{b}_3} T_{b_3,\mathbf{b}_2^2} \\
&\quad \times \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_2^{s-1}} \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}_2^s} \cdot v^{(1,1)}_{\mathbf{b}_2^{s+1},b_1} \cdot \mathcal{L}^{(s+2,p)}_{\mathbf{b}_1^s+1,\mathbf{b}_1^{s+1}}
\end{aligned}$$

by Lemma 2.2. Using Lemma 2.5 twice, and Lemma 2.4 many times, we find

$$\begin{aligned}
L(s+1) &= \mathcal{L}^{(1)}_{1,b_{s+2}} \cdot \mathcal{L}^{(2,s+1)}_{1,b_{s+2}} T_{b_{s+2},\mathbf{b}_2^{s+1}} \mathcal{L}^{(2,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_2^s} \dots \mathcal{L}^{(2,2)}_{1,\mathbf{b}_3} T_{b_3,\mathbf{b}_2^2} \\
&\quad \times \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_2^{s-1}} \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}_2^s} \cdot T^{\langle \mathbf{b}_2^{s+1}\rangle}_{b_{s+2},b_1} \cdot \mathcal{L}^{(s+2,p)}_{1,\mathbf{b}_2^{s+1}} \cdot v^{(1,1)}_{\mathbf{b}_2^{s+1},b_1} \\
&= \mathcal{L}^{(1)}_{1,b_{s+2}} \cdot v^{(2,s+1)}_{b_{s+2},\mathbf{b}_2^{s+1}} \cdot \mathcal{L}^{(2,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_2^s} \dots \mathcal{L}^{(2,2)}_{1,\mathbf{b}_3} T_{b_3,\mathbf{b}_2^2} \\
&\quad \times \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_2^{s-1}} \mathcal{L}^{(s+1)}_{1,\mathbf{b}_2^s} \cdot T^{\langle \mathbf{b}_2^{s+1}\rangle}_{b_{s+2},b_1} T_{\mathbf{b}_2^{s+1},b_1} \mathcal{L}^{(2,p)}_{1,b_1} \\
&= v^{(2,s+1)}_{b_{s+2},\mathbf{b}_2^{s+1}} \cdot \mathcal{L}^{(1)}_{\mathbf{b}_2^{s+1}+1,\mathbf{b}_2^{s+2}} \cdot \mathcal{L}^{(2,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}_2^s} \dots \mathcal{L}^{(2,2)}_{1,\mathbf{b}_3} T_{b_3,\mathbf{b}_2^2} \\
&\quad \times \mathcal{L}^{(3)}_{1,\mathbf{b}_2^2} \dots \mathcal{L}^{(s)}_{1,\mathbf{b}_2^{s-1}} \mathcal{L}^{(s+1)}_{1,\mathbf{b}_2^s} \cdot T_{\mathbf{b}_2^{s+2},b_1} \mathcal{L}^{(2,p)}_{1,b_1},
\end{aligned}$$

which, after some more rearranging using Lemma 2.4, is equal to $R(s+1)$. This proves the claim and hence completes the proof of part (a). To prove part (b), looking at the second last equality we see that there exists an $h \in \mathscr{H}_{r,n}$ such that

$$\begin{aligned}
L(s+1) &= \mathcal{L}^{(1)}_{1,b_{s+2}} v^{(2,s+1)}_{b_{s+2},\mathbf{b}_2^{s+1}} h \qquad = v^{(1,s+1)}_{b_{s+2},\mathbf{b}_2^{s+1}} \mathcal{L}^{(1)}_{1,\mathbf{b}_2^{s+1}} h \\
&= \mathcal{L}^{(1)}_{b_{s+2}+1,\mathbf{b}_2^{s+2}} v^{(1,s+1)}_{b_{s+2},\mathbf{b}_2^{s+1}} h = \mathcal{L}^{(1)}_{1,\mathbf{b}_2^{s+2}} \mathcal{L}^{(2,s+1)}_{1,b_{s+2}} T_{b_{s+2},\mathbf{b}_2^{s+1}} \mathcal{L}^{(s+2,p)}_{1,\mathbf{b}_2^{s+1}} h,
\end{aligned}$$

so that $L(s+1) \in \mathcal{L}^{(1)}_{1,\mathbf{b}_2^{s+2}} \mathscr{H}_{r,n}$. Taking $s = p-2$ proves (b) and so completes the proof of the Lemma. □

2.8. **Corollary.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $t\in\mathbb{Z}$. *Then* $v_{\mathbf{b}}^{(t)}\in\mathcal{L}_{1,\mathbf{b}_2^p}^{(t+1)}\mathscr{H}_{r,n}$.

Proof. It is enough to consider the case when $t=0$ and this is exactly part (b) of Lemma 2.7. □

For $t=1,\dots,p$, let $Y_t=\mathcal{L}_{1,b_t}^{(t+1,t+p-1)}T_{b_t,n-b_t}$.

2.9. **Corollary.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $1\le t\le p$. *Then*

$$Y_t v_{\mathbf{b}\langle t-1\rangle}^{(t-1)}=v_{\mathbf{b}\langle t\rangle}^{(t)}Y_t^*.$$

Proof. It is enough to consider the case $t=1$. If $t=1$ then by Lemma 2.7(a)

$$\begin{aligned}Y_t v_{\mathbf{b}}&=\mathcal{L}_{1,b_1}^{(2,p)}T_{b_1,\mathbf{b}_2^p}\cdot\mathcal{L}_{1,b_p}^{(2,p-1)}T_{b_p,\mathbf{b}_2^{p-1}}\dots\mathcal{L}_{1,b_3}^{(2,2)}T_{b_3,\mathbf{b}_2^2}\mathcal{L}_{1,\mathbf{b}_2^2}^{(3)}\dots\mathcal{L}_{1,\mathbf{b}_2^{p-1}}^{(p)}\\&\quad\times\mathcal{L}_{1,\mathbf{b}_2^p}^{(1)}T_{\mathbf{b}_2^p,b_1}\mathcal{L}_{1,b_1}^{(2,p)}\\&=v_{\mathbf{b}\langle 1\rangle}^{(1)}T_{\mathbf{b}_2^p,b_1}\mathcal{L}_{1,b_1}^{(2,p)},\end{aligned}$$

as required. □

The point of Corollary 2.9 is that left multiplication by Y_t defines an $\mathscr{H}_{r,n}$-module homomorphism from $V_{\mathbf{b}\langle t-1\rangle}^{(t-1)}=v_{\mathbf{b}\langle t-1\rangle}^{(t-1)}\mathscr{H}_{r,n}$ to $V_{\mathbf{b}\langle t\rangle}^{(t)}=v_{\mathbf{b}\langle t\rangle}^{(t)}\mathscr{H}_{r,n}$

2.10. **Definition.** *Suppose that* $1\le t\le p$ *and* $\mathbf{b}\in\mathscr{C}_{p,n}$. *Then* θ'_t *is the* $\mathscr{H}_{r,n}$*-module homomorphism*

$$\theta'_t: V_{\mathbf{b}\langle t-1\rangle}^{(t-1)}\longrightarrow V_{\mathbf{b}\langle t\rangle}^{(t)}; x\mapsto Y_t x,$$

for all $x\in V_{\mathbf{b}\langle t-1\rangle}^{(t-1)}$.

Since $v_{\mathbf{b}}=v_{\mathbf{b}\langle p\rangle}^{(p)}$, composing the maps $\theta'_p\circ\dots\circ\theta'_1$ gives an $\mathscr{H}_{r,n}$-module endomorphism of $v_{\mathbf{b}}\mathscr{H}_{r,n}$. We need to describe this map.

2.11. **Lemma.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$. *Then* $Y_pY_{p-1}\dots Y_2Y_1=v_{\mathbf{b}}T_{\mathbf{b}}$.

Proof. To prove the Lemma it is enough to show by induction on t that

$$Y_t\dots Y_1=\mathcal{L}_{1,b_t}^{(1,t-1)}T_{b_t,\mathbf{b}_1^{t-1}}\dots\mathcal{L}_{1,\mathbf{b}_2}^{(1,1)}T_{b_2,\mathbf{b}_1^1}\mathcal{L}_{1,b_1}^{(2)}\dots\mathcal{L}_{1,\mathbf{b}_1^{t-1}}^{(t)}\prod_{t<s\le p}\mathcal{L}_{1,\mathbf{b}_1^t}^{(s)}\cdot T_{b_t,\mathbf{b}_{t+1}^p}^{\langle\mathbf{b}_1^{t-1}\rangle}\dots T_{b_1,\mathbf{b}_2^p}.$$

When $t=1$ the right hand side of this equation is just Y_1 so there is nothing to prove. Now suppose that $1<t<p-1$. Then, by induction and Lemma 2.4,

$$\begin{aligned}Y_{t+1}\dots Y_1&=\mathcal{L}_{1,b_{t+1}}^{(t+2,t+p)}T_{b_{t+1},n-b_{t+1}}\cdot\mathcal{L}_{1,b_t}^{(1,t-1)}T_{b_t,\mathbf{b}_1^{t-1}}\dots\mathcal{L}_{1,\mathbf{b}_2}^{(1,1)}T_{b_2,\mathbf{b}_1^1}\\&\quad\times\mathcal{L}_{1,b_1}^{(2)}\dots\mathcal{L}_{1,\mathbf{b}_1^{t-1}}^{(t)}\prod_{t<s\le p}\mathcal{L}_{1,\mathbf{b}_1^t}^{(s)}\cdot T_{b_t,\mathbf{b}_{t+1}^p}^{\langle\mathbf{b}_1^{t-1}\rangle}\dots T_{b_2,\mathbf{b}_3^p}^{\langle\mathbf{b}_1^1\rangle}T_{b_1,\mathbf{b}_2^p}\\&=\mathcal{L}_{1,b_{t+1}}^{(t+2,t+p)}\cdot T_{b_{t+1},\mathbf{b}_1^t}T_{b_{t+1},\mathbf{b}_{t+2}^p}^{\langle\mathbf{b}_1^t\rangle}\cdot\mathcal{L}_{1,b_t}^{(1,t-1)}T_{b_t,\mathbf{b}_1^{t-1}}\dots\mathcal{L}_{1,\mathbf{b}_2}^{(1,1)}T_{b_2,\mathbf{b}_1^1}\\&\quad\times\mathcal{L}_{1,b_1}^{(2)}\dots\mathcal{L}_{1,\mathbf{b}_1^{t-1}}^{(t)}\prod_{t<s\le p}\mathcal{L}_{1,\mathbf{b}_1^t}^{(s)}\cdot T_{b_t,\mathbf{b}_{t+1}^p}^{\langle\mathbf{b}_1^{t-1}\rangle}\dots T_{b_2,\mathbf{b}_3^p}^{\langle\mathbf{b}_1^1\rangle}T_{b_1,\mathbf{b}_2^p}\\&=\mathcal{L}_{1,b_{t+1}}^{(t+2,p)}\cdot v_{b_{t+1},\mathbf{b}_1^t}^{(1,t)}\cdot\mathcal{L}_{1,b_t}^{(1,t-1)}T_{b_t,\mathbf{b}_1^{t-1}}\dots\mathcal{L}_{1,\mathbf{b}_2}^{(1,1)}T_{b_2,\mathbf{b}_1^1}\\&\quad\times\mathcal{L}_{1,b_1}^{(2)}\dots\mathcal{L}_{1,\mathbf{b}_1^{t-1}}^{(t)}\cdot T_{b_{t+1},\mathbf{b}_{t+2}^p}^{\langle\mathbf{b}_1^t\rangle}\cdot T_{b_t,\mathbf{b}_{t+1}^p}^{\langle\mathbf{b}_1^{t-1}\rangle}\dots T_{b_2,\mathbf{b}_3^p}^{\langle\mathbf{b}_1^1\rangle}T_{b_1,\mathbf{b}_2^p}\end{aligned}$$

Therefore, by Lemma 2.5 we have

$$\begin{aligned}
Y_{t+1}\dots Y_1 &= v^{(1,t)}_{b_{t+1},\mathbf{b}_1^t}\cdot\mathcal{L}^{(t+2,p)}_{\mathbf{b}_1^t+1,\mathbf{b}_1^{t+1}}\cdot\mathcal{L}^{(1,t-1)}_{1,b_t}T_{b_t,\mathbf{b}_1^{t-1}}\dots\mathcal{L}^{(1,1)}_{1,\mathbf{b}_2}T_{b_2,\mathbf{b}_1^1}\\
&\qquad\times\mathcal{L}^{(2)}_{1,b_1}\dots\mathcal{L}^{(t)}_{1,\mathbf{b}_1^{t-1}}\cdot T^{\langle\mathbf{b}_1^t\rangle}_{b_{t+1},\mathbf{b}^p_{t+2}}\cdot T^{\langle\mathbf{b}_1^{t-1}\rangle}_{b_t,\mathbf{b}^p_{t+1}}\dots T^{\langle\mathbf{b}_1^1\rangle}_{b_2,\mathbf{b}^p_3}T_{b_1,\mathbf{b}^p_2}\\
&= \mathcal{L}^{(1,t)}_{1,b_{t+1}}T_{b_{t+1},\mathbf{b}_1^t}\dots\mathcal{L}^{(1,1)}_{1,\mathbf{b}_2}T_{b_2,\mathbf{b}_1^1}\mathcal{L}^{(2)}_{1,b_1}\dots\mathcal{L}^{(t+1)}_{1,\mathbf{b}_1^t}\prod_{t+1<s\le p}\mathcal{L}^{(s)}_{1,\mathbf{b}_1^{t+1}}\\
&\qquad\times T^{\langle\mathbf{b}_1^t\rangle}_{b_{t+1},\mathbf{b}^p_{t+2}}\dots T_{b_1,\mathbf{b}^p_2},
\end{aligned}$$

completing the proof of our claim. Taking $t=p$ in the claim proves the Lemma. □

We close this subsection by generalizing Lemma 2.7 to show that $v_{\mathbf{b}}$ can be written in many other forms.

2.12. **Lemma.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $1\le j\le s\le p$. *Then*

$$\begin{aligned}
&\prod_{j\le k<s}\mathcal{L}^{(j,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}_j^k}\cdot\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}_j^{k-1}}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\\
&\qquad=\prod_{j+1\le k<s}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\mathcal{L}^{(j)}_{1,\mathbf{b}^s_{j+1}}T_{\mathbf{b}^s_{j+1},b_j}\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\cdot\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}_j^{k-1}},
\end{aligned}$$

where all products are read from left to right with <u>*decreasing*</u> *values of* i *and* k.

Proof. Let $L(s)$ and $R(s)$, respectively, be the left and right hand side of the formula in the statement of the Lemma. We show that $L(s)=R(s)$ by induction on s. To start the induction observe that, by our conventions,

$$L(j)=\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,b_j^{k-1}}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}=R(j).$$

Hence, the Lemma is true when $s=j$. If $j\le s<p$ then, by induction,

$$\begin{aligned}
L(s+1)&=\mathcal{L}^{(j,s)}_{1,b_{s+1}}T_{b_{s+1},\mathbf{b}^s_j}L(s)=\mathcal{L}^{(j,s)}_{1,b_{s+1}}T_{b_{s+1},\mathbf{b}^s_j}R(s)\\
&=\mathcal{L}^{(j,s)}_{1,b_{s+1}}T_{b_{s+1},\mathbf{b}^s_j}\prod_{j+1\le k<s}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\mathcal{L}^{(j)}_{1,\mathbf{b}^s_{j+1}}T_{\mathbf{b}^s_{j+1},b_j}\prod_{1\le i<j}\mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}}\cdot\prod_{j<k\le p}\mathcal{L}^{(k)}_{1,\mathbf{b}_j^{k-1}}\\
&=\mathcal{L}^{(j,s)}_{1,b_{s+1}}T_{b_{s+1},\mathbf{b}^s_j}\prod_{j+1\le k<s}\mathcal{L}^{(j+1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}^k_{j+1}}\cdot\prod_{1\le i<j}\mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad\times v^{(1,j)}_{\mathbf{b}^s_{j+1},b_j}\prod_{j+1<k\le p}\mathcal{L}^{(k)}_{b_j+1,\mathbf{b}_j^{k-1}},
\end{aligned}$$

since $T_{a,b}$ commutes with $\mathcal{L}^{(i)}_{1,k}$ by Lemma 2.3 whenever $a+b\le k$ and $1\le i\le p$ (we use this fact several times below). Therefore, using Lemma 2.2 and Lemma 2.5,

$$\begin{aligned}
L(s+1) &= \mathcal{L}^{(j,s)}_{1,b_{s+1}} T_{b_{s+1},\mathbf{b}^s_{j+1}} T^{\langle \mathbf{b}^s_{j+1}\rangle}_{b_{s+1},b_j} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}^s_{j+1}} v^{(1,j)}_{\mathbf{b}^s_{j+1},b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= \mathcal{L}^{(j)}_{1,b_{s+1}} v^{(j+1,s)}_{b_{s+1},\mathbf{b}^s_{j+1}} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times T^{\langle \mathbf{b}^s_{j+1}\rangle}_{b_{s+1},b_j} T_{\mathbf{b}^s_{j+1},b_j} \mathcal{L}^{(j+1,p)}_{1,b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= v^{(j+1,s)}_{b_{s+1},\mathbf{b}^s_{j+1}} \mathcal{L}^{(j)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^{s+1}_{j+1}} \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^s_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times T_{\mathbf{b}^{s+1}_{j+1},b_j} \mathcal{L}^{(j+1,p)}_{1,b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= \prod_{j+1\le k<s+1} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^{s+1}_{j+1}+1,\mathbf{b}^p_{i+1}}\\
&\qquad \times \mathcal{L}^{(s+1,p)}_{1,\mathbf{b}^s_{j+1}} v^{(1,j)}_{\mathbf{b}^{s+1}_{j+1},b_j} \prod_{j+1<k\le s} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \cdot \prod_{s+1<k\le p} \mathcal{L}^{(k)}_{\mathbf{b}^s_j+1,\mathbf{b}^{k-1}_j}\\
&= R(s+1),
\end{aligned}$$

where the two lines we have, in essence, reversed some of the previous steps. This completes the proof. □

The following result includes the definition of $v_{\mathbf{b}}$ and Lemma 2.7(a) as special cases (corresponding to $j=1$ and $j=2$, respectively). We proved Lemma 2.7 first because its proof is considerably easier than the proof of Proposition 2.13, even though the underlying argument is very similar.

2.13. **Proposition.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $1\le j\le p$. *Then*

$$v_{\mathbf{b}} = \prod_{j\le k<p} \mathcal{L}^{(j,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_j} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} \cdot \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j} \cdot \prod_{1<i\le j} T_{\mathbf{b}^p_i,b_{i-1}} \mathcal{L}^{(i,p)}_{1,b_{i-1}},$$

where all products are read from left to right with decreasing *values of* i *and* k.

Proof. We argue by induction on j. When $j=1$ the Lemma is a restatement of Definition 2.6, so there is nothing to prove. Suppose now that $1\le j<p$ and that the formula in the Proposition holds. Then by induction and Lemma 2.12 (with

$s=p$), we see that

$$\begin{aligned}
v_{\mathbf{b}} &= \prod_{j+1\le k<p} \mathcal{L}^{(j,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_j} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} \cdot \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j} \cdot \prod_{1<i\le j} T_{\mathbf{b}^p_i,b_{i-1}} \mathcal{L}^{(i,p)}_{1,b_{i-1}} \\
&= \prod_{j+1\le k<s} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \mathcal{L}^{(j)}_{1,\mathbf{b}^p_{j+1}} T_{\mathbf{b}^p_{j+1},b_j} \prod_{1\le i<j} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} \cdot \prod_{j<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_j} \\
&\quad \times \prod_{1<i\le j} T_{\mathbf{b}^p_i,b_{i-1}} \mathcal{L}^{(i,p)}_{1,b_{i-1}} \\
&= \prod_{j+1\le k<p} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^p_{j+1}+1,\mathbf{b}^p_{i+1}} \cdot v^{(1,j)}_{\mathbf{b}^p_{j+1},b_j} \cdot \prod_{j+1<k\le p} \mathcal{L}^{(k)}_{b_j+1,\mathbf{b}^{k-1}_j} \\
&\quad \times \prod_{1<i\le j} T_{\mathbf{b}^p_i,b_{i-1}} \mathcal{L}^{(i,p)}_{1,b_{i-1}} \\
&= \prod_{j+1\le k<p} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j} \mathcal{L}^{(i)}_{\mathbf{b}^p_{j+1}+1,\mathbf{b}^p_{i+1}} \cdot \prod_{j+1<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_{j+1}} \\
&\quad \times v^{(1,j)}_{\mathbf{b}^p_{j+1},b_j} \prod_{1<i\le j} T_{\mathbf{b}^p_i,b_{i-1}} \mathcal{L}^{(i,p)}_{1,b_{i-1}} \\
&= \prod_{j+1\le k<p} \mathcal{L}^{(j+1,k)}_{1,b_{k+1}} T_{b_{k+1},\mathbf{b}^k_{j+1}} \cdot \prod_{1\le i<j+1} \mathcal{L}^{(i)}_{1,\mathbf{b}^p_{i+1}} \cdot \prod_{j+1<k\le p} \mathcal{L}^{(k)}_{1,\mathbf{b}^{k-1}_{j+1}} \cdot \prod_{1<i\le j+1} T_{\mathbf{b}^p_i,b_{i-1}} \mathcal{L}^{(i,p)}_{1,b_{i-1}},
\end{aligned}$$

which is precisely the statement of the Proposition for $j+1$. □

2.4. **The central element $z_{\mathbf{b}}$.** . We now want to study the modules $Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \mathscr{H}_{r,n}$. To do this we first need to recall the following important property of $v_{\mathbf{b}}$ which was established in [**23**]. Before we can state this result, for $\mathbf{b}\in\mathscr{C}_{p,n}$ set

$$w_{\mathbf{b}} = w^{\langle \mathbf{b}^{p-2}_1\rangle}_{b_{p-1},\mathbf{b}^p_p} w^{\langle \mathbf{b}^{p-3}_1\rangle}_{b_{p-2},\mathbf{b}^p_{p-1}} \dots w^{\langle \mathbf{b}^{1}_1\rangle}_{b_2,\mathbf{b}^p_3} w_{b_1,\mathbf{b}^p_2}.$$

In two-line notation, $w_{\mathbf{b}}$ is the permutation

$$\begin{pmatrix} 1 & \dots & \mathbf{b}^1_1 & \mathbf{b}^1_1+1 & \dots & \mathbf{b}^2_1 & \mathbf{b}^2_1+1 & \dots & \mathbf{b}^{p-1}_1+1 & \dots & \mathbf{b}^p_1 \\ \mathbf{b}^p_2+1 & \dots & \mathbf{b}^p_1 & \mathbf{b}^p_3+1 & \dots & \mathbf{b}^p_2 & \mathbf{b}^p_4+1 & \dots & 1 & \dots & \mathbf{b}^p_p \end{pmatrix}.$$

Note that $b_1=\mathbf{b}^1_1$, $b_p=\mathbf{b}^p_p$ and $n=\mathbf{b}^p_1$. Also, if $\mathbf{b}=(a,b)$ then $w_{\mathbf{b}}=w_{a,b}$.

For convenience we set $T_{\mathbf{b}}=T_{w_{\mathbf{b}}}$. For example, $T_{a,b}=T_{w_{a,b}}$.

For any $\mathbf{b}=(b_1,\dots,b_p)\in\mathscr{C}_{p,n}$ we define $\mathbf{b}'=(b_p,\dots,b_1)$. Since $w^{-1}_{a,b}=w_{b,a}$ it follows that $w_{\mathbf{b}'}=w^{-1}_{\mathbf{b}}$.

Finally, set $\mathfrak{S}_b=\mathfrak{S}_{b_1}\times\mathfrak{S}_{b_2}\times\dots\times\mathfrak{S}_{b_p}$, which we consider as a subgroup of $\mathfrak{S}_n$ in the obvious way. Similarly, $\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}})$ is a subalgebra of $\mathscr{H}_q(\mathfrak{S}_n)$ via the natural embedding.

2.14. **Lemma** ([**23**, Proposition 2.5])**.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $1\le i,j\le n$, *with* $i\ne\mathbf{b}^p_t$ *for* $1\le t\le p$. *Then*

a) $T_i v_{\mathbf{b}} = v_{\mathbf{b}} T_{(i)w^{-1}_{\mathbf{b}}}$, *and*
b) $L_j v_{\mathbf{b}} = v_{\mathbf{b}} L_{(j)w^{-1}_{\mathbf{b}}}$.

Using this fact we can prove the following two results.

2.15. **Lemma.** *Suppose that $1 \le t \le p$ and let i and j be integers such that $1 \le i, j \le n$ and $i \ne \mathbf{b}^t_\alpha$ for $\alpha = t-p+1, t-p+2, \dots, t$. Then*

$$T_i\Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) = \begin{cases} \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) T_i, & \text{if } 1 \le i < b_t; \\ \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) T_{(i)w_{\mathbf{b}\langle t-1\rangle'}}, & \text{if } b_t + 1 \le i < n, \end{cases}$$

$$L_j\Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) = \begin{cases} \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) L_j, & \text{if } 1 \le j \le b_t; \\ \Big(Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}\Big) L_{(j)w_{\mathbf{b}\langle t-1\rangle'}}, & \text{if } b_t + 1 \le j \le n, \end{cases}$$

Proof. For the first equality, if $i \ne b_t$ then using Lemmas 2.2 and 2.4

$$\begin{aligned} T_i Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} &= T_i \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_{b_t,n-b_t} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \\ &= \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_i T_{b_t,n-b_t} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \\ &= \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_{b_t,n-b_t} T_{(i)w_{b_t,n-b_t}} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}. \end{aligned}$$

The first claim now follows using Lemma 2.14. For the second claim observe that by Corollary 2.8(b) there exists an $h \in \mathscr{H}_{r,n}$ such that

$$\begin{aligned} L_j Y_t v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} &= L_j v^{(t+1,t+p-1)}_{1,b_t} h = v^{(t+1,t+p-1)}_{b_t,n-b_t} L_{(j)w_{b_t,n-b_t}} h \\ &= \mathcal{L}^{(t+1,t+p-1)}_{1,b_t} T_{b_t,n-b_t} L_{(j)w_{b_t,n-b_t}} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle} \\ &= Y_t L_{(j)w_{b_t,n-b_t}} v^{(t-1)}_{\mathbf{b}\langle t-1\rangle}. \end{aligned}$$

So the result again follows using Lemma 2.14. □

2.16. **Lemma.** *Suppose that $1 \le t \le p$ and let i and j be integers such that $1 \le i, j \le n$ and $i \ne \mathbf{b}^t_\alpha$ whenever $t-p+1 \le \alpha \le t$. Then*

$$T_i(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) = \begin{cases} (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{i+\mathbf{b}^{t-1}_1}, & \text{if } 1 \le i < b_t; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{i-b_t+b_1+\dots+b_{t-2}}, & \text{if } b_t + 1 \le i < \mathbf{b}^t_{t-1}; \\ \quad\vdots & \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{i-\mathbf{b}^t_2}, & \text{if } \mathbf{b}^t_2 + 1 \le i < \mathbf{b}^t_1; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(i-\mathbf{b}^t_1)w_{\mathbf{b}'}}, & \text{if } \mathbf{b}^t_1 + 1 \le i < n; \end{cases}$$

$$L_j(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) = \begin{cases} (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{j+\mathbf{b}^{t-1}_1}, & \text{if } 1 \le j \le b_t; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{j-b_t+b_1+\dots+b_{t-2}}, & \text{if } b_t + 1 \le j \le \mathbf{b}^t_{t-1}; \\ \quad\vdots & \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{j-\mathbf{b}^t_2}, & \text{if } \mathbf{b}^t_2 + 1 \le j \le \mathbf{b}^t_1; \\ (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) L_{(j-\mathbf{b}^t_1)w_{\mathbf{b}'}}, & \text{if } \mathbf{b}^t_1 + 1 \le j \le n. \end{cases}$$

$$\begin{aligned} T_i(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(i)w_{(b_t,\dots,b_1)}} \\ L_j(Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_t \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(j)w_{(b_t,\dots,b_1)}} \end{aligned}$$

In particular, taking $t = p$, we have

$$\begin{aligned} T_i(Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) T_{(i)w_{\mathbf{b}'}}, \\ L_j(Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) &= (Y_p \dots Y_2 Y_1 v_{\mathbf{b}}) L_{(j)w_{\mathbf{b}'}}. \end{aligned}$$

Proof. This can be proved in exactly the same way as Lemma 2.15. Note that the final claim also follows from Lemma 2.11 using Lemma 2.2. □

All the results we have obtained so far are valid for the cyclotomic Hecke algebra $\mathscr{H}_{r,n}$ defined over an arbitrary ring.

For the rest of this paper we make the following assumption. This definition is repeated from the introduction.

2.17. **Definition.** *Suppose that R is a commutative ring with 1 and set*

$$A(\varepsilon, q, \mathbf{Q}) = \prod_{1\le i,j\le d} \prod_{-n<k<n} \prod_{1\le t<p} \big(Q_i - \varepsilon^t q^k Q_j\big).$$

*Then $\mathbf{Q}$ is (ε, q)-**separated** in R if $A(\varepsilon, q, \mathbf{Q})$ is invertible in R.*

Observe that, even though our notation does not reflect this, whether or not $\mathbf{Q}$ is (ε, q)-separated also depends on n and the ring R.

Fix $\mathbf{b} \in \mathscr{C}_{p,n}$ and set $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ and $\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q}) \otimes \cdots \otimes \mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q})$. Then an important result from [**23**] is the following.

2.18. **Lemma** ([**23**, Prop. 2.15])**.** *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and that $\mathbf{Q}$ is (ε, q)-separated if $d > 1$. Then $\mathscr{H}_{d,\mathbf{b}}$ acts faithfully on $V_{\mathbf{b}}$ from the left and $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}) \cong \mathscr{H}_{d,\mathbf{b}}$.*

2.19. *Remark.* When $d = 1$, the algebra $\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}})$ can be naturally embedded into $\mathscr{H}_{r,n}$ as a subalgebra; see [**20**]. In that case, the condition of being (ε, q)-separated means that $\prod_{|k|<n,1\le t<p}\big(1 - \varepsilon^t q^k\big)$ is invertible.

To describe the action of $\mathscr{H}_{d,\mathbf{b}}$ on $V_{\mathbf{b}}$ given a permutation $w = s_{i_1} \dots s_{i_k} \in \mathfrak{S}_n$ and an integer $c \in \mathbb{N}$ such that $i_j + c < n$, for $1 \le j \le k$, define $w^{\langle c\rangle} = s_{i_1+c} \dots s_{i_k+c}$. Then $w^{\langle c\rangle} \in \mathfrak{S}_n$. Note that this is compatible with our previous definition of $w_{a,b}^{\langle c\rangle}$.

Define $\Theta_{\mathbf{b}}$ to be the 'natural inclusion map' $\mathscr{H}_{d,\mathbf{b}} \hookrightarrow \mathscr{H}_{r,n}$. That is, $\Theta_{\mathbf{b}}$ is the R-linear map determined by

$$\begin{aligned}
\Theta_{\mathbf{b}}&\Big(\big(L_1^{a_{1,1}} \dots L_{b_1}^{a_{1,b_1}} T_{x_1}\big) \otimes \big(L_1^{a_{2,1}} \dots L_{b_2}^{a_{2,b_2}} T_{x_2}\big) \otimes \cdots \otimes \big(L_1^{a_{p,1}} \dots L_{b_p}^{a_{p,b_p}} T_{x_p}\big)\Big)\\
&= \big(L_1^{a_{1,1}} \dots L_{b_1}^{a_{1,b_1}} T_{x'_1}\big)\big(L_{\mathbf{b}_1^1+1}^{a_{2,1}} \dots L_{\mathbf{b}_1^1+b_2}^{a_{2,b_2}} T_{x'_2}\big) \cdots \big(L_{\mathbf{b}_1^{p-1}+1}^{a_{p,1}} \dots L_{\mathbf{b}_1^{p-1}+b_p}^{a_{p,b_p}} T_{x'_p}\big)\\
&= \big(L_1^{a_{1,1}} \dots L_{b_1}^{a_{1,b_1}}\big)\big(L_{\mathbf{b}_1^1+1}^{a_{2,1}} \dots L_{\mathbf{b}_1^1+b_2}^{a_{2,b_2}}\big) \cdots \big(L_{\mathbf{b}_1^{p-1}+1}^{a_{p,1}} \dots L_n^{a_{p,b_p}}\big) T_{x'_1} T_{x'_2} \dots T_{x'_p},
\end{aligned}$$

for all $x_t \in \mathfrak{S}_{b_t}$ and $0 \le a_{j,t} < d$, for $1 \le t \le p$ and $1 \le j \le b_t$, and where $x'_t := x_t^{\langle \mathbf{b}_1^{t-1}\rangle}$, for $1 \le t \le p$. The second equality follows because all of the terms commute. Thus, we have $x'_1 = x_1$ and $\Theta_{\mathbf{b}}(T_{x_1} \otimes \cdots \otimes T_{x_p}) = T_w$, where $w = x_1 x_2^{\langle \mathbf{b}_1^1\rangle} \dots x_p^{\langle \mathbf{b}_1^{p-1}\rangle} \in \mathfrak{S}_{\mathbf{b}}$, for $x_t \in \mathfrak{S}_{b_t}$. We emphasize that $\Theta_{\mathbf{b}}$ is an R-module homomorphism but *not* a ring homomorphism.

Similarly, define $\widehat{\Theta}_{\mathbf{b}}$ to be the R-linear map $\widehat{\Theta}_{\mathbf{b}} : \mathscr{H}_{d,\mathbf{b}} \longrightarrow \mathscr{H}_{r,n}$ determined by

$$\begin{aligned}
\widehat{\Theta}_{\mathbf{b}}&\Big(\big(L_1^{a_{1,1}} \dots L_{b_1}^{a_{1,b_1}} T_{x_1}\big) \otimes \big(L_1^{a_{2,1}} \dots L_{b_2}^{a_{2,b_2}} T_{x_2}\big) \otimes \cdots \otimes \big(L_1^{a_{p,1}} \dots L_{b_p}^{a_{p,b_p}} T_{x_p}\big)\Big)\\
&= \big(L_1^{a_{p,1}} \dots L_{b_p}^{a_{p,b_p}} T_{x''_p}\big) \cdots \big(L_{\mathbf{b}_3^p+1}^{a_{2,1}} \dots L_{\mathbf{b}_2^p}^{a_{2,b_2}} T_{x''_2}\big)\big(L_{\mathbf{b}_2^p+1}^{a_{1,1}} \dots L_{\mathbf{b}_1^p}^{a_{1,b_1}} T_{x''_1}\big)\\
&= \big(L_1^{a_{p,1}} \dots L_{b_p}^{a_{p,b_p}}\big) \dots \big(L_{\mathbf{b}_3^p+1}^{a_{2,1}} \dots L_{\mathbf{b}_2^p}^{a_{2,b_2}}\big)\big(L_{\mathbf{b}_2^p+1}^{a_{1,1}} \dots L_{\mathbf{b}_1^p}^{a_{1,b_1}}\big) T_{x''_1} T_{x''_2} \dots T_{x''_p},
\end{aligned}$$

where the x_t and $a_{t,j}$ are as before and $x''_t := w_{\mathbf{b}}^{-1} x_t^{\langle \mathbf{b}_1^{t-1}\rangle} w_{\mathbf{b}} = w_{\mathbf{b}}^{-1} x'_t w_{\mathbf{b}}$. In particular, $x''_p = x_p$ and $x''_1 x''_2 \dots x''_p = w_{\mathbf{b}}^{-1}\big(x_1 x_2^{\langle \mathbf{b}_1^1\rangle} \dots x_p^{\langle \mathbf{b}_1^{p-1}\rangle}\big) w_{\mathbf{b}} \in \mathfrak{S}_{\mathbf{b}'}$.

Given these definitions, the proof of Lemma 2.18 (that is, of [**23**, Prop. 2.15]), shows that $h \in \mathscr{H}_{d,\mathbf{b}}$ acts on $V_{\mathbf{b}}$ as left multiplication by $\widehat{\Theta}_{\mathbf{b}}(h)$. Moreover,

$$\widehat{\Theta}_{\mathbf{b}}(h) v_{\mathbf{b}} = v_{\mathbf{b}} \Theta_{\mathbf{b}}(h), \qquad \text{for all } h \in \mathscr{H}_{d,\mathbf{b}}, \tag{2.20}$$

by Lemma 2.14. Typically, if $h \in \mathscr{H}_{d,\mathbf{b}}$ then we will write $h \cdot v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(h)v_{\mathbf{b}}$ in what follows. Thus, we have

$$h \cdot v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(h), \qquad \text{for all } h \in \mathscr{H}_{d,\mathbf{b}},$$

for $h \in \mathscr{H}_{d,\mathbf{b}}$.

2.21. **Lemma.** *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated and let* $\mathbf{b} \in \mathscr{C}_{p,n}$*. Then there exists a unique element* $z_{\mathbf{b}}$ *in* $\mathscr{H}_{d,\mathbf{b}}$ *such that*

$$z_{\mathbf{b}} \cdot v_{\mathbf{b}} = Y_pY_{p-1}\dots Y_2Y_1v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(z_{\mathbf{b}}).$$

Moreover, $z_{\mathbf{b}}$ *belongs to the centre of* $\mathscr{H}_{d,\mathbf{b}}$*.*

Proof. By Lemma 2.11, left multiplication by $Y_p \dots Y_2Y_1$ defines a homomorphism in $\operatorname{End}_{\mathscr{H}_{r,n}}\big(V_{\mathbf{b}}\big)$. Therefore, there exists a unique element $z_{\mathbf{b}}$ in $\mathscr{H}_{d,\mathbf{b}}$ such that

$$Y_pY_{p-1}\dots Y_2Y_1v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})v_{\mathbf{b}} = v_{\mathbf{b}}\Theta_{\mathbf{b}}(z_{\mathbf{b}})$$

by Lemma 2.18 and (2.20).

