AN ABSTRACT APPROACH TO DOMAIN PERTURBATION FOR PARABOLIC EQUATIONS AND PARABOLIC VARIATIONAL INEQUALITIES

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Abstract. We study the behaviour of solutions of linear non-autonomous parabolic equations subject to Dirichlet or Neumann boundary conditions under perturbation of the domain. We prove that Mosco convergence of function spaces for non-autonomous parabolic problems is equivalent to Mosco convergence of function spaces for the corresponding elliptic problems. As a consequence, we obtain convergence of solutions of non-autonomous parabolic equations under domain perturbation by variational methods using the same characterisation of domains as in elliptic case. A similar technique can be applied to obtain convergence of weak solutions of parabolic variational inequalities when the underlying convex set is perturbed.

1. Introduction

The primary aim of this paper is to study convergence properties of solutions of linear *non-autonomous* parabolic equations under perturbation of the domain. We consider a sequence of bounded open sets $(\Omega_n)_{n=1}^{\infty}$ in $\mathbb{R}^N$ converging to a bounded open set Ω and investigate the behaviour of solutions of the following parabolic equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n(t)u = f_n(x,t) & \text{in } \Omega_n \times (0,T] \\ \mathcal{B}_n(t)u = 0 & \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{1}$$

where $\mathcal{A}_n$ is an elliptic operator of the form

$$\mathcal{A}_n(t)u := -\partial_i[a_{ij}(x,t)\partial_j u + a_i(x,t)u] + b_i(x,t)\partial_i u + c_0(x,t)u,$$

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and $\mathcal{B}_n(t)$ is one of the following boundary conditions

$$\begin{aligned} &\mathcal{B}_n(t)u := u && \text{Dirichlet boundary condition} \\ &\mathcal{B}_n(t)u := [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\,\nu_i && \text{Neumann boundary condition.} \end{aligned}$$

In abstract form, (1) can be written as

$$\begin{cases} u'(t) + A_n(t)u = f(t) & \text{for } t \in (0,T] \\ \qquad\qquad u(0) = u_{0,n} \end{cases} \tag{2}$$

in a Banach space V_n, where $V_n := H_0^1(\Omega_n)$ for Dirichlet problems or $V_n := H^1(\Omega_n)$ for Neumann problems. We refer to Section 2 for the precise framework of these parabolic equations. We are particularly interested in singular domain perturbation so that change of variables is not possible on these domains. Typically, the common examples include a sequence of dumbbell shape domains with shrinking handle, and a sequence of domains with cracks. Moreover, we mostly do not assume any smoothness of Ω_n and Ω. The second aim of this paper is to study a similar convergence properties of solutions of parabolic variational inequalities

$$\begin{cases} \langle u'(t), v - u(t)\rangle + \langle A(t)u(t), v - u(t)\rangle - \langle f(t), v - u(t)\rangle \geq 0, & \forall v \in K \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(0) = u_0. \end{cases} \tag{3}$$

on $(0,T)$ when we perturb the underlying convex set K in the problem.

To deal with non-autonomous parabolic equations, it is common to apply variational methods. In this paper we prove that under suitable assumptions on domains, a sequence of solutions u_n of (1) converges to the solution u of a linear non-autonomous parabolic equation on the limit domain Ω ((1) with n deleted). This result sometimes refers to *stability* of solutions under domain perturbation or *continuity* of solutions with respect to the domain. The method presented in this work is rather an abstract approach. In particular, it can be applied to obtain stability of solutions under domain perturbation for both Dirichlet problems (Theorem 4.6) and Neumann problems (Theorem 4.13). In general, it is more difficult when handling Neumann boundary condition. We cannot simply consider the trivial extension by zero outside the domain for functions in the sobolev space $H^1(\Omega)$ because the extended function does not belong to $H^1(\mathbb{R}^N)$. Moreover there is no smooth extension from $H^1(\Omega)$ to $H^1(\mathbb{R}^N)$ as we do not impose any regularity of the domain. This means the compactness result for a sequence of solutions u_n in [8, Lemma 2.1] cannot be applied in the case of Neumann problems. However, our abstract approach can be applied to Neumann problems. We refer to Section 4 for the study on stability of solutions of non-autonomous parabolic equations under domain perturbation.

The key result that enables us to determine a sufficient condition on domains for which solutions converge under domain perturbation is Theorem 3.4. In particular, Theorem 3.4 shows that continuity of solutions for non-autonomous problems can be deduced from the corresponding elliptic problems via *Mosco convergence*. We refer to Section 3 for the definition and results on Mosco convergence. A similar deduction is well-known for autonomous parabolic equations. This is simply because we can apply semigroup methods together with convergence result of degenerate semigroups due to Arendt [2, Theorem 5.2]. In Section 6 of the same paper, stability of solutions of Dirichlet heat equation is given as an example. Further examples on other boundary conditions including Neumann and Robin boundary conditions can be found in [10, Section 6]. Indeed, for quasilinear parabolic equations, Simondon [17] also obtained continuity of solutions of parabolic equations under Dirichlet boundary condition using a similar equivalence of Mosco convergences between certain Banach spaces. However, Theorem 3.4 can be seen as an abstract generalisation of [17]. We show equivalence between Mosco convergences of various *closed and convex subsets* of a Banach space rather than Mosco convergences of a particular choice of closed subspaces of a Banach space. The obvious reason for this generalisation is that Mosco convergence was originally introduced in [16] for convex sets and was the main tool to establish convergence properties of solutions of elliptic variational inequalities when the convex set is perturbed. The second advantage of Theorem 3.4 is that we do not only show the equivalence between Mosco convergence of convex subsets of the Bochner-Lebesgue space $L^2((0,T),V)$ and Mosco convergence of convex subsets of the corresponding Banach space V but also show that they are equivalent to Mosco convergence of convex subsets of the Bochner-Sobolev spaces $W((0,T),V,V')$. Hence a similar technique can be applied to obtain stability of solutions of parabolic variational inequalities when the underlying convex set is perturbed (Theorem 5.3). We study convergence of solutions of parabolic variational inequalities in Section 5.