It remains to show that $z_{\mathbf{b}}$ is central in $\mathscr{H}_{d,\mathbf{b}}$. As $\mathscr{H}_{d,\mathbf{b}}$ acts faithfully on $V_{\mathbf{b}}$, it is enough to show that $\widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}}h)v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(hz_{\mathbf{b}})v_{\mathbf{b}}$, for all $h \in \mathscr{H}_{d,\mathbf{b}}$. By Lemma 2.14,

$$\widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}}h)v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})\widehat{\Theta}_{\mathbf{b}}(h)v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})v_{\mathbf{b}}\Theta_{\mathbf{b}}(h) = Y_p\dots Y_2Y_1v_{\mathbf{b}}\Theta_{\mathbf{b}}(h).$$

Applying (the last statements in) Lemma 2.16, we see that

$$Y_p\dots Y_2Y_1v_{\mathbf{b}}\Theta_{\mathbf{b}}(h) = \widehat{\Theta}_{\mathbf{b}}(h)Y_p\dots Y_2Y_1v_{\mathbf{b}}, = \widehat{\Theta}_{\mathbf{b}}(h)\widehat{\Theta}_{\mathbf{b}}(z_{\mathbf{b}})v_{\mathbf{b}} = \widehat{\Theta}_{\mathbf{b}}(hz_{\mathbf{b}})v_{\mathbf{b}},$$

as required. □

2.5. **A Morita equivalence for** $\mathscr{H}_{r,n}$**.** By [**23**, Prop. 2.15], $V_{\mathbf{b}}$ is a projective $\mathscr{H}_{r,n}$-module. Let $\mathscr{H}_{r,n}(\mathbf{b})$ be the smallest two-sided ideal of $\mathscr{H}_{r,n}$ which contains $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ as a direct summand. By [**10**, Theorem 1.1] there is a Morita equivalence

$$\mathtt{H}_{\mathbf{b}} : \text{Mod-}\mathscr{H}_{d,\mathbf{b}} \xrightarrow[\text{Morita}]{\simeq} \text{Mod-}\mathscr{H}_{r,n}(\mathbf{b})$$

given by $\mathtt{H}_{\mathbf{b}}(X) = X \otimes_{\mathscr{H}_{d,\mathbf{b}}} V_{\mathbf{b}}$. Hence, by Lemma 2.18 and the general theory of Morita equivalences (*cf.* [**4**, §2.2]), we have the following.

2.22. **Lemma** (*cf.* [**10**, Corollary 4.9])**.** *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated in* R *and let* X *be a right ideal of* $\mathscr{H}_{d,\mathbf{b}}$*. Then, as right* $\mathscr{H}_{r,n}$*-modules,*

$$\mathtt{H}_{\mathbf{b}}(X) \cong \widehat{\Theta}_{\mathbf{b}}(X)V_{\mathbf{b}}.$$

We next show that $\mathtt{H}_{\mathbf{b}}$ can be realised as induction from a subalgebra of $\mathscr{H}_{r,n}$. To do this we need to produce a subalgebra of $\mathscr{H}_{r,n}$ which is isomorphic to $\mathscr{H}_{d,\mathbf{b}}$.

Before we state this result, given a sequence $\mathbf{b} = (b_1, \dots, b_p) \in \mathscr{C}_{p,n}$ define

$$u^+_{\mathbf{b}}(\mathbf{Q}) = \mathcal{L}^{(2)}_{1,\mathbf{b}^1_1}\mathcal{L}^{(3)}_{1,\mathbf{b}^2_1}\dots\mathcal{L}^{(p)}_{1,\mathbf{b}^{p-1}_1} \quad \text{and} \quad u^-_{\mathbf{b}}(\mathbf{Q}) = \mathcal{L}^{(p-1)}_{1,\mathbf{b}^p_p}\dots\mathcal{L}^{(2)}_{1,\mathbf{b}^p_3}\mathcal{L}^{(1)}_{1,\mathbf{b}^p_2}. \tag{2.23}$$

In the notation of [9, Definition 3.1], $u^+_{\mathbf{b}}(\mathbf{Q}) = u^+_{\boldsymbol{\omega}_{\mathbf{b}}}$, where $\boldsymbol{\omega}_{\mathbf{b}} = ({\boldsymbol{\omega}_{\mathbf{b}}}^{(1)}, \cdots, {\boldsymbol{\omega}_{\mathbf{b}}}^{(r)})$ is the multipartition

$${\boldsymbol{\omega}_{\mathbf{b}}}^{(s)} = \begin{cases} (1^{b_\alpha}), & \text{if } s = d\alpha \text{ for some } \alpha, \\ (0), & \text{otherwise.} \end{cases}$$

Hereafter, we write $u^{\pm}_{\mathbf{b}} = u^{\pm}_{\mathbf{b}}(\mathbf{Q})$.

By Proposition 2.13, we can write $v_{\mathbf{b}} = v^+_{\mathbf{b}}u^+_{\mathbf{b}} = u^-_{\mathbf{b}}v^-_{\mathbf{b}}$, where

$$v^+_{\mathbf{b}} = \mathcal{L}^{(1,p-1)}_{1,b_p}T_{b_p,\mathbf{b}^{p-1}_1}\mathcal{L}^{(1,p-2)}_{1,b_{p-1}}T_{b_{p-1},\mathbf{b}^{p-2}_1}\dots\mathcal{L}^{(1,1)}_{1,b_2}T_{b_2,\mathbf{b}^1_1}$$

and

$$v_{\mathbf{b}}^- = T_{\mathbf{b}_p^p,b_{p-1}}\mathcal{L}_{1,b_{p-1}}^{(p,p)}\dots T_{\mathbf{b}_3^p,b_2}\mathcal{L}_{1,b_2}^{(3,p)}T_{\mathbf{b}_2^p,b_1}\mathcal{L}_{1,b_1}^{(2,p)}.$$

(Take $j=1$ and $j=p$ in Proposition 2.13, respectively.)

The following key lemma plays an important role throughout this paper.

2.24. **Lemma.** *Suppose that* $\mathbf{Q}$ *is* (ε,q)*-separated in* R. *Let* $\mathbf{b}\in\mathscr{C}_{p,n}$. *Then* $z_{\mathbf{b}}$ *is invertible in* $\mathscr{H}_{d,\mathbf{b}}$.

Proof. The module $V_{\mathbf{b}} = v_{\mathbf{b}}\mathscr{H}_{r,n}$ is a projective submodule of $\mathscr{H}_{r,n}$-module by [**23**, Prop. 2.15], so $V_{\mathbf{b}} = e\mathscr{H}_{r,n}$ for some idempotent $e\in\mathscr{H}_{r,n}$. Therefore, $V_{\mathbf{b}} = e\mathscr{H}_{r,n} = e^2\mathscr{H}_{r,n}\subseteq e\mathscr{H}_{r,n}e\mathscr{H}_{r,n} = V_{\mathbf{b}}^2$ so that $V_{\mathbf{b}} = V_{\mathbf{b}}^2$. Therefore, using the formulae for $v_{\mathbf{b}}$ given before the Lemma,

$$\begin{aligned} V_{\mathbf{b}} = (V_{\mathbf{b}})^2 &= v_{\mathbf{b}}\mathscr{H}_{r,n}v_{\mathbf{b}}\mathscr{H}_{r,n} = v_{\mathbf{b}}^+\big(u_{\mathbf{b}}^+\mathscr{H}_{r,n}u_{\mathbf{b}}^-\big)v_{\mathbf{b}}^-\mathscr{H}_{r,n}\\ &= v_{\mathbf{b}}^+\big(u_{\mathbf{b}}^+T_{\mathbf{b}}u_{\mathbf{b}}^-\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}})\big)v_{\mathbf{b}}^-\mathscr{H}_{r,n},\end{aligned}$$

where the last equality follows by Du and Rui [**11**, Prop. 3.1(a)]. Lemma 2.4 shows that $\mathscr{H}_q(\mathfrak{S}_b)v_{\mathbf{b}}^- = v_{\mathbf{b}}^-\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}'})$. Hence,

$$V_{\mathbf{b}} = v_{\mathbf{b}}T_{\mathbf{b}}u_{\mathbf{b}}^-v_{\mathbf{b}}^-\mathscr{H}_q(\mathfrak{S}_{\mathbf{b}'})\mathscr{H}_{r,n}\subseteq v_{\mathbf{b}}T_{\mathbf{b}}v_{\mathbf{b}}\mathscr{H}_{r,n} = z_{\mathbf{b}}\cdot V_{\mathbf{b}},$$

by Lemma 2.11 and Lemma 2.21. Therefore, the endomorphism of $V_{\mathbf{b}}$ given by left multiplication by $z_{\mathbf{b}}$ has a right inverse in $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}})$. Consequently, $z_{\mathbf{b}}$ has a right inverse in $\mathscr{H}_{d,\mathbf{b}}$ by Lemma 2.18. Hence, $z_{\mathbf{b}}$ is invertible in $\mathscr{H}_{d,\mathbf{b}}$ since it is central. □

Under the conditions of Lemma 2.24 we can make the following definition.

2.25. **Definition.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $\mathbf{Q}$ *is* (ε,q)*-separated in* R. *Let* $e_{\mathbf{b}} = z_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}}\in V_{\mathbf{b}}$ *and define*

$$\widehat{\mathscr{H}_{d,\mathbf{b}}} = \{\, h\cdot e_{\mathbf{b}} \mid h\in\mathscr{H}_{d,\mathbf{b}}\,\} = \{\, e_{\mathbf{b}}\Theta_{\mathbf{b}}(h)\mid h\in\mathscr{H}_{d,b}\,\}\subseteq V_b.$$

Quite surprisingly, $\widehat{\mathscr{H}_{d,\mathbf{b}}}$ is something like a 'parabolic' subalgebra of $\mathscr{H}_{r,n}$.

2.26. **Theorem.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $\mathbf{Q}$ *is* (ε,q)*-separated. Then:*

a) $e_{\mathbf{b}}$ *is an idempotent in* $\mathscr{H}_{r,n}$ *and* $V_{\mathbf{b}} = e_{\mathbf{b}}\mathscr{H}_{r,n}$.
b) $\widehat{\mathscr{H}_{d,\mathbf{b}}}$ *is a unital subalgebra of* $\mathscr{H}_{r,n}$ *with identity element* $e_{\mathbf{b}}$.
c) *The map* $\mathscr{H}_{d,\mathbf{b}}\longrightarrow\widehat{\mathscr{H}_{d,\mathbf{b}}}; h\mapsto h\cdot e_{\mathbf{b}}$ *is an algebra isomorphism.*

Proof. Suppose that $x,y\in\mathscr{H}_{d,\mathbf{b}}$. Then using the definitions, (2.20) and Lemma 2.21 we have that

$$\begin{aligned}(x\cdot e_{\mathbf{b}})(y\cdot e_{\mathbf{b}}) &= (xz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}})(yz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}}) = xz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}}v_{\mathbf{b}}\Theta_{\mathbf{b}}(yz_{\mathbf{b}}^{-1})T_{\mathbf{b}}\\ &= xz_{\mathbf{b}}^{-1}z_{\mathbf{b}}\cdot v_{\mathbf{b}}\Theta_{\mathbf{b}}(yz_{\mathbf{b}}^{-1})T_{\mathbf{b}} = x\cdot v_{\mathbf{b}}\Theta_{\mathbf{b}}(yz_{\mathbf{b}}^{-1})T_{\mathbf{b}}\\ &= xyz_{\mathbf{b}}^{-1}\cdot v_{\mathbf{b}}T_{\mathbf{b}} = (xy)\cdot e_{\mathbf{b}}.\end{aligned}$$

Taking $x=y=1_{\mathscr{H}_{d,\mathbf{b}}}$ shows that $e_{\mathbf{b}}$ is an idempotent in $\mathscr{H}_{r,n}$. As $\mathscr{H}_{d,\mathbf{b}}$ acts faithfully on $V_{\mathbf{b}}$ by Lemma 2.18, all of the claims now follow. □

Theorem 2.26 says that the natural inclusion map $\Theta_{\mathbf{b}}:\mathscr{H}_{d,\mathbf{b}}\hookrightarrow\mathscr{H}_{r,n}$ is an inclusion of algebras when it is composed with left multiplication by $e_{\mathbf{b}}$. Note that the image of $\Theta_{\mathbf{b}}$ is *not* a subalgebra of $\mathscr{H}_{r,n}$.

Combining Theorem 2.26 and Lemma 2.22 gives a second description of the Morita equivalence $\mathtt{H}_{\mathbf{b}}$. If A is a subalgebra of an algebra B then let $\uparrow_A^B$ be the corresponding induction functor.

2.27. **Corollary.** *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated and that* X *is a right* $\mathscr{H}_{d,\mathbf{b}}$ *module, where* $\mathbf{b} \in \mathscr{C}_{p,n}$*. Then*

$$\mathtt{H}_{\mathbf{b}}(X) \cong (X \cdot e_{\mathbf{b}}) \uparrow_{\widehat{\mathscr{H}}_{d,\mathbf{b}}}^{\mathscr{H}_{r,n}} = X \cdot e_{\mathbf{b}} \otimes_{\widehat{\mathscr{H}}_{d,\mathbf{b}}} \mathscr{H}_{r,n}.$$

2.6. **Comparing trace forms on $V_{\mathbf{b}}$.** Recall that a trace form on an R-algebra A is a linear map $\operatorname{tr} : A \longrightarrow R$ such that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$, for all $a, b \in A$. The form tr is non-degenerate if whenever $0 \neq a \in A$ then $\operatorname{tr}(ab) \neq 0$ for some $b \in A$.

By [**26**] the Hecke algebras $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$ are both equipped with 'canonical' non-degenerate trace forms $\operatorname{Tr}_{\mathbf{b}}$ and Tr, respectively. The aim of this subsection is to compare these two trace forms. More precisely, we show that

$$\operatorname{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}_{\mathbf{b}}(h) \operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}),$$

for all $h \in \mathscr{H}_{d,\mathbf{b}}$. This result will be used in the next section to compute the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ from the introduction.

The trace form $\operatorname{Tr} : \mathscr{H}_{r,n} \longrightarrow R$ on $\mathscr{H}_{r,n}$ is the R-linear map determined by

$$\operatorname{Tr}(L_1^{a_1} \dots L_n^{a_n} T_x T_y) = \begin{cases} q^{\ell(x)}, & \text{if } a_1 = \dots = a_n = 0 \text{ and } x = y^{-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.28}$$

The trace form $\operatorname{Tr}_{\mathbf{b}}$ on $\mathscr{H}_{d,\mathbf{b}}$ is defined similarly.

Comparing these two trace forms requires some preparation. Before Lemma 2.24 we noted that $v_{\mathbf{b}} = v_{\mathbf{b}}^+ u_{\mathbf{b}}^+$, for some element $v_{\mathbf{b}}^+$. We need to understand $v_{\mathbf{b}}^+$ better in order to compare Tr and $\operatorname{Tr}_{\mathbf{b}}$.

Let $\mathcal{H}_m^L$ be the R-submodule of $\mathscr{H}_{r,n}$ spanned by the elements

$$\{\, T_w L_1^{a_1} \dots L_{m-1}^{a_{m-1}} \mid 0 \le a_1, \dots, a_{m-1} < r \text{ and } w \in \mathfrak{S}_m \,\}.$$

Note that $\mathcal{H}_m^L$ is not, in general, a subalgebra of $\mathscr{H}_{r,n}$.

2.29. **Lemma.** *Suppose that* a, b, k *and* l *are positive integers such that* $k \le l \le a$ *and* $1 \le s \le t \le p$*. Then*

$$\mathcal{L}_{k,l}^{(s,t)} T_{a,b} = T_{a,b} \Big(\mathcal{L}_{b+k,b+l}^{(s,t)} + \sum_{m=b+k}^{b+l} \sum_{e=1}^{d(t-s+1)} h_{m,e} L_m^e \Big),$$

for some $h_{m,e} \in \mathcal{H}_m^L$*.*

Proof. For the duration of this proof let $L_{k,l}(Q) = \prod_{m=k}^{l} (L_m - Q)$, for $Q \in R$. Then $\mathcal{L}_{k,l}^{(s,t)} = \prod_{i=1}^{d} \prod_{u=s}^{t} L_{k,l}(\varepsilon^u Q_i)$. By the right handed version of [**27**, Lemma 5.6],

$$L_{k,l}(Q) T_{a,b} = T_{a,b} \Big(L_{b+k,b+l}(Q) + \sum_{m=b+k}^{b+l} h_m L_m \Big),$$

for some $h_m \in \mathcal{H}_m^L$. Therefore, there exist elements $h_{m,i,t} \in \mathcal{H}_m^L$ such that

$$\mathcal{L}_{k,l}^{(s,t)} T_{a,b} = T_{a,b} \prod_{i=1}^{d} \prod_{u=s}^{t} \Big(L_{b+k,b+l}(\varepsilon^u Q_i) + \sum_{m=b+k}^{b+l} h_{m,i,u} L_m \Big).$$

Collecting the terms in the product, we obtain $\mathcal{L}_{b+k,b+l}^{(s,t)}$, as the leading term, plus a linear combination of terms which are products of $d(t-s+1)$ elements, each of which is equal to either $L_{b+k,b+l}(\varepsilon^u Q_i)$ or $h_{m,i,u} L_m$, for some m, i, u as above. Expand the factors $L_{b+k,b+l}(\varepsilon^u Q_i)$ into a sum of monomials in $L_{b+k}, \dots, L_{b+l}$ and consider the resulting linear combination of products of these summands with the terms $h_{m,i,u} L_m$ above. Fix one of these products of $d(t-s+1)$ terms, say X, and let m be maximal such that L_m appears in X. By assumption the rightmost L_m which appears in X cannot have both T_m and T_{m-1} to its right, so using

Lemma 2.3 we can rewrite X as a linear combination of terms of the form $h_{X,e}L_m^e$, where $1 \le e \le d(t-s+1)$ and $h_{X,e} \in \mathcal{H}_m^L$. Note that when we rewrite X in this form some of the $L_{m'}$, with $m' < m$, are changed into L_m when we move them to the right. However, T_m never appears to the right of these newly created L_m. The final exponent of L_m is at most $d(t-s+1)$ because no factor can increase the exponent of L_m by more than one. The result follows. □

2.30. **Lemma.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$. *Then*

$$v_{\mathbf{b}}^+ = T_{\mathbf{b}'}\Big(\mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n}\mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n}\dots\mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n} + \sum_{l=1}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl} h_{l,m,e}L_m^e\Big)$$

for some $h_{l,m,e} \in \mathcal{H}_m^L$.

Proof. Recall that $v_{\mathbf{b}}^+ = \mathcal{L}^{(1,p-1)}_{1,b_p}T_{b_p,\mathbf{b}_1^{p-1}}\mathcal{L}^{(1,p-2)}_{1,b_{p-1}}T_{b_{p-1},\mathbf{b}_1^{p-2}}\dots\mathcal{L}^{(1,1)}_{1,b_2}T_{b_2,\mathbf{b}_1^1}$. To prove the lemma let $v_{\mathbf{b},p}^+ = 1$ and set $v_{\mathbf{b},k}^+ = v_{\mathbf{b},k+1}^+\mathcal{L}^{(1,k)}_{1,b_{k+1}}T_{b_{k+1},\mathbf{b}_1^k}$, for $1 \le k < p$. We claim that if $1 \le k \le p$ then

$$v_{\mathbf{b},k}^+ = T_{(b_p,\dots,b_{k+1},\mathbf{b}_1^k)}\Big(\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p}\dots\mathcal{L}^{(1,k)}_{\mathbf{b}_1^k+1,\mathbf{b}_1^{k+1}} + \sum_{l=k}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl} h'_{l,m,e}L_m^e\Big),$$

for some $h'_{l,m,e} \in \mathcal{H}_m^L$. When $k = p$ there is nothing to prove, so we may assume that $1 \le k < p$ and, by induction, that the claim is true for $v_{\mathbf{b},k+1}^+$. Therefore, by Lemma 2.29,

$$\begin{aligned} v_{\mathbf{b},k}^+ = {} & T_{(b_p,\dots,b_{k+2},\mathbf{b}_1^{k+1})}\Big(\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p}\dots\mathcal{L}^{(1,k+1)}_{\mathbf{b}_1^{k+1}+1,\mathbf{b}_1^{k+2}} + \sum_{l=k+1}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl} h'_{l,m,e}L_m^e\Big) \\ & \times \quad T_{b_{k+1},\mathbf{b}_1^k}\Big(\mathcal{L}^{(1,k)}_{\mathbf{b}_1^k+1,\mathbf{b}_1^{k+1}} + \sum_{m=\mathbf{b}_1^k+1}^{\mathbf{b}_l^{k+1}}\sum_{e=1}^{dk} h''_{m,e}L_m^e\Big), \end{aligned}$$

for some $h'_{l,m,e}, h''_{m,e} \in \mathcal{H}_m^L$. Now, by Lemma 2.3, $T_{b_{k+1},\mathbf{b}_1^k}$ commutes with L_m whenever $m > \mathbf{b}_1^{k+1}$. Moreover, if $m > \mathbf{b}_1^{k+1}$ then

$$h'_{l,m,e}L_m^eT_{b_{k+1},\mathbf{b}_1^k} = h'_{l,m,e}T_{b_{k+1},\mathbf{b}_1^k}L_m^e = T_{b_{k+1},\mathbf{b}_1^k}h''_{l,m,e}L_m^e,$$

where $h''_{l,m,e} = T^{-1}_{b_{k+1},\mathbf{b}_1^k}h'_{l,m,e}T_{b_{k+1},\mathbf{b}_1^k}$. It is easy to check that $h''_{l,m,e} \in \mathcal{H}_m^L$. Next note that $T_{(b_p,\dots,b_{k+2},\mathbf{b}_1^{k+1})}T_{b_{k+1},\mathbf{b}_1^k} = T_{(b_p,\dots,b_{k+1},\mathbf{b}_1^k)}$. Therefore, $v_{\mathbf{b},k}^+$ is equal to

$$\begin{aligned} v_{\mathbf{b},k}^+ = {} & T_{(b_p,\dots,b_{k+1},\mathbf{b}_1^k)}\Big(\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p}\dots\mathcal{L}^{(1,k+1)}_{\mathbf{b}_1^{k+1}+1,\mathbf{b}_1^{k+2}} + \sum_{l=k+1}^{p-1}\sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}}\sum_{e=1}^{dl} h''_{l,m,e}L_m^e\Big) \\ & \times \quad \Big(\mathcal{L}^{(1,k)}_{\mathbf{b}_1^k+1,\mathbf{b}_1^{k+1}} + \sum_{m=\mathbf{b}_1^k+1}^{\mathbf{b}_1^{k+1}}\sum_{e=1}^{dk} h''_{m,e}L_m^e\Big). \end{aligned}$$

To complete the proof of the claim observe that

$$\mathcal{L}^{(1,p-1)}_{\mathbf{b}_1^{p-1}+1,\mathbf{b}_1^p}\dots\mathcal{L}^{(1,k+1)}_{\mathbf{b}_1^{k+1}+1,\mathbf{b}_1^{k+2}} = \mathcal{L}^{(1)}_{\mathbf{b}_1^{k+1}+1,n}\dots\mathcal{L}^{(k+1)}_{\mathbf{b}_1^{k+1}+1,n}\mathcal{L}^{(k+2)}_{\mathbf{b}_1^{k+2}+1,n}\dots\mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}.$$

Therefore, when we write this element as a polynomial in $L_{\mathbf{b}_1^{k+1}+1},\dots,L_n$, the exponent of L_m is at most dl if $\mathbf{b}_1^l < m \le \mathbf{b}_1^{l+1}$ for some $k+1 \le l \le p-1$. Using this observation it is now a straightforward exercise to expand the formula for $v_{\mathbf{b},k}^+$

above and show that $v^+_{\mathbf{b},k}$ can be written in the required form, thus completing the proof of the claim.

Returning to the proof of the lemma, observe that $v^+_{\mathbf{b}} = v^+_{\mathbf{b},1}$ and that the statement of the lemma is the special case of the claim above when $k = 1$ (and setting $k = 0$ in the last displayed equation). ☐

We can now prove the promised comparison theorem for Tr and $\mathrm{Tr}_{\mathbf{b}}$.

2.31. **Theorem.** *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated and that* $b \in \mathscr{C}_{p,n}$*. Then*

$$\mathrm{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \mathrm{Tr}_{\mathbf{b}}(h)\,\mathrm{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}),$$

for all $h \in \mathscr{H}_{d,\mathbf{b}}$*.*

Proof. By linearity, it is enough to let h run over a basis of $\mathscr{H}_{d,\mathbf{b}}$. Let

$$\mathfrak{B}_{\mathbf{b}} = \{\, L_1^{a_{1,1}} \dots L_{b_1}^{a_{1,b_1}} T_{x_1} \otimes \dots \otimes L_1^{a_{p,1}} \dots L_{b_p}^{a_{p,b_p}} T_{x_p} \mid 0 \le a_{i,t} < d \text{ and } x_t \in \mathfrak{S}_{b_t} \,\}$$

be the 'Ariki-Koike basis' of $\mathscr{H}_{d,\mathbf{b}}$. Then it is enough to show that

$$\mathrm{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \mathrm{Tr}_{\mathbf{b}}(h)\,\mathrm{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}), \qquad \text{for all } h \in \mathfrak{B}_{\mathbf{b}}.$$

If $h = 1_{\mathscr{H}_{d,\mathbf{b}}}$ there is nothing to prove. Therefore, by (2.28) it remains to show that $\mathrm{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = 0$ whenever $1_{\mathscr{H}_{d,\mathbf{b}}} \ne h \in \mathfrak{B}_{\mathbf{b}}$. For the rest of the proof fix such an h. Write $h = L_1^{a_{1,1}} \dots L_{b_1}^{a_{1,b_1}} T_{x_1} \otimes \dots \otimes L_1^{a_{p,1}} \dots L_{b_p}^{a_{p,b_p}} T_{x_p}$, where $0 \le a_{j,t} < d$ and $x_t \in \mathfrak{S}_{b_t}$, and set $h' = \Theta_{\mathbf{b}}(h)$. Then

$$h' = L_1^{a_{1,1}} \dots L_{\mathbf{b}_1^1}^{a_{1,b_1}} L_{\mathbf{b}_1^1+1}^{a_{2,1}} \dots L_{\mathbf{b}_1^2}^{a_{2,b_2}} \dots L_{\mathbf{b}_1^{p-1}+1}^{a_{p,1}} \dots L_{\mathbf{b}_1^p}^{a_{p,b_p}} T_x,$$

where $x = x_1 x_2^{\langle \mathbf{b}_1^1 \rangle} \dots x_p^{\langle \mathbf{b}_1^{p-1} \rangle}$.

Recall from before Lemma 2.24 that $v_{\mathbf{b}} = v^+_{\mathbf{b}} u^+_{\mathbf{b}}$. Therefore, using Lemma 2.30 and the fact that Tr is a trace form,

$$\begin{aligned}
\mathrm{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) &= \mathrm{Tr}(v_{\mathbf{b}} h' T_{\mathbf{b}}) = \mathrm{Tr}(v^+_{\mathbf{b}} u^+_{\mathbf{b}} h' T_{\mathbf{b}}) \\
&= \mathrm{Tr}(T_{\mathbf{b}'} \hat{u}^-_{\mathbf{b}} u^+_{\mathbf{b}} h' T_{\mathbf{b}}) + \sum_{l=1}^{p-1} \sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}} \sum_{e=1}^{dl} \mathrm{Tr}(T_{\mathbf{b}'} h_{l,m,e} L_m^e u^+_{\mathbf{b}} h' T_{\mathbf{b}}) \\
&= \mathrm{Tr}(\hat{u}^-_{\mathbf{b}} u^+_{\mathbf{b}} h' T_{\mathbf{b}} T_{\mathbf{b}'}) + \sum_{l=1}^{p-1} \sum_{m=\mathbf{b}_1^l+1}^{\mathbf{b}_1^{l+1}} \sum_{e=1}^{dl} \mathrm{Tr}(L_m^e u^+_{\mathbf{b}} h' T_{\mathbf{b}} T_{\mathbf{b}'} h_{l,m,e}),
\end{aligned}$$

where $h_{l,m,e} \in \mathcal{H}_m^L$ and $\hat{u}^-_{\mathbf{b}} := \mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n} \mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n} \dots \mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}$. Fix a triple (l, m, e), from the sum, with $1 \le l < p$, $\mathbf{b}_1^l < m \le \mathbf{b}_1^{l+1}$ and $1 \le e \le dl$. By assumption, L_m appears in h' with exponent $0 \le a_{l+1,m'} < d$, where $m = \mathbf{b}_1^l + m'$. Therefore, $L_m^e u^+_{\mathbf{b}} h' T_{\mathbf{b}} T_{\mathbf{b}'} h_{l,m,e}$ is a linear combination of terms of the form $L_m^e u^+_{\mathbf{b}} f_1(L) T_w f_2(L)$, where $w \in \mathfrak{S}_n$, $f_1(L)$ is a a polynomial in $L_1, \dots, L_n$ of degree at most $a_{l+1,m'} < d$ as a polynomial in L_m, and where $f_2(L)$ is a polynomial in $L_1, \dots, L_{m-1}$. As Tr is a trace form,

$$\mathrm{Tr}(L_m^e u^+_{\mathbf{b}} f_1(L) T_w f_2(L)) = \mathrm{Tr}(f_2(L) L_m^e u^+_{\mathbf{b}} f_1(L) T_w).$$

Now, considered as a polynomial in L_m, $f_2(L) L_m^e u^+_{\mathbf{b}} f_1(L)$ is a polynomial with zero constant term (since $e > 0$) and degree

$$0 < f := e + d(p - l - 1) + a_{l+1,m'} < d(p-1) + d = r.$$

By the same argument, if $m < k \le n$ then L_k appears in $f_2(L) L_m^e u^+_{\mathbf{b}} f_1(L)$ with exponent at most $d(p - l'_k - 1) + a_{l_k+1,k'} < d(p-1) < r$, where $k = \mathbf{b}_1^{l_k-1} + k'$ and $1 \le k' \le b_{l_k}$. If $k < m$ then L_k could appear in $f_2(L) L_m^e u^+_{\mathbf{b}} f_1(L)$ with exponent greater than $r-1$, however, by Lemma 2.3 this will not affect the exponents of $L_m, \dots, L_n$ when rewrite this term as a linear combination of Ariki-Koike basis elements. Hence, L_m^f is a left divisor of $f_2(L)L_m^e u_{\mathbf{b}}^+ f_1(L)$ when it is written as a linear combination of Ariki-Koike basis elements. Consequently, $\operatorname{Tr}(f_2(L)L_m^e u_{\mathbf{b}}^+ f_1(L)) = 0$ by (2.28). Therefore, $\operatorname{Tr}(L_m^e u_{\mathbf{b}}^+ h' T_{\mathbf{b}} T_{\mathbf{b}'} h_{l,m,e}) = 0$ so that $\operatorname{Tr}(h \cdot v_{\mathbf{b}} h' T_{\mathbf{b}}) = \operatorname{Tr}(v_{\mathbf{b}} h' T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+ h' T_{\mathbf{b}} T_{\mathbf{b}'})$.

Now consider $\operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+ h' T_{\mathbf{b}} T_{\mathbf{b}'})$. By definition,

$$\begin{aligned}\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+ h' &= \mathcal{L}^{(1)}_{\mathbf{b}_1^1+1,n}\mathcal{L}^{(2)}_{\mathbf{b}_1^2+1,n}\dots\mathcal{L}^{(p-1)}_{\mathbf{b}_1^{p-1}+1,n}\cdot\mathcal{L}^{(2)}_{1,\mathbf{b}_1^1}\mathcal{L}^{(3)}_{1,\mathbf{b}_1^2}\dots\mathcal{L}^{(p)}_{1,\mathbf{b}_1^{p-1}}h'\\ &= \prod_{i=1}^p \mathcal{L}^{(i)}_{1,\mathbf{b}_1^{i-1}}\mathcal{L}^{(i)}_{\mathbf{b}_1^i+1,n}\cdot h'.\end{aligned}$$

If $a_{l,m'} \neq 0$, for some l and m', then $L_m^{a_{l,m'}}$ divides h', where $m = \mathbf{b}_1^{l-1} + m'$ as above. By the argument above $\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+ h'$, when considered as a polynomial in L_m, is a polynomial with zero constant term and degree strictly less than r. Therefore,

$$\operatorname{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+ h' T_{\mathbf{b}} T_{\mathbf{b}'}) = 0,$$

as required. It remains, then, to consider the cases when $a_{l,m'} = 0$, for $1 \le l \le p$ and $1 \le m' \le b_l$. That is, when $h' = T_x$ for some $1 \neq x \in \mathfrak{S}_{\mathbf{b}}$. By (2.28), in this case we have

$$\operatorname{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+ T_x T_{\mathbf{b}} T_{\mathbf{b}'}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+) \operatorname{Tr}(T_x T_{\mathbf{b}} T_{\mathbf{b}'}) \tag{2.32}$$

Recall that $w_{\mathbf{b}}$ is a distinguished coset representative for $\mathfrak{S}_{\mathbf{b}}$, so that $\ell(x w_{\mathbf{b}}) = \ell(x) + \ell(w_{\mathbf{b}})$. Therefore, $\operatorname{Tr}(T_x T_{\mathbf{b}} T_{\mathbf{b}'}) = \operatorname{Tr}(T_{x w_{\mathbf{b}}} T_{\mathbf{b}'}) = 0$ by (2.28) since $x \neq 1$. Hence, $\operatorname{Tr}(h \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = 0$, completing the proof. □

We can improve on Theorem 2.31 by explicitly computing $\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}})$. In fact, in proving the theorem we have essentially already done this. To state the result, given $\mathbf{b} \in \mathscr{C}_{p,n}$ set $\alpha(\mathbf{b}) = \sum_{i=1}^p i b_i \in \mathbb{N}$.

2.33. **Corollary.** *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated and that* $b \in \mathscr{C}_{p,n}$*. Then*

$$\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) = (-1)^{dn(p-1)} q^{\ell(w_{\mathbf{b}})} \varepsilon^{\frac12 rn(p-1) - d\alpha(\mathbf{b})} (Q_1 \dots Q_d)^{n(p-1)}.$$

Proof. By (2.32), and (2.28), we have that

$$\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}(1_{\mathscr{H}_{d,\mathbf{b}}} \cdot v_{\mathbf{b}} T_{\mathbf{b}}) = \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+) \operatorname{Tr}(T_{\mathbf{b}'} T_{\mathbf{b}}) = q^{\ell(w_{\mathbf{b}})} \operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+).$$

Now, $\operatorname{Tr}(\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+)$ is just the constant term of $\hat{u}_{\mathbf{b}}^- u_{\mathbf{b}}^+$ by (2.28). Therefore,

$$\begin{aligned}\operatorname{Tr}(v_{\mathbf{b}} T_{\mathbf{b}}) &= q^{\ell(w_{\mathbf{b}})} \prod_{t=1}^p \Big((-1)^d \varepsilon^{td} Q_1 \dots Q_d\Big)^{n-b_t}\\ &= (-1)^{dn(p-1)} q^{\ell(w_{\mathbf{b}})} \varepsilon^{\frac12 rn(p-1) - d\alpha(\mathbf{b})} (Q_1 \dots Q_d)^{n(p-1)},\end{aligned}$$

since $b_1 + \dots + b_p = n$. □

2.34. *Remark.* Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$. Then it is not difficult to see that

$$\ell(w_{\mathbf{b}}) = \sum_{1 \le i < j \le p} b_i b_j.$$

3. Shifting homomorphisms and Specht modules

In this section we begin to apply the results of the last section to the representation theory of $\mathscr{H}_{r,p,n}$. First, we recall a construction of the Specht modules for the algebras $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$ and use this to define the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ from the introduction. Next we explicitly compute these scalars. Building on these results, and using Clifford theory, we then define analogues of the Specht modules for the algebras $\mathscr{H}_{r,p,n}$. As a consequence we construct the simple modules of $\mathscr{H}_{r,p,n}$ over a field.

Throughout this section we maintain our assumption that $\mathbf{Q}$ is (ε, q)-separated over R (see Assumption 2.17).

3.1. **Specht modules for $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$.** The algebras $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,n}$ are both cellular algebras [**9**, **18**] with the cell modules of both algebras being called Specht modules. In this subsection we quickly recall the construction of these modules and the relationship between the Specht modules of these algebras.

First, recall that a **partition** of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of weakly decreasing non-negative integers which sum to $|\lambda| = n$. The **conjugate** of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$, where $\lambda'_i = \#\{\, j \geq 1 \mid \lambda_j \geq i \,\}$.

A **multipartition** of n is an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. Let $\mathscr{P}_{r,n}$ be the set of r-partitions of n. The partitions $\lambda^{(s)}$ are the **components** of $\boldsymbol{\lambda}$ and we call $\boldsymbol{\lambda}$ a multipartition when r is understood. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multipartition then its **conjugate** is the multipartition $\boldsymbol{\lambda}' = (\lambda^{(r)\prime}, \dots, \lambda^{(1)\prime})$. To each multipartition $\boldsymbol{\lambda}$ we also associate a Young subgroup $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ of $\mathfrak{S}_n$ in the obvious way.