An important consequence of Theorem 3.4 is that the same conditions for a sequence of domains give stability of solutions under domain perturbation for both parabolic and elliptic equations. We refer to [5–7, 9] for the study of domain perturbation for elliptic equations using Mosco convergence.

2. Preliminaries on parabolic equations and parabolic variational inequalities

In this section we state some basic results on variational methods for parabolic equations and a variational formulation for parabolic inequalities.

Suppose V is a real separable and reflexive Banach space and H is a separable Hilbert space such that V is dense in H. By identifying H with its dual space H', we consider the following *evolution triple*

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'.$$

Throughout this paper, we denote by $(\cdot|\cdot)$, the scalar product in H and $\langle\cdot,\cdot\rangle$, the duality paring between V' and V. For an interval $(a,b)\subset\mathbb{R}$, we denote by $L^2((a,b),V)$ the Bochner-Lebesgue space. We define the Bochner-Sobolev space

$$W((a,b),V,V') := \{u \in L^2((a,b),V) : u' \in L^2((a,b),V')\},$$

where u' is the derivative in the sense of distributions taking values in V'. The space $W((a,b),V,V')$ is a Banach space when equipped with the following norm

$$\|u\|_W := \Big(\int_a^b \|u(t)\|_V^2\, dt + \int_a^b \|u'(t)\|_{V'}^2\, dt\Big)^{1/2}.$$

It is well known that $W((a,b),V,V') \hookrightarrow C([a,b],H)$, where the space of H-valued continuous functions $C([a,b],H)$ equipped with the uniform norm ([11, Theorem I1.3.1]). Moreover, for $u,v \in W((a,b),V,V')$ and $a_0, b_0 \in [a,b]$ with $a_0 < b_0$ we have the integration by parts formula

$$(u(b_0)|v(b_0)) - (u(a_0)|v(a_0)) = \int_{a_0}^{b_0} \langle u'(t), v(t)\rangle + \langle v'(t), u(t)\rangle\, dt. \tag{4}$$

Let I, J be two sets, we write $J \subset\subset I$ if $\bar{J} \subset I$. For a subset X of a Banach space V, we define the closed convex hull by

$$\overline{\text{conv}}(X) := \overline{\Big\{\sum_{i=1}^k \alpha_i x_i \mid x_i \in X, \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1, k = 1,2,\dots\Big\}}.$$

For each $t \in [0,T]$, suppose $a(t;\cdot,\cdot)$ is a continuous bilinear form on V satisfying the following hypothesis:

- for every $u,v \in V$, the map $t \mapsto a(t;u,v)$ is measurable.
- there exists a constant $M > 0$ independent of $t \in [0,T]$ such that
$$|a(t;u,v)| \leq M\|u\|_V\|v\|_V, \tag{5}$$
for all $u,v \in V$.
- there exist $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that
$$a(t;u,u) + \lambda\|u\|_H^2 \geq \alpha\|u\|_V^2, \tag{6}$$
for all $u \in V$.

It follows that for each $t \in [0,T]$ and $u \in V$ the bilinear form $a(t;\cdot,\cdot)$ induces a continuous linear operator $A(t) \in \mathscr{L}(V,V')$ with

$$\langle A(t)u, v\rangle = a(t;u,v),$$

for all $u,v \in V$. We easily see from (5) that $\sup_{t\in[0,T]} \|A(t)\|_{\mathscr{L}(V,V')} \leq M$.

2.1. **Parabolic equations.** Let us consider the abstract parabolic equation

$$\begin{cases} u'(t) + A(t)u = f(t) & \text{for } t \in (0,T] \\ u(0) = u_0, \end{cases} \tag{7}$$

where $u_0 \in H$ and $f \in L^2((0,T),V')$. A function $u \in W(0,T,V,V')$ satisfying (7) is called a *variational solution*. It is well known that u is a variational solution of

(7) if and only if $u \in L^2((0,T),V)$ and

$$\begin{aligned} -\int_0^T (u(t)|v)\phi'(t)\,dt &+ \int_0^T a(t;u(t),v)\phi(t)\,dt \\ &= (u_0|v)\phi(0) + \int_0^T \langle f(t),v\rangle \phi(t)\,dt, \end{aligned} \tag{8}$$

for all $v \in V$ and for all $\phi \in \mathscr{D}([0,T))$. The existence and uniqueness of solution is given in the following theorem (see, for example, [11, XVIII §3] and [18, §23.7]).

Theorem 2.1. *Given $f \in L^2((0,T),V')$ and $u_0 \in H$, there exists a unique variational solution of* (7) *satisfies*

$$\|u\|_{W(0,T,V,V')} \leq C\Big(\|u_0\|_H + \|f\|_{L^2((0,T),V')}\Big). \tag{9}$$

Moreover, if $\lambda = 0$ in (6) *the variational solution satisfies*

$$\|u(t)\|_H^2 + \alpha \int_0^t \|u(s)\|_V^2\,ds \leq \|u_0\|_H^2 + \alpha^{-1}\int_0^t \|f(s)\|_{V'}^2\,ds, \tag{10}$$

for all $t \in [0,T]$.