The **diagram** of $\boldsymbol{\lambda}$ is the set $[\boldsymbol{\lambda}] = \{\, (i,j,s) \mid 1 \leq j \leq \lambda_i^{(s)} \text{ and } 1 \leq s \leq r \,\}$. A **$\boldsymbol{\lambda}$-tableau** is a map $\mathfrak{t}\colon [\boldsymbol{\lambda}] \longrightarrow \{1, 2, \dots, n\}$, which we think of as a labeling of the diagram of $\boldsymbol{\lambda}$. Thus we write $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ and we talk of the rows, columns and components of $\mathfrak{t}$.

By [**9**, Theorem 3.26], $\mathscr{H}_{r,n}$ is a cellular algebra with a cellular basis of the form

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}), \text{ for } \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \,\}.$$

Hence, the cell modules of $\mathscr{H}_{r,n}$ are indexed by $\mathscr{P}_{r,n}$ and if $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ then the corresponding cell module $S(\boldsymbol{\lambda})$ has a basis of the form $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$.

3.1. **Definition.** a) *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$. Then the **Specht module** $S(\boldsymbol{\lambda})$ for $\mathscr{H}_{r,n}$ is the cell module indexed by $\boldsymbol{\lambda}$ defined in* [**9**, Defn 3.28].

b) *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Then the **Specht module** for $\mathscr{H}_{d,\mathbf{b}}$ is the module $S_{\mathbf{b}}(\boldsymbol{\lambda}) \cong S(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes S(\boldsymbol{\lambda}^{[p]})$.*

We write $S^R(\boldsymbol{\lambda})$ when we want to emphasize that $S(\boldsymbol{\lambda})$ is an R-module. We will give a more explicit construction of these modules in Section 4.2.

When $\mathscr{H}_{d,\mathbf{b}}$ is semisimple the modules Specht modules $\{\, S_{\mathbf{b}}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}} \,\}$ give a complete set of pairwise non-isomorphic simple $\mathscr{H}_{d,\mathbf{b}}$-modules. Similarly, the modules $\{\, S(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,n} \,\}$ give a complete set of pairwise non-isomorphic simple $\mathscr{H}_{r,n}$-modules when $\mathscr{H}_{r,n}$ is semisimple.

More generally, the cellular basis of $\mathscr{H}_{r,n}$ endows each Specht module $S(\boldsymbol{\lambda})$ with an associative bilinear form and the radical $\operatorname{rad} S(\boldsymbol{\lambda})$ of this form is an $\mathscr{H}_{r,n}$-module. Define $D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda}) / \operatorname{rad} S(\boldsymbol{\lambda})$. Let $\mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon}) = \{\, \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \mid D(\boldsymbol{\lambda}) \neq 0 \,\}$. Then a multipartition $\boldsymbol{\lambda}$ is **Kleshchev** if $\boldsymbol{\lambda} \in \mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ and

$$\{\, D(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon}) \,\}$$

is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{r,n}$-modules. Typically we write $\mathscr{K}_{r,n} = \mathscr{K}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ in what follows.

If A is an algebra and M is an A-module let $\operatorname{Head}(M)$ be the **head** of M. That is, M is the largest semisimple quotient of M. For example, if $\boldsymbol{\lambda} \in \mathscr{K}_{r,n}$ then

$D(\boldsymbol{\lambda}) = \operatorname{Head}(S(\boldsymbol{\lambda}))$. If S and D are modules for an algebra, with D irreducible, let $[S : D]$ be the multiplicity of D as a composition factor of S.

If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two multipartitions then $\boldsymbol{\lambda}$ **dominates** μ, and we write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if

$$\sum_{s=1}^{t-1} |\boldsymbol{\lambda}^{(s)}| + \sum_{j=1}^{i} \boldsymbol{\lambda}_j^{(t)} \geq \sum_{s=1}^{t-1} |\boldsymbol{\mu}^{(s)}| + \sum_{j=1}^{i} \boldsymbol{\mu}_j^{(t)}$$

for $1 \leq t \leq r$ and $i \geq 0$. We write $\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. The dominance partial order on $\mathscr{P}_{r,n}$ is useful because of the following fact.

3.2. **Lemma** ([**9**, §3]). *Suppose that $[S(\boldsymbol{\lambda}) : D(\boldsymbol{\mu})] \neq 0$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{r,n}$. Then $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$. Moreover, if $\boldsymbol{\mu} \in \mathscr{K}_{r,n}$ then $[S(\boldsymbol{\mu}) : S(\boldsymbol{\mu})] = 1$ and $D(\boldsymbol{\mu}) = \operatorname{Head} S(\boldsymbol{\mu})$.*

Let $\mathscr{K}_{d,\mathbf{b}} = \{\, \boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}} \mid \boldsymbol{\lambda}^{[t]} \in \mathscr{K}_{d,b_i}(\varepsilon^t \mathbf{Q}) \text{ for } 1 \leq i \leq p \,\}$. If $\boldsymbol{\lambda} \in \mathscr{K}_{d,\mathbf{b}}$ let

$$D_{\mathbf{b}}(\boldsymbol{\lambda}) = S_{\mathbf{b}}(\boldsymbol{\lambda}) / \operatorname{rad} S_{\mathbf{b}}(\boldsymbol{\lambda}) \cong D(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes D(\boldsymbol{\lambda}^{[p]}).$$

The remarks above imply that $\{\, D_{\mathbf{b}}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{K}_{d,\mathbf{b}} \text{ for } 1 \leq t \leq p \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_{d,\mathbf{b}}$-modules.

Recall the functor $\mathtt{H}_{\mathbf{b}}$ from §2.5. By [**10**, Prop. 4.11] (see also [**23**, Prop. 2.13]), we have the following.

3.3. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Then*

a) $\mathtt{H}_{\mathbf{b}}\big(S_{\mathbf{b}}(\boldsymbol{\lambda})\big) \cong S(\boldsymbol{\lambda})$ *as $\mathscr{H}_{r,n}$-modules.*
b) $\mathtt{H}_{\mathbf{b}}\big(D_{\mathbf{b}}(\boldsymbol{\lambda})\big) \cong D(\boldsymbol{\lambda})$ *as $\mathscr{H}_{r,n}$-modules.*
c) $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{[1]}, \dots, \boldsymbol{\lambda}^{[p]}) \in \mathscr{K}_{d,\mathbf{b}}(\mathbf{Q}^{\vee\varepsilon})$ *is Kleshchev if and only if $\boldsymbol{\lambda}^{[t]} \in \mathscr{K}_{d,b_t}(\varepsilon^t \mathbf{Q})$, for $1 \leq t \leq p$.*

In particular, we can consider $S(\boldsymbol{\lambda}) \cong \mathtt{H}_{\mathbf{b}}\big(S_{\mathbf{b}}(\boldsymbol{\lambda})\big) = S_b(\boldsymbol{\lambda}) \cdot V_{\mathbf{b}}$ to be a submodule of $V_{\mathbf{b}}$.

3.2. **The scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$.** We are now ready to define and compute the scalars $\mathfrak{f}_{\boldsymbol{\lambda}}$ which play an important part in all of the main results of this paper.

Recall from the introduction that $\mathcal{A} = \mathbb{Z}[\dot{\varepsilon}, \dot{q}^{\pm 1}, \dot{Q}_1^{\pm 1}, \dots, \dot{Q}_d^{\pm 1}, A(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}})^{-1}]$, where $\dot{\varepsilon}$ is a primitive pth root of unity in $\mathbb{C}$ and $\dot{q}$ and $\dot{\mathbf{Q}} = (\dot{Q}_1, \dots, \dot{Q}_d)$ are indeterminates over $\mathbb{Z}[\dot{\varepsilon}]$. Let $\mathcal{F}$ be the field of fractions of $\mathcal{A}$. If $\mathbf{Q}$ is (ε, q)-separated over R then R can be considered as an $\mathcal{A}$-module by letting $\dot{\varepsilon}$ act on R as multiplication by ε, $\dot{q}$ act as multiplication by q and $\dot{Q}_i$ act as multiplication by Q_i, for $1 \leq i \leq d$. Therefore, $\mathscr{H}_{r,n}^R(q, \mathbf{Q}) \cong \mathscr{H}_{r,n}^{\mathcal{A}}(\dot{q}, \dot{\mathbf{Q}}) \otimes_{\mathcal{A}} R$ are isomorphic R-algebras. In particular, $\mathscr{H}_{r,n}^{\mathcal{F}} \cong \mathscr{H}_{r,n}^{\mathcal{A}}(\dot{q}, \dot{\mathbf{Q}}) \otimes_{\mathcal{A}} \mathcal{F}$. The algebra $\mathscr{H}_{r,n}^{\mathcal{F}}$ is semisimple by Ariki's semisimplicity criteria [**1**]. The algebra $\mathscr{H}_{r,n}^{\mathcal{F}}$ is split semisimple because $\mathscr{H}_{r,n}^{\mathcal{F}}$ is a cellular algebra (and every field is a splitting field for a cellular algebra).

Abusing notation, we call the elements of $\mathcal{A}$ *polynomials* and if $f(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) \in \mathcal{A}$ then we define $f(\varepsilon, q, \mathbf{Q}) = f(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) 1_R$ to be the value of $f(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}})$ at $(\varepsilon, q, \mathbf{Q})$.

The scalar $\mathfrak{f}_{\boldsymbol{\lambda}}$ in the next Proposition plays a key role in all of the main results, Theorems A–D, from the introduction.

3.4. **Proposition.** *Suppose that $\mathbf{Q}$ is (ε, q)-separated in R and that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. Then there exists a non-zero scalar $\mathfrak{f}_{\boldsymbol{\lambda}} \in R$ such that*

$$z_{\mathbf{b}} \cdot x = \mathfrak{f}_{\boldsymbol{\lambda}} x,$$

for all $x \in S(\boldsymbol{\lambda})$. Moreover, there exists a non-zero polynomial $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}} = \mathfrak{f}_{\boldsymbol{\lambda}}(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) \in \mathcal{A}$ such that $\mathfrak{f}_{\boldsymbol{\lambda}} = \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}(\varepsilon, q, \mathbf{Q}) \in R$.

Proof. The Specht module $S_{\mathbf{b}}(\boldsymbol{\lambda})$ is free as an R-module so, by the remarks above, $S_{\mathbf{b}}(\boldsymbol{\lambda}) \cong S_{\mathbf{b}}^{\mathcal{A}}(\boldsymbol{\lambda}) \otimes_{\mathcal{A}} R$. Therefore, so to show that such a scalar exists it is enough to consider the case when $R = \mathcal{A}$. Similarly, since $S_{\mathbf{b}}^{\mathcal{A}}(\boldsymbol{\lambda})$ embeds into $S_{\mathbf{b}}^{\mathcal{F}}(\boldsymbol{\lambda}) \cong$

$S^{\mathcal{A}}_{\mathbf{b}}(\boldsymbol{\lambda})\otimes_{\mathcal{A}}\mathcal{F}$ we may assume that $R=\mathcal{F}$. By the remarks above, the algebra $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ is split semisimple and the module $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda})$ is an irreducible $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$-module, so by Schur's Lemma the homomorphism of $S_{\mathbf{b}}(\boldsymbol{\lambda})$ given by left multiplication by $z_{\mathbf{b}}$ is equal to multiplication by some scalar $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$. Notice that $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ is an element of $\mathcal{A}$ because $z_{\mathbf{b}}z_{\mathbf{b}}(\boldsymbol{\lambda})\cdot v_{\mathbf{b}}T_{\mathbf{b}}\in\mathscr{H}^{\mathcal{A}}_{r,n}$. By specialization, the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}\in R$ in the statement of the Lemma is given by evaluating the polynomial $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}(\dot{\varepsilon},\dot{q},\dot{\mathbf{Q}})$ at $(\varepsilon,q,\mathbf{Q})$. Finally, observe that $\mathfrak{f}_{\boldsymbol{\lambda}}\neq 0$ since $z_{\mathbf{b}}$ acts invertibly on $V_{\mathbf{b}}$ by Lemma 2.24. □

We will determine the scalar $\mathfrak{f}_{\boldsymbol{\lambda}}\in R$ by computing the polynomial $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ in $\mathcal{A}$. In fact, we have already done all of the work needed to determine $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$. To describe $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ we only need one definition.

Abusing notation slightly, let Tr be the trace form of $\mathscr{H}^{\mathcal{F}}_{r,n}$ given by (2.28). Let $\chi^{\boldsymbol{\lambda}}$ be the character of $S^{\mathcal{F}}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$. Then $\{\,\chi^{\boldsymbol{\lambda}}\mid\boldsymbol{\lambda}\in\mathscr{P}_{r,n}\,\}$ is a complete set of pairwise inequivalent irreducible characters for $\mathscr{H}^{\mathcal{F}}_{r,n}$, so it is a basis for the space of trace functions on $\mathscr{H}^{\mathcal{F}}_{r,n}$. In particular, Tr can be written in a unique way as a linear combination of the irreducible characters. Moreover, it is easy to see that every character $\chi^{\boldsymbol{\lambda}}$ must appear in Tr with *non-zero coefficient* because Tr is non-degenerate; see, for example, [**14**, Example 7.1.3]. Consequently, the following definition makes sense.

3.5. **Definition.** *The* ***Schur elements*** *of* $\mathscr{H}^{\mathcal{F}}_{r,n}$ *are the scalars* $\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}=\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}(\dot{\varepsilon},\dot{q},\dot{\mathbf{Q}})\in\mathcal{F}$*, for* $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$*, such that*

$$\operatorname{Tr}=\sum_{\boldsymbol{\lambda}\in\mathscr{P}_{r,n}}\frac{1}{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}}\chi^{\boldsymbol{\lambda}}.$$

For $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$ fix $F_{\boldsymbol{\lambda}}$ a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{r,n}$ such that $F_{\boldsymbol{\lambda}}\mathscr{H}^{\mathcal{F}}_{r,n}\cong S^{\mathcal{F}}(\boldsymbol{\lambda})$. Using, for example seminormal forms $\mathscr{H}^{\mathcal{F}}_{r,n}$ [**27**, Theorem 2.11], it is easy to see that $\chi^{\boldsymbol{\lambda}}(F_{\boldsymbol{\mu}})=\delta_{\boldsymbol{\lambda\mu}}$, for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{r,n}$. Hence, a second characterisation of the Schur elements is that

$$\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}=\frac{1}{\operatorname{Tr}(F_{\boldsymbol{\lambda}})}.$$

Similarly, for each $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ the trace form $\operatorname{Tr}_{\mathbf{b}}$ determines Schur elements $\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}\in\mathcal{F}$ for $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$, for $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. By the remarks above, the Schur elements of $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ satisfy

$$\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}=\prod_{t=1}^{p}\dot{\mathfrak{s}}_{\boldsymbol{\lambda}^{[t]}}(\dot{\varepsilon},\dot{q},\dot{\varepsilon}^{t}\dot{\mathbf{Q}})=\frac{1}{\operatorname{Tr}_{\mathbf{b}}(F_{\mathbf{b}}(\boldsymbol{\lambda}))},$$

where $F_{\mathbf{b}}(\boldsymbol{\lambda})$ is a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ such that $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda})\cong F_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$.

3.6. **Theorem.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$*. Then*

$$\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}=\frac{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}}{\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}}\operatorname{Tr}(v_{\mathbf{b}}T_{\mathbf{b}}).$$

Consequently, $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}=(-1)^{n(r-d)}\dot{q}^{\ell(w_{\mathbf{b}})}\dot{\varepsilon}^{\frac{1}{2}rn(p-1)-d\alpha(\mathbf{b})}(\dot{Q}_1\dots\dot{Q}_d)^{n(p-1)}\dfrac{\dot{\mathfrak{s}}_{\boldsymbol{\lambda}}}{\dot{\mathfrak{s}}^{\mathbf{b}}_{\boldsymbol{\lambda}}}.$

Proof. To compute $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ we may assume that $R=\mathcal{F}$ and work in $\mathscr{H}^{\mathcal{F}}_{r,n}$. Let $F_{\mathbf{b}}(\boldsymbol{\lambda})$ be a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$ such that $S^{\mathcal{F}}_{\mathbf{b}}(\boldsymbol{\lambda})\cong F_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}^{\mathcal{F}}_{d,\mathbf{b}}$. Then $F_{\mathbf{b}}(\boldsymbol{\lambda})\cdot e_{\mathbf{b}}$ is a primitive idempotent in $\mathscr{H}^{\mathcal{F}}_{r,n}$ such that $F_{\mathbf{b}}(\boldsymbol{\lambda})\cdot e_{\mathbf{b}}\mathscr{H}^{\mathcal{F}}_{r,n}\cong S^{\mathcal{F}}(\boldsymbol{\lambda})$ by Theorem 2.26

and Lemma 3.3. Therefore, using the remarks above,

$$\begin{aligned}\frac{1}{\dot{\mathfrak{s}}_{\boldsymbol\lambda}} &= \operatorname{Tr}(F_{\mathbf b}(\boldsymbol\lambda)\cdot e_{\mathbf b}) = \operatorname{Tr}(z_{\mathbf b}^{-1}F_{\mathbf b}(\boldsymbol\lambda)\cdot v_{\mathbf b}T_{\mathbf b}), && \text{since } z_{\mathbf b} \text{ is central in } \mathscr{H}_{d,\mathbf b},\\ &= \frac{1}{\dot{\mathfrak{f}}_{\boldsymbol\lambda}}\operatorname{Tr}(F_{\mathbf b}(\boldsymbol\lambda)\cdot v_{\mathbf b}T_{\mathbf b}), && \text{by Proposition 3.4},\\ &= \frac{1}{\dot{\mathfrak{f}}_{\boldsymbol\lambda}}\operatorname{Tr}_{\mathbf b}(F_{\mathbf b}(\boldsymbol\lambda))\operatorname{Tr}(\mathfrak{v}_{\mathbf b}T_{\mathbf b}), && \text{by Theorem 2.31},\\ &= \frac{1}{\dot{\mathfrak{f}}_{\boldsymbol\lambda}\dot{\mathfrak{s}}^{\mathbf b}_{\boldsymbol\lambda}}\operatorname{Tr}(v_{\mathbf b}T_{\mathbf b}).\end{aligned}$$

Rearranging this equation gives the first formula for $\dot{\mathfrak{f}}_{\boldsymbol\lambda}$. Applying Corollary 2.33 proves the second. □

3.7. *Remark.* The proof of Theorem 3.6 is deceptively easy: all of the hard work is done in proving Theorem 2.26 and Theorem 2.31.

We want to make the formula for $\dot{\mathfrak{f}}_{\boldsymbol\lambda}$ more explicit. To do this we recall one of the formulas for the Schur elements obtained in [**27**]. First, given $1 \le i \le j \le p$ and $1 \le a, b \le d$ such that $(i-1)d + a < (j-1)d + b$, define

$$\begin{aligned}\dot{\mathfrak{s}}^{ia}_{jb}(\boldsymbol\lambda) &= \dot\varepsilon^{j|\lambda^{(d(j-1)+b)}|+i|\lambda^{(d(i-1)+a)}|}\prod_{(x,y)\in[\lambda^{(d(j-1)+b)}]}(\dot q^{y-x}\dot Q_b - \dot\varepsilon^{i-j}\dot Q_a)\\ &\times\prod_{(u,v)\in[\lambda^{(d(i-1)+a)}]}(\dot q^{v-u}\dot Q_a - \dot\varepsilon^{j-i}\dot Q_b)\prod_{k=1}^{\lambda_1^{(d(j-1)+b)}}\frac{\dot q^{v-u}\dot Q_a - \dot q^{k-1-\lambda_k^{(d(j-1)+b)'}}\dot\varepsilon^{j-i}\dot Q_b}{\dot q^{v-u}\dot Q_a - \dot q^{k-\lambda_k^{(d(j-1)+b)'}}\dot\varepsilon^{j-i}\dot Q_b}.\end{aligned}$$

Then by [**27**, Cor. 6.3],

$$\dot{\mathfrak{s}}_{\boldsymbol\lambda} = \dot{\mathfrak{q}}_{\boldsymbol\lambda}\prod_{x\in[\boldsymbol\lambda]}[h^{\boldsymbol\lambda}_x]_q\cdot\prod_{1\le i\le j\le p}\prod_{\substack{1\le a,b\le d\\(i-1)d+a<(j-1)d+b}}\dot{\mathfrak{s}}^{ia}_{jb}(\boldsymbol\lambda),$$

where $h^{\boldsymbol\lambda}_x$ is the *hook length* of $x\in[\boldsymbol\lambda]$ (see, for example, [**25**, §3.2]), and

$$\dot{\mathfrak{q}}_{\boldsymbol\lambda} = (-1)^{n(r-1)}\dot q^{-\alpha(\boldsymbol\lambda')}\prod_{t=1}^{p}\prod_{i=1}^{d}(\dot\varepsilon^t\dot Q_i)^{|\lambda^{(d(t-1)+i)}|-n}.$$

with $\alpha(\boldsymbol\mu) = \sum_{s=1}^r\sum_{i\ge1}\binom{\mu_i^{(s)}}{2}$, for $\boldsymbol\mu\in\mathscr{P}_{r,n}$. There is an analogous formula for $\dot{\mathfrak{s}}^{\mathbf b}_{\boldsymbol\lambda} = \prod_{t=1}^p\dot{\mathfrak{s}}_{\boldsymbol\lambda^{[t]}}$ involving the scalar $\dot{\mathfrak{q}}^{\mathbf b}_{\boldsymbol\lambda} = \prod_{t=1}^p\dot{\mathfrak{q}}_{\boldsymbol\lambda^{[t]}}$, which equals

$$\dot{\mathfrak{q}}^{\mathbf b}_{\boldsymbol\lambda} = \prod_{t=1}^{p}\Big((-1)^{b_t(d-1)}\dot q^{-\alpha(\boldsymbol\lambda^{[t]\prime})}\prod_{i=1}^{d}(\dot\varepsilon^t\dot Q_t)^{|\lambda^{(d(t-1)+i)}|-b_t}\Big).$$

Miraculously, as the reader may check using Corollary 2.33, $\dot q^{\ell(w_{\mathbf b})}\dot{\mathfrak{q}}^{\mathbf b}_{\boldsymbol\lambda} = \dot{\mathfrak{q}}_{\boldsymbol\lambda}\operatorname{Tr}(v_{\mathbf b}T_{\mathbf b})$. Hence, by Theorem 3.6 and the equations above, we have the following.

3.8. **Corollary.** *Suppose that* $\mathbf b\in\mathscr{C}_{p,n}$ *and that* $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf b}$. *Then*

$$\dot{\mathfrak{f}}_{\boldsymbol\lambda} = \dot q^{\ell(w_{\mathbf b})}\prod_{1\le i<j\le p}\prod_{1\le a,b\le d}\dot{\mathfrak{s}}^{ia}_{jb}(\boldsymbol\lambda).$$

It is evident in the formulae above that $\dot{\mathfrak{f}}_{\boldsymbol\lambda}\in\mathcal{F}$. We remind the reader that, in fact, $\dot{\mathfrak{f}}_{\boldsymbol\lambda}\in\mathcal{A}$ by Proposition 3.4, for $\boldsymbol\lambda\in\mathscr{P}_{r,n}$. Hence, we can evaluate these expressions for the polynomials $\dot{\mathfrak{f}}_{\boldsymbol\lambda}$ at $(\varepsilon, q, \mathbf Q)$ whenever $\mathbf Q$ is (ε, q)-separated over R.

3.9. **Corollary.** *Let* $\mathbf b\in\mathscr{C}_{p,n}$, $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf b}$ *and* $\mathfrak{t}\in\mathbb{Z}$. *Suppose that* $\mathbf Q$ *is* (ε, q)*-separated over the field* K. *Then* $V^{(t)}_{\mathbf b\langle t\rangle}\cong V^{(t+1)}_{\mathbf b\langle t+1\rangle}$.

Proof. It is enough to consider the case where $t = 0$. By Lemma 2.21 left multiplication by Y_1 induces an $\mathscr{H}_{r,n}$-module homomorphism. This map is an isomorphism because left multiplication by $Y_p \dots Y_1$ is invertible by Lemma 2.24 (and Lemma 2.21). □

4. A Specht module theory for $\mathscr{H}_{r,p,n}$

We are now ready to start studying the algebras $\mathscr{H}_{r,p,n}$. In this section we show that the scalars $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}$ from the last section have p/mth roots in $\mathcal{A}$ whenever $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$. Using this we then define analogue of the Specht modules for the algebras $\mathscr{H}_{r,p,n}$.

The relations in $\mathscr{H}_{r,n}$ imply that there is a unique algebra automorphism σ of $\mathscr{H}_{r,n}$ such that $\sigma(T_0) = \varepsilon T_0$ and $\sigma(T_i) = T_i$, for $1 \le i < n$. By definition, σ is an automorphism of order p. Further, applying the definitions

$$\mathscr{H}_{r,p,n} = \mathscr{H}_{r,n}^{\sigma} = \{\, h \in \mathscr{H}_{r,n} \mid \sigma(h) = h \,\}.$$

That is, $\mathscr{H}_{r,p,n}$ is the fixed point subalgebra of $\mathscr{H}_{r,n}$ under σ. As we will see, this gives $\mathscr{H}_{r,n}$ the structure of a graded Clifford system.

4.1. **Graded Clifford systems.** Let A be a finitely generated R-algebra. Recall that a family of R-submodules $\{\, A_s \mid s \in \mathbb{Z}/p\mathbb{Z} \,\}$ is a $\mathbb{Z}/p\mathbb{Z}$**-graded Clifford** system if the following conditions are satisfied:

a) $A_sA_t = A_{st}$ for any $s, t \in \mathbb{Z}/p\mathbb{Z}$;
b) For each $s \in \mathbb{Z}/p\mathbb{Z}$, there is a unit $a_s \in A$ such that $A = a_sA_1 = A_1a_s$;
c) $A = \oplus_{s\in\mathbb{Z}/p\mathbb{Z}} A_s$;
d) $1 \in A_1$.

Recall that any automorphism α of an R-algebra A induces an equivalence $\mathtt{F}^\alpha\colon \text{Mod-}A \longrightarrow \text{Mod-}A$. Explicitly, if M is an A-module then $\mathtt{F}^\alpha(M) = M^\alpha$ is the A-module which is equal to M as an R-module but with the action *twisted* by α so that if $m \in M$ and $x \in A$ then $m \cdot x = mx^\alpha = m\alpha(x)$, where on the right hand side we have the usual (untwisted) action of A.

The following general result is proved in [**15**, Prop. 2.2], together with [**21**, Appendix] which corrects a gap in the original argument. Recall that we have assumed that R contains a primitive pth root of unity ε.

4.1. **Lemma.** *Suppose that A and B finitely generated R-free R-algebras such that $A = \bigoplus_{t=0}^{p-1} B\theta^t$ where θ is a unit in A such that $\theta^p \in B$ and $\theta B = B\theta$. Then there is an isomorphism of (A,A)-bimodules*

$$A \otimes_B A \cong \bigoplus_{t=0}^{p-1} A^{\theta^t}; b\theta^i \otimes \theta^j \mapsto \sum_{t=0}^{p-1} (\varepsilon^{jt} b\theta^{i+j})_{(t)},$$

for $b \in B$ and $0 \le i, j < p$ and where $(\varepsilon^{jt} b\theta^{i+j})_{(t)} \in A^{\theta^t}$. Here we view $\bigoplus_t A^{\theta^t}$ as an (A,A)-bimodule by making A act from the left as left multiplication and from the right on A^{θ^t} as right multiplication twisted by θ^t, for $0 \le t < p$.

The explicit isomorphism in the lemma is constructed in [**21**, p. 3391].

In the setup of Lemma 4.1 the subspaces $\{\, B\theta^s \mid s \in \mathbb{Z}/p\mathbb{Z} \,\}$ form a $\mathbb{Z}/p\mathbb{Z}$-graded Clifford system in A. Now we assume that $R = K$ is a field. Let α be the automorphism of B given by $\alpha(b) = \theta b\theta^{-1}$, for $b \in B$. Let β be the automorphism of A given by $\beta(b\theta^j) = \varepsilon^j b\theta^j$, for $b \in B$ and $j \in \mathbb{Z}/p\mathbb{Z}$. Let $\operatorname{Irr}(A)$ and $\operatorname{Irr}(B)$ be the sets of isomorphism classes of simple A-modules and simple B-modules, respectively. For each $D(\lambda) \in \operatorname{Irr}(A)$ fix a simple B-submodule D^λ of $D(\lambda)\downarrow_B^A$. It is clear that $D(\lambda)^\alpha \cong D(\lambda)$ and $(D^\lambda)^\beta \cong D^\lambda$. Let o_λ be the smallest positive integer such that

$D(\lambda)^{\beta^{\mathsf{o}_\lambda}} \cong D(\lambda)$. Then o_λ divides p so we set $p_\lambda = p/\mathsf{o}_\lambda$. Define an equivalence relation $\sim_\beta$ on $\mathrm{Irr}(A)$ by declaring that

$$D(\lambda) \sim_\beta D(\mu) \iff D(\lambda) \cong D(\mu)^{\beta^t}, \quad \text{for some } t \in \mathbb{Z}/p\mathbb{Z}.$$

Similarly, let $\sim_\alpha$ be the equivalence relation on $\mathrm{Irr}(B)$ given by

$$D^\lambda \sim_\alpha D^\mu \iff D^\lambda \cong (D^\mu)^{\alpha^t}, \quad \text{for some } t \in \mathbb{Z}/p\mathbb{Z}.$$

If D is an A-module let $\mathrm{Soc}_A(M)$ be its **socle**; that is the maximal semisimple submodule of A. Similarly, let $\mathrm{Head}_A(M)$ be the maximal semisimple quotient of M.

The following result is similar to [**16**, Lemma 2.2]. The result in [**16**] is proved only in the case $R = \mathbb{C}$. As we now show, the argument applies over any algebraically closed field.

4.2. **Lemma** (cf. [**16**, Lemma 2.2]). *Suppose that $R = K$ is an algebraically closed field and that $A = \bigoplus_{t=0}^{p-1} B\theta^t$ as in Lemma 4.1.*

a) *Suppose that $D(\lambda) \in \mathrm{Irr}(A)$. Then p_λ is the smallest positive integer such that $D^\lambda \cong \big(D^\lambda\big)^{\alpha^{p_\lambda}}$.*
b) *Suppose that $D^\lambda \in \mathrm{Irr}(B)$. Then $D^\lambda\!\uparrow_B^A \cong D(\lambda) \oplus D(\lambda)^\beta \oplus \cdots \oplus D(\lambda)^{\beta^{\mathsf{o}_\lambda - 1}}$ and $D(\lambda)\!\downarrow_B^A \cong D^\lambda \oplus (D^\lambda)^\alpha \oplus \cdots \oplus \big(D^\lambda\big)^{\alpha^{(p_\lambda - 1)}}$.*
c) *$\{\, (D^\lambda)^{\alpha^i} \mid D(\lambda) \in \mathrm{Irr}(A)/{\sim_\beta}$ for $1 \le i \le p_\lambda \,\}$ is a complete set of pairwise non-isomorphic absolutely irreducible B-modules.*
d) *$\{\, D(\lambda)^{\beta^i} \mid D^\lambda \in \mathrm{Irr}(B)/{\sim_\alpha}$ for $1 \le i \le o_\lambda \,\}$ is a complete set of pairwise non-isomorphic absolutely irreducible A-modules.*

Proof. Let $D(\lambda) \in \mathrm{Irr}(A)$. Let p'_λ be the smallest positive integer such that $D^\lambda \cong \big(D^\lambda\big)^{\alpha^{p'_\lambda}}$. By [**7**, Proposition 11.16], the module $D(\lambda)\!\downarrow_B^A$ is semisimple. Now,

$$\mathrm{Hom}_A\big(D^\lambda, D(\lambda)\!\downarrow_B^A\big) \cong \mathrm{Hom}_A\big((D^\lambda)^{\alpha^t}, D(\lambda)\!\downarrow_B^A\big), \quad \text{for any } t \in \mathbb{Z}.$$

Therefore, there exists an integer $c > 0$ such that

$$D(\lambda)\!\downarrow_B^A \cong \big(D^\lambda \oplus (D^\lambda)^\alpha \oplus \cdots \oplus (D^\lambda)^{\alpha^{p'_\lambda - 1}}\big)^{\oplus c}. \tag{4.3}$$

By Frobenius Reciprocity [**7**, Proposition (11.13)(ii)], we have that

$$\mathrm{Hom}_B\big(D(\lambda)\!\downarrow_B^A, D^\lambda\big) \cong \mathrm{Hom}_A\big(D(\lambda), D^\lambda\!\uparrow_B^A\big).$$

Since K is algebraically closed, both A and B are split over K. It follows that

$$\big(D(\lambda) \oplus D(\lambda)^\beta \oplus \cdots \oplus D(\lambda)^{\beta^{\mathsf{o}_\lambda - 1}}\big)^{\oplus c} \subseteq \mathrm{Soc}_A\big(D^\lambda\!\uparrow_B^A\big). \tag{4.4}$$

By (4.3) and (4.4), we have that

$$\dim D(\lambda) = c p'_\lambda \dim D^\lambda \quad \text{and} \quad p \dim D^\lambda \ge c\mathsf{o}_\lambda \dim D(\lambda). \tag{4.5}$$

Hence

$$p \ge c^2 p'_\lambda \mathsf{o}_\lambda. \tag{4.6}$$

On the other hand, since R contains a primitive pth root of unity, the integer p and all of its divisors must be invertible in R. Let π_λ be a linear endomorphism of $D(\lambda)$ which induces an A-module isomorphism $D(\lambda) \cong D(\lambda)^{\beta^{o_\lambda}}$. Then $(\pi_\lambda)^{p_\lambda} \in \mathrm{End}_A\big(D(\lambda)\big) = K$. Renormalising π_λ, if necessary, we can assume that $(\pi_\lambda)^{p_\lambda} = \mathrm{id}_\lambda$, where id_λ is the identity map on $D(\lambda)$.

Let X be an indeterminate over K and suppose then o divides p. Differentiating the identity $X^{p/\mathsf{o}}-1=\prod_{j=1}^{p/\mathsf{o}}(X-\varepsilon^{j\mathsf{o}})$ and setting $X=\pi_\lambda$ and $\mathsf{o}=\mathsf{o}_\lambda$, shows that

$$p_\lambda\pi_\lambda^{p_\lambda-1}=\sum_{j=1}^{p_\lambda}\prod_{\substack{1\le t\le p_\lambda\\ t\ne j}}(\pi_\lambda-\varepsilon^{t\mathsf{o}_\lambda}).$$

Thus,

$$\mathrm{id}_\lambda=\frac{1}{p_\lambda}\sum_{j=1}^{p_\lambda}\prod_{\substack{1\le t\le p_\lambda\\ t\ne j}}(\pi_\lambda-\varepsilon^{t\mathsf{o}_\lambda})\pi_\lambda^{1-p_\lambda}.$$

For each integer $1\le j\le p_{\boldsymbol\lambda}$, we define

$$D_j(\lambda):=\frac{1}{p_\lambda}\prod_{\substack{1\le t\le p_\lambda\\ t\ne j}}(\pi_\lambda-\varepsilon^{t\mathsf{o}_\lambda})\,\pi_\lambda^{1-p_\lambda}\,D(\lambda).$$

It is easy to check that each $D_j(\lambda)$ is a B-submodule of $D(\lambda)\downarrow_B^A$ and $D_j(\lambda)\theta=D_{j+1}(\lambda)$ for each $j\in\mathbb{Z}/p\mathbb{Z}$. In particular, this implies that $D(\lambda)\downarrow_B^A$ can be decomposed into a direct sum of $p_{\boldsymbol\lambda}$ nonzero B-submodules. Comparing this with (4.3), we can deduce that $p_{\boldsymbol\lambda}=p/\mathsf{o}_{\boldsymbol\lambda}\le cp'_\lambda$. Combining this with (4.6), we get that $c^2p'_\lambda\mathsf{o}_\lambda\le p\le cp'_\lambda\mathsf{o}_\lambda$, which forces that $c=1$, $p=\mathsf{o}_\lambda p'_\lambda$, and

$$D^\lambda\uparrow_B^A=\mathrm{Soc}_A\big(D^\lambda\uparrow_B^A\big)=D(\lambda)\oplus D(\lambda)^\beta\oplus\cdots\oplus D(\lambda)^{\beta^{\mathsf{o}_\lambda-1}}.$$

This proves the first two statements of the lemma. The last two statements follow by Frobenius reciprocity using the first two statements. □

We now apply these results to $\mathscr{H}_{r,p,n}$. It is straightforward to check that, as a right $\mathscr{H}_{r,p,n}$-module,

$$\mathscr{H}_{r,n}=\mathscr{H}_{r,p,n}\oplus T_0\mathscr{H}_{r,p,n}\oplus\cdots\oplus T_o^{p-1}\mathscr{H}_{r,p,n}.$$

(For example, use [**23**, Lemma 3.1].) Hence, $\mathscr{H}_{r,n}$ is a $\mathbb{Z}/p\mathbb{Z}$-graded Clifford system over $\mathscr{H}_{r,p,n}$. Applying Lemma 4.1 to $\mathscr{H}_{r,n}=\bigoplus_{i=0}^{p-1}\mathscr{H}_{r,p,n}T_0^i$ we obtain the following useful result.