Note that $v(t) := e^{-\lambda t}u(t)$ is a variational solution of (7) with $A(t)$ replaced by $A(t)+\lambda$. Hence we can assume without loss of generality that $\lambda = 0$ in (6).

Let Ω be an open bounded set in $\mathbb{R}^N$. Let $D \subset \mathbb{R}^N$ be a ball such that $\Omega \subset D$. We shall consider a closed subspace V of $H^1(\Omega)$ with $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. We take $H := L^2(\Omega)$ and consider the evolution triple $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$. In this paper, we study bilinear forms $a(t;\cdot,\cdot)$ for $t \in [0,T]$ given by

$$a(t;u,v) := \int_\Omega [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\partial_i v + b_i(x,t)\partial_i uv + c_0(x,t)uv\,dx, \tag{11}$$

for $u, v \in V$. In the above, we use summation convention with i,j running from 1 to N. Also, we assume a_{ij}, a_i, b_i, c_0 are functions in $L^\infty(D \times (0,T))$ and there exists a constant $\alpha > 0$ independent of $(x,t) \in \Omega \times (0,T)$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2,$$

for all $\xi \in \mathbb{R}^N$. It is clear that the map $t \mapsto a(t;u,v)$ is measurable for all $u, v \in V$. Moreover, it can be verified that the form $a(t;\cdot,\cdot)$ defined above satisfies (5) and (6) (see [11]). Let $\mathcal{A}(t)$ be a differential operator on V defined by

$$\mathcal{A}(t)u := -\partial_i[a_{ij}(x,t)\partial_j u + a_i(x,t)u] + b_i(x,t)\partial_i u + c_0(x,t)u. \tag{12}$$

Given $u_0 \in L^2(D)$ and $f \in L^2(D \times (0,T))$, we consider the following parabolic boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = f(x,t) & \text{in } \Omega \times (0,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{13}$$

where $\mathcal{B}(t)$ is one of the following boundary conditions

$$\mathcal{B}(t)u := u \qquad \text{Dirichlet boundary condition}$$
$$\mathcal{B}(t)u := [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\,\nu_i \qquad \text{Neumann boundary condition}$$

It is well known that we can consider the boundary value problem (13) as an abstract equation (7) by taking $V = H^1_0(\Omega)$ for Dirichlet boundary problem or $V = H^1(\Omega)$ for Neumann boundary problem ([18, Corollary 23.24]).

2.2. **Parabolic variational inequalities.** Suppose that K is a closed and convex subset of V. We denote by

$$L^2((0,T),K) := \{u \in L^2((0,T),V) \mid u(t) \in K \text{ a.e.}\}.$$

For each $t \in (0,T)$, suppose $a(t;\cdot,\cdot)$ is a continuous bilinear form on V satisfying (5) and (6). As before, we denote the induced linear operator by $A(t)$. Given $u_0 \in K$ and $f \in L^2((0,T),V')$, we wish to find u such that for a.e. $t \in (0,T)$, $u(t) \in K$ and

$$\begin{cases} \langle u'(t), v-u(t)\rangle + \langle A(t)u(t), v-u(t)\rangle - \langle f(t), v-u(t)\rangle \geq 0, \quad \forall v \in K \\ \hfill u(0) = u_0. \end{cases} \tag{14}$$

A function $u \in W((0,T),V,V')$ satisfying (14) is called a *strong solution* of parabolic variational inequality (14). In this paper, we are mainly interested in a weak formulation of the problem. There are various (slightly different) definitions of weak solution of parabolic variational inequalities (see e.g. [4], [13], [14], [15]). We shall define a weak notion of solution similar to the one in [13] as follows.

Definition 2.2. A function u is a *weak solution* of parabolic variational inequality (14) if $u \in L^2((0,T),K)$ and

$$\begin{aligned} \int_0^T \langle v'(t), v(t)-u(t)\rangle + \langle A(t)u(t), v(t)-u(t)\rangle - \langle f(t), v(t)-u(t)\rangle \, dt \\ + \frac{1}{2}\|v(0)-u_0\|_H^2 \geq 0, \end{aligned} \tag{15}$$

for all $v \in W((0,T),V,V') \cap L^2((0,T),K)$.

The existence and uniqueness of weak solutions of parabolic variational inequalities have been studied by various authors according to their definitions. In our case, we can state the result in the following theorem.

Theorem 2.3. *Given $u_0 \in K$ and $f \in L^2((0,T),V')$. There exists a unique weak solution u of the parabolic variational inequality (14) satisfying $u \in L^\infty((0,T),H)$.*

Note that the existence of our weak solution follows immediately from the existence results in [14, Theorem 6.2]. The uniqueness can be proved in the same way as in [15, Theorem 2.3].

3. Mosco convergence

We often consider Mosco convergence as introduced in [16] when dealing with a sequence of functions belonging to a sequence of function spaces. In this section, we establish the key result which enables us to study domain perturbation for parabolic problems via the corresponding elliptic problems. We prove that Mosco convergence of function spaces for non-autonomous parabolic problems is equivalent to Mosco convergence of function spaces for the corresponding elliptic problems. Throughout this section, we assume that V is a reflexive and separable Banach space, and K_n, K are closed and convex subsets of V. We start by giving a definition of Mosco convergence in various spaces including V, $L^2((0,T),V)$ and $W((0,T),V,V')$.

Definition 3.1. We say that K_n converges to K in the sense of Mosco if the following conditions hold

- $(M1)$ For every $u \in K$ there exists a sequence $u_n \in K_n$ such that $u_n \to u$ strongly in V.
- $(M2)$ If (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in K_{n_k}$ for every k and $u_n \rightharpoonup u$ weakly in V, then $u \in K$.

There is an alternative definition of Mosco convergence defined in terms of Kuratowski limits. A general result on Mosco convergence and equivalence of these definitions can be found in [3, Chapter 3].

As discussed in Section 2, solutions of parabolic equations and parabolic variational inequalities are functions in $L^2((0,T),V)$. Thus, it is worthwhile to study Mosco convergence in $L^2((0,T),V)$. We denote by

$$L^2((0,T),K) := \{u \in L^2((0,T),V) \mid u(t) \in K \text{ a.e.}\},$$

and

$$C([0,T],K) := \{u \in C([0,T],V) \mid u(t) \in K \quad \forall t \in [0,T]\}.$$

It can be verified that $L^2((0,T),K)$ is a closed and convex subset of $L^2((0,T),V)$. We next state Mosco convergence of function spaces for parabolic problems.

Definition 3.2. We say that $L^2((0,T),K_n)$ converges to $L^2((0,T),K)$ in the sense of Mosco if the following conditions hold

- $(M1')$ For every $u \in L^2((0,T),K)$ there exists a sequence $u_n \in L^2((0,T),K_n)$ such that $u_n \to u$ strongly in $L^2((0,T),V)$.
- $(M2')$ If (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in L^2((0,T),K_{n_k})$ for every k and $u_k \rightharpoonup u$ weakly in $L^2((0,T),V)$, then $u \in L^2((0,T),K)$.

It is also useful to define a similar Mosco convergence in $W((0,T),V,V')$ when studying domain perturbation for parabolic variational inequalities.

Definition 3.3. We say that $W((0,T),V,V') \cap L^2((0,T),K_n)$ converges to $W((0,T),V,V') \cap L^2((0,T),K)$ in the sense of Mosco if the following conditions hold

($M1$”) for every $u \in W((0,T),V,V') \cap L^2((0,T),K)$ there exists a sequence $u_n \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that u_n converges strongly to u in $W((0,T),V,V')$.

($M2$”) if (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in W((0,T),V,V') \cap L^2((0,T),K_{n_k})$ for every k, $u_k \rightharpoonup u$ weakly in $L^2((0,T),V)$ and $u'_k \rightharpoonup w$ weakly in $L^2((0,T),V')$, then $u' = w$ and $u \in W((0,T),V,V') \cap L^2((0,T),K)$.

The following theorem is the key result of this paper.

Theorem 3.4. *The following assertions are equivalent:*

(i) K_n *converges to* K *in the sense of Mosco.*
(ii) $L^2((0,T),K_n)$ *converges to* $L^2((0,T),K)$ *in the sense of Mosco.*
(iii) $W((0,T),V,V') \cap L^2((0,T),K_n)$ *converges to* $W((0,T),V,V') \cap L^2((0,T),K)$ *in the sense of Mosco.*

Before proving the equivalence of Mosco convergences in Theorem 3.4, we require some technical lemmas.

Lemma 3.5. *For a bounded open interval* $(a,b) \subset \mathbb{R}$*, let* $u \in L^2((a,b),K)$*. If* $\phi \in \mathscr{D}((a,b))$ *such that* $\int_a^b \phi(t)\,dt = 1$ *then* $\int_a^b u(t)\phi(t)\,dt \in K$*.*

Proof. Since K is closed and convex, $\int_a^b u(t)\phi(t)\,dt \in \overline{\operatorname{conv}}\{u(t) \mid t \in (a,b)\} \subset K$ for all $u \in L^2((a,b),K)$. □

Lemma 3.6. *Let* $I = (a,b)$ *be a bounded open interval in* $\mathbb{R}$*. If* $u \in L^2(I,V)$ *and* $\int_I u(t)\phi(t)\,dt \in K$ *for all* $\phi \in \mathscr{D}(I)$ *with* $\int_I \phi(t)\,dt = 1$*, then* $u \in L^2(J,K)$ *for all* $J = (c,d) \subset\subset I$*.*

Proof. Let $\eta \in \mathscr{D}(\mathbb{R})$ be the standard mollifier. For $\epsilon > 0$, we define $\eta_\epsilon(t) = \frac{1}{\epsilon}\eta(\frac{t}{\epsilon})$ so that $\eta_\epsilon \in \mathscr{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \eta_\epsilon(t)\,dt = 1$ and $\operatorname{supp}(\eta_\epsilon) \subset (-\epsilon,\epsilon)$. Consider the mollified function $u_\epsilon := \eta_\epsilon * u$. For a.e. $t \in I$, we have

$$\begin{aligned}
\|u_\epsilon(t) - u(t)\|_V &= \left\| \int_{t-\epsilon}^{t+\epsilon} \eta_\epsilon(t-s)[u(s)-u(t)]\,ds \right\|_V \\
&\leq \frac{1}{\epsilon}\int_{t-\epsilon}^{t+\epsilon} \eta\Big(\frac{t-s}{\epsilon}\Big)\|u(s)-u(t)\|_V\,ds \\
&\leq C\frac{1}{\epsilon}\int_{t-\epsilon}^{t+\epsilon} \|u(s)-u(t)\|_V\,ds.
\end{aligned}$$