4.7. **Proposition.** *There is a natural isomorphism of $(\mathscr{H}_{r,n},\mathscr{H}_{r,n})$-bimodules*

$$\mathscr{H}_{r,n}\otimes_{\mathscr{H}_{r,p,n}}\mathscr{H}_{r,n}\cong\bigoplus_{m=0}^{p-1}\big(\mathscr{H}_{r,n}\big)^{\sigma^m},$$

where $\mathscr{H}_{r,n}$ acts from the left on $\big(\mathscr{H}_{r,n}\big)^{\sigma^m}$ as left multiplication and from the right with its action twisted by σ^m.

4.8. **Corollary.** *Suppose that M is an $\mathscr{H}_{r,n}$-module. Then, as $\mathscr{H}_{r,n}$-modules,*

$$M\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\cong\bigoplus_{i=0}^{p-1}M^{\sigma^i}.$$

Proof. By definition, $M\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}=M\otimes_{\mathscr{H}_{r,n}}\mathscr{H}_{r,n}\otimes_{\mathscr{H}_{r,p,n}}\mathscr{H}_{r,n}$. Now apply Proposition 4.7. □

4.2. **Twisting modules by σ.** It is easy to check that $\sigma(T_w) = T_w$ and that $\sigma(L_m) = \varepsilon L_m$, for $w \in \mathfrak{S}_n$ and for $1 \le m \le n$. Hence, using the definitions we obtain the following.

4.9. **Lemma.** *Suppose that $1 \le b \le n$ and $1 \le s \le t \le p$. Then*

$$\sigma(\mathcal{L}_{1,b}^{(s,t)}) = \varepsilon^{bd(t-s+1)}\mathcal{L}_{1,b}^{(s-1,t-1)}.$$

Consequently, if $\mathbf{b} \in \mathscr{C}_{p,n}$ then $\sigma(v_{\mathbf{b}}) = \varepsilon^{-nd}v_{\mathbf{b}}^{(-1)}$ and $\sigma(Y_t) = \varepsilon^{-db_t}Y_{t-1}$, for $1 \le t \le p$.

By the remarks in Section 4.1, the automorphism σ induces a functor $\mathtt{F}^\sigma$ on the category of $\mathscr{H}_{r,n}$-modules. We want to compare $\mathtt{F}^\sigma$ with the functors $\mathtt{H}_{\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$, which appear in the Morita equivalences discussed in Section 2.5.

4.10. **Lemma.** *Let $\mathbf{b} \in \mathscr{C}_{p,n}$ and $t \in \mathbb{Z}$. Suppose that $\mathbf{Q}$ is (ε, q)-separated over K. Then $V_{\mathbf{b}\langle t\rangle} \cong V^\sigma_{\mathbf{b}\langle t+1\rangle}$.*

Proof. It is enough to show that $V_{\mathbf{b}}^{\sigma^{-1}} \cong V_{\mathbf{b}\langle 1\rangle}$ which is equivalent to the statement in the Lemma when $t = 0$. By Corollary 3.9, there is an isomorphism $V_{\mathbf{b}} \xrightarrow{\simeq} V^{(1)}_{\mathbf{b}\langle 1\rangle}$. On the other hand, $V_{\mathbf{b}}^{\sigma^{-1}} \cong \sigma(V_{\mathbf{b}}) \cong V_{\mathbf{b}}^{(-1)}$ by Lemma 4.9. Therefore, the map $v \mapsto (Y_1 v)^{\sigma^{-1}}$, for $v \in V_{\mathbf{b}}$, gives the required isomorphism $V_{\mathbf{b}}^{\sigma^{-1}} \xrightarrow{\simeq} V_{\mathbf{b}\langle 1\rangle}$. □

Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and recall that, by definition,

$$\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,\mathbf{b}}(\mathbf{Q}^{\vee\varepsilon}) = \mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q}) \otimes \cdots \otimes \mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q}).$$

Suppose that $h = h_1 \otimes \cdots \otimes h_p \in \mathscr{H}_{d,\mathbf{b}}$ and set $h\langle -1\rangle = h_p \otimes h_1 \otimes \cdots \otimes h_{p-1}$. It is trivial to see that there is an isomorphism of algebras

$$\mathscr{H}_{d,\mathbf{b}} \xrightarrow{\simeq} \mathscr{H}_{d,\mathbf{b}\langle -1\rangle}; h_1 \otimes \cdots \otimes h_p \mapsto h\langle -1\rangle = h_p^\sigma \otimes h_1^\sigma \otimes \cdots \otimes h_{p-1}^\sigma, \tag{4.11}$$

where we abuse notation slightly and define $\sigma(T_0^{(t)}) = \varepsilon^{-1}T_0^{(t+1)}$ and $\sigma(T_i^{(t)}) = T_i^{(t+1)}$, for $1 \le i < b_t$ and where we equate superscripts modulo p. It follows that there is an equivalence $\mathtt{F}^\sigma_{\mathbf{b}}\colon \text{Mod-}\mathscr{H}_{d,\mathbf{b}} \longrightarrow \text{Mod-}\mathscr{H}_{d,\mathbf{b}\langle -1\rangle}$ given by

$$\mathtt{F}^\sigma_{\mathbf{b}}(M_1 \otimes \cdots \otimes M_p) = M_p \otimes M_1 \otimes \cdots \otimes M_{p-1},$$

for an $\mathscr{H}_{d,\mathbf{b}}$-module $M_1 \otimes \cdots \otimes M_p$ and where $\mathscr{H}_{d,\mathbf{b}\langle -1\rangle}$ acts via the isomorphism above.

4.12. **Proposition.** *Let $\mathbf{b} \in \mathscr{C}_{p,n}$. Suppose that $\mathbf{Q}$ is (ε, q)-separated over K. Then the following diagram commutes*

$$\begin{array}{ccc} \text{Mod-}\mathscr{H}_{d,\mathbf{b}} & \xrightarrow{\mathtt{F}^\sigma_{\mathbf{b}}} & \text{Mod-}\mathscr{H}_{d,\mathbf{b}\langle -1\rangle} \\ {\scriptstyle \mathtt{H}_{\mathbf{b}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathtt{H}_{\mathbf{b}\langle -1\rangle}} \\ \text{Mod-}\mathscr{H}_{r,n} & \xrightarrow[\mathtt{F}^\sigma]{} & \text{Mod-}\mathscr{H}_{r,n} \end{array}$$

Proof. Let M be an $\mathscr{H}_{d,\mathbf{b}}$-module. Then we have to prove that

$$\big(M \otimes_{\mathscr{H}_{d,\mathbf{b}}} V_{\mathbf{b}}\big)^\sigma \cong \mathtt{F}^\sigma_{\mathbf{b}}(M) \otimes_{\mathscr{H}_{d,\mathbf{b}\langle -1\rangle}} V_{\mathbf{b}\langle -1\rangle}$$

as right $\mathscr{H}_{r,n}$-modules. Mimicking the proof of Lemma 4.10, the required isomorphism is the map $m \otimes v \mapsto m\langle -1\rangle \otimes (Y_1 v)^\sigma$, for $m \otimes v \in M \otimes_{\mathscr{H}_{d,\mathbf{b}}} V_{\mathbf{b}}$. □

We want to use this result to determine the σ-twists of various $\mathscr{H}_{r,n}$-modules. To this end set $a^{\boldsymbol\lambda}_{s,t} = |\lambda^{(dt-d+1)}| + \dots + |\lambda^{(dt-d+s-1)}|$, for $1 \le i \le d$ and $1 \le t \le p$ and define

$$u^+_{\boldsymbol\lambda^{[t]}} = u^+_{\boldsymbol\lambda^{[t]}}(\varepsilon^t\mathbf{Q}) = \prod_{s=2}^{d}\prod_{j=1}^{a^{\boldsymbol\lambda}_{s,t}}(L_j - \varepsilon^t Q_s) \qquad \text{and} \qquad x_{\boldsymbol\lambda^{[t]}} = \sum_{w\in\mathfrak{S}_{\boldsymbol\lambda^{[t]}}} T_w,$$

$$y_{\boldsymbol\lambda^{[t]}} = \sum_{w\in\mathfrak{S}_{\boldsymbol\lambda^{[t]}}} (-1)^{\ell(w)}T_w,$$

which we think of as elements of $\mathscr{H}_{d,b_t}(\varepsilon^t\mathbf{Q})$ in the natural way. Now set $u^+_{\boldsymbol\lambda,\mathbf{b}} = u^+_{\boldsymbol\lambda^{[1]}}\otimes\dots\otimes u^+_{\boldsymbol\lambda^{[p]}}$ and $x_{\boldsymbol\lambda,\mathbf{b}} = x_{\boldsymbol\lambda^{[1]}}\otimes\dots\otimes x_{\boldsymbol\lambda^{[p]}}$. We remark that it is easy to check that $u^+_{\boldsymbol\lambda,\mathbf{b}}$ and $x_{\boldsymbol\lambda,\mathbf{b}}$ commute using Lemma 2.3.

By [**12**, Theorem 2.9], there exists an element $s_{\mathbf{b}}(\boldsymbol\lambda) = s(\boldsymbol\lambda^{[1]})\otimes\dots\otimes s(\boldsymbol\lambda^{[p]}) \in \mathscr{H}_{d,\mathbf{b}}$ such that $S_{\mathbf{b}}(\boldsymbol\lambda) \cong s_{\mathbf{b}}(\boldsymbol\lambda)\mathscr{H}_{d,\mathbf{b}}$. Explicitly, $s(\boldsymbol\lambda^{[i]}) = u^+_{\boldsymbol\mu^{[i]}}(\varepsilon^i\mathbf{Q}')y_{\boldsymbol\mu^{[i]}}T_{w(\boldsymbol\mu)}x_{\boldsymbol\lambda^{[i]}}u^+_{\boldsymbol\lambda^{[i]}}$ where $\boldsymbol\mu^{[i]}$ is the multipartition conjugate to $\boldsymbol\lambda^{[i]}$, for $1 \le i \le p$. By Lemma 3.3, we have that

$$S(\boldsymbol\lambda) \cong \mathtt{H}_{\mathbf{b}}(S_{\mathbf{b}}(\boldsymbol\lambda)) \cong s_{\mathbf{b}}(\boldsymbol\lambda)\cdot v_{\mathbf{b}}\mathscr{H}_{r,n}.$$

Henceforth, we identify $S(\boldsymbol\lambda)$ with $s_{\mathbf{b}}(\boldsymbol\lambda)\cdot V_{\mathbf{b}}$ and $S_{\mathbf{b}}(\boldsymbol\lambda)$ with $s_{\mathbf{b}}(\boldsymbol\lambda)\mathscr{H}_{d,\mathbf{b}}$ via these isomorphisms. Observe that $\mathtt{H}_{\mathbf{b}}(S_{\mathbf{b}}(\boldsymbol\lambda)) = S(\boldsymbol\lambda)$ with these identifications.

4.13. **Definition.** *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$. Define*

$$M_{\mathbf{b}}(\boldsymbol\lambda) = u^+_{\boldsymbol\lambda,\mathbf{b}}x_{\boldsymbol\lambda,\mathbf{b}}\mathscr{H}_{d,\mathbf{b}} \qquad \textit{and} \qquad M^{\boldsymbol\lambda}_{\mathbf{b}} = \mathtt{H}_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol\lambda)\big).$$

The definitions above apply equally well to $\mathscr{H}_{r,n}$-modules by taking $p = 1$. In particular, we have elements $u^+_{\boldsymbol\lambda}$ and $x_{\boldsymbol\lambda}$ in $\mathscr{H}_{r,n}$ and an $\mathscr{H}_{r,n}$-module $M(\boldsymbol\lambda) = u^+_{\boldsymbol\lambda}x_{\boldsymbol\lambda}\mathscr{H}_{r,n}$. Using the definitions it is easy to check that $x_{\boldsymbol\lambda} = \Theta_{\mathbf{b}}(x_{\boldsymbol\lambda,\mathbf{b}})$ and that $u^+_{\boldsymbol\lambda} = u^+_{\mathbf{b}}\Theta_{\mathbf{b}}(u^+_{\boldsymbol\lambda,\mathbf{b}})$, where $u^+_{\mathbf{b}}$ is the element introduced in (2.23). It follows that $M^{\boldsymbol\lambda}_{\mathbf{b}} = v^+_{\mathbf{b}}M(\boldsymbol\lambda)$. Hence, in general, $M^{\boldsymbol\lambda}_{\mathbf{b}}$ is a proper submodule of $V_{\mathbf{b}}$.

We can now prove the promised result about σ-twisted modules.

4.14. **Proposition.** *Let $\mathbf{b}\in\mathscr{C}_{p,n}$ and $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$. Suppose that $\mathbf{Q}$ is (ε,q)-separated over K. Then*

$$\big(M^{\boldsymbol\lambda}_{\mathbf{b}}\big)^\sigma \cong M^{\boldsymbol\lambda\langle-1\rangle}_{\mathbf{b}\langle-1\rangle} \quad \textit{and} \quad S(\boldsymbol\lambda)^\sigma \cong S(\boldsymbol\lambda\langle-1\rangle).$$

Moreover, if $\boldsymbol\lambda\in\mathscr{K}_{r,n}$ then $D(\boldsymbol\lambda)^\sigma \cong D(\boldsymbol\lambda\langle-1\rangle)$.

Proof. We have that $\sigma\big(u^+_{\boldsymbol\lambda^{[t]}}(\varepsilon^t\mathbf{Q})\big) = \varepsilon^{k_t}u^+_{\boldsymbol\lambda^{[t]}}(\varepsilon^{t-1}\mathbf{Q})$, for some integer k_t, exactly as in Lemma 4.9. From the definitions, $F^\sigma_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol\lambda)\big) \cong M_{\mathbf{b}\langle-1\rangle}(\boldsymbol\lambda\langle-1\rangle)$. Therefore, using Proposition 4.12,

$$\begin{aligned}\big(M^{\boldsymbol\lambda}_{\mathbf{b}}\big)^\sigma &= \mathtt{F}^\sigma\big(\mathtt{H}_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol\lambda)\big)\big) \cong \mathtt{H}_{\mathbf{b}\langle-1\rangle}\big(F^\sigma_{\mathbf{b}}\big(M_{\mathbf{b}}(\boldsymbol\lambda)\big)\big)\\ &\cong \mathtt{H}_{\mathbf{b}\langle-1\rangle}\big(M_{\mathbf{b}\langle-1\rangle}(\boldsymbol\lambda\langle-1\rangle)\big) \cong M^{\boldsymbol\lambda\langle-1\rangle}_{\mathbf{b}\langle-1\rangle},\end{aligned}$$

giving the first isomorphism. A similar argument shows that $S(\boldsymbol\lambda)^\sigma \cong S(\boldsymbol\lambda\langle-1\rangle)$.

Finally, if $\boldsymbol\lambda$ is Kleshchev then $D(\boldsymbol\lambda)\ne 0$ and there is a short exact sequence

$$0 \longrightarrow \operatorname{rad} S(\boldsymbol\lambda) \longrightarrow S(\boldsymbol\lambda) \longrightarrow D(\boldsymbol\lambda) \longrightarrow 0.$$

The functor $\mathtt{F}^\sigma$ is exact, and $D(\boldsymbol\lambda\langle-1\rangle)$ is the head of $S(\boldsymbol\lambda\langle-1\rangle)$, so $D(\boldsymbol\lambda)^\sigma \cong D(\boldsymbol\lambda\langle-1\rangle)$ because $S(\boldsymbol\lambda)^\sigma \cong S(\boldsymbol\lambda\langle-1\rangle)$ by the last paragraph. (Note that $\boldsymbol\lambda$ is Kleshchev if and only if $\boldsymbol\lambda\langle-1\rangle$ is Kleshchev by Lemma 3.3(c).) □

As σ is trivial on $\mathscr{H}_{r,p,n}$, Lemma 4.10 and Proposition 4.14 imply the following.

4.15. **Corollary.** *Suppose that $\mathbf{Q}$ is (ε,q)-separated over K and that $\mathbf{b}\in\mathscr{C}_{p,n}$, $\boldsymbol\lambda\in\mathscr{P}_{d,\mathbf{b}}$ and $t\in\mathbb{Z}$. Then:*

a) $V_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong V_{\mathbf{b}\langle t\rangle}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$,

b) $M^{\boldsymbol{\lambda}}_{\mathbf{b}}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong M^{\boldsymbol{\lambda}\langle t\rangle}_{\mathbf{b}\langle t\rangle}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$,

c) $S(\boldsymbol{\lambda})\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong S(\boldsymbol{\lambda}\langle t\rangle)\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$, *and,*

d) *if* $\boldsymbol{\lambda}\in\mathscr{K}_{r,n}$ *then* $D(\boldsymbol{\lambda})\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong D(\boldsymbol{\lambda}\langle t\rangle)\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$.

4.3. **Shifting homomorphisms.** Extending the notation that we used for the modules $V^{(t)}_{\mathbf{b}}$, for each multipartition $\boldsymbol{\lambda}\in\mathscr{P}_{r,n}$ let $S(\boldsymbol{\lambda})^{(t)}$ be the Specht module for $\mathscr{H}_{r,n}$ which is defined with respect to the ordered parameters $\varepsilon^t\mathbf{Q}^{\vee\boldsymbol{\varepsilon}}$ (rather than $\mathbf{Q}^{\vee\boldsymbol{\varepsilon}}$). Then $S(\boldsymbol{\lambda})\cong S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$ as $\mathscr{H}_{r,n}$-modules and $S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$ is a submodule of $V^{(t)}_{\mathbf{b}\langle t\rangle}$. The following result makes this more explicit.

4.16. **Lemma.** *Suppose that* $\mathbf{Q}$ *is* (ε,q)*-separated over* K *and that* $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$*, for* $\mathbf{b}\in\mathscr{C}_{p,n}$*, and* $1\le t\le p$*. Then*

$$Y_t\dots Y_1S(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$$

as subsets of $\mathscr{H}_{r,n}$*.*

Proof. As we have already observed, left multiplication by $Y_p\dots Y_1$ is invertible by Lemma 2.24 and Lemma 2.21. Therefore, $Y_t\dots Y_1S(\boldsymbol{\lambda})\cong S(\boldsymbol{\lambda})$ as a right $\mathscr{H}_{r,n}$-modules, so it is enough to show that $Y_t\dots Y_1S(\boldsymbol{\lambda})\subseteq S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$. Recall from before Definition 4.13 that we are identifying $S_{\mathbf{b}}(\boldsymbol{\lambda})$ with the ideal $S_{\mathbf{b}}(\boldsymbol{\lambda}) = s_{\mathbf{b}}(\boldsymbol{\lambda})\mathscr{H}_{d,\mathbf{b}}$ and $S(\boldsymbol{\lambda}) = s_{\mathbf{b}}(\boldsymbol{\lambda})\cdot V_{\mathbf{b}}$. Using Lemma 2.16 we compute

$$\begin{aligned} Y_t\dots Y_1\big(s_{\mathbf{b}}(\boldsymbol{\lambda})\cdot v_{\mathbf{b}}\big) &= Y_t\dots Y_1 v_{\mathbf{b}}\Theta_{\mathbf{b}}\big(s_{\mathbf{b}}(\boldsymbol{\lambda})\big)\\ &= \widehat{\Theta}_{\mathbf{b}\langle t\rangle}\big(s_{\mathbf{b}\langle t\rangle}(\boldsymbol{\lambda}\langle t\rangle)\big)Y_t\dots Y_1 v_{\mathbf{b}}\\ &= s_{\mathbf{b}\langle t\rangle}(\boldsymbol{\lambda}\langle t\rangle)\cdot v^{(t)}_{\mathbf{b}\langle t\rangle}Y_t^*\dots Y_1^*, \end{aligned}$$

the last equality following from Corollary 2.9. Hence, $Y_t\dots Y_1S(\boldsymbol{\lambda})\subseteq S(\boldsymbol{\lambda}\langle t\rangle)^{(t)}$ as we needed to show. □

Fix $\mathbf{b}\in\mathscr{C}_{p,n}$ and $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and suppose that $\boldsymbol{\lambda}=\boldsymbol{\lambda}\langle m\rangle$, for some integer $1\le m\le p$ with m dividing p. Then $\mathbf{b}=\mathbf{b}\langle m\rangle$ and σ^m is an automorphism of $\mathscr{H}_{r,n}$ of order $\frac{p}{m}$. Set

$$\check{\mathbf{Q}} = \big(Q_1, Q_1\varepsilon,\cdots,Q_1\varepsilon^{m-1},Q_2,\cdots,Q_2\varepsilon^{m-1},\cdots,Q_d,\cdots,Q_d\varepsilon^{m-1}\big).$$

Then $\mathscr{H}_{r,n}=\mathscr{H}_{r,n}(\mathbf{Q}^{\vee\boldsymbol{\varepsilon}})=\mathscr{H}_{r,n}(\check{\mathbf{Q}}^{\vee\boldsymbol{\varepsilon}^m})$. By definition, $\mathscr{H}_{r,\frac{p}{m},n}=\mathscr{H}_{r,\frac{p}{m},n}(\check{\mathbf{Q}})$ is the subalgebra of $\mathscr{H}_{r,n}$ generated by $T_0^{p/m},T_1,\cdots,T_{n-1}$, so that

$$\mathscr{H}_{r,\frac{p}{m},n}\cong\{\,h\in\mathscr{H}_{r,n}\mid h=\sigma^m(h)\,\}.$$

This observation will be useful below.

For $0\le t<\frac{p}{m}$ we now consider the modules $V^{(tm)}_{\mathbf{b}}$ and $S(\boldsymbol{\lambda})^{(tm)}$. Then, by definition, $S(\boldsymbol{\lambda})^{(tm)}$ is a submodule of $V^{(tm)}_{\mathbf{b}}$, $\big(V^{(tm+m)}_{\mathbf{b}}\big)^{\sigma^{-m}} = V^{(tm)}_{\mathbf{b}}$ and $\big(S(\boldsymbol{\lambda})^{(tm+m)}\big)^{\sigma^{-m}} = S(\boldsymbol{\lambda})^{(tm)}$, by Lemma 4.9 and Proposition 4.14, respectively. Motivated by Definition 2.10, define

$$Y_{t,m} = Y_{tm+m}\dots Y_{tm+2}Y_{tm+1},$$

for $0\le t<\frac{p}{m}$, and let $\theta'_{t,m}\colon V^{(tm)}_{\mathbf{b}}\longrightarrow V^{(tm+m)}_{\mathbf{b}}$ be the map $\theta'_{t,m}(v)=Y_{t,m}v$, for $v\in V^{(tm)}_{\mathbf{b}}$.

4.17. **Definition** (Shifting homomorphisms)**.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and that* $\mathbf{b}=\mathbf{b}\langle m\rangle$ *for some* $1\le m\le p$ *with* m *dividing* p*. For* $0\le t<\frac{p}{m}$ *define* $\theta_{t,m}=\sigma^m\circ\theta'_{t,m}$*.*

4.18. **Lemma.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$*, with* $\mathbf{b}=\mathbf{b}\langle m\rangle$ *for some* $1\le m\le p$ *with* m *dividing* p*, and suppose that* $0\le t<\frac{p}{m}$*. Then* $\theta_{t,m}\in\operatorname{End}_{\mathscr{H}_{r,p/m,n}}\big(V_{\mathbf{b}}^{(tm)}\big)$.

Proof. By Definition 2.10 and the remarks above, $\theta_{t,m}\in\operatorname{End}_R\big(V_{\mathbf{b}}^{(tm)}\big)$ since $\mathbf{b}=\mathbf{b}\langle m\rangle$. Moreover, if $v\in V_{\mathbf{b}}^{(tm)}$ and $h\in\mathscr{H}_{r,n}$ then

$$\theta_{t,m}(vh)=\sigma^m\big(\theta'_{t,m}(vh)\big)=\sigma^m\big(\theta'_{t,m}(v)\big)\sigma^m(h),$$

since $\theta'_{t,m}$ is an $\mathscr{H}_{r,n}$-module homomorphism by Definition 2.10. Therefore, $\theta_{t,m}(vh)$ is an $\mathscr{H}_{r,p/m,n}$-module homomorphism since $\mathscr{H}_{r,\frac{p}{m},n}=\mathscr{H}_{r,n}^{\sigma^m}$. □

4.4. **Seminormal forms and roots of $\mathfrak{f}_{\boldsymbol\lambda}$.** In this section we show that if $\boldsymbol\lambda=\boldsymbol\lambda\langle m\rangle$, for some integer m dividing p such that $1\le m\le p$, then there exists a scalar $\mathfrak{f}_{\boldsymbol\lambda}^{(1)}$ such that $\mathfrak{f}_{\boldsymbol\lambda}=\varepsilon^{mnd(l-1)/2}\big(\mathfrak{f}_{\boldsymbol\lambda}^{(1)}\big)^l$, where $l=p/m$. We are not able to prove this directly, however, and instead argue via the semisimple case using seminormal forms.

Recall that $\mathcal{A}=\mathbb{Z}[\dot\varepsilon,\dot q^{\pm1},\dot Q_1^{\pm1},\dots,\dot Q_d^{\pm1},A(\dot\varepsilon,\dot q,\dot{\mathbf{Q}})^{-1}]$ and that $\mathcal{F}$ is the field of fractions of $\mathcal{A}$. As we noted in Section 3.2, the algebra $\mathscr{H}_{r,n}^{\mathcal{F}}$ is semisimple. Note that $\dot{\mathbf{Q}}$ is $(\dot\varepsilon,\dot q)$-separated over $\mathcal{F}$ so we can apply all of our previous results.

Fix $\boldsymbol\lambda\in\mathscr{P}_{r,n}$ and an integer m such that $\boldsymbol\lambda=\boldsymbol\lambda\langle m\rangle$ and $1\le m\le p$ and $m\mid p$. Let $l=p/m$. Since $\mathscr{H}_{r,n}^{\mathcal{F}}$ is semisimple the Specht module $S(\boldsymbol\lambda)=S^{\mathcal{F}}(\boldsymbol\lambda)$ is irreducible and has, as we recall, a seminormal representation over $\mathcal{F}$. First we need some notation.

Recall from Section 3.2 that $\operatorname{Std}(\boldsymbol\lambda)$ is the set of standard $\boldsymbol\lambda$-tableaux. Each tableau $\mathfrak{s}\in\operatorname{Std}(\boldsymbol\lambda)$ is an r-tuple $\mathfrak{s}=(\mathfrak{s}^{(1)},\dots,\mathfrak{s}^{(r)})$ of standard tableaux. Extending the notation for $\boldsymbol\lambda=(\boldsymbol\lambda^{[1]},\dots,\boldsymbol\lambda^{[p]})$ write $\mathfrak{s}=(\mathfrak{s}^{[1]},\dots,\mathfrak{s}^{[p]})$, where $\mathfrak{s}^{[j]}=(\mathfrak{s}^{[jd-d+1]},\dots,\mathfrak{s}^{[jd]})$ is a $\boldsymbol\lambda^{[j]}$-tableau for $1\le j\le p$. Similarly, if $z\in\mathbb{Z}$ define $\mathfrak{s}\langle z\rangle=(\mathfrak{s}^{[z+1]},\dots,\mathfrak{s}^{[z+p]})$ where, as usual, we set $\mathfrak{s}^{[j+kp]}=\mathfrak{s}^{[j]}$ for $1\le j\le p$ and $k\in\mathbb{Z}$.

Finally, if $1\le k\le n$ and $\mathfrak{s}\in\operatorname{Std}(\boldsymbol\lambda)$ define the **content** of k in $\mathfrak{t}$ to be

$$\operatorname{cont}_{\mathfrak{s}}(k)=\dot\varepsilon^j\dot q^{b-a}\dot Q_c\in\mathcal{F},$$

if k appears in row a and column b of $\mathfrak{s}^{(c+jd)}$. The following useful fact is easily proved by induction on n.

4.19. **Lemma** (cf. [**24**, Lemma 3.12])**.** *Suppose that* $\mathfrak{s}\in\operatorname{Std}(\boldsymbol\lambda)$ *and* $\mathfrak{t}\in\operatorname{Std}(\boldsymbol\mu)$*, for* $\boldsymbol\lambda,\boldsymbol\mu\in\mathscr{P}_{r,n}$*. Then* $\mathfrak{s}=\mathfrak{t}$ *if and only if* $\operatorname{cont}_{\mathfrak{s}}(k)=\operatorname{cont}_{\mathfrak{t}}(k)$*, for* $1\le k\le n$.

If $\mathfrak{s}$ is a standard $\boldsymbol\lambda$-tableau and $1\le i<n$ let $\mathfrak{s}(i,i+1)$ be the tableau obtained by interchanging the positions of i and $i+1$ in $\mathfrak{s}$. Then $\mathfrak{s}(i,i+1)$ is a standard $\boldsymbol\lambda$-tableau unless i and $i+1$ are either in the same row or in the same column.

4.20. **Lemma** (Ariki-Koike [**3**, Theorem 3.7])**.** *Let* $V(\boldsymbol\lambda)$ *be the* $\mathcal{F}$*-vector space with basis* $\{\,v_{\mathfrak{s}}\mid\mathfrak{s}\in\operatorname{Std}(\boldsymbol\lambda)\,\}$*. Then* $V(\boldsymbol\lambda)$ *becomes an* $\mathscr{H}_{r,n}^{\mathcal{F}}$*-module with* $\mathscr{H}_{r,n}^{\mathcal{F}}$*-action, for* $1\le k\le n$ *and* $1\le i<n$*, given by*

$$v_{\mathfrak{s}}L_k=\operatorname{cont}_{\mathfrak{s}}(k)v_{\mathfrak{s}}\quad\text{and}\quad v_{\mathfrak{s}}T_i=\beta_{\mathfrak{s}}(i)v_{\mathfrak{s}}+\big(1+\beta_{\mathfrak{s}}(i)\big)v_{\mathfrak{t}},$$

where $\mathfrak{t}=\mathfrak{s}(i,i+1)$*,* $v_{\mathfrak{t}}=0$ *if* $\mathfrak{t}$ *is not standard and*

$$\beta_{\mathfrak{s}}(i)=\frac{(\dot q-1)\operatorname{cont}_{\mathfrak{t}}(i)}{(\operatorname{cont}_{\mathfrak{t}}(i)-\operatorname{cont}_{\mathfrak{s}}(i))}.$$

Moreover, $V(\boldsymbol\lambda)\cong S^{\mathcal{F}}(\boldsymbol\lambda)$ *as* $\mathscr{H}_{r,n}^{\mathcal{F}}$*-modules.*

The module $V(\boldsymbol\lambda)$ is a **seminormal form** for $S^{\mathcal{F}}(\boldsymbol\lambda)$.

Recall that we have fixed integers m and $l = p/m$ such that $m \mid p$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$. Thus, $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)} \cong S^{\mathcal{F}}(\boldsymbol{\lambda})$, for $0 \le t < l = p/m$. By definition,

$$S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)} = s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v_{\mathbf{b}}^{(tm)} \mathscr{H}_{r,n}^{\mathcal{F}}.$$

For convenience, we set $v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)} = s_{\mathbf{b}}(\boldsymbol{\lambda}) \cdot v_{\mathbf{b}}^{(tm)} \in \mathscr{H}_{r,n}^{\mathcal{F}}$.

Recall from Section 3.2 that $\mathfrak{t}^{\boldsymbol{\lambda}}$ is the standard $\boldsymbol{\lambda}$-tableau which have the numbers $1, 2, \dots, n$ entered in order from left to right along the rows of its first component, then its second component and so on.

4.21. **Lemma.** *Suppose that* $0 \le t < l$. *Then*

$$v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)} L_k = \operatorname{cont}_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}(k) v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)},$$

for $1 \le k \le n$.

Proof. It suffices to consider the case $t = 0$ when the result is effectively a restatement of [**27**, Prop. 3.13]. Alternatively, this can be proved using Du and Rui's proof [**12**, Theorem 2.9] that the Specht module $S(\boldsymbol{\lambda})$ is isomorphic to the corresponding cell module from [**9**] together with the description of the action of $L_1, \dots, L_n$ on the standard basis of the cell modules from [**24**, Prop. 3.7]. □

4.22. **Corollary.** *Suppose that* $0 \le t < l$. *Then there exists a unique* $\mathscr{H}_{r,n}^{\mathcal{F}}$*-module isomorphism*

$$\varphi_{\boldsymbol{\lambda}}^{(tm)} : V(\boldsymbol{\lambda}) \xrightarrow{\simeq} S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$$

such that $\varphi_{\boldsymbol{\lambda}}^{(tm)}(v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}) = v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)}$.

Proof. By the Lemma, $v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)}$ is a simultaneous eigenvector for $L_1, \dots, L_n$ with the eigenvalues being given by the contents $\operatorname{cont}_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle tm\rangle}(k)$, for $1 \le k \le n$. By Proposition 4.20 the corresponding simultaneous eigenspace in $V(\boldsymbol{\lambda})$ is $\mathcal{F} v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}$, so any $\mathscr{H}_{r,n}^{\mathcal{F}}$-module isomorphism from $V(\boldsymbol{\lambda})$ to $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ must send $v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}$ to a scalar multiple of $v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)}$. As $V(\boldsymbol{\lambda}) \cong S^{\mathcal{F}}(\boldsymbol{\lambda}) \cong S^{\mathcal{F}}(\boldsymbol{\lambda}\langle tm\rangle)^{(tm)} = S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ by renormalizing any isomorphism $V(\boldsymbol{\lambda}) \longrightarrow S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ we get the result. □

Suppose that $0 \le t < l$. For each standard $\boldsymbol{\lambda}$-tableau $\mathfrak{s}$ set $v_{\mathfrak{s}}^{(tm)} = \varphi_{\boldsymbol{\lambda}}^{(tm)}(v_{\mathfrak{s}\langle -tm\rangle})$. Then $\{\, v_{\mathfrak{s}}^{(tm)} \mid \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$ is a Young seminormal basis of $S^{\mathcal{F}}(\boldsymbol{\lambda})^{(tm)}$ and, by construction,

$$v_{\mathfrak{s}}^{(tm)} L_k = \varphi_{\boldsymbol{\lambda}}^{(tm)}(v_{\mathfrak{s}\langle tm\rangle}) L_k = \operatorname{cont}_{\mathfrak{s}\langle tm\rangle}(k) v_{\mathfrak{s}}^{(tm)},$$

for $1 \le k \le n$. Recall from Lemma 4.16 that $Y_{t,m} S(\boldsymbol{\lambda})^{(tm)} = S(\boldsymbol{\lambda})^{(tm+m)}$. Finally, we are able to describe this map more concretely.

4.23. **Proposition.** *Suppose that* $0 \le t \le m$ *and* $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$. *Then there exists a scalar* $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t+1:m)}(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}}) \in \mathcal{F}$ *such that*

$$Y_{t,m} v_{\mathfrak{s}\langle m\rangle}^{(tm)} = \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t+1:m)} v_{\mathfrak{s}}^{(tm+m)},$$

for all $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$.

Proof. By definition, if $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $v_{\mathfrak{s}}^{(tm+m)} L_k = \operatorname{cont}_{\mathfrak{s}\langle tm+m\rangle}(k) v_{\mathfrak{s}}^{(tm+m)}$, for $1 \le k \le n$. The same statement holds true for $Y_{t,m} v_{\mathfrak{s}\langle m\rangle}^{(tm)}$, so by construction $Y_{t,m} v_{\mathfrak{s}\langle m\rangle}^{(tm)}$ must be a scalar multiple of $v_{\mathfrak{s}}^{(tm+m)}$. By direct verification, we know that the map which sends $v_{\mathfrak{s}\langle m\rangle}^{(tm)}$ to $v_{\mathfrak{s}}^{(tm+m)}$ (for each $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$) defines an $\mathscr{H}_{r,n}^{\mathcal{F}}$-isomorphism. By Schur's Lemma this scalar is independent of $\mathfrak{s}$ so the Lemma follows. □

We write $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t)} = \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t:m)}(\dot{\varepsilon}, \dot{q}, \dot{\mathbf{Q}})$ if m is clear from context. It is tempting to say that $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t)} \in \mathcal{A}$ since left multiplication by $Y_{t,m}$ is defined over $\mathcal{A}$, however, the construction of the basis $\{v_{\mathfrak{s}}^{(tm)}\}$ is only valid over $\mathcal{F}$. Nonetheless, we will show below, using the fact that $\mathcal{A}$ is integrally closed in $\mathcal{F}$, that $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t)} \in \mathcal{A}$.

4.24. **Lemma.** *Let $\varphi\colon V(\boldsymbol{\lambda}) \longrightarrow V(\boldsymbol{\lambda})$ be the $\mathcal{F}$-linear map such that*

$$\varphi(v_{\mathfrak{s}}) = v_{\mathfrak{s}\langle m\rangle}, \qquad \text{for all } \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}).$$

Then φ is an $\mathscr{H}_{\dot{q}}^{\mathcal{F}}(\mathfrak{S}_n)$-module homomorphism. Moreover, $\varphi(vx) = \varphi(v)\sigma^m(x)$, for all $v \in V(\boldsymbol{\lambda})$ and $x \in \mathscr{H}_{r,n}^{\mathcal{F}}$. Hence, φ is an $\mathscr{H}_{r,p/m,n}^{\mathcal{F}}$-module homomorphism.