By Lebesgue’s differentiation theorem for vector valued functions (Theorem III.12.8 of [12]), $u_\epsilon(t) \to u(t)$ in V a.e. $t \in I$. By the definition of u_ϵ,

$$u_\epsilon(t) = \int_I \eta_\epsilon(t-s)u(s)\,ds =: \int_I u(s)\phi_\epsilon(s)\,ds,$$

where we set $\phi_\epsilon(s) := \eta_\epsilon(t-s)$. Let $J \subset\subset I$. For $t \in J$, we can choose ϵ sufficiently small so that $\operatorname{supp}(\phi_\epsilon) = (t-\epsilon, t+\epsilon) \subset I$. It follows from the assumption that $u_\epsilon(t) \in K$ for all $t \in J$. Since K is a closed subset of V, the limit point $u(t) \in K$ a.e. $t \in J$. Hence $u \in L^2(J,K)$ as required. □

Lemma 3.7. *The set $C([0,T],K)$ is dense in $L^2((0,T),K)$.*

Proof. Note first that the lemma is trivial if K is a subspace of V (i.e. K is a Banach space) [18, Theorem 23.2 (c)]. Let $u \in L^2((0,T),K)$. We choose a function $\phi \in \mathscr{D}((0,T))$ with $\int_0^T \phi(t)\,dt = 1$. It follows from Lemma 3.5 that $\xi := \int_0^T u(t)\phi(t)\,dt \in K$. Define the extended function $\tilde{u} \in L^2((-1,T+1),K)$ by

$$\tilde{u}(t) := \begin{cases} \xi & \text{on } (-1,0)\cup(T,T+1) \\ u(t) & \text{on } (0,T). \end{cases}$$

By a mollification argument, the function $u_\epsilon := \eta_\epsilon * \tilde{u}$ belongs to $C(\mathbb{R},V)$. Moreover, u_ϵ converges to $\tilde{u}$ in $L^2((-1,T+1),V)$. By Choosing $0<\epsilon<1$, $u_\epsilon(t) \in K$ for all $t\in[0,T]$. Therefore, the restriction of u_ϵ on $[0,T]$ belongs to $C([0,T],K)$ and converges to u in $L^2((0,T),V)$ as $\epsilon \to 0$. ☐

Lemma 3.8. *The set $C^\infty([0,T],V)\cap C([0,T],K)$ is dense in $W((0,T),V,V')\cap L^2((0,T),K)$.*

Proof. Let $u \in W((0,T),V,V')\cap L^2((0,T),K)$. For $\delta>0$, we define the stretching map $S_\delta : [0,T] \to [-\delta, T+\delta]$ by

$$S_\delta(t) := \Big(\frac{T+2\delta}{T}\Big)t - \delta. \tag{16}$$

We define $u_\delta \in W((-\delta,T+\delta),V,V')\cap L^2((-\delta,T+\delta),K)$ by $u\circ S_\delta^{-1}$. It can be shown that the restriction of u_δ on $(0,T)$ converges to u in $W((0,T),V,V')$ as $\delta \to 0$. Let η_ϵ be a mollifier. For $t\in[0,T]$ and $\epsilon<\delta$, the translation of η_ϵ by t (denoted by $\eta_{\epsilon,t}$) belongs to $\mathscr{D}((-\delta,T+\delta))$. Hence if $\epsilon<\delta$, $\eta_\epsilon * u_\delta$ belongs to $C^\infty([0,T],V)\cap C([0,T],K)$. Moreover, a mollification argument shows that $\eta_\epsilon * u_\delta$ converges to u_δ in $W((0,T),V,V')$ as $\epsilon\to 0$. The result then follows. ☐

Proposition 3.9. *Suppose Mosco condition $(M1)$ is satisfied. For $\delta \geq 0$, let $A_{\delta,n} := \Big\{\sum_{i=1}^m \phi_i(t)v_i, m\in\mathbb{N}\Big\}$, where*

$$\begin{cases} v_i \in K_n, \phi_i \in C^\infty([-\delta,T+\delta]) & \textit{for all } i=1,\dots,m, \\ 0\le \phi_i(t) \le 1 & \textit{for all } t\in[-\delta,T+\delta] \textit{ and for all } i=1,\dots,m, \\ \displaystyle\sum_{i=1}^m \phi_i(t) = 1 & \textit{for all } t\in[-\delta,T+\delta]. \end{cases} \tag{17}$$

If $u_\delta \in C([-\delta,T+\delta],K)$, then there exists a sequence of functions $u_{\delta,n}\in A_{\delta,n}$ such that $u_{\delta,n}(t) \to u_\delta(t)$ in V uniformly on $[-\delta,T+\delta]$ as $n\to\infty$.

Proof. Let $u_\delta \in C([-\delta,T+\delta],K)$. We extend u_δ to $\tilde{u}_\delta \in C(\mathbb{R},K)$ by

$$\tilde{u}_\delta(t) := \begin{cases} u_\delta(-\delta) & \text{on } (-\infty,-\delta) \\ u_\delta(t) & \text{on } [-\delta,T+\delta] \\ u_\delta(T+\delta) & \text{on } (T+\delta,\infty). \end{cases}$$

Let $\epsilon > 0$ be arbitrary. We denote by $B(t) := B_V(\tilde{u}_\delta(t), \epsilon/2)$ the open ball in V about $\tilde{u}_\delta(t)$ of radius $\epsilon/2$. Let us construct an open covering $\mathscr{O}$ of $(-\delta-1, T+\delta+1)$ by

$$\mathscr{O} = \{\tilde{u}_\delta^{-1}(B(t)) \cap (-\delta-1, T+\delta+1)\}_{t\in[-\delta,T+\delta]}.$$

Since $\mathscr{O}$ is also an open covering of the compact set $[-\delta, T+\delta]$, there exists a finite subcovering

$$\tilde{\mathscr{O}} = \{\tilde{u}_\delta^{-1}(B(t_i)) \cap (-\delta-1, T+\delta+1)\}_{i=1,\dots,m},$$

where $t_i \in [-\delta, T+\delta]$ for all $i = 1, \dots, m$. We can assume that $t_1 < t_2 < \dots < t_m$ and $t_1 = -\delta$, $t_m = T+\delta$ (add them if required) so that $\tilde{\mathscr{O}}$ is an open covering of $[-\delta-1/2, T+\delta+1/2]$. For each $i \in \{1, \dots, m\}$, we have $u_\delta(t_i) \in K$. Thus, by Mosco condition (M1), there exists $v_{i,n} \in K_n$ such that $\|v_{i,n} - u_\delta(t_i)\|_V < \epsilon/2$ if $n > N_i$ for some $N_i \in \mathbb{N}$. Let $N := \max_{i=1,\dots,m} N_i$. It follows that $\|v_{i,n} - u_\delta(t_i)\|_V < \epsilon/2$ if $n > N$ for all $i \in \{1, \dots, m\}$.

Choose a smooth partition of unity $\{\phi_i\}_{i=1,\dots m}$ for $[-\delta-1/2, T+\delta+1/2]$ subordinate to $\tilde{\mathscr{O}}$. Precisely, we choose ϕ_i such that $\phi_i \in C_0^\infty(\tilde{u}_\delta^{-1}(B(t_i)) \cap (-\delta-1, T+\delta+1))$ and $\sum_{i=1}^m \phi_i(t) = 1$ for all $t \in [-\delta-1/2, T+\delta+1/2]$. Define a function $u_{\delta,n}$ on $(-\delta-1, T+\delta+1)$ by

$$u_{\delta,n}(t) := \sum_{i=1}^m \phi_i(t) v_{i,n}.$$

It is clear that the restriction of $u_{\delta,n}$ on $[-\delta, T+\delta]$ belongs to $A_{\delta,n}$ if $n > N$. Moreover, for $t \in [-\delta, T+\delta]$,

$$\begin{aligned}
\|u_{\delta,n}(t) - u_\delta(t)\|_V &\le \sum_{i=1}^m \phi_i(t)\|v_{i,n} - u_\delta(t)\|_V \\
&\le \sum_{i=1}^m \phi_i(t)\|v_{i,n} - u_\delta(t_i)\|_V + \sum_{i=1}^m \phi_i(t)\|u_\delta(t_i) - u_\delta(t)\|_V \\
&< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

if $n > N$. Note that m and N chosen above depend on ϵ. As the above argument holds for each fixed ϵ, we conclude that for every $\epsilon > 0$, there exists a sequence $u_{\delta,n}^\epsilon \in A_{\delta,n}$ and $N(\epsilon) \in \mathbb{N}$ such that

$$\|u_{\delta,n}^\epsilon(t) - u_\delta(t)\|_V \le \epsilon,$$

for all $t \in [-\delta, T+\delta]$ if $n > N(\epsilon)$.

In particular, for every $k \in \mathbb{N}$ we can find a sequence $u_{\delta,n}^k \in A_{\delta,n}$ and $N_k \in \mathbb{N}$ such that

$$\|u_{\delta,n}^k(t) - u_\delta(t)\|_V \le \frac{1}{k}, \tag{18}$$

for all $t \in [-\delta, T+\delta]$ if $n > N_k$. By choosing inductively we can assume that $N_k < N_{k+1}$ for all $k \in \mathbb{N}$. We extract a sequence of the form

$$u_{\delta,1}^1, u_{\delta,2}^1, \dots, u_{\delta,(N_1+1)}^1, \dots, u_{\delta,N_2}^1, u_{\delta,(N_2+1)}^2, \dots, u_{\delta,N_3}^2, u_{\delta,(N_3+1)}^3, \dots, u_{\delta,N_4}^3, \dots$$

so that the n-th element of this sequence belongs to $A_{\delta,n}$ for all $n \in \mathbb{N}$. Moreover, by (18), we see that this sequence converges to u_δ uniformly with respect to $t \in [-\delta, T+\delta]$ as $n \to \infty$. This proves the statement of the proposition. □

We are now in a position to prove our main result.

Proof of Theorem 3.4. The proof is divided into four parts including $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (i)$, $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$. For $(i) \Rightarrow (ii)$, we actually show that $(M1) \Rightarrow (M1')$ and $(M2) \Rightarrow (M2')$. The other three directions are proved in the same way.

$(i) \Rightarrow (ii)$: Let $u \in L^2((0,T), K)$. By the density of $C([0,T], K)$ in $L^2((0,T), K)$ (Lemma 3.7), we may assume that $u \in C([0,T], K)$. We apply Proposition 3.9 with $\delta = 0$ to obtain a sequence of functions $u_n \in L^2((0,T), K_n)$ such that $u_n(t) \to u(t)$ in V uniformly on $[0,T]$. The uniform convergence on $[0,T]$ implies that $u_n \to u$ in $L^2((0,T), V)$, showing $(M1')$. To prove condition $(M2')$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in L^2((0,T), K_{n_k})$ for every k and $u_k \rightharpoonup u$ in $L^2(0,T), V)$. By the definition of weak convergence,

$$\int_0^T \langle w(t), u_k(t) \rangle \, dt \to \int_0^T \langle w(t), u(t) \rangle \, dt, \tag{19}$$

for all $w \in L^2((0,T), V')$. By taking w of the form $w = \xi\phi(t)$ where $\xi \in V'$ and $\phi \in \mathscr{D}((0,T))$ in (19) and applying a basic property of Bochner-Lebesgue space [18, Proposition 23.9(a)], it follows that