Proof. Suppose that $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $1 \le i < n$ and let $\mathfrak{t} = \mathfrak{s}(i, i+1)$. Then, by Lemma 4.20,

$$\begin{aligned}\varphi(v_{\mathfrak{s}}T_i) &= \beta_{\mathfrak{s}}(i)\varphi(v_{\mathfrak{s}}) + (1+\beta_{\mathfrak{s}}(i))\varphi(v_{\mathfrak{t}}) = \beta_{\mathfrak{s}}(i)v_{\mathfrak{s}\langle m\rangle} + (1+\beta_{\mathfrak{s}}(i))v_{\mathfrak{t}\langle m\rangle}\\ &= v_{\mathfrak{s}\langle m\rangle}T_i = \varphi(v_{\mathfrak{s}})T_i,\end{aligned}$$

where the second last equality follows because $\beta_{\mathfrak{s}}(i) = \beta_{\mathfrak{s}\langle m\rangle}(i)$. Hence, φ is a $\mathscr{H}_q(\mathfrak{S}_n)$-homomorphism. To prove the second claim it is enough to show that $\varphi(v_{\mathfrak{s}}L_k) = \dot{\varepsilon}^m v_{\mathfrak{s}\langle m\rangle}L_k$, for all $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $1 \le k \le n$. This is immediate because $\operatorname{cont}_{\mathfrak{s}\langle m\rangle}(k) = \dot{\varepsilon}^{-m}\operatorname{cont}_{\mathfrak{s}}(k)$. □

4.25. **Corollary.** *Suppose that $0 \le t < l$ and that $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then*

$$\sigma^m\big(v_{\mathfrak{s}}^{(tm)}\big) = \dot{\varepsilon}^{-dmn}v_{\mathfrak{s}}^{(tm-m)}.$$

Proof. First note that $\sigma^m(v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)}) = \dot{\varepsilon}^{-dmn}v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm-m)}$ because

$$\sigma^m(v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)}) = \sigma^m\big(s_{\mathbf{b}}(\boldsymbol{\lambda})\cdot v_{\mathbf{b}}^{(tm)}\big) = \dot{\varepsilon}^{-dmn}s_{\mathbf{b}}(\boldsymbol{\lambda})\cdot v_{\mathbf{b}}^{(tm-m)} = \dot{\varepsilon}^{-dmn}v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm-m)},$$

by Lemma 4.9. Therefore, writing $v_{\mathfrak{s}}^{(tm)} = v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)}h = \varphi_{\boldsymbol{\lambda}}^{(tm)}(v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}h)$ we have $v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}h = v_{\mathfrak{s}\langle -tm\rangle}$, and so

$$\begin{aligned}\sigma^m\big(v_{\mathfrak{s}}^{(tm)}\big) &= \sigma^m(v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm)})\sigma^m(h) = \dot{\varepsilon}^{-dmn}v_{\mathfrak{t}^{\boldsymbol{\lambda}}}^{(tm-m)}\sigma^m(h)\\ &= \dot{\varepsilon}^{-dmn}\varphi_{\boldsymbol{\lambda}}^{(tm-m)}\big(v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle m-tm\rangle}\sigma^m(h)\big) = \dot{\varepsilon}^{-dmn}\varphi_{\boldsymbol{\lambda}}^{(tm-m)}\big(\varphi(v_{\mathfrak{t}^{\boldsymbol{\lambda}}\langle -tm\rangle}h)\big)\\ &= \dot{\varepsilon}^{-dmn}\varphi_{\boldsymbol{\lambda}}^{(tm-m)}\big(\varphi(v_{\mathfrak{s}\langle -tm\rangle})\big) = \dot{\varepsilon}^{-dmn}\varphi_{\boldsymbol{\lambda}}^{(tm-m)}\big(v_{\mathfrak{s}\langle m-tm\rangle}\big)\\ &= \dot{\varepsilon}^{-dmn}v_{\mathfrak{s}}^{(tm-m)},\end{aligned}$$

as required. □

4.26. **Theorem.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ be a multipartition such that $\boldsymbol{\lambda} = \boldsymbol{\lambda}\langle m\rangle$, for some $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le m \le p$ with $m \mid p$. Set $l = p/m$. Then*

$$\dot{\mathfrak{f}}_{\boldsymbol{\lambda}} = \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(1)}\dots\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(l)} = \dot{\varepsilon}^{\frac{1}{2}dmn(1-l)}\big(\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(1)}\big)^l.$$

Consequently, $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(t)} \in \mathcal{A}$ for $1 \le t \le l$.

Proof. By Lemma 3.4 and Proposition 4.23, if $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$\begin{aligned}\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}v_{\mathfrak{s}}^{(0)} &= Y_p\dots Y_1 v_{\mathfrak{s}}^{(0)} = Y_{l-1,m}\dots Y_{0,m}v_{\mathfrak{s}}^{(0)}\\ &= \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(1)}Y_{l-1,m}\dots Y_{1,m}v_{\mathfrak{s}\langle -m\rangle}^{(m)} = \dots = \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(1)}\dots\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(l)}v_{\mathfrak{s}}^{(p)}.\end{aligned}$$

Therefore, $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}} = \dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(1)}\dots\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(l)}$, since $v_{\mathfrak{s}}^{(p)} = v_{\mathfrak{s}}^{(0)}$. This proves the first claim.

For the second claim, observe that by Lemma 4.9

$$\sigma^m(Y_{t,m}) = \dot{\varepsilon}^{(p-1)dm\mathbf{b}_1^m}Y_{t-1,m} = \dot{\varepsilon}^{-dmn/l}Y_{t-1,m},$$

since $\dot\varepsilon^p = 1$ and $l\mathbf{b}_1^m = \mathbf{b}_1^p = n$. Therefore,

$$\begin{aligned}\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t+1)} v_{\mathfrak{t}^{\boldsymbol\lambda}}^{(tm+m)} &= Y_{t,m} v_{\mathfrak{t}^{\boldsymbol\lambda}\langle m\rangle}^{(tm)} = \sigma^{-m}\big(\sigma^m(Y_{t,m} v_{\mathfrak{t}^{\boldsymbol\lambda}\langle m\rangle}^{(tm)}\big)\\ &= \dot\varepsilon^{-dmn(1+1/l)}\sigma^{-m}\big(Y_{t-1,m} v_{\mathfrak{t}^{\boldsymbol\lambda}\langle m\rangle}^{(tm-m)}\big)\\ &= \dot\varepsilon^{-dmn(1+1/l)}\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)}\sigma^{-m}\big(v_{\mathfrak{t}^{\boldsymbol\lambda}}^{(tm)}\big)\\ &= \dot\varepsilon^{-dmn/l}\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)} v_{\mathfrak{t}^{\boldsymbol\lambda}}^{(tm+m)}\end{aligned}$$

Therefore, $\mathfrak{f}_{\boldsymbol\lambda}^{(t+1)} = \dot\varepsilon^{-dmn/l}\mathfrak{f}_{\boldsymbol\lambda}^{(t)} = \dot\varepsilon^{-tdmn/l}\mathfrak{f}_{\boldsymbol\lambda}^{(1)}$. The second claim now follows.

Finally, by [**13**, page 138, Exercise 4.18 and 4.21], the ring $\mathcal{A}$ is an integrally closed domain. Therefore, since $\dot{\mathfrak{f}}_{\boldsymbol\lambda} \in \mathcal{A}$ and $\big(\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(1)}\big)^l = \dot\varepsilon^{\frac12 dmn(l-1)}\dot{\mathfrak{f}}_{\boldsymbol\lambda} \in \mathcal{A}$ by Proposition 3.4, we deduce that $\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(1)} \in \mathcal{A}$. (Hence, $\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)} \in \mathcal{A}$, for $1 \le t \le m$.) This completes the proof. □

Henceforth, let $\mathfrak{f}_{\boldsymbol\lambda}^{(t)}$ be the value of $\dot{\mathfrak{f}}_{\boldsymbol\lambda}^{(t)}$ at $(\dot\varepsilon, \dot q, \dot{\mathbf{Q}}) = (\varepsilon, q, \mathbf{Q})$, for each integer $1 \le t \le l$.

4.27. **Corollary.** *Suppose that $\mathbf{Q}$ is (ε, q)-separated over R and let $\boldsymbol\lambda \in \mathscr{P}_{d,\mathbf{b}}$ be a multipartition such that $\boldsymbol\lambda = \boldsymbol\lambda\langle m\rangle$, for some $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le m \le p$ with $m \mid p$. Set $l = p/m$. Then*

$$\mathfrak{f}_{\boldsymbol\lambda} = \mathfrak{f}_{\boldsymbol\lambda}^{(1)} \dots \mathfrak{f}_{\boldsymbol\lambda}^{(l)} = \varepsilon^{\frac12 dmn(1-l)}\big(\mathfrak{f}_{\boldsymbol\lambda}^{(1)}\big)^l.$$

Combining Corollary 4.27 with Proposition 3.4 and Theorem 3.6 we have proved Theorem B from the introduction.

4.5. Specht modules for $\mathscr{H}_{r,p,n}$.

We can now construct analogue of the Specht modules for $\mathscr{H}_{r,p,n}$ using the shifting homomorphisms $\theta_{t,m}$. As a consequence we construct and classify the irreducible $\mathscr{H}_{r,p,n}$-modules over a field and show that the decomposition matrix of $\mathscr{H}_{r,p,n}$ is unitriangular.

4.28. **Lemma.** *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and that $\mathbf{b} = \mathbf{b}\langle m\rangle$, for some $1 \le m \le p$ with m dividing p. Let $l = p/m$. Then $\theta'_{0,m} = \varepsilon^{dmnt/l}\sigma^{tm} \circ \theta'_{t,m} \circ \sigma^{-tm}$, for $0 \le t < l$.*

Proof. We first show that $\theta'_{t,m} = \varepsilon^{dmn/l}\sigma^m \circ \theta'_{t+1,m} \circ \sigma^{-m}$ whenever $0 \le t < l$. It is clear that both maps belong to $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}^{(tm)}, V_{\mathbf{b}}^{(tm+m)})$. By Lemma 4.9, $\sigma^m(Y_{t+1,m}) = \varepsilon^{-dmn/l}Y_{t,m}$. Consequently, if $v \in V_{\mathbf{b}}^{(tm)}$ then

$$\big(\sigma^m \circ \theta'_{t+1,m} \circ \sigma^{-m}\big)(v) = \sigma^m\big(Y_{t+1,m}\sigma^{-m}(v)\big) = \varepsilon^{-dmn/l}Y_{t,m}v = \varepsilon^{-dmn/l}\theta'_{t,m}(v)$$

Hence, $\theta'_{t,m} = \varepsilon^{dmn/l}\sigma^m \circ \theta'_{t+1,m} \circ \sigma^{-m}$ as claimed. Therefore, if $0 \le t < l$ then $\theta'_{0,m} = \varepsilon^{dmnt/l}\sigma^{tm} \circ \theta'_{t,m} \circ \sigma^{-tm}$ by induction on t. □

By Lemma 4.18, we have that $\theta_{t,m} = \sigma^m \circ \theta'_{t,m} \in \operatorname{End}_{\mathscr{H}_{r,p/m,n}}\big(V_{\mathbf{b}}^{(mt)}\big)$, for $0 \le t < p/m$. In particular, $\theta_{0,m} \in \operatorname{End}_{\mathscr{H}_{r,p/m,n}}\big(V_{\mathbf{b}}\big)$.

4.29. **Lemma.** *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and that $\mathbf{b} = \mathbf{b}\langle m\rangle$, for some $1 \le m \le p$ with m dividing p. Let $l = p/m$. Then*

$$(\theta_{0,m})^l(v) = \varepsilon^{\frac12 dmn(l-1)}z_{\mathbf{b}} \cdot v,$$

for all $v \in V_{\mathbf{b}}$. That is, $\big(\theta_{0,m}\big)^l = \varepsilon^{\frac12 dmn(l-1)}z_{\mathbf{b}}$ as elements of $\operatorname{End}_{\mathscr{H}_{r,n}}\big(V_{\mathbf{b}}\big)$.

Proof. By Lemma 4.28, $\theta'_{0,m} = \varepsilon^{dmnt/l}\sigma^{tm}\circ\theta'_{t,m}\circ\sigma^{-tm}$ for $1\le t<l$. Therefore,

$$\begin{aligned}\big(\theta_{0,m}\big)^l &= \big(\sigma^m\circ\theta'_{0,m}\big)\circ\big(\sigma^m\circ\theta'_{0,m}\big)\circ\cdots\circ\big(\sigma^m\circ\theta'_{0,m}\big)\\ &= \sigma^m\circ\varepsilon^{dmn(l-1)/l}\sigma^{(l-1)m}\circ\theta'_{l-1,m}\circ\sigma^{(1-l)m}\circ\sigma^m\circ\varepsilon^{dmn(l-2)/l}\sigma^{(l-2)m}\\ &\qquad\circ\theta'_{l-2,m}\circ\qquad\cdots\circ\sigma^m\circ\varepsilon^{dmn/l}\sigma^m\circ\theta'_{1,m}\circ\sigma^{-m}\circ\sigma^m\circ\theta'_{0,m}\\ &= \varepsilon^{\frac12 dmn(l-1)}\theta'_{l,m}\theta'_{l-1,m}\circ\cdots\circ\theta'_{0,m},\end{aligned}$$

since $\sigma^{lm}=\sigma^p$ is the identity map on $\mathscr{H}_{r,n}$. By Lemma 2.21 and the definitions, if $v\in V_{\mathbf{b}}$ then $\big(\theta'_{l-1,m}\circ\theta'_{l-2,m}\circ\cdots\circ\theta'_{0,m}\big)(v) = Y_p\dots Y_1 v = z_{\mathbf{b}}\cdot v$, so the result follows. □

Recall from the introduction that if $\mathbf{a}=(a_1,a_2,\dots,a_p)$ is any sequence then

$$\mathsf{o}_p(\mathbf{a}) = \min\{\,1\le k\le p\mid \mathbf{a}=\mathbf{a}\langle k\rangle\,\}\qquad\text{and}\qquad o^p(\mathbf{a}) = p/\mathsf{o}_p(\mathbf{a}).$$

In particular, if $\mathbf{b}\in\mathscr{C}_{p,n}$ and $\boldsymbol{\lambda}=(\lambda^{[1]},\dots,\lambda^{[p]})\in\mathscr{P}_{d,\mathbf{b}}$ then this defines integers $\mathsf{o}_p(\mathbf{b})$ and $\mathsf{o}_p(\boldsymbol{\lambda})$. By definition, $o_p(\mathbf{b})$ and $\mathsf{o}_p(\boldsymbol{\lambda})$ both divide p, so $o^p(\mathbf{b})$ and $o^p(\boldsymbol{\lambda})$ are both integers. Further, $\mathsf{o}_p(\mathbf{b})$ divides $\mathsf{o}_p(\boldsymbol{\lambda})$.

For convenience, set $\mathsf{o}_{\boldsymbol{\lambda}}=\mathsf{o}_p(\boldsymbol{\lambda})$, $p_{\boldsymbol{\lambda}}=p/\mathsf{o}_{\boldsymbol{\lambda}}$, $\mathsf{o}_{\mathbf{b}}=\mathsf{o}_p(\mathbf{b})$ and $p_{\mathbf{b}}=p/\mathsf{o}_{\mathbf{b}}$.

4.30. **Definition.** *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. Let $\theta_{\boldsymbol{\lambda}}$ be the restriction of $\theta_{0,\mathsf{o}_{\boldsymbol{\lambda}}}$ to $S(\boldsymbol{\lambda})$ and set $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}=\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}^{(1:\mathsf{o}_{\boldsymbol{\lambda}})}$. Let $\mathfrak{g}_{\boldsymbol{\lambda}}$ be the specialization of $\dot{\mathfrak{g}}_{\boldsymbol{\lambda}}$ at $\varepsilon,q,\mathbf{Q}$.*

As in Lemma 4.18, the image of $\theta_{\boldsymbol{\lambda}}$ is contained in $S(\boldsymbol{\lambda})$ so we can consider $\theta_{\boldsymbol{\lambda}}$ to be an $\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}$-module endomorphism of $S(\boldsymbol{\lambda})$.

4.31. **Corollary.** *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$ and $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$. Then*

$$\big(\theta_{\boldsymbol{\lambda}}\big)^{p_{\boldsymbol{\lambda}}} = \mathfrak{g}_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}}}1_{S(\boldsymbol{\lambda})},$$

where $1_{S(\boldsymbol{\lambda})}$ is the identity map on $S(\boldsymbol{\lambda})$.

Proof. Proposition 3.4 and Lemma 4.29 show that $(\theta_{\boldsymbol{\lambda}})^{p_{\boldsymbol{\lambda}}} = \varepsilon^{\frac12 dn\mathsf{o}_{\boldsymbol{\lambda}}(p_{\boldsymbol{\lambda}}-1)}\mathfrak{f}_{\boldsymbol{\lambda}}1_{S(\boldsymbol{\lambda})}$. Now apply Theorem 4.26. □

4.32. **Definition.** *Suppose that $\mathbf{b}\in\mathscr{C}_{p,n}$, $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and $1\le t\le p_{\boldsymbol{\lambda}}$. Define*

$$S_t^{\boldsymbol{\lambda}} = \{\,x\in S(\boldsymbol{\lambda})\mid \theta_{\boldsymbol{\lambda}}(x)=\varepsilon^{t\mathsf{o}_{\boldsymbol{\lambda}}}\mathfrak{g}_{\boldsymbol{\lambda}}x\,\} = \ker\big(\theta_{\boldsymbol{\lambda}}-\varepsilon^{t\mathsf{o}_{\boldsymbol{\lambda}}}\mathfrak{g}_{\boldsymbol{\lambda}}1_{S(\boldsymbol{\lambda})}\big).$$

Set $\pi_t^{\boldsymbol{\lambda}} = \prod_{1\le s\le p_{\boldsymbol{\lambda}},s\ne t}\big(\theta_{\boldsymbol{\lambda}}-\varepsilon^{s\mathsf{o}_{\boldsymbol{\lambda}}}\mathfrak{g}_{\boldsymbol{\lambda}}\big)$, so that $\pi_t^{\boldsymbol{\lambda}}\in\operatorname{End}_{\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}}\big(S(\boldsymbol{\lambda})\big)$.

By definition, $S_t^{\boldsymbol{\lambda}}$ is an $\mathscr{H}_{r,p_{\boldsymbol{\lambda}},n}$-submodule of $S(\boldsymbol{\lambda})$, for $1\le t\le p_{\boldsymbol{\lambda}}$. By restriction, we consider $S_t^{\boldsymbol{\lambda}}$ to be an $\mathscr{H}_{r,p,n}$-module. Recall that τ is the automorphism of $\mathscr{H}_{r,n}$ given by $\tau(h)=T_0^{-1}hT_0$, for $h\in\mathscr{H}_{r,n}$.

4.33. **Theorem.** *Let $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$, and that $1\le t\le p_{\boldsymbol{\lambda}}$. Suppose that $\mathfrak{f}_{\boldsymbol{\lambda}}$ is invertible in R. Then*

a) $S_t^{\boldsymbol{\lambda}}T_0 = S_{t+1}^{\boldsymbol{\lambda}}$. *Equivalently,* $\big(S_{t+1}^{\boldsymbol{\lambda}}\big)^\tau\cong S_t^{\boldsymbol{\lambda}}$.
b) $S_t^{\boldsymbol{\lambda}} = \pi_t^{\boldsymbol{\lambda}}\big(S(\boldsymbol{\lambda})\big)$;
c) $S(\boldsymbol{\lambda})\!\downarrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\cong S_1^{\boldsymbol{\lambda}}\oplus\cdots\oplus S_{p_{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}}$;
d) $\dim S_t^{\boldsymbol{\lambda}} = \frac{1}{p_{\boldsymbol{\lambda}}}\dim S(\boldsymbol{\lambda})$;
e) $S_t^{\boldsymbol{\lambda}}\!\uparrow_{\mathscr{H}_{r,p,n}}^{\mathscr{H}_{r,n}}\cong S(\boldsymbol{\lambda})\oplus S(\boldsymbol{\lambda})^\sigma\oplus\cdots\oplus S(\boldsymbol{\lambda})^{\sigma^{(\mathsf{o}_{\boldsymbol{\lambda}}-1)}}$.

Proof. Suppose that $x\in S_t^{\boldsymbol{\lambda}}$ and let $m=\mathsf{o}_{\boldsymbol{\lambda}}$. By definition,

$$\theta_{\boldsymbol{\lambda}}(xT_0) = \big(\sigma^m\circ\theta'_{0,m}\big)(xT_0) = \sigma^m\big(\theta'_{0,\mathsf{o}_p(\boldsymbol{\lambda})}(x)T_0\big),$$

since $\theta'_{0,m}$ is an $\mathscr{H}_{r,n}$-module homomorphism. Therefore,

$$\theta_{\boldsymbol\lambda}(xT_0) = \theta_{\boldsymbol\lambda}(x)\sigma^m(T_0) = \varepsilon^{(t+1)m}\mathfrak{g}_{\boldsymbol\lambda} xT_0.$$

Hence, $xT_0 \in S^{\boldsymbol\lambda}_{t+1}$, proving the first half of (a). That $S^{\boldsymbol\lambda}_{t+1} \cong (S^{\boldsymbol\lambda}_t)^\tau$ is now immediate because if $x \in S^{\boldsymbol\lambda}_{t+1}$ then $x = x'T_0$ for some $x' \in S^{\boldsymbol\lambda}_t$. Therefore, if $h \in \mathscr{H}_{r,n}$ then $xh = x'T_0h = x'\tau(h)T_0$. Hence, we have proved (a).

By Corollary 4.31, the map $\theta^{p_{\boldsymbol\lambda}}_{\boldsymbol\lambda} - \mathfrak{g}^{p_{\boldsymbol\lambda}}_{\boldsymbol\lambda}$ kills every element of $S(\boldsymbol\lambda)$. Thus, on $S(\boldsymbol\lambda)$ we have

$$0 = \theta^{p_{\boldsymbol\lambda}}_{\boldsymbol\lambda} - \mathfrak{g}^{p_{\boldsymbol\lambda}}_{\boldsymbol\lambda} = \prod_{1\le s\le p_{\boldsymbol\lambda}} \big(\theta_{\boldsymbol\lambda} - \varepsilon^{so_{\boldsymbol\lambda}}\mathfrak{g}_{\boldsymbol\lambda}\big) = \pi^{\boldsymbol\lambda}_t \circ \big(\theta_{\boldsymbol\lambda} - \varepsilon^{to_{\boldsymbol\lambda}}\mathfrak{g}_{\boldsymbol\lambda}\big).$$

Hence, the image of $\pi^{\boldsymbol\lambda}_t$ is contained in $S^{\boldsymbol\lambda}_t$ and $\ker\pi^{\boldsymbol\lambda}_t = \sum_{s\ne t} S^{\boldsymbol\lambda}_s$. Note that the assumption $\mathfrak{f}_{\boldsymbol\lambda}$ is invertible in R implies that $\mathfrak{g}_{\boldsymbol\lambda}$ is also invertible in R. If $x \in S^{\boldsymbol\lambda}_t$ then $\pi^{\boldsymbol\lambda}_t(x) = \alpha_t x$, where $\alpha_t = \mathfrak{g}_{\boldsymbol\lambda}\prod_{s\ne t}(\varepsilon^{to_{\boldsymbol\lambda}} - \varepsilon^{so_{\boldsymbol\lambda}})$ is invertible in R. It follows that if we set $\hat\pi^{\boldsymbol\lambda}_s = \frac{1}{\alpha_s}\pi^{\boldsymbol\lambda}_s$ then

$$1_{S(\boldsymbol\lambda)} = \hat\pi^{\boldsymbol\lambda}_1 + \hat\pi^{\boldsymbol\lambda}_2 + \dots + \hat\pi^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda}},$$

and $\hat\pi^{\boldsymbol\lambda}_t$ is the projection map from $S(\boldsymbol\lambda)$ onto $S^{\boldsymbol\lambda}_t$. Hence, (b) and (c) now follow. Moreover, since $\dim S^{\boldsymbol\lambda}_t = \dim S^{\boldsymbol\lambda}_{t+1}$ by (a), we obtain (d) from (c).

It remains then to prove (e). First observe that by part (a),

$$S^{\boldsymbol\lambda}_t \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong (S^{\boldsymbol\lambda}_{t+1})^\tau \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong \big(S^{\boldsymbol\lambda}_{t+1} \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\big)^\tau \cong S^{\boldsymbol\lambda}_{t+1} \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}.$$

Therefore, $S^{\boldsymbol\lambda}_1 \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong \dots \cong S^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda}} \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$. Hence, using part (c), which we have already proved, and applying Corollary 4.8 we see that

$$\Big(S^{\boldsymbol\lambda}_t \uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\Big)^{\oplus p_{\boldsymbol\lambda}} \cong \big(S^{\boldsymbol\lambda}_1 \oplus\dots\oplus S^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda}}\big)\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong S(\boldsymbol\lambda)\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong \bigoplus_{j=0}^{p-1} S(\boldsymbol\lambda)^{\sigma^j}$$
$$\cong \Big(\bigoplus_{j=0}^{o_{\boldsymbol\lambda}-1} S(\boldsymbol\lambda)^{\sigma^j}\Big)^{\oplus p_{\boldsymbol\lambda}},$$

where the last isomorphism follows because $S(\boldsymbol\lambda)^{\sigma^t} \cong S(\boldsymbol\lambda\langle -t\rangle)$ by Proposition 4.14. Applying the Krull–Schmidt theorem we deduce

$$S^{\boldsymbol\lambda}_t \uparrow^{\mathscr{H}_{r,n}} \cong S(\boldsymbol\lambda) \oplus S(\boldsymbol\lambda)^\sigma \oplus \dots \oplus S(\boldsymbol\lambda)^{\sigma^{(o_{\boldsymbol\lambda}-1)}},$$

proving (e). This completes the proof of Theorem 4.33. □

As in the introduction, let $\sim_\sigma$ be the equivalence relation on $\mathscr{P}_{r,n}$ where $\boldsymbol\mu \sim_\sigma \boldsymbol\lambda$ whenever $\boldsymbol\lambda = \boldsymbol\mu\langle m\rangle$, for some $m \in \mathbb{Z}$. Let $\mathscr{P}^\sigma_{r,n}$ be the set of $\sim_\sigma$-equivalence classes in $\mathscr{P}_{r,n}$. By Proposition 4.14, the set $\mathscr{K}_{r,n}$ of Kleshchev multipartitions is closed under $\sim_\sigma$-equivalence. Let $\mathscr{K}^\sigma_{r,n}$ be the set of $\sim_\sigma$-equivalence classes of Kleshchev multipartitions. We will abuse notation and think of the elements of $\mathscr{P}^\sigma_{r,n}$ as multipartitions so that when we write $\boldsymbol\mu \in \mathscr{P}^\sigma_{r,n}$ we will really mean that $\boldsymbol\mu$ is a representative of an equivalence class in $\mathscr{P}^\sigma_{r,n}$. Similarly, $\boldsymbol\mu \in \mathscr{K}^\sigma_{r,n}$ means that $\boldsymbol\mu$ is a representative for an equivalence class in $\mathscr{K}^\sigma_{r,n}$.

Let $R = K$ be a field. We call the modules $\{\, S^{\boldsymbol\lambda}_i \mid \boldsymbol\lambda \in \mathscr{P}^\sigma_{r,n} \text{ and } 1 \le i \le p_{\boldsymbol\lambda}\,\}$ the **Specht modules** of $\mathscr{H}_{r,p,n}$. Using these modules we can now construct the irreducible $\mathscr{H}_{r,p,n}$-modules.

4.34. **Definition.** *Suppose that $\boldsymbol\lambda \in \mathscr{K}_{r,n}$ and $1 \le t \le p_{\boldsymbol\lambda}$. Define $D^{\boldsymbol\lambda}_t = \mathrm{Head}(S^{\boldsymbol\lambda}_t)$.*

Although this is not clear from the definition, the module $D_i^{\boldsymbol\lambda}$ is irreducible when $\boldsymbol\lambda \in \mathscr{K}_{r,n}$ and, moreover every irreducible $\mathscr{H}_{r,p,n}$-module arises in this way.

This following result establishes of Theorem C from the introduction and, in fact, proves quite a bit more.

4.35. **Theorem.** *Suppose that* $\mathbf{Q}$ *is* (ε, q)*-separated over the field* K*. Let* $\boldsymbol\lambda \in \mathscr{K}_{r,n}$*. Then:*

a) *The module* $D_i^{\boldsymbol\lambda} = \operatorname{Head}(S_i^{\boldsymbol\lambda})$ *is an irreducible* $\mathscr{H}_{r,p,n}$*-module, for* $1 \le i \le p_{\boldsymbol\lambda}$*. Moreover,* $(D_{i+1}^{\boldsymbol\lambda})^\tau \cong D_i^{\boldsymbol\lambda}$*, for* $1 \le i \le p_{\boldsymbol\lambda}$*.*
b) *If* $1 \le i, j \le p_{\boldsymbol\lambda}$ *then* $[S_i^{\boldsymbol\lambda} : D_j^{\boldsymbol\lambda}] = \delta_{ij}$*.*
c) *The integer* $p_{\boldsymbol\lambda}$ *is the smallest positive integer such that* $D_i^{\boldsymbol\lambda} \cong \big(D_i^{\boldsymbol\lambda}\big)^{\tau^{p_{\boldsymbol\lambda}}}$*.*
d) *The integer* $\mathsf{o}_{\boldsymbol\lambda}$ *is the smallest positive integer such that* $D(\boldsymbol\lambda) \cong D(\boldsymbol\lambda)^{\sigma^{\mathsf{o}_{\boldsymbol\lambda}}}$*.*
e) $(D_i^{\boldsymbol\lambda})\!\uparrow^{\mathscr{H}_{r,n}} \cong D(\boldsymbol\lambda) \oplus D(\boldsymbol\lambda)^\sigma \oplus \cdots \oplus D(\boldsymbol\lambda)^{\sigma^{\mathsf{o}_{\boldsymbol\lambda}-1}}$ *and* $D(\boldsymbol\lambda)\!\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong D_i^{\boldsymbol\lambda} \oplus (D_i^{\boldsymbol\lambda})^\tau \oplus \cdots \oplus \big(D_i^{\boldsymbol\lambda}\big)^{\tau^{p_{\boldsymbol\lambda}-1}}$*.*

Furthermore, the Hecke algebra $\mathscr{H}_{r,p,n}$ *is split over* K *and*

$$\{\, D_i^{\boldsymbol\mu} \mid \boldsymbol\mu \in \mathscr{K}_{r,n}^\sigma \text{ and } 1 \le i \le p_{\boldsymbol\mu} \,\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathscr{H}_{r,p,n}$*-modules.*

Proof. By Proposition 4.14, $D(\boldsymbol\lambda)^\sigma \cong D(\boldsymbol\lambda\langle -1\rangle)$, so it is clear that $\mathsf{o}_{\boldsymbol\lambda}$ is the smallest positive integer such that $D(\boldsymbol\lambda) \cong D(\boldsymbol\lambda)^{\sigma^{\mathsf{o}_{\boldsymbol\lambda}}}$. Similarly, once we know that $D_i^{\boldsymbol\lambda} = \operatorname{Head}(S_i^{\boldsymbol\lambda})$ is irreducible then $(D_{i+1}^{\boldsymbol\lambda})^\tau \cong D_i^{\boldsymbol\lambda}$ by Theorem 4.33(a) since twisting by τ induces an exact functor on Mod-$\mathscr{H}_{r,p,n}$.

For the other statements, we first consider the case where $K = \overline{K}$ is algebraically closed so that $\mathscr{H}_{r,p,n}^{\overline{K}}$ splits over $\overline{K}$. The algebra $\mathscr{H}_{r,n}$ is cellular over any ring and so, in particular, it is split over K. For each Kleshchev multipartition $\boldsymbol\mu \in \mathscr{K}_{r,n}^\sigma$ fix an irreducible $\mathscr{H}_{r,p,n}^{\overline{K}}$-submodule $D_{\overline{K}}^{\boldsymbol\mu}$ of $D^{\overline{K}}(\boldsymbol\mu) = D(\boldsymbol\mu) \otimes_K \overline{K}$. By Lemma 4.2, the integer $p_{\boldsymbol\lambda}$ is the smallest positive integer such that $D_{\overline{K}}^{\boldsymbol\lambda} \cong (D_{\overline{K}}^{\boldsymbol\lambda})^{\tau^{p_{\boldsymbol\lambda}}}$ and, further,

$$\begin{aligned} D^{\overline{K}}(\boldsymbol\lambda)\!\downarrow_{\mathscr{H}_{r,p,n}^{\overline{K}}} &\cong D_{\overline{K}}^{\boldsymbol\lambda} \oplus (D_{\overline{K}}^{\boldsymbol\lambda})^\tau \oplus \cdots \oplus (D_{\overline{K}}^{\boldsymbol\lambda})^{\tau^{p_{\boldsymbol\lambda}-1}}; \\ D_{\overline{K}}^{\boldsymbol\lambda}\!\uparrow^{\mathscr{H}_{r,n}^{\overline{K}}} &\cong D^{\overline{K}}(\boldsymbol\lambda) \oplus D^{\overline{K}}(\boldsymbol\lambda)^\sigma \oplus \cdots \oplus D^{\overline{K}}(\boldsymbol\lambda)^{\sigma^{\mathsf{o}_{\boldsymbol\lambda}-1}}. \end{aligned}$$

Moreover, $\{\, \big(D_{\overline{K}}^{\boldsymbol\mu}\big)^{\tau^i} \mid \boldsymbol\mu \in \mathscr{K}_{r,n}^\sigma$ and $1 \le i \le p_{\boldsymbol\mu} \,\}$ is a complete set of pairwise non-isomorphic simple $\mathscr{H}_{r,p,n}^{\overline{K}}$-modules.

Suppose that $\boldsymbol\mu \in \mathscr{P}_{r,n}$ and let $S_{\overline{K},i}^{\boldsymbol\mu} = S_i^{\boldsymbol\mu} \otimes_K \overline{K}$, for $1 \le j \le p_{\boldsymbol\mu}$. We claim that $D_{\overline{K}}^{\boldsymbol\lambda} \cong \operatorname{Head}(S_{\overline{K},i}^{\boldsymbol\mu})$, for some i, if and only if $\boldsymbol\lambda \sim_\sigma \boldsymbol\mu$ and in this case i is uniquely determined. Using the restriction formula for $D^{\overline{K}}(\boldsymbol\lambda)$ given above, Frobenius reciprocity [**7**, Proposition 11.13(ii)] and Theorem 4.33 we find that

$$\begin{aligned} \bigoplus_{i=0}^{p_{\boldsymbol\mu}-1} \operatorname{Hom}_{\mathscr{H}_{r,p,n}^{\overline{K}}}\big(S_{\overline{K},i}^{\boldsymbol\mu}, D_{\overline{K}}^{\boldsymbol\lambda}\big) &\cong \operatorname{Hom}_{\mathscr{H}_{r,p,n}^{\overline{K}}}\big(S^{\overline{K}}(\boldsymbol\mu)\!\downarrow_{\mathscr{H}_{r,p,n}^{\overline{K}}}, D_{\overline{K}}^{\boldsymbol\lambda}\big) \\ &\cong \operatorname{Hom}_{\mathscr{H}_{r,n}^{\overline{K}}}\big(S^{\overline{K}}(\boldsymbol\mu), D_{\overline{K}}^{\boldsymbol\lambda}\!\uparrow^{\mathscr{H}_{r,n}^{\overline{K}}}\big) \\ &\cong \bigoplus_{j=0}^{\mathsf{o}_{\boldsymbol\lambda}-1} \operatorname{Hom}_{\mathscr{H}_{r,n}^{\overline{K}}}\big(S^{\overline{K}}(\boldsymbol\mu), D^{\overline{K}}(\boldsymbol\lambda)^{\sigma^j}\big) \\ &\cong \begin{cases} \overline{K}, & \text{if } \boldsymbol\mu \sim_\sigma \boldsymbol\lambda, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where the last line follows because $D^{\overline{K}}(\boldsymbol{\mu}) = \mathrm{Head}(S^{\overline{K}}(\boldsymbol{\mu}))$, by Lemma 3.2, and because $D^{\overline{K}}(\boldsymbol{\lambda})^{\sigma^j} \cong D^{\overline{K}}(\boldsymbol{\lambda}\langle -j\rangle)$ by Proposition 4.14. This proves our claim. Without loss of generality, we can take $\boldsymbol{\mu} = \boldsymbol{\lambda}$. Note that $\mathrm{Head}\, S^{\overline{K}}(\boldsymbol{\lambda}) = D^{\overline{K}}(\boldsymbol{\lambda})$ is simple. The above isomorphisms imply that $\mathrm{Head}(S^{\boldsymbol{\lambda}}_{\overline{K},i}) = D^{\boldsymbol{\lambda}}_{\overline{K},i} = D^{\boldsymbol{\lambda}}_{\overline{K}}$ is also simple. By Lemma 3.2, $[S^{\overline{K}}(\boldsymbol{\lambda}) : D^{\overline{K}}(\boldsymbol{\lambda})] = 1$ and $D^{\overline{K}}(\boldsymbol{\lambda})$ is the simple head of $S^{\overline{K}}(\boldsymbol{\lambda})$. By considering the restriction of the composition series of $S^{\overline{K}}(\boldsymbol{\lambda})$ to $\mathscr{H}_{r,p,n}$, it is easy to see that $[S^{\boldsymbol{\lambda}}_{\overline{K},i} : D^{\boldsymbol{\lambda}}_{\overline{K},j}] = \delta_{ij}$. This proves all the statements in the Theorem when $K = \overline{K}$.