$$\int_0^T u_k(t)\phi(t) \, dt \rightharpoonup \int_0^T u(t)\phi(t) \, dt \tag{20}$$

weakly in V for all $\phi \in \mathscr{D}((0,T))$. Let $\phi_0 \in \mathscr{D}((0,T))$ with $\int_0^T \phi_0(t) \, dt = 1$ and define $\zeta_k := \int_0^T u_k(t)\phi_0(t) \, dt$. Lemma 3.5 implies that $\zeta_k \in K_{n_k}$ for all $k \in \mathbb{N}$. Since $\zeta_k \rightharpoonup \zeta := \int_0^T u(t)\phi_0(t) \, dt$ by (20), Mosco condition $(M2)$ implies that $\zeta \in K$. We now extend u_k to $\tilde{u}_k \in L^2((-1, T+1), K_{n_k})$ by

$$\tilde{u}_k(t) := \begin{cases} \zeta_k & \text{on } (-1,0) \cup (T, T+1) \\ u_k(t) & \text{on } (0,T). \end{cases} \tag{21}$$

It can be easily seen that $\tilde{u}_k \rightharpoonup \tilde{u}$ weakly in $L^2((-1, T+1), V)$, where $\tilde{u}$ defined as (21) with k deleted. Using the definition of weak convergence in $L^2((-1, T+1), V)$ and a similar argument as above, we obtain $\int_{-1}^{T+1} \tilde{u}_k(t)\phi(t) \, dt \rightharpoonup \int_{-1}^{T+1} \tilde{u}(t)\phi(t) \, dt$ weakly in V for all $\phi \in \mathscr{D}((-1, T+1))$. In particular, taking $\phi \in \mathscr{D}((-1, T+1))$ with $\int_{-1}^{T+1} \phi(t) \, dt = 1$, we have $\int_{-1}^{T+1} \tilde{u}_k(t)\phi(t) \, dt \in K_{n_k}$ converges weakly to $\int_{-1}^{T+1} \tilde{u}(t)\phi(t) \, dt$ in V. Thus, Mosco condition $(M2)$ implies $\int_{-1}^{T+1} \tilde{u}(t)\phi(t) \, dt \in K$ for all $\phi \in \mathscr{D}((-1, T+1))$ with $\int_{-1}^{T+1} \phi(t) \, dt = 1$. By Lemma 3.6, we conclude that $u \in L^2((0,T), K)$ and Mosco condition $(M2')$ follows.

$(ii) \Rightarrow (i)$: Let $u \in K$. Define $v \in L^2((0,T), K)$ by the constant function $v(t) := u$ for $t \in (0,T)$. By condition $(M1')$, there exists $(v_n)_{n \in \mathbb{N}}$ with $v_n \in L^2((0,T), K_n)$ such that $v_n \to v$ in $L^2((0,T), V)$. Let $\phi_0 \in \mathscr{D}((0,T))$ with $\int_0^T \phi_0(t) \, dt = 1$. We show that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n := \int_0^T v_n(t)\phi_0(t)\,dt$ gives Mosco condition $(M1)$. First note that $u_n \in K_n$ for all $n \in \mathbb{N}$ by Lemma 3.5. Moreover,

$$\begin{aligned}\|u_n - u\|_V &= \Big\| \int_0^T v_n(t)\phi_0(t)\,dt - u \Big\|_V \\ &= \Big\| \int_0^T [v_n(t)\phi_0(t) - v(t)\phi_0(t)]\,dt \Big\|_V \\ &\leq \int_0^T |\phi_0(t)| \|v_n(t) - v(t)\|_V\,dt \\ &\leq \sqrt{T}\Big(\int_0^T \|v_n(t) - v(t)\|_V^2\,dt \Big)^{\frac{1}{2}} \|\phi_0\|_\infty \\ &\to 0,\end{aligned}$$

as $n \to \infty$. To prove condition $(M2)$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in K_{n_k}$ for every k and $u_k \rightharpoonup u$ in V. Define $v_k \in L^2((0,T), K_{n_k})$ by the constant function $v_k(t) := u_k$ for $t \in (0,T)$. It can be easily verified that $v_k \rightharpoonup v$ in $L^2((0,T), V)$, where v is the constant function $v(t) := u$ for $t \in (0,T)$. It follows from Mosco condition $(M2')$ that $v \in L^2((0,T), K)$. Hence $u \in K$ as required.

$(i) \Rightarrow (iii)$: Let $u \in W((0,T), V, V') \cap L^2((0,T), K)$. By Lemma 3.8, we may assume that $u \in C^\infty([0,T], V) \cap C([0,T], K)$. For $\delta > 0$, we define the stretched function $u_\delta \in C^\infty([-\delta, T+\delta], V) \cap C([-\delta, T+\delta], K)$ by $u_\delta = u \circ S_\delta^{-1}$, where S_δ is the stretching map given by (16). It can be shown that the restriction of u_δ on $[0,T]$ converges to u in $W((0,T), V, V')$ as $\delta \to 0$. By Proposition 3.9, there exists a sequence of functions $u_{\delta,n} \in A_{\delta,n}$ such that $u_{\delta,n}(t) \to u_\delta(t)$ uniformly on $[-\delta, T+\delta]$ as $n \to \infty$. Let $\eta_{1/j}$ be a mollifier. For $t \in [0,T]$ and $j > 1/\delta$, the translation of $\eta_{1/j}$ by t (denoted by $\eta_{1/j,t}$) belongs to $\mathscr{D}((-\delta, T+\delta))$. Hence if $j > 1/\delta$, we have $\eta_{1/j} * u_{\delta,n} \in C^\infty([0,T], V) \cap C([0,T], K_n)$. By continuity of convolution and the well known fact on the r-th order derivative that