We now return to the general case where K is an arbitrary field. By the last paragraph, $S^{\boldsymbol{\lambda}}_{\overline{K},i} \cong S^{\boldsymbol{\lambda}}_i \otimes_K \overline{K}$ has a simple head, so that $D^{\boldsymbol{\lambda}}_i = \mathrm{Head}(S^{\boldsymbol{\lambda}}_i)$ is indecomposable. Therefore, $D^{\boldsymbol{\lambda}}_i$ is irreducible (since it is also semisimple).

To complete the proof of the Theorem we show that $D^{\boldsymbol{\lambda}}_i \otimes_K \overline{K} \cong D^{\boldsymbol{\lambda}}_{\overline{K},i}$. Let $l \geq 1$ be the minimal positive integer such that $(D^{\boldsymbol{\lambda}}_i)^{\tau^l} \cong D^{\boldsymbol{\lambda}}_i$. Then $l \geq p_{\boldsymbol{\lambda}}$ since $D^{\boldsymbol{\lambda}}_{\overline{K},i} \cong \mathrm{Head}(D^{\boldsymbol{\lambda}}_i \otimes_K \overline{K})$. Similarly, $\dim_K D^{\boldsymbol{\lambda}}_i \geq \dim_{\overline{K}} D^{\boldsymbol{\lambda}}_{\overline{K},i}$. By [**7**, Proposition (11.16)], there exists an integer $c \geq 1$ such that

$$D(\boldsymbol{\lambda})\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong \Big(D^{\boldsymbol{\lambda}}_i \oplus (D^{\boldsymbol{\lambda}}_i)^{\tau} \oplus \cdots \oplus \big(D^{\boldsymbol{\lambda}}_i\big)^{\tau^{l-1}}\Big)^{\oplus c}.$$

Taking dimensions, $\dim_K D(\boldsymbol{\lambda}) = cl \dim_K D^{\boldsymbol{\lambda}}_i$. Hence, comparing dimensions on both sides of the restriction formula for $D^{\overline{K}}(\boldsymbol{\lambda})$ above shows that

$$cl \dim D^{\boldsymbol{\lambda}}_i = \dim_K D(\boldsymbol{\lambda}) = \dim_{\overline{K}} D^{\overline{K}}(\boldsymbol{\lambda}) = p_{\boldsymbol{\lambda}} \dim_{\overline{K}} D^{\boldsymbol{\lambda}}_{\overline{K}} \leq p_{\boldsymbol{\lambda}} \dim_K D^{\boldsymbol{\lambda}}_i.$$

Since $l \geq p_{\boldsymbol{\lambda}}$ this forces $c = 1$, $l = p_{\boldsymbol{\lambda}}$ and $\dim_K D^{\boldsymbol{\lambda}}_i = \dim_{\overline{K}} D^{\boldsymbol{\lambda}}_{\overline{K},i}$. Therefore, $D^{\boldsymbol{\lambda}}_{\overline{K},i} \cong D^{\boldsymbol{\lambda}}_i \otimes_K \overline{K}$, implying that $D^{\boldsymbol{\lambda}}_i$ is absolutely irreducible and hence that K is a splitting field for $\mathscr{H}_{r,p,n}$. All of the parts in the theorem now follow from the corresponding statements for $D^{\boldsymbol{\lambda}}_{\overline{K},i}$ using the isomorphism $D^{\boldsymbol{\lambda}}_{\overline{K},i} \cong D^{\boldsymbol{\lambda}}_i \otimes_K \overline{K}$. $\square$

The algebra $\mathscr{H}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ is not necessarily semisimple when $d > 1$. With a little more work it is possible to show that if $\mathbf{Q}$ is (ε, q)-separated over K then the following are equivalent:

a) $\mathscr{H}_{r,n}$ is (split) semisimple.
b) $\mathscr{H}_{r,p,n}$ is (split) semisimple.
c) $S^{\boldsymbol{\lambda}}_t = D^{\boldsymbol{\lambda}}_t$, for all $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ and $1 \leq t \leq p_{\boldsymbol{\lambda}}$.

We omit the details. If $d = 1$ then it is known that $\mathscr{H}_{p,p,n}$ is semisimple if and only if $\langle\varepsilon\rangle \cap \langle q\rangle = \{1\}$ and $e > n$ [**20**, Theorem 5.9].

Extend the dominance order to $\mathscr{P}^{\sigma}_{r,n} \times \mathbb{Z}$ by defining $(\boldsymbol{\lambda}, j) \rhd (\boldsymbol{\mu}, i)$ if $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$. Let $\mathbf{D}_{\mathscr{H}_{r,p,n}} = \big([S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j]\big)_{(\boldsymbol{\lambda},i),(\boldsymbol{\mu},j)}$ be the **decomposition matrix** of $\mathscr{H}_{r,p,n}$, where $\boldsymbol{\lambda} \in \mathscr{P}^{\sigma}_{r,n}$, $\boldsymbol{\mu} \in \mathscr{K}^{\sigma}_{r,n}$, $1 \leq i \leq p_{\boldsymbol{\lambda}}$ and $1 \leq j \leq p_{\boldsymbol{\mu}}$, and where the rows and columns of $\mathbf{D}_{\mathscr{H}_{r,p,n}}$ are ordered in a way that is compatible with dominance.

Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$, $\boldsymbol{\mu} \in \mathscr{K}^{\sigma}_{r,n}$ and $1 \leq i \leq p_{\boldsymbol{\lambda}}$ and $1 \leq i \leq p_{\boldsymbol{\mu}}$. If $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ then $[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j] \neq 0$ only if $(\boldsymbol{\lambda}, i) \rhd (\boldsymbol{\mu}, j)$ because, by Theorem 4.35 and Lemma 3.2,

$$[S^{\boldsymbol{\lambda}}_i : D^{\boldsymbol{\mu}}_j] \neq 0 \quad\Longrightarrow\quad [S(\boldsymbol{\lambda}) : D(\boldsymbol{\mu})] \neq 0 \quad\Longrightarrow\quad \boldsymbol{\lambda} \rhd \boldsymbol{\mu}.$$

On the other hand, $[S^{\boldsymbol{\mu}}_i : D^{\boldsymbol{\mu}}_j] = \delta_{ij}$ by Theorem 4.35. Hence, we have proved the following.

4.36. **Corollary.** *Suppose that $\mathbf{Q}$ is (ε, q)-separated over the field K. Then the decomposition matrix $\mathbf{D}_{\mathscr{H}_{r,p,n}}$ of $\mathscr{H}_{r,p,n}$ is unitriangular.*

4.6. **Morita equivalences of $\mathscr{H}_{r,p,n}$-modules.** In this section we prove an Morita equivalence theorem for the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}$ which is an analogue of the Morita equivalence theorem $\mathscr{H}_{r,n}$ which was discussed in section 2.5. Our main result is a generalization of the Morita equivalence theorem given by the first author for the Hecke algebras of type D [**19**].

We maintain our assumption that $\mathbf{Q}$ is (ε, q)-separated. Most of the results in this section hold over an arbitrary ring, however, for convenience we work over the field K throughout.

Fix a composition $\mathbf{b} \in \mathscr{C}_{p,n}$ and set $\mathsf{o}_\mathbf{b} = \mathsf{o}_p(\mathbf{b})$ and $p_\mathbf{b} = p/\mathsf{o}_\mathbf{b}$. Mirroring Definition 4.30 define

$$\theta_\mathbf{b} = \theta_{0,\mathsf{o}_p(\mathbf{b})}.$$

Then $\theta_\mathbf{b} \in \operatorname{End}_{\mathscr{H}_{r,p_\mathbf{b},n}}(V_\mathbf{b})$ by Lemma 4.18 and $\theta_\mathbf{b}(v) = \sigma^{\mathsf{o}_\mathbf{b}}(Y_{0,\mathsf{o}_\mathbf{b}}v)$, for all $v \in V_\mathbf{b}$. In particular, $\theta_\mathbf{b}$ is an $\mathscr{H}_{r,p,n}$-endomorphism of $V_\mathbf{b}$.

The module $V_\mathbf{b} = v_\mathbf{b}\mathscr{H}_{r,n}$ is an $\mathscr{H}_{r,p,n}$-module by restriction. For simplicity we will usually write $V_\mathbf{b}$, instead of $V_\mathbf{b}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}$, when we consider $V_\mathbf{b}$ as an $\mathscr{H}_{r,p,n}$-module.

4.37. **Definition.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$. *Define* $\mathscr{E}_{d,\mathbf{b}} = \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(V_\mathbf{b}\big)$.

Notice that $\mathscr{H}_{d,\mathbf{b}}$ is a subalgebra of $\mathscr{E}_{d,\mathbf{b}}$, by Lemma 2.18, and that $\theta_\mathbf{b}$ is an element of $\mathscr{E}_{d,\mathbf{b}}$ by the remarks above.

4.38. **Theorem.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$. *Then, as an algebra,* $\mathscr{E}_{d,\mathbf{b}}$ *is generated by* $\mathscr{H}_{d,\mathbf{b}}$ *and the endomorphism* $\theta_\mathbf{b}$. *Moreover, if* $\{\, x_i \mid i \in I \,\}$ *is a* K*-basis of* $\mathscr{H}_{d,\mathbf{b}}$ *then* $\{\, x_i\theta_\mathbf{b}^k \mid i \in I$ *and* $0 \le k < p_\mathbf{b} \,\}$ *in a* K*-basis of* $\operatorname{End}_{\mathscr{H}_{r,p,n}}(V_\mathbf{b})$. *In particular,* $\dim \mathscr{E}_{d,\mathbf{b}} = p_\mathbf{b} \dim \mathscr{H}_{d,\mathbf{b}}$.

Proof. We first compute the dimension of $\mathscr{E}_{d,\mathbf{b}}$. By Frobenius reciprocity [**7**, Proposition 11.13(ii)],

$$\begin{aligned}\mathscr{E}_{d,\mathbf{b}} &= \operatorname{Hom}_{\mathscr{H}_{r,p,n}}(V_\mathbf{b}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}, V_\mathbf{b}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}) \cong \operatorname{Hom}_{\mathscr{H}_{r,n}}(V_\mathbf{b}, V_\mathbf{b}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\uparrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}})\\ &\cong \bigoplus_{i=0}^{p-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_\mathbf{b}, V_\mathbf{b}^{\sigma^i}) \cong \bigoplus_{i=0}^{p-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_\mathbf{b}, V_{\mathbf{b}\langle i\rangle}),\end{aligned}$$

where the third isomorphism is Corollary 4.8 and the fourth isomorphism follows because $V_\mathbf{b}^{\sigma^i} \cong V_{\mathbf{b}\langle -i\rangle}$ by Proposition 4.10. By [**23**, Proposition 2.13] if $\mathbf{b} \ne \mathbf{c}$ then $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_\mathbf{b}, V_\mathbf{c}) = 0$ because $V_\mathbf{b}$ and $V_\mathbf{c}$ belong to different blocks. Therefore, as vector spaces,

$$\mathscr{E}_{d,\mathbf{b}} \cong \bigoplus_{i=0}^{p_\mathbf{b}-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}\big(V_\mathbf{b}, V_{\mathbf{b}\langle i\mathsf{o}_\mathbf{b}\rangle}\big) \cong \bigoplus_{i=0}^{p_\mathbf{b}-1}\operatorname{Hom}_{\mathscr{H}_{r,n}}\big(V_\mathbf{b}, V_\mathbf{b}\big) \cong \mathscr{H}_{d,\mathbf{b}}^{\oplus p_\mathbf{b}}$$

since $\operatorname{End}_{\mathscr{H}_{r,n}}(V_\mathbf{b}) \cong \mathscr{H}_{d,\mathbf{b}}$ by Lemma 2.18. Hence, $\dim \mathscr{E}_{d,\mathbf{b}} = p_\mathbf{b} \dim \mathscr{H}_{d,\mathbf{b}}$ as we wanted to show.

It remains to show that $\mathscr{H}_{d,\mathbf{b}}$ and $\theta_\mathbf{b}$ generate $\mathscr{E}_{d,\mathbf{b}}$ as a K-algebra. First observe that $\theta_\mathbf{b}$ is an invertible element of $\mathscr{E}_{d,\mathbf{b}}$ because $(\theta_\mathbf{b})^{p_\mathbf{b}} = \varepsilon^{dn(p-\mathsf{o}_\mathbf{b})/2}z_\mathbf{b}$ by Lemma 4.29. Therefore, since $\operatorname{End}_{\mathscr{H}_{r,n}}(V_\mathbf{b}) \cong \mathscr{H}_{d,\mathbf{b}}$ by Lemma 2.18, it suffices to show that every element of $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_\mathbf{b}, V_\mathbf{b}^{\sigma^{i\mathsf{o}_\mathbf{b}}})$ corresponds to $\theta_\mathbf{b}^{-i}x$, for some x in $\mathscr{H}_{d,\mathbf{b}}$. Let π_j be the projection from $\mathscr{E}_{d,\mathbf{b}}$ to $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_\mathbf{b}, V_\mathbf{b}^{\sigma^{j\mathsf{o}_\mathbf{b}}})$ under the vector space isomorphism above. Under Frobenius reciprocity [**7**, Proposition 11.13(ii)], the $\mathscr{H}_{r,p,n}$-endomorphism

$$\theta_\mathbf{b}^{-i} \in \operatorname{End}_{\mathscr{H}_{r,p,n}}\big(V_\mathbf{b}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}}\big)$$

corresponds to the $\mathscr{H}_{r,n}$-homomorphism $V_{\mathbf{b}} \longrightarrow V_{\mathbf{b}} \otimes_{\mathscr{H}_{r,p,n}} \mathscr{H}_{r,n}$ given by

$$v_{\mathbf{b}}h \mapsto \sum_{s=0}^{p-1} \theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}hT_0^{-s}) \otimes T_0^s,$$

for $h \in \mathscr{H}_{r,n}$. Using Proposition 4.2, and the explicit isomorphism given in Lemma 4.1,

$$\begin{aligned}\pi_j(\theta_{\mathbf{b}}^{-i})(v_{\mathbf{b}}) &= \sum_{s=0}^{p-1} \varepsilon^{jo_{\mathbf{b}}s}\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}T_0^{-s})T_0^s = \sum_{s=0}^{p-1} \varepsilon^{jo_{\mathbf{b}}s}\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}})\varepsilon^{-iso_{\mathbf{b}}}T_0^{-s}T_0^s \\ &= \sum_{s=0}^{p-1} \varepsilon^{(j-i)so_{\mathbf{b}}}\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}) = \delta_{ij}p\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}}).\end{aligned}$$

By assumption p does not divide the characteristic of K, so p is invertible in K. So we deduce that $\pi_i(\theta_{\mathbf{b}}^{-i})$ is actually an isomorphism from $V_{\mathbf{b}}$ onto $V_{\mathbf{b}}^{\sigma^{io_{\mathbf{b}}}}$. Essentially the same argument shows that if $x \in \mathscr{H}_{d,\mathbf{b}}$ then

$$\pi_j(x)(v_{\mathbf{b}}) = \delta_{j0}px \cdot v_{\mathbf{b}} = \delta_{j0}pv_{\mathbf{b}}\Theta_{\mathbf{b}}(x).$$

Therefore, $\pi_j(\theta_{\mathbf{b}}^{-i}x)(v_{\mathbf{b}}) = \delta_{ij}\delta_{j0}p^2\theta_{\mathbf{b}}^{-i}(v_{\mathbf{b}})\Theta_{\mathbf{b}}(x)$. Note that every homomorphism in $\operatorname{Hom}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}, V_{\mathbf{b}}^{\sigma^{io_{\mathbf{b}}}})$ can be decomposed into a composition of the isomorphism $\pi_i(\theta_{\mathbf{b}}^{-i})$ with an endomorphism in $\operatorname{End}_{\mathscr{H}_{r,n}}(V_{\mathbf{b}}) \cong \mathscr{H}_{d,\mathbf{b}}$. All of the claims in the theorem now follow. □

The algebra $\mathscr{E}_{d,\mathbf{b}}$ is generated by $\mathscr{H}_{d,\mathbf{b}}$ and $\theta_{\mathbf{b}}$ by Theorem 4.38. To make this more explicit, for $s = 1, 2, \dots, p$ let $T_i^{(s)}$ and $L_j^{(s)}$, for $1 \le i < b_s$ and $1 \le j \le b_s$, be the generators of $\mathscr{H}_{d,\mathbf{b}}$. That is,

$$T_i^{(s)} = 1^{\otimes s-1} \otimes T_i \otimes 1^{\otimes p-s} \quad \text{and} \quad L_j^{(s)} = 1^{\otimes s-1} \otimes L_j \otimes 1^{\otimes p-s},$$

interpreted as elements of $\mathscr{H}_{d,\mathbf{b}} = \mathscr{H}_{d,b_1}(\varepsilon\mathbf{Q}) \otimes \cdots \otimes \mathscr{H}_{d,b_p}(\varepsilon^p\mathbf{Q})$. The elements $T_i^{(s)}$ and $L_j^{(s)}$, for $1 \le s \le p$, $1 \le i < b_s$ and $1 \le j \le b_s$, generate $\mathscr{H}_{d,\mathbf{b}}$ subject to the relations implied by the defining relations for $\mathscr{H}_{r,n}$.

To determine relations these elements satisfy in $\mathscr{E}_{d,\mathbf{b}}$ we need to determine the commutation relations for these elements and $\theta_{\mathbf{b}}$. Using Lemma 2.16, it is easy to deduce the following result.

4.39. **Lemma.** *Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$, $1 \le s \le p$, $1 \le i < b_s$ and $1 \le j \le b_s$. Then*

$$T_i^{(s)}\theta_{\mathbf{b}} = \begin{cases}\theta_{\mathbf{b}}T_i^{(s+o_{\mathbf{b}})}, & \text{if } s+o_{\mathbf{b}} \le p,\\ \theta_{\mathbf{b}}T_i^{(s+o_{\mathbf{b}}-p)}, & \text{if } s+o_{\mathbf{b}} > p,\end{cases}$$

$$L_j^{(s)}\theta_{\mathbf{b}} = \begin{cases}\varepsilon^{-o_{\mathbf{b}}}\theta_{\mathbf{b}}L_j^{(s+o_{\mathbf{b}})}, & \text{if } s+o_{\mathbf{b}} \le p,\\ \varepsilon^{-o_{\mathbf{b}}}\theta_{\mathbf{b}}L_j^{(s+o_{\mathbf{b}}-p)}, & \text{if } s+o_{\mathbf{b}} > p.\end{cases}$$

This lemma, when combined with the relation that $\theta_{\mathbf{b}}^{o_{\mathbf{b}}} = \mathfrak{f}_{\boldsymbol{\lambda}}z_{\mathbf{b}}$ is central in $\mathscr{E}_{d,\mathbf{b}}$ and the relations coming from $\mathscr{H}_{d,\mathbf{b}}$ gives a complete set of commutator relations for the generators of $\mathscr{E}_{d,\mathbf{b}}$. It would be interesting to know whether or not this gives a presentation for the algebra $\mathscr{E}_{d,\mathbf{b}}$.

4.40. *Remark.* Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le s, t \le p$ and $s \equiv t \pmod{o_{\mathbf{b}}}$, so that $b_s = b_t$. Let π_{st} be the algebra isomorphism $\mathscr{H}_{d,b_s}^{(s)} \cong \mathscr{H}_{d,b_t}^{(t)}$ given by

$$T_i^{(s)} \mapsto T_i^{(t)} \quad \text{and} \quad T_0^{(s)} = L_1^{(s)} \mapsto \varepsilon^{s-t}T_0^{(t)}, \qquad \text{for } 1 \le i \le n-1.$$

Thus, π_{st} identifies the s^{th} tensor factor and the t^{th} tensor factor in $\mathscr{H}_{d,\mathbf{b}}$ and Lemma 4.39 says that conjugation by $\theta_{\mathbf{b}}$ coincides with the map π_{st}, where $t = s+o_{\mathbf{b}}$ if $s + o_{\mathbf{b}} \le p$; or $t = s + o_{\mathbf{b}} - p$ if $s + o_{\mathbf{b}} > p$.

Extend the equivalence relation $\sim_\sigma$ on $\mathscr{P}_{r,n}$ to $\mathscr{C}_{p,n}$ by defining $\mathbf{b} \sim_\sigma \mathbf{c}$ if $\mathbf{b} = \mathbf{c}\langle k\rangle$ for some $k \in \mathbb{Z}$, for $\mathbf{b}, \mathbf{c} \in \mathscr{C}_{p,n}$. Let $\mathscr{C}^\sigma_{p,n} = \mathscr{C}_{p,n}/{\sim_\sigma}$ be the set of $\sim_\sigma$-equivalence classes in $\mathscr{C}_{p,n}$. Once again, we write $\mathbf{b} \in \mathscr{C}^\sigma_{p,n}$ to indicate that $\mathbf{b}$ is a representative for an equivalence class in $\mathscr{C}^\sigma_{p,n}$.

Define $\mathscr{E} = \bigoplus_{\mathbf{b}\in\mathscr{C}^\sigma_{p,n}} \mathscr{E}_{d,\mathbf{b}}$. Note that $\mathscr{E}$ depends on the parameters q and $\mathbf{Q}^{\vee\boldsymbol{\varepsilon}}$ and on n. Further, by definition, $\text{Mod-}\mathscr{E} = \bigoplus_{\mathbf{b}\in\mathscr{C}^\sigma_{p,n}} \text{Mod-}\mathscr{E}_{d,\mathbf{b}}$.

4.41. **Corollary.** *Suppose that Suppose that $\mathbf{Q}$ is (ε, q)-separated over K. Then there is a Morita equivalence*

$$\mathtt{F}_\mathscr{E} : \text{Mod-}\mathscr{E} \longrightarrow \text{Mod-}\mathscr{H}_{r,p,n}; M \mapsto M \otimes_{\mathscr{E}_{d,\mathbf{b}}} V_\mathbf{b},$$

for $M \in \text{Mod-}\mathscr{E}_{d,\mathbf{b}}$, and $\mathbf{b} \in \mathscr{C}^\sigma_{p,n}$.

Proof. By [**23**, Proposition 2.15], $\bigoplus_{\mathbf{b}\in\mathscr{C}_{p,n}} V_\mathbf{b}$ is a progenerator for $\mathscr{H}_{r,n}$. Moreover, if $\mathbf{b} \in \mathscr{C}_{p,n}$ then

$$V_\mathbf{b}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong V_\mathbf{b}^{\sigma^t}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}} \cong V_{\mathbf{b}\langle -t\rangle}\downarrow^{\mathscr{H}_{r,n}}_{\mathscr{H}_{r,p,n}},$$

for any $t \in \mathbb{Z}$ by Lemma 4.10. Therefore, $\bigoplus_{\mathbf{b}\in\mathscr{C}^\sigma_{p,n}} V_\mathbf{b}$ is a progenerator for $\mathscr{H}_{r,p,n}$ and, by well-known arguments [**4**, §2.2], it induces the Morita equivalence $\mathtt{F}_\mathscr{E}$ as described above. □

We now describe the images of the Specht modules and simple modules of the algebra $\mathscr{H}_{r,p,n}$ under this Morita equivalence.

Let $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. By definition $\mathtt{o}_\mathbf{b} \mid \mathtt{o}_{\boldsymbol{\lambda}}$ and $\mathtt{o}_{\boldsymbol{\lambda}} \mid p$. Let $p_{\mathbf{b}/\boldsymbol{\lambda}} := p_\mathbf{b}/p_{\boldsymbol{\lambda}} = \mathtt{o}_{\boldsymbol{\lambda}}/\mathtt{o}_\mathbf{b} \in \mathbb{N}$. Then $p_\mathbf{b} = p_{\mathbf{b}/\boldsymbol{\lambda}} p_{\boldsymbol{\lambda}}$.

4.42. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}^\sigma_{p,n}$. Define*

$$S^{\boldsymbol{\lambda}} = S_\mathbf{b}(\boldsymbol{\lambda})\uparrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{H}_{d,\mathbf{b}}} \quad \textit{and} \quad D^{\boldsymbol{\lambda}} = D_\mathbf{b}(\boldsymbol{\lambda})\uparrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{H}_{d,\mathbf{b}}}.$$

Define $\mathscr{E}_{d,\boldsymbol{\lambda}}$ to be the subalgebra of $\mathscr{E}_{d,\mathbf{b}}$ generated by $\mathscr{H}_{d,\mathbf{b}}$ and $(\theta_\mathbf{b})^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$.

By definition $\mathscr{E}_{d,\boldsymbol{\lambda}} \cong \mathscr{E}_{d,\boldsymbol{\mu}}$ whenever $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ and $p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}}$. Further, $\dim \mathscr{E}_{d,\boldsymbol{\lambda}} = p_{\boldsymbol{\lambda}} \dim \mathscr{H}_{d,\mathbf{b}}$ by Theorem 4.38. Notice that the maps $(\theta_\mathbf{b})^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$ and $\theta_{\boldsymbol{\lambda}}$ agree when they are restricted to $S(\boldsymbol{\lambda})$.

Now fix generators $s_\mathbf{b}(\boldsymbol{\lambda})$ and $d_\mathbf{b}(\boldsymbol{\lambda})$ of $S_\mathbf{b}(\boldsymbol{\lambda})$ and $D_\mathbf{b}(\boldsymbol{\lambda})$, respectively, which we consider as elements of $\mathscr{E}_{d,\mathbf{b}}$. Motivated by Definition 4.32 and Theorem 4.33 define

$$S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} = s_\mathbf{b}(\boldsymbol{\lambda}) \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}} \big((\theta_\mathbf{b})^{p_{\mathbf{b}/\boldsymbol{\lambda}}} - \mathfrak{g}_{\boldsymbol{\lambda}}\big)\mathscr{H}_{d,\mathbf{b}} \hookrightarrow \mathscr{E}_{d,\boldsymbol{\lambda}}$$

$$D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}} = d_\mathbf{b}(\boldsymbol{\lambda}) \prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}} \big((\theta_\mathbf{b})^{p_{\mathbf{b}/\boldsymbol{\lambda}}} - \mathfrak{g}_{\boldsymbol{\lambda}}\big)\mathscr{H}_{d,\mathbf{b}} \hookrightarrow \mathscr{E}_{d,\boldsymbol{\lambda}}.$$

By Lemma 4.39, $S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ and $D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ are $\mathscr{E}_{d,\boldsymbol{\lambda}}$-submodules of $S^{\boldsymbol{\lambda}}$ and $D^{\boldsymbol{\lambda}}$, respectively. Moreover, it is easy to see that

$$S_\mathbf{b}(\boldsymbol{\lambda})\uparrow^{\mathscr{E}_{d,\boldsymbol{\lambda}}}_{\mathscr{H}_{d,\mathbf{b}}} \cong \oplus_{i=1}^{p_{\boldsymbol{\lambda}}} S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}, \;\; D_\mathbf{b}(\boldsymbol{\lambda})\uparrow^{\mathscr{E}_{d,\boldsymbol{\lambda}}}_{\mathscr{H}_{d,\mathbf{b}}} \cong \oplus_{i=1}^{p_{\boldsymbol{\lambda}}} D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}.$$

Now define

$$S^{\boldsymbol{\lambda}}_{i,p} = S^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\uparrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{E}_{d,\boldsymbol{\lambda}}} \quad \text{and} \quad D^{\boldsymbol{\lambda}}_{i,p} = D^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\uparrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{E}_{d,\boldsymbol{\lambda}}}.$$

Let $\sim_\mathbf{b}$ be the equivalence relation on $\mathscr{P}_{d,\mathbf{b}}$ where if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ then $\boldsymbol{\mu} \sim_\mathbf{b} \boldsymbol{\lambda}$ if $\boldsymbol{\lambda} = \boldsymbol{\mu}\langle k o_\mathbf{b}\rangle$, for some $k \in \mathbb{Z}$. Let $\mathscr{P}^\mathbf{b}_{d,\mathbf{b}}$ be the set of $\sim_\mathbf{b}$-equivalence classes in $\mathscr{P}_{d,\mathbf{b}}$ and let $\mathscr{K}^\mathbf{b}_{d,\mathbf{b}}$ be the equivalence classes in $\mathscr{K}_{d,\mathbf{b}}$. Once again, we blur the distinction between equivalence classes in $\mathscr{P}^\mathbf{b}_{d,\mathbf{b}}$ and the multipartitions in these equivalence classes.

4.43. **Lemma.** *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, $\boldsymbol{\mu} \in \mathscr{K}_{d,\mathbf{b}}^{\mathbf{b}}$, $1 \le i \le p_{\boldsymbol{\lambda}}$ *and* $1 \le j \le p_{\boldsymbol{\mu}}$. *Then* $\mathtt{F}_{\mathscr{E}} S_{i,p}^{\boldsymbol{\lambda}} \cong S_i^{\boldsymbol{\lambda}}$ *and* $\mathtt{F}_{\mathscr{E}} D_{j,p}^{\boldsymbol{\mu}} \cong D_j^{\boldsymbol{\mu}}$. *In particular,*

$$\{\, D_{j,p}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{K}_{d,\mathbf{b}}^{\mathbf{b}} \text{ and } 1 \le j \le p_{\boldsymbol{\mu}} \,\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathscr{E}_{d,\mathbf{b}}$*-modules.*

Proof. This follows directly from the definitions and standard properties of the Schur functor $\mathtt{F}_{\mathscr{E}}$. □

5. Cyclotomic Schur algebras and decomposition numbers

In this section we use the results so far to define analogues of the cyclotomic Schur algebras for $\mathscr{H}_{r,p,n}$. We then use the formal characters of these algebras to compute the p-splittable decomposition numbers of $\mathscr{H}_{r,p,n}$, extending the arguments of [**22**], and hence proving our main results from the introduction.

Many of the early results in this section apply over an integral domain, however, for convenience we work over a field $R = K$. We maintain our assumption that $\mathbf{Q}$ is (ε, q)-separated over K.

5.1. **Lifting to cyclotomic q-Schur algebras.** For each $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ we defined modules $M(\boldsymbol{\lambda})$, $M_{\mathbf{b}}(\boldsymbol{\lambda}) = M(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes M(\boldsymbol{\lambda}^{[p]})$ and

$$M_{\mathbf{b}}^{\boldsymbol{\lambda}} = \mathtt{H}_{\mathbf{b}}(M_{\mathbf{b}}(\boldsymbol{\lambda})) \cong v_{\mathbf{b}}^{+} M(\boldsymbol{\lambda})$$

in or after Definition 4.13. Using these modules we introduce analogues of the *Schur algebras* for the algebras $\mathscr{H}_{r,n}$, $\mathscr{H}_{d,\mathbf{b}}$ and $\mathscr{H}_{r,p,n}$.

5.1. **Definition.** a) *The* ***cyclotomic q-Schur algebra*** *of* $\mathscr{H}_{r,n}$ *is the endomorphism algebra*

$$\mathscr{S}_{r,n} = \operatorname{End}_{\mathscr{H}_{r,n}}\Big(\bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}_{r,n}} M(\boldsymbol{\lambda}) \Big).$$

b) *For* $\mathbf{b} \in \mathscr{C}_{p,n}$ *the* ***cyclotomic q-Schur algebra*** *of* $\mathscr{H}_{d,\mathbf{b}}$ *is the endomorphism algebra*

$$\mathscr{S}_{d,\mathbf{b}} = \operatorname{End}_{\mathscr{H}_{d,\mathbf{b}}}\Big(\bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}} M_{\mathbf{b}}(\boldsymbol{\lambda}) \Big).$$

c) *The* ***cyclotomic q-Schur algebra*** *of* $\mathscr{H}_{r,p,n}$ *is the endomorphism algebra* $\mathscr{S}_{r,p,n} = \bigoplus_{\mathbf{b} \in \mathscr{C}_{p,n}} \mathscr{S}_{r,p,n}(\mathbf{b})$, *where*

$$\mathscr{S}_{r,p,n}(\mathbf{b}) = \operatorname{End}_{\mathscr{H}_{r,p,n}}\Big(\bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}} M_{\mathbf{b}}^{\boldsymbol{\lambda}} \Big),$$

where $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ *is considered as an* $\mathscr{H}_{r,p,n}$*-module by restriction.*

The algebra $\mathscr{S}_{r,p,n}$ is new, generalizing the Schur algebras of type D introduced by the first author in [**22**]. The cyclotomic Schur algebra $\mathscr{S}_{r,n} = \mathscr{S}_{r,n}(\mathbf{Q}^{\vee\varepsilon})$ was introduced in [**9**]. By Definition 4.13, $M_{\mathbf{b}}(\boldsymbol{\lambda}) = M(\boldsymbol{\lambda}^{[1]}) \otimes \cdots \otimes M(\boldsymbol{\lambda}^{[p]})$ so that

$$\mathscr{S}_{d,\mathbf{b}} \cong \operatorname{End}_{\mathscr{H}_{d,\mathbf{b}}}\big(\bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}} M_{\mathbf{b}}(\boldsymbol{\lambda}) \big) \cong \mathscr{S}_{d,\mathbf{b}_1}(\varepsilon \mathbf{Q}) \otimes \cdots \otimes \mathscr{S}_{d,b_p}(\varepsilon^p \mathbf{Q}).$$

Moreover, applying the functor $\mathtt{H}_{\mathbf{b}}$ shows that

$$\mathscr{S}_{d,\mathbf{b}} \cong \operatorname{End}_{\mathscr{H}_{r,n}}\Big(\bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}} M_{\mathbf{b}}^{\boldsymbol{\lambda}} \Big). \tag{5.2}$$

Hence, we can — and do! — consider $\mathscr{S}_{d,\mathbf{b}}$ as a subalgebra of $\mathscr{S}_{r,p,n}$.

Recall that after Definition 4.37 we defined $\theta_{\mathbf{b}} = \theta_{0,\mathsf{o}_{\mathbf{b}}} \in \operatorname{End}_{\mathscr{H}_{r,p_{\mathbf{b}},n}}(V_{\mathbf{b}})$. By definition, $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ is a submodule of $V_{\mathbf{b}}$. We next show that $\theta_{\mathbf{b}}$ maps $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ to $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle \mathsf{o}_{\mathbf{b}} \rangle}$.

5.3. **Lemma.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$ *and* $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$*. Then* $\theta_{\mathbf{b}}$ *restricts to give an* $\mathscr{H}_{r,p,n}$*-homomorphism from* $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ *to* $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$*.*

Proof. Let $Y_{\mathbf{b}}=Y_{0,o_{\mathbf{b}}}=Y_{o_{\mathbf{b}}}\dots Y_1$. Then $\theta_{\mathbf{b}}(v)=\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}v)$, for all $v\in V_{\mathbf{b}}$. By construction, $M_{\mathbf{b}}^{\boldsymbol{\lambda}}=v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}\mathscr{H}_{r,n}$ and

$$v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}=v_{\mathbf{b}}\Theta_{\mathbf{b}}(u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}})=\widehat{\Theta}_{\mathbf{b}}(x_{\boldsymbol{\lambda},\mathbf{b}}u_{\boldsymbol{\lambda},\mathbf{b}}^{+})v_{\mathbf{b}},$$

where these elements are defined before Definition 4.13. Therefore, it is enough to prove that $\theta_{\mathbf{b}}(v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})=\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})$ belongs to $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$. Using (2.20) we compute

$$\begin{aligned}Y_{\mathbf{b}}v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}&=Y_{\mathbf{b}}v_{\mathbf{b}}\Theta_{\mathbf{b}}(u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}})\\&=\widehat{\Theta}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}(u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle}x_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle})Y_{\mathbf{b}}v_{\mathbf{b}}, &&\text{by Lemma 2.16,}\\&=\widehat{\Theta}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}(u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle}x_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle})v^{(o_{\mathbf{b}})}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}Y_{\mathbf{b}}^{*}, &&\text{by Corollary 2.9.}\end{aligned}$$

Hence, using Lemma 4.9 there exists a $c\in\mathbb{Z}$ such that

$$\begin{aligned}\theta_{\mathbf{b}}(v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})&=\varepsilon^{c}v_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}\Theta_{\mathbf{b}}(u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle}x_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle,\mathbf{b}\langle o_{\mathbf{b}}\rangle})\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}^{*})\\&\in v^{+}_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}u^{+}_{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}^{*}).\end{aligned}$$

Thus, $\theta_{\mathbf{b}}(v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}})\in M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$. Moreover, this map is surjective since $Y_{\mathbf{b}}$, and hence $\sigma^{o_{\mathbf{b}}}(Y_{\mathbf{b}}^{*})$, is invertible by Lemma 2.24 and Lemma 2.21. As $M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ and $M_{\mathbf{b}}^{\boldsymbol{\lambda}\langle o_{\mathbf{b}}\rangle}$ are both free and of the same rank the proof is complete. □

Recall from Lemma 2.21 that $z_{\mathbf{b}}$ is a central element of $\mathscr{H}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$. Consequently, if $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ then

$$z_{\mathbf{b}}\cdot v_{\mathbf{b}}^{+}u_{\boldsymbol{\lambda}}^{+}x_{\boldsymbol{\lambda}}=(z_{\mathbf{b}}u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}})\cdot v_{\mathbf{b}}=(u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}}z_{\mathbf{b}})\cdot v_{\mathbf{b}}=(u_{\boldsymbol{\lambda},\mathbf{b}}^{+}x_{\boldsymbol{\lambda},\mathbf{b}})\cdot v_{\mathbf{b}}\Theta_{\mathbf{b}}(z_{\mathbf{b}})\in M_{\mathbf{b}}^{\boldsymbol{\lambda}}.$$

Therefore, left multiplication by $z_{\mathbf{b}}$ induces a homomorphism in $\operatorname{End}_{\mathscr{H}_{r,n}}(M_{\mathbf{b}}^{\boldsymbol{\lambda}})$.

5.4. **Definition.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$*. Define maps* $\vartheta_{\mathbf{b}}$ *and* $\zeta_{\mathbf{b}}$ *in* $\mathscr{S}_{r,p,n}(\mathbf{b})$ *by*

$$\vartheta_{\mathbf{b}}(m)=\theta_{\mathbf{b}}(m)\quad\textit{and}\quad\zeta_{\mathbf{b}}(m)=z_{\mathbf{b}}\cdot m,$$

for $m\in M_{\mathbf{b}}^{\boldsymbol{\lambda}}$*, and* $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$*.*

Using this definition and Lemma 4.29 we obtain:

5.5. **Lemma.** *Suppose that* $\mathbf{b}\in\mathscr{C}_{p,n}$*. Then* $\zeta_{\mathbf{b}}$ *is central in* $\mathscr{S}_{r,p,n}$ *and*

$$\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}}=\varepsilon^{\frac12 do_{\mathbf{b}}n(p_{\mathbf{b}}-1)}\zeta_{\mathbf{b}}.$$

As remarked in (5.2) above, $\mathscr{S}_{d,\mathbf{b}}\cong\operatorname{End}_{\mathscr{H}_{r,n}}\big(\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}^{\boldsymbol{\lambda}}\big)$ so we can view $\mathscr{S}_{d,\mathbf{b}}$ as a subalgebra of $\mathscr{S}_{r,p,n}(\mathbf{b})$.

5.6. **Theorem.** *As a K-algebra,* $\mathscr{S}_{r,p,n}(\mathbf{b})$ *is generated by* $\mathscr{S}_{d,\mathbf{b}}$ *and the endomorphism* $\vartheta_{\mathbf{b}}$*. Moreover, if* $\{\,x_i\mid i\in I\,\}$ *is a K-basis of* $\mathscr{S}_{d,\mathbf{b}}$ *then*

$$\{\,x_i\vartheta_{\mathbf{b}}^{k}\mid i\in I\ \textit{and}\ 0\le k<p_{\mathbf{b}}\,\}$$

is a K-basis of $\mathscr{S}_{r,p,n}(\mathbf{b})$*. In particular,* $\dim\mathscr{S}_{r,p,n}(\mathbf{b})=p_{\mathbf{b}}\dim\mathscr{S}_{d,\mathbf{b}}$*.*

Proof. This can be proved by repeating the argument of Theorem 4.38. □

5.2. **Weyl modules, simple modules and Schur functors.** The cyclotomic Schur algebra $\mathscr{S}_{r,n}$ is a quasi-hereditary cellular algebra with basis

$$\{\, \varphi_{\mathsf{ST}} \mid \mathsf{S} \in \mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu}), \mathsf{T} \in \mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\nu}) \text{ for } \boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu} \in \mathscr{P}_{r,n} \,\},$$

where $\mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\tau})$ is the set of *semistandard* $\boldsymbol{\lambda}$*-tableaux of type* $\boldsymbol{\tau}$ for $\boldsymbol{\tau} \in \mathscr{P}_{r,n}$; see [**9**, Definition 4.4 and Theorem 6.6]. In this paper we do not need the precise combinatorial definition of semistandard tableaux. For our purposes it is enough to know that if $x = u^+_{\boldsymbol{\tau}} x_{\boldsymbol{\tau}} h \in M(\boldsymbol{\tau})$, and $\mathsf{S} \in \mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu})$ and $\mathsf{T} \in \mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\nu})$, then

$$\varphi_{\mathsf{ST}}(x) = \delta_{\boldsymbol{\nu}\boldsymbol{\tau}} m_{\mathsf{ST}} h,$$

where m_{ST} is a certain element of $M(\boldsymbol{\mu})$.

For each $\boldsymbol{\lambda} \in \mathscr{P}_{r,n}$ there is a **Weyl module** $\Delta(\boldsymbol{\lambda})$, which is a cell module for $\mathscr{S}_{r,n}$. Let $L(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})/\operatorname{rad}\Delta(\boldsymbol{\lambda})$, where $\operatorname{rad}\Delta(\boldsymbol{\lambda})$ is the Jacobson radical of $\Delta(\boldsymbol{\lambda})$. Then $\{\, L(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \mathscr{P}_{r,n} \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{S}_{r,n}$-modules. Further, if $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathscr{P}_{r,n}$ then $L(\boldsymbol{\mu})$ is the simple head of $\Delta(\boldsymbol{\mu})$ and

$$[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})] = \begin{cases} 1, & \text{if } \boldsymbol{\lambda} = \boldsymbol{\mu}, \\ 0, & \text{if } \boldsymbol{\lambda} \not\trianglerighteq \boldsymbol{\mu}. \end{cases} \tag{5.7}$$

All of these facts are proved in [**9**, §6].

Similarly, for $\mathbf{b} \in \mathscr{C}_{p,n}$ let $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ and $L_{\mathbf{b}}(\boldsymbol{\lambda})$ be the Weyl modules and the irreducible modules of $\mathscr{S}_{d,\mathbf{b}}$, for $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$. For $1 \le t \le p$, $\lambda,\nu,\mu \in \mathscr{P}_{d,b_t}$ and $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$, $\mathsf{T} \in \mathcal{T}_0(\lambda,\nu)$, let $\varphi^{(t)}_{\mathsf{ST}}$ be the corresponding element of $\mathscr{S}_{d,\mathbf{b}}$ given by

$$\varphi^{(t)}_{\mathsf{ST}}(x_1 \otimes \cdots \otimes x_p) = x_1 \otimes \cdots \otimes x_{t-1} \otimes \varphi_{\mathsf{ST}}(x_t) \otimes x_{t+1} \otimes \cdots \otimes x_p.$$

5.8. **Lemma.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$*,* $1 \le s \le p$ *and that* $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$*, and* $\mathsf{T} \in \mathcal{T}_0(\lambda,\nu)$*, where* $\lambda,\mu,\nu \in \mathscr{P}_{d,b_s}$*. Then*

$$\varphi^{(s)}_{\mathsf{ST}} \vartheta_{\mathbf{b}} = \begin{cases} \varepsilon^{-\mathsf{o}_{\mathbf{b}} k_{\lambda,\nu}} \vartheta_{\mathbf{b}} \varphi^{(s+\mathsf{o}_{\mathbf{b}})}_{\mathsf{ST}}, & \text{if } s + \mathsf{o}_{\mathbf{b}} \le p, \\ \varepsilon^{-\mathsf{o}_{\mathbf{b}} k_{\lambda,\nu}} \vartheta_{\mathbf{b}} \varphi^{(s+\mathsf{o}_{\mathbf{b}}-p)}_{\mathsf{ST}}, & \text{if } s + \mathsf{o}_{\mathbf{b}} > p. \end{cases}$$

where $k_{\lambda,\nu} = \sum_{s=1}^{d-1} \sum_{t=1}^{s} (|\lambda^{(t)}| - |\nu^{(t)}|)$.

Proof. We first note that $\mathbf{b}\langle \mathsf{o}_{\mathbf{b}} \rangle = \mathbf{b}$, so that the notations $\varphi^{(s+\mathsf{o}_{\mathbf{b}})}_{\mathsf{ST}}$ and $\varphi^{(s+\mathsf{o}_{\mathbf{b}}-p)}_{\mathsf{ST}}$ make sense. As the map φ_{ST} is given by left multiplication by an element of $\mathscr{H}_{d,\mathbf{b}}$, the result follows from Lemma 4.39. (In what follows we only need to know that the scalar $\varepsilon^{-\mathsf{o}_{\mathbf{b}} k_{\lambda,\nu}}$ above is equal to $\varepsilon^{\mathsf{o}_b k}$, for some $k \in \mathbb{Z}$. This is a consequence of Lemma 4.39. That $k = k_{\lambda,\nu}$ can be determined using the definition of m_{ST} from [**9**].) □

5.9. *Remark.* Suppose that $\mathbf{b} \in \mathscr{C}_{p,n}$ and $1 \le s,t \le p$ and $s \equiv t \pmod{\mathsf{o}_{\mathbf{b}}}$, so that $b_s = b_t$. Just as in Remark 4.40, if we let π'_{st} be the algebra isomorphism ${\mathscr{S}_{d,b_s}}^{(s)} \cong {\mathscr{S}_{d,b_t}}^{(t)}$ given by $\varphi^{(s)}_{\mathsf{ST}} \mapsto \varepsilon^{-\mathsf{o}_{\mathbf{b}} k_{\lambda,\nu}} \varphi^{(t)}_{\mathsf{ST}}$, for S and T as above. Then $\vartheta_{\mathbf{b}}$ coincides with π'_{st}, where $t = s + \mathsf{o}_{\mathbf{b}}$ if $s + \mathsf{o}_{\mathbf{b}} \le p$; or $t = s + \mathsf{o}_{\mathbf{b}} - p$ if $s + \mathsf{o}_{\mathbf{b}} > p$.

For each multipartition $\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ the identity map $\varphi_{\boldsymbol{\mu}} : M_{\mathbf{b}}(\boldsymbol{\mu}) \longrightarrow M_{\mathbf{b}}(\boldsymbol{\mu})$ belongs to $\mathscr{S}_{d,\mathbf{b}}$. Then φ_μ is an idempotent in $\mathscr{S}_{d,\mathbf{b}}$ and $\sum_{\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}} \varphi_{\boldsymbol{\mu}}$ is the identity element of $\mathscr{S}_{d,\mathbf{b}}$. If M is a $\mathscr{S}_{d,\mathbf{b}}$-module then M has a **weight space** decomposition

$$M = \bigoplus_{\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}} M_{\boldsymbol{\mu}}, \qquad \text{where } M_{\boldsymbol{\mu}} = M\varphi_{\boldsymbol{\mu}}.$$

Recall from (2.23) that $\boldsymbol{\omega}_{\mathbf{b}} = ({\boldsymbol{\omega}_{\mathbf{b}}}^{[1]}, \cdots, {\boldsymbol{\omega}_{\mathbf{b}}}^{[p]})$ is the unique multipartition in $\mathscr{P}_{d,\mathbf{b}}$ such that $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\omega}_{\mathbf{b}}$ for all $\mu \in \mathscr{P}_{d,\mathbf{b}}$. By definition, $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ is the identity map on $\mathscr{H}_{d,\mathbf{b}}$ so that $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}} \mathscr{S}_{d,\mathbf{b}} \varphi_{\boldsymbol{\omega}_{\mathbf{b}}} \cong \mathscr{H}_{d,\mathbf{b}}$. Hence, we have a **Schur functor**

$$\mathsf{F}_{\boldsymbol{\omega}_{\mathbf{b}}} : \text{Mod-}\mathscr{S}_{d,\mathbf{b}} \longrightarrow \text{Mod-}\mathscr{H}_{d,\mathbf{b}}; M \mapsto M_{\boldsymbol{\omega}_{\mathbf{b}}}, \qquad \text{for } M \in \text{Mod-}\mathscr{S}_{d,\mathbf{b}}.$$

By [**9**, Corollary 6.14], the Weyl module $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ has a basis

$$\{\,\varphi_{\mathsf{S}} \mid \mathsf{S}\in\mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu}) \text{ for } \boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}\,\}$$

such that $\{\,\varphi_{\mathsf{S}} \mid \mathsf{S}\in\mathcal{T}_0(\boldsymbol{\lambda},\boldsymbol{\mu})\,\}$ is a basis for the $\boldsymbol{\mu}$-weight space of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. This implies that $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}(\Delta_{\mathbf{b}}(\boldsymbol{\lambda}))\cong S_{\mathbf{b}}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$; see [**24**, Proposition 2.17]. Hence, $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}(L_{\mathbf{b}}(\boldsymbol{\lambda}))\cong D_{\mathbf{b}}(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda}\in\mathscr{K}_{d,\mathbf{b}}$, since $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}$ is exact.

There is a unique semistandard $\boldsymbol{\lambda}$-tableau $\mathsf{T}^{\boldsymbol{\lambda}}$ of type $\boldsymbol{\lambda}$ and $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ is a "highest weight vector" in $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$. In particular, $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ generates $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$.

5.10. **Lemma.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$. Then*

$$\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\zeta_{\mathbf{b}}=\mathfrak{f}_{\boldsymbol{\lambda}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\quad\textit{and}\quad \varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}}=(\mathfrak{g}_{\boldsymbol{\lambda}})^{p_{\mathbf{b}}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}.$$

Proof. By [**24**, (2.18)], the Weyl module $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ can be identified with a set of maps from $\bigoplus_{\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}(\boldsymbol{\mu})$ to $S_{\mathbf{b}}(\boldsymbol{\lambda})$ in such a way that $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ is the natural projection map $M_{\mathbf{b}}(\boldsymbol{\lambda})\longrightarrow S_{\mathbf{b}}(\boldsymbol{\lambda})$. Hence, $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\zeta_{\mathbf{b}}=\mathfrak{f}_{\boldsymbol{\lambda}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ by Proposition 3.4 and $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}}=(\mathfrak{g}_{\boldsymbol{\lambda}})^{p_{\mathbf{b}}}\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ by Corollary 4.31 □

By Theorem 5.6, the subspaces $\{\mathscr{S}_{d,\mathbf{b}},\vartheta_{\mathbf{b}}\mathscr{S}_{d,\mathbf{b}},\dots,(\vartheta_{\mathbf{b}})^{p_{\mathbf{b}}-1}\mathscr{S}_{d,\mathbf{b}}\}$ define a $\mathbb{Z}/p_{\mathbf{b}}\mathbb{Z}$-graded Clifford system for $\mathscr{S}_{r,p,n}(\mathbf{b})$. In particular, conjugation with $\vartheta_{\mathbf{b}}$ defines an algebra automorphism of $\mathscr{S}_{d,\mathbf{b}}$. For any $\mathscr{S}_{d,\mathbf{b}}$-module M let $M^{\vartheta_{\mathbf{b}}}$ be the $\mathscr{S}_{d,\mathbf{b}}$-module obtained by twisting the action of $\mathscr{S}_{d,\mathbf{b}}$ by $\vartheta_{\mathbf{b}}$.

5.11. **Lemma.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$. Then*

$$\Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}}\cong\Delta_{\mathbf{b}}(\boldsymbol{\lambda}\langle\mathsf{o}_{\mathbf{b}}\rangle)\quad\textit{and}\quad L_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}}\cong L_{\mathbf{b}}(\boldsymbol{\lambda}\langle\mathsf{o}_{\mathbf{b}}\rangle)$$

as $\mathscr{S}_{d,\mathbf{b}}$-modules.

Proof. This follows directly from Lemma 5.8 and Remark 5.9. □

The following definitions mirror the constructions for $\mathscr{E}_{d,\mathbf{b}}$ in Definition 4.42.

5.12. **Definition.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b}\in\mathscr{C}_{p,n}$. Define*

$$\Delta^{\boldsymbol{\lambda}}=\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow_{\mathscr{S}_{d,\mathbf{b}}}^{\mathscr{S}_{r,p,n}(\mathbf{b})}\quad\textit{and}\quad L^{\boldsymbol{\lambda}}=L_{\mathbf{b}}(\boldsymbol{\lambda})\uparrow_{\mathscr{S}_{d,\mathbf{b}}}^{\mathscr{S}_{r,p,n}(\mathbf{b})}.$$

Let $\hat\sigma$ be the automorphism of $\mathscr{S}_{r,p,n}(\mathbf{b})$ which, using Theorem 5.6, is defined on generators by

$$(x\vartheta_{\mathbf{b}}^k)^{\hat\sigma}x=\varepsilon^{ko_{\mathbf{b}}}x\vartheta_{\mathbf{b}}^k,\qquad\text{for all } x\in\mathscr{S}_{d,\mathbf{b}}\text{ and } 0\le k<p_{\mathbf{b}}.$$

By definition, $\hat\sigma$ restricts to the identity map on $\mathscr{S}_{d,\mathbf{b}}$. By Lemma 4.1 there is an isomorphism of $\mathscr{S}_{r,p,n}(\mathbf{b})$-$\mathscr{S}_{r,p,n}(\mathbf{b})$-bimodules,

$$\mathscr{S}_{r,p,n}(\mathbf{b})\otimes_{\mathscr{S}_{d,\mathbf{b}}}\mathscr{S}_{r,p,n}(\mathbf{b})\cong\bigoplus_{j=1}^{p_{\mathbf{b}}}\big(\mathscr{S}_{r,p,n}(\mathbf{b})\big)^{\hat\sigma^j},\tag{5.13}$$

such that the left $\mathscr{S}_{r,p,n}(\mathbf{b})$-module structure on $\big(\mathscr{S}_{r,p,n}(\mathbf{b})\big)^{\hat\sigma^j}$ is given by left multiplication and the right action is twisted by $\hat\sigma^j$.

Recall that if $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ then $p_{\mathbf{b}/\boldsymbol{\lambda}}=p_{\mathbf{b}}/p_{\boldsymbol{\lambda}}$. Let $\mathscr{S}_{d,\boldsymbol{\lambda}}$ be the subalgebra of $\mathscr{S}_{r,p,n}$ generated by $\mathscr{S}_{d,\mathbf{b}}$ and $\vartheta_{\boldsymbol{\lambda}}=\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol{\lambda}}}$. Let $\overline{\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}}$ be the image of $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ in $L_{\mathbf{b}}(\boldsymbol{\lambda})$ and for $1\le i\le p_{\boldsymbol{\lambda}}$ define

$$\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}=\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}}\Big(\vartheta_{\boldsymbol{\lambda}}-\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}t}\Big)\mathscr{S}_{d,\mathbf{b}}\hookrightarrow\mathscr{S}_{d,\boldsymbol{\lambda}},$$

$$L^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}=\overline{\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}}\prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}}\Big(\vartheta_{\boldsymbol{\lambda}}-\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}t}\Big)\mathscr{S}_{d,\mathbf{b}}\hookrightarrow\mathscr{S}_{d,\boldsymbol{\lambda}}.$$

Then, by Lemma 5.8 and Lemma 5.10, $\Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}$ and $L^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}$ are $\mathscr{S}_{d,\boldsymbol\lambda}$-submodules of $\Delta^{\boldsymbol\lambda}$ and $L^{\boldsymbol\lambda}$, respectively. Next, for $1 \le i \le p_{\boldsymbol\lambda}$ define

$$\Delta^{\boldsymbol\lambda}_{i,p} = \Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}} \uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\boldsymbol\lambda}} \quad \text{and} \quad L^{\boldsymbol\lambda}_{i,p} = L^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}} \uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\boldsymbol\lambda}}.$$

5.14. **Proposition.** *Suppose that $\boldsymbol\lambda \in \mathscr{P}_{d,\mathbf{b}}$, for $\mathbf{b} \in \mathscr{C}_{p,n}$, and let $\hat\sigma_{\boldsymbol\lambda} = (\hat\sigma)^{p_{\mathbf{b}}/\boldsymbol\lambda}$. Then:*

a) *if $1 \le i \le p_{\boldsymbol\lambda}$ then*

$$\big(\Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}\big)^{\hat\sigma_{\boldsymbol\lambda}} \cong \Delta^{\boldsymbol\lambda}_{i+1,p_{\boldsymbol\lambda}}, \qquad \big(\Delta^{\boldsymbol\lambda}_{i,p}\big)^{\hat\sigma_{\boldsymbol\lambda}} \cong \Delta^{\boldsymbol\lambda}_{i+1,p},$$
$$\big(L^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}\big)^{\hat\sigma_{\boldsymbol\lambda}} \cong L^{\boldsymbol\lambda}_{i+1,p_{\boldsymbol\lambda}} \qquad \big(L^{\boldsymbol\lambda}_{i,p}\big)^{\hat\sigma_{\boldsymbol\lambda}} \cong L^{\boldsymbol\lambda}_{i+1,p}.$$

b) $\Delta_{\mathbf{b}}(\boldsymbol\lambda)\uparrow^{\mathscr{S}_{d,\boldsymbol\lambda}}_{\mathscr{S}_{d,\mathbf{b}}} \cong \oplus_{i=1}^{p_{\boldsymbol\lambda}} \Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}$, $L_{\mathbf{b}}(\boldsymbol\lambda)\uparrow^{\mathscr{S}_{d,\boldsymbol\lambda}}_{\mathscr{S}_{d,\mathbf{b}}} \cong \oplus_{i=1}^{p_{\boldsymbol\lambda}} L^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}$, *and there is a unique $\mathscr{S}_{d,\mathbf{b}}$-module isomorphism $\Delta_{\mathbf{b}}(\boldsymbol\lambda) \longrightarrow \Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}\downarrow^{\mathscr{S}_{d,\boldsymbol\lambda}}_{\mathscr{S}_{d,\mathbf{b}}}$ such that*

$$\varphi_{\mathsf{T}^{\boldsymbol\lambda}} \mapsto \varphi_{\mathsf{T}^{\boldsymbol\lambda}} \prod_{\substack{1\le t\le p_{\boldsymbol\lambda}\\ t\ne i}} \Big(\vartheta_{\boldsymbol\lambda} - \mathfrak{g}_{\boldsymbol\lambda}\varepsilon^{o_{\boldsymbol\lambda} t}\Big).$$

This latter map also induces an isomorphism $L_{\mathbf{b}}(\boldsymbol\lambda) \longrightarrow L^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}\downarrow^{\mathscr{S}_{d,\boldsymbol\lambda}}_{\mathscr{S}_{d,\mathbf{b}}}$.

c) $\Delta^{\boldsymbol\lambda} = \Delta^{\boldsymbol\lambda}_{1,p} \oplus \cdots \oplus \Delta^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda},p}$ *and* $L^{\boldsymbol\lambda} = L^{\boldsymbol\lambda}_{1,p} \oplus \cdots \oplus L^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda},p}$ *as $\mathscr{S}_{d,\mathbf{b}}$-modules.*

d) $\Delta^{\boldsymbol\lambda} \cong \Delta^{\boldsymbol\lambda\langle o_{\mathbf{b}}\rangle}$ *and* $L^{\boldsymbol\lambda} \cong L^{\boldsymbol\lambda\langle o_{\mathbf{b}}\rangle}$ *as $\mathscr{S}_{r,p,n}$-modules.*

Proof. We only prove the results for the Weyl modules. The other cases follow either using similar arguments or because twisting by $\hat\sigma$ is an exact functor, so we leave the details to the reader.

By Lemma 5.11, we know that $(\Delta_{\mathbf{b}}(\boldsymbol\lambda))^{\vartheta_{\boldsymbol\lambda}} \cong \Delta_{\mathbf{b}}(\boldsymbol\lambda\langle o_{\boldsymbol\lambda}\rangle) = \Delta_{\mathbf{b}}(\boldsymbol\lambda)$. Therefore,

$$\begin{aligned}\Delta^{\boldsymbol\lambda\langle o_{\mathbf{b}}\rangle} &= \Delta_{\mathbf{b}}(\boldsymbol\lambda\langle o_{\mathbf{b}}\rangle)\uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong \Delta_{\mathbf{b}}(\boldsymbol\lambda)^{\vartheta_{\mathbf{b}}}\uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\\ &\cong \big(\Delta_{\mathbf{b}}(\boldsymbol\lambda)\uparrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\big)^{\vartheta_{\mathbf{b}}} = \big(\Delta^{\boldsymbol\lambda}\big)^{\vartheta_{\mathbf{b}}} \cong \Delta^{\boldsymbol\lambda}.\end{aligned}$$

This proves (d).

Arguing as in Theorem 4.33, it is easy to see that $\varphi_{\mathsf{T}^{\boldsymbol\lambda}} \in \Delta^{\boldsymbol\lambda}_{1,p} + \cdots + \Delta^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda},p}$. Hence, $\Delta^{\boldsymbol\lambda} = \Delta^{\boldsymbol\lambda}_{1,p} + \cdots + \Delta^{\boldsymbol\lambda}_{p_{\boldsymbol\lambda},p}$. On the other hand, if $1 \le i \le p_{\boldsymbol\lambda}$ and $f \in \mathscr{S}_{d,\mathbf{b}}$ then the isomorphisms in Remark 5.9 and the fact that $\boldsymbol\lambda\langle o_{\boldsymbol\lambda}\rangle = \boldsymbol\lambda$ imply that $\varphi_{\mathsf{T}^{\boldsymbol\lambda}} f = 0$ if and only if $\varphi_{\mathsf{T}^{\boldsymbol\lambda}}(\vartheta^i_{\boldsymbol\lambda} f \vartheta^{-i}_{\boldsymbol\lambda}) = 0$. It follows that the map

$$\varphi_{\mathsf{T}^{\boldsymbol\lambda}} \mapsto \varphi_{\mathsf{T}^{\boldsymbol\lambda}}\Bigg(\prod_{\substack{1\le t\le p_{\boldsymbol\lambda}\\ t\ne i}} \Big(\vartheta_{\boldsymbol\lambda} - \mathfrak{g}_{\boldsymbol\lambda}\varepsilon^{o_{\boldsymbol\lambda} t}\Big)\Bigg)$$

extends uniquely to a $\mathscr{S}_{d,\mathbf{b}}$-module surjection $\rho_i : \Delta_{\mathbf{b}}(\boldsymbol\lambda) \twoheadrightarrow \Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}\downarrow^{\mathscr{S}_{d,\boldsymbol\lambda}}_{\mathscr{S}_{d,\mathbf{b}}}$. In particular, $\dim \Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}} \le \dim \Delta_{\mathbf{b}}(\boldsymbol\lambda)$. By construction, however, $\dim \Delta^{\boldsymbol\lambda} = p_{\boldsymbol\lambda} \dim \Delta_{\mathbf{b}}(\boldsymbol\lambda)$. Therefore, the maps ρ_i, for $1 \le i \le p_{\boldsymbol\lambda}$, are all isomorphisms. This proves (b), while (c) follows easily from definitions and (b).

It remains to prove part (a). Suppose that $1 \le i \le p_{\boldsymbol\lambda}$. The definition of $\hat\sigma$ implies that if $f \in \mathscr{S}_{r,p,n}(\mathbf{b})$ then $\varphi_{\mathsf{T}^{\boldsymbol\lambda}} f = 0$ if and only if $\varphi_{\mathsf{T}^{\boldsymbol\lambda}} f^{\hat\sigma_{\boldsymbol\lambda}} = 0$. Therefore, the map

$$\varphi_{\mathsf{T}^{\boldsymbol\lambda}} \prod_{\substack{1\le t\le p_{\boldsymbol\lambda}\\ t\ne i+1}} \Big(\vartheta_{\boldsymbol\lambda} - \mathfrak{g}_{\boldsymbol\lambda}\varepsilon^{o_{\boldsymbol\lambda} t}\Big) f \mapsto \varphi_{\mathsf{T}^{\boldsymbol\lambda}} \prod_{\substack{1\le t\le p_{\boldsymbol\lambda}\\ t\ne i}} \Big(\vartheta_{\boldsymbol\lambda} - \mathfrak{g}_{\boldsymbol\lambda}\varepsilon^{o_{\boldsymbol\lambda} t}\Big) f^{\hat\sigma_{\boldsymbol\lambda}}$$

is a well-defined $\mathscr{S}_{r,p,n}(\mathbf{b})$-module homomorphism from $\Delta^{\boldsymbol\lambda}_{i+1,p}$ onto $\big(\Delta^{\boldsymbol\lambda}_{i,p}\big)^{\hat\sigma_{\boldsymbol\lambda}}$. Similarly, one can prove that $\big(\Delta^{\boldsymbol\lambda}_{i,p_{\boldsymbol\lambda}}\big)^{\hat\sigma_{\boldsymbol\lambda}} \cong \Delta^{\boldsymbol\lambda}_{i+1,p_{\boldsymbol\lambda}}$. □

The proof of Proposition 5.14(a) yields the following.

5.15. **Corollary.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and that $1\le i\le p_{\boldsymbol{\lambda}}$. Then, as a K-vector space*

$$\Delta^{\boldsymbol{\lambda}}_{i,p}\cong\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\oplus\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\vartheta_{\mathbf{b}}\oplus\cdots\oplus\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}/\boldsymbol{\lambda}-1},$$

Moreover, the action of $\mathscr{S}_{r,p,n}(\mathbf{b})$ on $\Delta^{\boldsymbol{\lambda}}_{i,p}$ is uniquely determined by

a) $\Delta^{\boldsymbol{\lambda}}_{i,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\cong\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\oplus\Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{-1}}\oplus\cdots\oplus\Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}}/\boldsymbol{\lambda}}}$;
b) $(x\vartheta_{\mathbf{b}}^{j})\vartheta_{\mathbf{b}}^{t}=x\vartheta_{\mathbf{b}}^{j+t}$, *for all $x\in\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ and $j,t\in\mathbb{Z}$;*
c) *$\vartheta_{\boldsymbol{\lambda}}$ acts as the scalar $\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{io_{\boldsymbol{\lambda}}}$ on the highest weight vector of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\hookrightarrow\Delta^{\boldsymbol{\lambda}}_{i,p}$.*

Analogous statements hold for the simple module $L^{\boldsymbol{\lambda}}_{i,p}$.

Proof. By definition,

$$\Delta^{\boldsymbol{\lambda}}_{i,p}\cong\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\oplus\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}\oplus\cdots\oplus\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}/\boldsymbol{\lambda}-1}.$$

As in the proof of Proposition 5.14, we can identify $\Delta^{\boldsymbol{\lambda}}_{i,p_{\boldsymbol{\lambda}}}$ with $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ using the isomorphism ρ_i, for $1\le i\le p_{\boldsymbol{\lambda}}$. Then the highest weight vector $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}$ of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ corresponds to the vector $\varphi_{\mathsf{T}^{\boldsymbol{\lambda}}}\Big(\prod_{\substack{1\le t\le p_{\boldsymbol{\lambda}}\\ t\ne i}}\big(\vartheta_{\boldsymbol{\lambda}}-\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{o_{\boldsymbol{\lambda}}t}\big)\Big)$. This implies that $\vartheta_{\boldsymbol{\lambda}}=\vartheta_{\mathbf{b}}^{p_{\mathbf{b}}/\boldsymbol{\lambda}}$ acts as the scalar $\mathfrak{g}_{\boldsymbol{\lambda}}\varepsilon^{io_{\boldsymbol{\lambda}}}$ on the highest weight vector of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})\hookrightarrow\Delta^{\boldsymbol{\lambda}}_{i,p}$. All of the claims in the Corollary now follow. □

5.16. **Corollary.** *Suppose that $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$.*

a) *If $1\le i\le p_{\boldsymbol{\lambda}}$ then $L^{\boldsymbol{\lambda}}_{i,p}$ is the simple head of $\Delta^{\boldsymbol{\lambda}}_{i,p}$.*
b) *If $1\le i\le p_{\boldsymbol{\lambda}}$ and $1\le j\le p_{\boldsymbol{\mu}}$ then*

$$[\Delta^{\boldsymbol{\lambda}}_{i,p}:L^{\boldsymbol{\mu}}_{j,p}]=\begin{cases}\delta_{ij}, & \text{if } \boldsymbol{\lambda}=\boldsymbol{\mu},\\ 0, & \text{if } \boldsymbol{\lambda}\not\trianglerighteq\boldsymbol{\mu}.\end{cases}$$

Proof. By (5.7) $L_{\mathbf{b}}(\boldsymbol{\lambda})$ is the simple head of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ and

$$[\Delta_{\mathbf{b}}(\boldsymbol{\lambda}):L_{\mathbf{b}}(\boldsymbol{\mu})]=\begin{cases}1, & \text{if } \boldsymbol{\mu}=\boldsymbol{\lambda},\\ 0, & \text{if } \boldsymbol{\lambda}\not\trianglerighteq\boldsymbol{\mu}.\end{cases}$$

Hence, the result follows from Proposition 5.14 and Frobenius reciprocity. □

Recall that $\sim_{\mathbf{b}}$ is the equivalence relation on $\mathscr{P}_{d,\mathbf{b}}$ such that $\boldsymbol{\lambda}\sim_{\mathbf{b}}\boldsymbol{\mu}$ if $\boldsymbol{\mu}=\boldsymbol{\lambda}\langle ko_{\mathbf{b}}\rangle$ for some $k\in\mathbb{Z}$.

5.17. **Corollary.** *The algebra $\mathscr{S}_{r,p,n}(\mathbf{b})$ is split over K and*

$$\{\,L^{\boldsymbol{\lambda}}_{i,p}\mid\boldsymbol{\lambda}\in\mathscr{P}^{\mathbf{b}}_{d,\mathbf{b}}\text{ and }1\le i\le p_{\boldsymbol{\lambda}}\,\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible $\mathscr{S}_{r,p,n}(\mathbf{b})$-modules.

Proof. Just as in Section 5, this follows from Corollary 5.16, Frobenius reciprocity and some general arguments in Clifford theory. □

Recall from subsection §5.2 that the Schur functor $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}:\text{Mod-}\mathscr{S}_{d,\mathbf{b}}\longrightarrow\text{Mod-}\mathscr{H}_{d,\mathbf{b}}$ is given by $\mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}(M)=M\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$, where $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ is the identity map on $\mathscr{H}_{d,\mathbf{b}}$. Using the embedding $\mathscr{S}_{d,\mathbf{b}}\hookrightarrow\mathscr{S}_{r,p,n}(\mathbf{b})$, and the fact that $v_{\mathbf{b}}=v_{\mathbf{b}}^{+}u_{\boldsymbol{\omega}_{\mathbf{b}}}^{+}$, it is easy to check that $\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ corresponds to the natural projection from $\bigoplus_{\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}}M_{\mathbf{b}}^{\boldsymbol{\lambda}}$ onto $V_{\mathbf{b}}=M_{\mathbf{b}}^{\boldsymbol{\omega}_{\mathbf{b}}}$. In particular,

$$\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}\mathscr{S}_{r,p,n}(\mathbf{b})\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}=\mathscr{E}_{d,\mathbf{b}}\quad\text{and}\quad\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}\mathscr{S}_{d,\mathbf{b}}\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}=\mathscr{H}_{d,\mathbf{b}}.$$

Hence, we have a second Schur functor $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}:\text{Mod-}\mathscr{S}_{r,p,n}(\mathbf{b})\longrightarrow\text{Mod-}\mathscr{E}_{d,\mathbf{b}}$ which is given by $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(M)=M\varphi_{\boldsymbol{\omega}_{\mathbf{b}}}$ and if $\varphi\in\operatorname{Hom}_{\mathscr{S}_{r,p,n}(\mathbf{b})}(M,N)$ then $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(\varphi)(x\varphi_{\boldsymbol{\omega}_{\mathbf{b}}})=$

$\varphi(x)$, for all $x \in M$. It is straightforward to check that we have the following commutative diagram of functors:

$$
\begin{array}{ccc}
\text{Mod-}\mathscr{S}_{r,p,n}(\mathbf{b}) & \xrightarrow{\;?\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}\;} & \text{Mod-}\mathscr{S}_{d,\mathbf{b}} \\
{\scriptstyle \mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathtt{F}_{\boldsymbol{\omega}_{\mathbf{b}}}} \\
\text{Mod-}\mathscr{E}_{d,\mathbf{b}} & \xrightarrow[\;?\downarrow^{\mathscr{E}_{d,\mathbf{b}}}_{\mathscr{H}_{d,\mathbf{b}}}\;]{} & \text{Mod-}\mathscr{H}_{d,\mathbf{b}}
\end{array}
\tag{5.18}
$$

5.19. **Lemma.** *Suppose that* $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ *and* $1 \le i \le p_{\boldsymbol{\lambda}}$. *Then*

$$
\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(\Delta^{\boldsymbol{\lambda}}_{i,p}) \cong S^{\boldsymbol{\lambda}}_{i,p} \quad \textit{and} \quad \mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}(L^{\boldsymbol{\lambda}}_{i,p}) \cong \begin{cases} D^{\boldsymbol{\lambda}}_{i,p}, & \textit{if } \boldsymbol{\lambda} \in \mathscr{K}_{d,\mathbf{b}}, \\ 0, & \textit{otherwise.} \end{cases}
$$

Proof. This follow directly from (5.18) and Lemma 4.43. □

5.20. **Corollary.** *Suppose that* $\mathbf{b} \in \mathscr{C}_{p,n}$, $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$, $\boldsymbol{\mu} \in \mathscr{K}_{d,\mathbf{b}}$, $1 \le i \le p_{\boldsymbol{\lambda}}$ *and that* $1 \le j \le p_{\boldsymbol{\mu}}$. *Then*

$$
[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}] = [S^{\boldsymbol{\lambda}}_{i,p} : D^{\boldsymbol{\mu}}_{j,p}] = [S^{\boldsymbol{\lambda}}_{i} : D^{\boldsymbol{\mu}}_{j}].
$$

Proof. This follows directly from Lemma 5.19 and Lemma 4.43 together with the easily checked fact that the functors $\mathtt{F}^{(p)}_{\boldsymbol{\omega}_{\mathbf{b}}}$ and $\mathtt{F}_{\mathscr{E}}$ are exact. □

Therefore, in order to compute the decomposition number $[S^{\boldsymbol{\lambda}}_{i} : D^{\boldsymbol{\mu}}_{j}]$ it is enough to determine the decomposition number $[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}]$ for $\mathscr{S}_{r,p,n}$. In [**23**, §4] we defined a decomposition number $[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}]$ to be **l-splittable** if $p_{\boldsymbol{\lambda}} = l = p_{\boldsymbol{\mu}}$ for some integer l and we showed that all decomposition numbers of algebras like $\mathscr{S}_{r,p,n}$ are determined by their l-splittable decomposition numbers. We compute the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ in the next section.

5.3. **Splittable decomposition numbers.** In this section we derive explicit formulae for the l-splittable decomposition numbers of the algebras $\mathscr{S}_{r,p,n}(\mathbf{b})$ in characteristic zero. By Corollary 5.20 this will determine all of the l-splittable decomposition numbers of the cyclotomic Hecke algebras $\mathscr{H}_{r,p,n}$ in characteristic zero. By the main results of [**23**], this will determine all of the decomposition numbers of $\mathscr{H}_{r,p,n}$. We show that the splittable decomposition numbers depend, in an explicit way, on the decomposition numbers of certain Ariki–Koike algebras and on the scalars $\mathfrak{g}_{\boldsymbol{\lambda}}$ introduced in Lemma 4.30.

Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions in $\mathscr{P}_{d,\mathbf{b}}$. We want to compute the decomposition numbers $[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}]$ for $1 \le i \le p_{\boldsymbol{\lambda}}$ and $1 \le j \le p_{\boldsymbol{\mu}}$. By Corollary 5.15 and the exactness of $\vartheta_{\mathbf{b}}$, if $p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}}$ then

$$
[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}] = [\Delta^{\boldsymbol{\lambda}}_{i+1,p} : L^{\boldsymbol{\mu}}_{j+1,p}], \tag{5.21}
$$

where we read $i+1$ and $j+1$ modulo $p_{\boldsymbol{\lambda}}$. Therefore, these decomposition numbers are determined by the decomposition numbers

$$
d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} = [\Delta^{\boldsymbol{\lambda}}_{0,p} : L^{\boldsymbol{\mu}}_{j,p}],
$$

for $1 \le j \le p_{\boldsymbol{\mu}}$. In fact, as noted above, it is enough to compute the splittable decomposition numbers. That is, the $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ such that $p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}}$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$.

Before we start to compute the decomposition numbers $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ we introduce some new notation. If A is any finite dimensional algebra let $\mathcal{R}(A)$ be the Grothendieck group of finitely generated A-modules. If M is an A-module let $[M]$ be the image of M in $\mathcal{R}(A)$. In particular, note that the Grothendieck group of $\mathcal{R}(\mathscr{S}_{r,n})$ is equipped with two distinguished bases:

$$\{\,[\Delta(\boldsymbol{\lambda})]\mid\boldsymbol{\lambda}\in\mathscr{P}_{r,n}\,\}\quad\text{and}\quad\{\,[L(\boldsymbol{\lambda})\mid\boldsymbol{\lambda}\in\mathscr{P}_{r,n}\,\}.$$

Similar remarks apply to the Grothendieck groups of the cyclotomic Schur algebras $\mathscr{S}_{d,\mathbf{b}}$ and $\mathscr{S}_{r,p,n}(\mathbf{b})$, for $\mathbf{b}\in\mathscr{C}_{p,n}$.

Fix integers l and m such that $p=lm$ and suppose that $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$, for some $\mathbf{b}\in\mathscr{C}_{p,n}$. Then a multipartition $\boldsymbol{\mu}$ is **l-symmetric** if

$$\boldsymbol{\mu}=\boldsymbol{\nu}^{\,l}:=(\underbrace{\boldsymbol{\nu},\dots,\boldsymbol{\nu}}_{l\text{ times}}),$$

for some multipartition $\boldsymbol{\nu}\in\mathscr{P}_{r/l,n/l}$. Note that if $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is an l-splittable decomposition number then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are both l-symmetric multipartitions.

Let $\mathscr{P}^{\,l}_{d,\mathbf{b}}$ be the set of l-symmetric multipartitions in $\mathscr{P}_{d,\mathbf{b}}$. It is easy to see that

$$\mathscr{P}^{\,l}_{d,\mathbf{b}}=\{\,\boldsymbol{\mu}\mid\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}\text{ and }\mathsf{o}_{\boldsymbol{\mu}}|m\,\}.$$

If $\mathscr{P}^{\,l}_{d,\mathbf{b}}$ is non-empty then $\mathsf{o}_{\mathbf{b}}|m$ and we define $\mathbf{b}_m=(b_1,\dots,b_m)$. If $\boldsymbol{\mu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ define $\boldsymbol{\mu}_m=(\boldsymbol{\mu}^{[1]},\dots,\boldsymbol{\mu}^{[m]})$. Then $\boldsymbol{\mu}_m\in\mathscr{P}_{r/l,\mathbf{b}_m}\subseteq\mathscr{P}_{r/l,n/l}$. It is easy to check that the map $\boldsymbol{\nu}\mapsto\boldsymbol{\nu}^{\,l}$ defines a bijection from $\mathscr{P}_{r/l,\mathbf{b}_m}$ to $\mathscr{P}^{\,l}_{d,\mathbf{b}}$, with the inverse map being given by $\boldsymbol{\mu}\mapsto\boldsymbol{\mu}_m$.

We now return to our main task of computing splittable decomposition numbers. We will do this by deriving a system of equations which uniquely determine the decomposition numbers $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$, for $1\le j\le l=p_{\boldsymbol{\lambda}}$.

For the rest of this subsection fix $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$ and set $m=\mathsf{o}_{\boldsymbol{\lambda}}$ and $l=p_{\boldsymbol{\lambda}}$. Then $\mathbf{b}_m=(b_1,\dots,b_m)\in\mathscr{C}_{r/l,n/l}$ and $\boldsymbol{\lambda}_m\in\mathscr{P}_{r/l,\mathbf{b}_m}$. By (5.2) the cyclotomic Schur algebras $\mathscr{S}_{r/l,\mathbf{b}_m}$ and $\mathscr{S}_{d,\mathbf{b}}$ are related by

$$\mathscr{S}_{r/l,\mathbf{b}_m}\cong\mathscr{S}_{d,b_1}\otimes\cdots\otimes\mathscr{S}_{d,b_m}\quad\text{and}\quad\mathscr{S}_{d,\mathbf{b}}\cong\big(\mathscr{S}_{r/l,\mathbf{b}_m}\big)^{\otimes l}.$$

For $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$ let $d_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}=[\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m):L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)]$ be the corresponding decomposition number for the cyclotomic Schur algebra $\mathscr{S}_{r/l,\mathbf{b}_m}$. Since

$$\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m)\cong\Delta(\boldsymbol{\lambda}^{[1]})\otimes\cdots\otimes\Delta(\boldsymbol{\lambda}^{[m]})\quad\text{and}\quad L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)\cong L(\boldsymbol{\mu}^{[1]})\otimes\cdots\otimes L(\boldsymbol{\mu}^{[m]})$$

we have that

$$d_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}=\prod_{i=1}^{m}[\Delta(\boldsymbol{\lambda}^{[i]}):L(\boldsymbol{\mu}^{[i]})]=d_{\boldsymbol{\lambda}^{[1]}\boldsymbol{\mu}^{[1]}}\dots d_{\boldsymbol{\lambda}^{[m]}\boldsymbol{\mu}^{[m]}},\tag{5.22}$$

where $d_{\boldsymbol{\lambda}^{[i]}\boldsymbol{\mu}^{[i]}}=[\Delta(\boldsymbol{\lambda}^{[i]}):L(\boldsymbol{\mu}^{[i]})]$, for $1\le i\le m=\mathsf{o}_{\boldsymbol{\lambda}}$.

Recall that if $\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$ then $p_{\mathbf{b}/\boldsymbol{\mu}}=p_{\mathbf{b}}/p_{\boldsymbol{\mu}}=\mathsf{o}_{\boldsymbol{\mu}}/\mathsf{o}_{\mathbf{b}}$. If $\boldsymbol{\mu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ is l-symmetric then $\mathsf{o}_{\boldsymbol{\mu}}$ divides m, so we define $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}=p_{\boldsymbol{\mu}}/p_{\boldsymbol{\lambda}}$. Then $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}\in\mathbb{N}$ and $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}=\mathsf{o}_{\boldsymbol{\lambda}}/\mathsf{o}_{\boldsymbol{\mu}}=p_{\mathbf{b}/\boldsymbol{\lambda}}/p_{\mathbf{b}/\boldsymbol{\mu}}$.

5.23. **Lemma.** *Suppose that $\boldsymbol{\lambda}\in\mathscr{P}_{d,\mathbf{b}}$, $l=p_{\boldsymbol{\lambda}}$ and $m=\mathsf{o}_{\boldsymbol{\lambda}}$. Then:*

a) $[\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m)]=\displaystyle\sum_{\boldsymbol{\nu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}}d_{\boldsymbol{\lambda}_m\boldsymbol{\nu}_m}[L_{\mathbf{b}_m}(\boldsymbol{\nu}_m)]$.

b) $[\Delta^{\boldsymbol{\lambda}}_{0,p}]=\displaystyle\sum_{\boldsymbol{\nu}\in\mathscr{P}_{d,\mathbf{b}}}\sum_{1\le j\le p_{\boldsymbol{\nu}}}d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\nu}}[L^{\boldsymbol{\nu}}_{j,p}]$.

c) *If $\boldsymbol{\mu}\in\mathscr{P}^{\,l}_{d,\mathbf{b}}$ then $d^{(1)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}+d^{(2)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}+\cdots+d^{(l)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}=p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}d^{\,l}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}$.*

Proof. Part (a) is just a rephrasing of the definition of decomposition numbers combined with the bijection $\mathscr{P}^{\,l}_{d,\mathbf{b}}\xrightarrow{\simeq}\mathscr{P}_{r/l,\mathbf{b}_m};\boldsymbol{\mu}\mapsto\boldsymbol{\mu}_m$. Part (b) follows similarly.

Suppose that $\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$. We prove (c) by computing the decomposition multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ on both sides of part (b) upon restriction to $\mathscr{S}_{d,\mathbf{b}}$. By Corollary 5.15,

$$\Delta^{\boldsymbol{\lambda}}_{0,p}{\downarrow}^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong \Delta_{\mathbf{b}}(\boldsymbol{\lambda}) \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{-1}} \oplus \cdots \oplus \Delta_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}/\boldsymbol{\lambda}}}}.$$

Now, every composition factor of $\Delta_{\mathbf{b}}(\boldsymbol{\lambda})$ is isomorphic to $L_{\mathbf{b}}(\boldsymbol{\nu})$, for some $\boldsymbol{\nu} \in \mathscr{P}_{d,\mathbf{b}}$, and $L_{\mathbf{b}}(\boldsymbol{\nu})^{\vartheta_{\mathbf{b}}} \cong L_{\mathbf{b}\langle o_{\mathbf{b}}\rangle}(\boldsymbol{\nu})$ by Lemma 5.11. Therefore, the decomposition multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ in $\Delta^{\boldsymbol{\lambda}}_{0,p}{\downarrow}^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}}$ is

$$\frac{p_{\mathbf{b}/\boldsymbol{\lambda}}}{p_{\mathbf{b}/\boldsymbol{\mu}}}[\Delta_{\mathbf{b}}(\boldsymbol{\lambda}) : L_{\mathbf{b}}(\boldsymbol{\mu})] = p_{\boldsymbol{\mu}/\boldsymbol{\lambda}} d^{l}_{\boldsymbol{\lambda}_m,\boldsymbol{\mu}_m},$$

where the second equality follows from (5.22).

Now consider the multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ on the right hand side of (b). If $\boldsymbol{\nu} \in \mathscr{P}_{d,\mathbf{b}}$ and $1 \le j \le p_{\boldsymbol{\nu}}$ then, using Corollary 5.15 again,

$$L^{\boldsymbol{\lambda}}_{j,p}{\downarrow}^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong L_{\mathbf{b}}(\boldsymbol{\lambda}) \oplus L_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{-1}} \oplus \cdots \oplus L_{\mathbf{b}}(\boldsymbol{\lambda})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}/\boldsymbol{\nu}}}}.$$

Therefore, $[L^{\boldsymbol{\lambda}}_{j,p}{\downarrow}^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} : L_{\mathbf{b}}(\boldsymbol{\mu})] = 1$ by Lemma 5.11. Equating the multiplicity of $L_{\mathbf{b}}(\boldsymbol{\mu})$ on both sides of (b) now gives (c). ☐

Lemma 5.23 gives our first relation satisfied by the decomposition numbers $d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$. We now use formal characters to find more relations. Let $K[\mathscr{P}_{r,n}]$ be the K-vector space with basis $\{\, e^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_{r,n}\,\}$. The ($K$-valued) **formal character** of the $\mathscr{S}_{d,\mathbf{b}}$-module M is

$$\operatorname{ch} M = \sum_{\boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}} (\dim M_{\boldsymbol{\mu}}) e^{\boldsymbol{\mu}},$$

an element of $K[\mathscr{P}_{r,n}]$. The coefficients appearing in the formal characters are the traces of the identity maps on the weight spaces. We need a more general version of the formal character which records the traces of powers of $\vartheta^t_{\boldsymbol{\lambda}}$, for $1 \le t < l = p_{\boldsymbol{\lambda}}$, on certain weight spaces.

Fix an integer t with $1 \le t < p_{\boldsymbol{\lambda}}$. Let $l_t = \gcd(t, l)$ be the greatest common divisor of t and l and set $\ell_t = l/l_t$. By convention, we set $l_0 = l$. Then $r/\ell_t = dml_t$ so that $K[\mathscr{P}_{dml_t,n/\ell_t}] = K[\mathscr{P}_{r/\ell_t,n/\ell_t}]$.

Now suppose that M is an $\mathscr{S}_{r,p,n}(\mathbf{b})$-module and that $\boldsymbol{\gamma} = \gamma^{\ell_t}$ is an ℓ_t-symmetric multipartition. Since $p/\ell_t = ml_t$ divides tm, it is possible to show that the map $\vartheta^t_{\boldsymbol{\lambda}}$ stabilizes the ℓ_t-symmetric weight space $M_{\gamma^{\ell_t}}$ using Lemma 5.8 and Remark 5.9; see the proof of Lemma 5.24 below. Define the **twining character** of M to be

$$\mathrm{ch}^{\mathrm{l}}_{\mathrm{t}} M = \sum_{\gamma \in \mathscr{P}_{r/\ell_t,n/\ell_t}} \operatorname{Tr}\left(\vartheta^t_{\boldsymbol{\lambda}}, M_{\gamma^{\ell_t}}\right) e^{\gamma} \in K[\mathscr{P}_{r/\ell_t,n/\ell_t}].$$

It is easy to see that, just like the usual character, the twining character lifts to a well-defined map $\mathrm{ch}^{\mathrm{l}}_{\mathrm{t}} : \mathcal{R}(\mathscr{S}_{r,p,n}(\mathbf{b})) \longrightarrow K[\mathscr{P}_{r/\ell_t,n/\ell_t}]$ on the Grothendieck group of $\mathscr{S}_{r,p,n}(\mathbf{b})$.

The following Lemma will allow us to compute the twining character $\mathrm{ch}^{\mathrm{l}}_{\mathrm{t}}$ on both sides of Lemma 5.23(b).

5.24. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_{d,\mathbf{b}}$ and $1 \le t < l = p_{\boldsymbol{\lambda}}$. Then*

$$ch^{\mathrm{l}}_{\mathrm{t}}\, \Delta^{\boldsymbol{\lambda}}_{i,p} = \varepsilon^{itm} p_{\mathbf{b}/\boldsymbol{\lambda}} \mathfrak{g}^t_{\boldsymbol{\lambda}} \operatorname{ch} \Delta_{\mathbf{b}_{l_t m}}(\boldsymbol{\lambda}_{l_t m}),$$

for $1 \le i \le p_{\boldsymbol{\lambda}}$. Moreover, if $\boldsymbol{\mu} \in \mathscr{P}^{\,l}_{d,\mathbf{b}}$ and $1 \le j \le p_{\boldsymbol{\mu}}$ then

$$ch^{\mathrm{l}}_{\mathrm{t}}\, L^{\boldsymbol{\mu}}_{j,p} = \varepsilon^{jtm} p_{\mathbf{b}/\boldsymbol{\mu}} \mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}} \operatorname{ch} L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m}).$$

Proof. We only prove the formula for $\operatorname{ch}^{\mathsf{l}}_{\mathsf{t}} L^{\boldsymbol{\mu}}_{j,p}$ and leave the almost identical calculation of $\operatorname{ch}^{\mathsf{l}}_{\mathsf{t}} \Delta^{\boldsymbol{\lambda}}_{0,p}$ to the reader. To ease the notation let $m' = \mathsf{o}_{\boldsymbol{\mu}}$ so that $\mathbf{b}_{m'} = (\mathbf{b}^{[1]}, \dots, \mathbf{b}^{[m']})$ and $\boldsymbol{\mu}_{m'} = (\boldsymbol{\mu}^{[1]}, \dots, \boldsymbol{\mu}^{[m']}) \in \mathscr{P}_{r/p_{\boldsymbol{\mu}}, \mathbf{b}_{m'}}$.

To determine $\operatorname{ch}^{\mathsf{l}}_{\mathsf{t}} L^{\boldsymbol{\mu}}_{j,p}$ for each $\gamma \in \mathscr{P}_{r/\ell_t, n/\ell_t}$ we need to compute

$$\begin{aligned}\operatorname{Tr}\big(\vartheta^t_{\boldsymbol{\lambda}}, (L^{\boldsymbol{\mu}}_{j,p})_{\gamma^{\ell_t}}\big) &= \operatorname{Tr}\big(\vartheta_{\mathbf{b}}^{tp_{\mathbf{b}/\boldsymbol{\lambda}}}, (L^{\boldsymbol{\mu}}_{j,p})_{\gamma^{\ell_t}}\big) = \operatorname{Tr}\big((\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol{\mu}}})^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}, (L^{\boldsymbol{\mu}}_{j,p})_{\gamma^{\ell_t}}\big)\\ &= \operatorname{Tr}\big(\vartheta_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}, (L^{\boldsymbol{\mu}}_{j,p})_{\gamma^{\ell_t}}\big).\end{aligned}$$

By Corollary 5.15 we can identify $L^{\boldsymbol{\mu}}_{j,p}$ with the K-vector space

$$L_{\mathbf{b}}(\boldsymbol{\mu}) \oplus L_{\mathbf{b}}(\boldsymbol{\mu})\vartheta_{\mathbf{b}} \oplus \cdots \oplus L_{\mathbf{b}}(\boldsymbol{\mu})\vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol{\mu}}-1},$$

where the action of $\mathscr{S}_{r,p,n}(\mathbf{b})$ on $L^{\boldsymbol{\mu}}_{j,p}$ is determined by

a) $L^{\boldsymbol{\mu}}_{j,p}\downarrow^{\mathscr{S}_{r,p,n}(\mathbf{b})}_{\mathscr{S}_{d,\mathbf{b}}} \cong L_{\mathbf{b}}(\boldsymbol{\mu}) \oplus L_{\mathbf{b}}(\boldsymbol{\mu})^{\vartheta_{\mathbf{b}}^{-1}} \oplus \cdots \oplus L_{\mathbf{b}}(\boldsymbol{\mu})^{\vartheta_{\mathbf{b}}^{1-p_{\mathbf{b}/\boldsymbol{\mu}}}}$,
b) $(x\vartheta^a_{\mathbf{b}})\vartheta^c_{\mathbf{b}} = x\vartheta^{a+c}_{\mathbf{b}}$, for all $x \in L_{\mathbf{b}}(\boldsymbol{\mu})$ and $a, c \in \mathbb{Z}$,
c) $\vartheta_{\boldsymbol{\mu}}$ acts as the scalar $\varepsilon^{j\mathsf{o}_{\boldsymbol{\mu}}}\mathfrak{g}_{\boldsymbol{\mu}}$ on the highest weight vector of $L_{\mathbf{b}}(\boldsymbol{\mu})$.

Note that $p_{\boldsymbol{\mu}/\boldsymbol{\lambda}} = m/m' \in \mathbb{N}$, since $\boldsymbol{\mu} \in \mathscr{P}^l_{d,\mathbf{b}}$, and $\vartheta_{\boldsymbol{\mu}} = \vartheta_{\mathbf{b}}^{p_{\mathbf{b}/\boldsymbol{\mu}}} = \vartheta_{\boldsymbol{\lambda}}^{p_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}$. Therefore,

$$\operatorname{Tr}\big(\vartheta^t_{\boldsymbol{\lambda}}, (L^{\boldsymbol{\mu}}_{j,p})_{\gamma^{\ell_t}}\big) = \operatorname{Tr}\big(\vartheta_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}, (L^{\boldsymbol{\mu}}_{j,p_{\boldsymbol{\mu}}})_{\gamma^{\ell_t}}\big) = p_{\mathbf{b}/\boldsymbol{\mu}} \operatorname{Tr}\big(\vartheta_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}, L_{\mathbf{b}}(\boldsymbol{\mu})_{\gamma^{\ell_t}}\big).$$

To compute this trace first observe that if $\overline{\varphi}_{\mathfrak{t}^{\boldsymbol{\mu}}}$ is the highest weight vector of $L_{\mathbf{b}}(\boldsymbol{\mu})$ then, by (c) above (which comes from Corollary 5.15),

$$\overline{\varphi}_{\mathfrak{t}^{\boldsymbol{\mu}}}\vartheta^t_{\boldsymbol{\lambda}} = \varepsilon^{jtm}\mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}\overline{\varphi}_{\mathfrak{t}^{\boldsymbol{\mu}}}. \tag{5.25}$$

Now, $p = \ell_t l_t m = \ell_t l_t p_{\boldsymbol{\mu}/\boldsymbol{\lambda}} m'$ so we can identify the two modules $L_{\mathbf{b}}(\boldsymbol{\mu})$ and $L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m})^{\otimes \ell_t}$. Using Lemma 5.8, if $1 \le j \le p/\ell_t$ then

$$\varphi^{(j)}_{\mathsf{ST}}\vartheta^t_{\boldsymbol{\lambda}} = \varepsilon^{-mtk}\vartheta^t_{\boldsymbol{\lambda}}\varphi^{(tm+j)}_{\mathsf{ST}} \tag{5.26}$$

for some $k \in \mathbb{Z}$, where we identity $\varphi^{(j)}_{\mathsf{ST}}$ and $\varphi^{(j')}_{\mathsf{ST}}$ if $j \equiv j' \pmod p$. Therefore, since $\overline{\varphi}_{\mathfrak{t}^{\boldsymbol{\mu}}}$ generates $L_{\mathbf{b}}(\boldsymbol{\mu})$, it follows from (5.25) and (5.26) that each simple p-tensor

$$\boldsymbol{\beta} = (x^{(1)}_1 \otimes \cdots \otimes x^{(1)}_{l_t m}) \otimes \cdots \otimes (x^{(\ell_t)}_1 \otimes \cdots \otimes x^{(\ell_t)}_{l_t m})$$

in $L_{\mathbf{b}}(\boldsymbol{\mu})_{\gamma^{\ell_t}}$ is mapped by $\vartheta^t_{\boldsymbol{\lambda}} = \vartheta_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}$ to a scalar multiple of

$$(x^{(tm+1)}_1 \otimes \cdots \otimes x^{(tm+1)}_{l_t m}) \otimes \cdots \otimes (x^{(tm+\ell_t)}_1 \otimes \cdots \otimes x^{(tm+\ell_t)}_{l_t m}),$$

where we identity $x^{(j)}_i = x^{(j')}_i$ whenever $j \equiv j' \pmod{\ell_t}$ for $1 \le i \le l_t m$. Thus, to calculate $\operatorname{Tr}\big(\vartheta^t_{\boldsymbol{\lambda}}, L_{\mathbf{b}}(\boldsymbol{\mu})\big)$ we only need to consider the case when $x^{(s)}_i = x^{(tm+s)}_i$, for all $1 \le i \le l_t m$ and all $1 \le s \le \ell_t$. By construction, $(tm)/(l_t m) \not\equiv 0 \pmod{\ell_t}$, so this can only happen if

$$x^{(s)}_i = x^{(s')}_i, \quad \text{whenever } 1 \le i \le l_t m \text{ and } 1 \le s, s' \le \ell_t.$$

Consequently, $\boldsymbol{\beta}$ contributes to the twining character only if $\boldsymbol{\beta} = \beta \otimes \cdots \otimes \beta$ (ℓ_t times), for some $\beta \in L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m})$. Notice that if $\beta \in L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m})_\gamma$, for some $\gamma \in \mathscr{P}_{r/\ell_t, n/\ell_t}$ then $\boldsymbol{\beta} \in L_{\mathbf{b}}(\boldsymbol{\mu})_{\gamma^{\ell_t}}$. In particular, this shows that $\vartheta^t_{\boldsymbol{\lambda}}$ stabilizes $L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m})_\gamma$ as we claimed when introducing the twining character.

In (5.25) we have already shown that $\vartheta^t_{\boldsymbol{\lambda}}$ acts as multiplication by $\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}$ on the highest weight vector of $L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m})^{\otimes \ell_t}$. On the other hand, by (5.26) and abusing the notation of Lemma 5.8 slightly, if $1 \le j \le \ell_t$ then

$$\big(\varphi^{(j)}_{\mathsf{ST}}\big)^{\otimes \ell_t}\vartheta^t_{\boldsymbol{\lambda}} = \varepsilon^{-mt\ell_t k}\vartheta^t_{\boldsymbol{\lambda}}\big(\varphi^{(j)}_{\mathsf{ST}}\big)^{\otimes \ell_t} = \vartheta^t_{\boldsymbol{\lambda}}\big(\varphi^{(j)}_{\mathsf{ST}}\big)^{\otimes \ell_t},$$

where the last equality follows because $mt\ell_t = p(t/l_t)$ is divisible by p. Therefore, writing $\beta^{\otimes \ell_t} = \overline{\varphi}_{\mathfrak{t}\boldsymbol{\mu}}\varphi^{\otimes \ell_t}$, for some $\varphi \in \mathscr{S}_{l_t m, \mathbf{b}_{l_t m}}$, we have that

$$\beta^{\otimes \ell_t}\vartheta^t_{\boldsymbol{\lambda}} = \overline{\varphi}_{\mathfrak{t}\boldsymbol{\mu}}\varphi^{\otimes \ell_t}\vartheta^t_{\boldsymbol{\lambda}} = \overline{\varphi}_{\mathfrak{t}\boldsymbol{\mu}}\vartheta^t_{\boldsymbol{\lambda}}\varphi^{\otimes \ell_t} = \varepsilon^{jtm}\mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}\overline{\varphi}_{\mathfrak{t}\boldsymbol{\mu}}\varphi^{\otimes \ell_t} = \varepsilon^{jtm}\mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}\beta^{\otimes \ell_t},$$

where the third equality uses (5.25). Consequently,

$$\operatorname{Tr}\left(\vartheta^t_{\boldsymbol{\lambda}}, \left(L^{\boldsymbol{\mu}}_{j,p}\right)_{\gamma^{\ell_t}}\right) = p_{\mathbf{b}/\boldsymbol{\mu}}\varepsilon^{jtm}\mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}\dim L_{\mathbf{b}_{l_t m}}(\boldsymbol{\mu}_{l_t m})_\gamma.$$

Summing over $\mathscr{P}^l_{d,\mathbf{b}}$ gives the desired formula for $\operatorname{ch}^{\mathrm{l}}_{\mathrm{t}}\left(L^{\boldsymbol{\mu}}_{j,p}\right)$ and completes the proof. □

5.27. **Corollary.** *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^l_{d,\mathbf{b}}$, and $0 \le t < l = p_{\boldsymbol{\lambda}}$, $l' = p_{\boldsymbol{\mu}}$. Then in K*

$$p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}\left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}^{p_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}}\right)^t d^{l_t}_{\boldsymbol{\lambda}_m,\boldsymbol{\mu}_m} = \varepsilon^{tm}d^{(1)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} + \varepsilon^{2tm}d^{(2)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} + \cdots + \varepsilon^{l'tm}d^{(l')}_{\boldsymbol{\lambda}\boldsymbol{\mu}}.$$

Proof. If $t = 0$ then the result is just Lemma 5.23(c). If $t \ne 1$ then combining Lemma 5.24 and Lemma 5.23(b) shows that

$$\operatorname{ch}\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m)^{\otimes l_t} = \sum_{\boldsymbol{\mu}\in\mathscr{P}^l_{d,\mathbf{b}}}\sum_{1\le j\le p_{\boldsymbol{\mu}}} \varepsilon^{jmt}d^{(j)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}\frac{p_{\mathbf{b}/\boldsymbol{\mu}}\mathfrak{g}_{\boldsymbol{\mu}}^{tp_{\boldsymbol{\mu}/\boldsymbol{\lambda}}}}{p_{\mathbf{b}/\boldsymbol{\lambda}}\mathfrak{g}^t_{\boldsymbol{\lambda}}}\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)^{\otimes l_t}.$$

On the other hand, by Lemma 5.23(a),

$$\operatorname{ch}\Delta_{\mathbf{b}_m}(\boldsymbol{\lambda}_m)^{\otimes l_t} = \sum_{\boldsymbol{\mu}\in\mathscr{P}^l_{d,\mathbf{b}}} d^{l_t}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)^{\otimes l_t}.$$

As the characters $\{\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol{\nu}_m)\}$ are linearly independent, comparing the coefficient of $\operatorname{ch}L_{\mathbf{b}_m}(\boldsymbol{\mu}_m)$ on both sides gives the result. □

5.28. **Corollary.** *Suppose that l divides p, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^l_{d,\mathbf{b}}$, $0 \le t < l$ and that $p_{\boldsymbol{\lambda}} = p_{\boldsymbol{\mu}} = l$. Then in K*

$$\left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^t d^{l_t}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m} = \varepsilon^{tm}d^{(1)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} + \varepsilon^{2tm}d^{(2)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} + \cdots + \varepsilon^{ltm}d^{(l)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}.$$

We can now complete the proof of the main results of this paper. Recall from just before Theorem D in the introduction that we defined matrices $V(l)$ and $V_i(l)$, whenever l divides p and $1 \le i \le l$. Let $\operatorname{char}K$ be the characteristic of the field K.

5.29. **Theorem.** *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{d,\mathbf{b}}$ and $p_\lambda = l = p_{\boldsymbol{\lambda}}$, for some $\mathbf{b} \in \mathscr{C}_{p,n}$. Then, for $1 \le j \le p_{\boldsymbol{\lambda}}$,*

$$[\Delta^{\boldsymbol{\lambda}}_{0,p} : L^{\boldsymbol{\mu}}_{j,p}] \equiv \frac{\det V_j(l)}{\det V(l)} \pmod{\operatorname{char}K}.$$

In particular, $[\Delta^{\boldsymbol{\lambda}}_{0,p} : L^{\boldsymbol{\mu}}_{j,p}] = \frac{\det V_j(l)}{\det V(l)}$ if K is a field of characteristic zero.

Proof. By Corollary 5.28 the decomposition numbers $d^{(1)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}, \dots, d^{(l)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ satisfy the matrix equation

$$V(l)\begin{pmatrix} d^{(1)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \\ \vdots \\ d^{(l)}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m}\end{pmatrix} = \begin{pmatrix} \left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^0 d^{l_0}_{\boldsymbol{\lambda}_m\boldsymbol{\mu}_m} \\ \vdots \\ \left(\frac{\mathfrak{g}_{\boldsymbol{\lambda}}}{\mathfrak{g}_{\boldsymbol{\mu}}}\right)^{l-1} d^{l_{(l-1)}}_{\boldsymbol{\lambda}_m,\boldsymbol{\mu}_m}\end{pmatrix}$$

Hence, the theorem follows by Cramer's rule. □

Recall that the decomposition number $[\Delta^{\boldsymbol{\lambda}}_{i,p} : L^{\boldsymbol{\mu}}_{j,p}]$ is l-splittable if $p_{\boldsymbol{\lambda}} = l = p_{\boldsymbol{\mu}}$, for $1 \le i, j \le p_{\boldsymbol{\lambda}}$. Combining Corollary 5.20, (5.21) and Theorem 5.29 we can now compute the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}(\mathbf{b})$ and $\mathscr{H}_{r,p,n}$.

5.30. **Corollary.** *Suppose that* $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{d,\mathbf{b}}$*, for some* $\mathbf{b}\in\mathscr{C}_{p,n}$*, and that* $p_{\boldsymbol{\lambda}}=p_{\boldsymbol{\mu}}$*. Then, for* $1\le i,j\le p_{\boldsymbol{\lambda}}$*,*

$$[S_i^{\boldsymbol{\lambda}}:D_j^{\boldsymbol{\mu}}]=[\Delta_{i,p}^{\boldsymbol{\lambda}}:L_{j,p}^{\boldsymbol{\mu}}]\equiv\frac{\det V_{j-i}(l)}{\det V(l)}\pmod{\operatorname{char}K}.$$

In particular, this establishes Theorem D from the introduction. Finally, we are able to prove Theorem A, our Main Theorem from the introduction.

Proof of Theorem A. By [**23**, Theorem B] the decomposition numbers of $\mathscr{H}_{r,p,n}$ are completely determined by the l-splittable decomposition numbers of the Hecke algebras $\mathscr{H}_{s,l,m}$, where l divides p, $1\le s\le r$ and $1\le m\le n$. Hence, Theorem A follows from Corollary 5.30. □

We remind the reader that the polynomials $\dot{\mathfrak{f}}_{\boldsymbol{\lambda}}=\dot{\varepsilon}^{\frac12 dmn(1-p_{\boldsymbol{\lambda}})}(\dot{\mathfrak{g}}_{\boldsymbol{\lambda}})^{p_{\boldsymbol{\lambda}}}$ are completely determined by Theorem 3.6. Hence, this result explicitly determines the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ (and of $\mathscr{H}_{r,p,n}$).

When K is a field of positive characteristic the results above only determine the l-splittable decomposition numbers of $\mathscr{S}_{r,p,n}$ and $\mathscr{H}_{r,p,n}$ modulo the characteristic of K.

Acknowledgment

The research was supported by the Australian Research Council.

References

[1] S. Ariki, *On the semi–simplicity of the Hecke algebra of* $(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_n$, J. Algebra, **169** (1994), 216–225.
[2] ———, *On the decomposition numbers of the Hecke algebra of* $G(m,1,n)$, J. Math. Kyoto Univ., **36** (1996), 789–808.
[3] S. Ariki and K. Koike, *A Hecke algebra of* $(\mathbf{Z}/r\mathbf{Z})\wr\mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math., **106** (1994), 216–243.
[4] D. J. Benson, *Representations and cohomology*, Cambridge studies in advanced mathematics, **30**, CUP, 1991.
[5] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Asterisque, **212** (1993), 119–189.
[6] M. Broué, G. Malle, and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math., **500** (1998), 127–190.
[7] C. W. Curtis and I. Reiner, *Methods of Representation theory, I*, Wiley-Interscience, New York, 1981.
[8] R. Dipper and G. James, *Representations of Hecke algebras of type* B_n, J. Algebra, **146** (1992), 454–481.
[9] R. Dipper, G. James, and A. Mathas, *Cyclotomic q–Schur algebras*, Math. Z., **229** (1999), 385–416.
[10] R. Dipper and A. Mathas, *Morita equivalences of Ariki–Koike algebras*, Math. Zeit., **240** (2002), 579–610.
[11] J. Du and H. Rui, *Ariki-Koike algebras with semisimple bottoms*, Math. Z., **234** (2000), 807–830.
[12] ———, *Specht modules for Ariki–Koike algebras*, Comm. Alg., **29** (2001), 4710–4719.
[13] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, **150**, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
[14] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Oxford University Press, New York, 2000.
[15] G. Genet, *On decomposition matrices for graded algebras*, J. Algebra, **274** (2004), 523–542.
[16] G. Genet and N. Jacon, *Modular representations of cyclotomic Hecke algebras of type* $G(r,p,n)$, Int. Math. Res. Not., (2006), Art. ID 93049, 18.
[17] V. Ginzbug, N. Guay, E. Opdam, and R. Rouquier, *On the category* $\mathcal{O}$ *for the rational Cherednik algebras*, Invent. Math., **154** (2003), 617–651.
[18] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math., **123** (1996), 1–34.
[19] J. Hu, *A Morita equivalence theorem for Hecke algebra* $\mathscr{H}_q(D_n)$ *when* n *is even*, Manuscripta Math., **108** (2002), 409–430.

[20] ———, *Modular representations of Hecke algebras of type* $G(p, p, n)$, J. Algebra, **274** (2004), 446–490.
[21] ———, *The number of simple modules for the Hecke algebras of type* $G(r, p, n)$, J. Algebra, **321** (2009), 3375–3396. With an appendix by Xiaoyi Cui.
[22] ———, *On the decomposition numbers of the Hecke algebra of type* D_n *when* n *is even*, J. Algebra, **321** (2009), 1016–1038.
[23] J. Hu and A. Mathas, *Morita equivalences of cyclotomic Hecke algebras of type* $G(r, p, n)$, J. Reine. Angew Math., **628** (2009), 169–194.
[24] G. D. James and A. Mathas, *The Jantzen sum formula for cyclotomic q–Schur algebras*, Trans. Amer. Math. Soc., **352** (2000), 5381–5404.
[25] S. Lyle and A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math., **216** (2007), 854–878.
[26] G. Malle and A. Mathas, *Symmetric cyclotomic Hecke algebras*, J. Algebra, **205** (1998), 275–293.
[27] A. Mathas, *Matrix units and generic degrees for the Ariki-Koike algebras*, J. Algebra, **281** (2004), 695–730.

School of Mathematics and Statistics F07, University of Sydney, NSW 2006, Australia
E-mail address: junhu303@yahoo.com.cn

School of Mathematics and Statistics F07, University of Sydney, NSW 2006, Australia
E-mail address: a.mathas@usyd.edu.au