$$\frac{d^r}{dt^r}(\eta_{1/j} * u_{\delta,n}) = \frac{d^r}{dt^r}\eta_{1/j} * u_{\delta,n} = \eta_{1/j} * \frac{d^r}{dt}u_{\delta,n},$$

we deduce that $\eta_{1/j} * u_{\delta,n} \to \eta_{1/j} * u_\delta$ in $C^\infty([0,T], V)$ as $n \to \infty$. Similarly, $\eta_{1/j} * u_\delta \to u_\delta$ in $C^\infty([0,T], V)$ as $j \to \infty$. The above shows that we can construct a function of the form $\eta_{1/j} * u_{\delta,n} \in W((0,T), V, V') \cap L^2((0,T), K_n)$ converging to u in $W((0,T), V, V')$. Hence Mosco condition $(M1")$ follows. To prove condition $(M2")$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in W((0,T), V, V') \cap L^2((0,T), K_{n_k})$ for every k, $u_k \rightharpoonup u$ in $L^2((0,T), V)$ and $u_k' \rightharpoonup w$ in $L^2((0,T), V')$. Since V is continuously embedded in V', it follows immediately that $u' = w$ and hence $u \in W((0,T), V, V')$ (see [18, Proposition 23.19]). Using $(i) \Rightarrow (ii)$, specifically Mosco condition $(M2')$, we conclude that $u \in W((0,T), V, V') \cap L^2((0,T), K)$.

$(iii) \Rightarrow (i)$: Let $u \in K$. Define $v \in W((0,T), V, V') \cap L^2((0,T), K)$ by the constant function $v(t) := u$ for $t \in (0,T)$. By condition $(M1")$, there exists $(v_n)_{n\in\mathbb{N}}$

with $v_n \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $v_n \to v$ in $W((0,T),V,V')$. In particular, v_n converges strongly to v in $L^2((0,T),V)$. By the same argument as in the proof of $(ii) \Rightarrow (i)$, we can show that $u_n := \int_0^T v_n(t)\phi_0(t)\, dt$, for some $\phi_0 \in \mathscr{D}((0,T))$ with $\int_0^T \phi_0(t)\, dt = 1$ establishes Mosco condition $(M1)$. To prove condition $(M2)$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in K_{n_k}$ for every k and $u_k \rightharpoonup u$ in V. Define $v_k \in W((0,T),V,V') \cap L^2((0,T),K_{n_k})$ by the constant function $v_k(t) := u_k$ for $t \in (0,T)$. By the same argument as in the proof of $(ii) \Rightarrow (i)$, we have $v_k \rightharpoonup v$ in $L^2((0,T),V)$, where $v(t) := u$ for $t \in (0,T)$. Moreover, it is clear that $v_k' = 0$ for all $k \in \mathbb{N}$ and hence $v_k' \rightharpoonup v' = 0$ in $L^2((0,T),V')$. We apply $(M2")$ to deduce that $v \in W((0,T),V,V') \cap L^2((0,T),K)$. Hence $u \in K$. □

4. Application in domain perturbation for parabolic equations

In this section, we study the behaviour of solutions of parabolic equations subject to Dirichlet boundary condition and Neumann boundary condition under domain perturbation. Let Ω_n, Ω be bounded open sets in $\mathbb{R}^N$ and $D \subset \mathbb{R}^N$ be a ball such that $\Omega_n, \Omega \subset D$ for all $n \in \mathbb{N}$. Suppose a_{ij}, a_i, b_i, c_0 are functions in $L^\infty(D \times (0,T))$ and a_{ij} satisfies ellipticity condition. More precisely, there exists $\alpha > 0$ such that $a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^N$. We consider the evolution triple $V_n \overset{d}{\hookrightarrow} H_n \overset{d}{\hookrightarrow} V_n'$, where we choose

- $V_n = H_0^1(\Omega_n)$ and $H_n = L^2(\Omega_n)$ for Dirichlet problem
- $V_n = H^1(\Omega_n)$ and $H_n = L^2(\Omega_n)$ for Neumann problem.

For $t \in (0,T)$, suppose $a_n(t;\cdot,\cdot)$ is a bilinear form on V_n defined by

$$a_n(t;u,v) := \int_{\Omega_n} [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\partial_i v + b_i(x,t)\partial_i uv + c_0(x,t)uv\, dx. \tag{22}$$

It follows that for all $n \in \mathbb{N}$, there exist three constants $M > 0, \alpha > 0$ and $\lambda \in \mathbb{R}$ independent of $t \in [0,T]$ such that

$$|a_n(t;u,v)| \leq M\|u\|_{V_n}\|v\|_{V_n}, \tag{23}$$

for all $u, v \in V_n$ and

$$a_n(t;u,u) + \lambda\|u\|_{H_n}^2 \geq \alpha\|u\|_{V_n}^2, \tag{24}$$

for all $u \in V_n$. Given $u_{0,n} \in L^2(D)$ and $f_n \in L^2(D \times (0,T))$, let us consider the following boundary value problem in $\Omega_n \times (0,T]$.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n(t)u = f_n(x,t) & \text{in } \Omega_n \times (0,T] \\ \mathcal{B}_n(t)u = 0 & \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{25}$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are operators on V_n given by

$$\mathcal{A}_n(t)u := -\partial_i[a_{ij}(x,t)\partial_j u + a_i(x,t)u] + b_i(x,t)\partial_i u + c_0(x,t)u,$$

and $\mathcal{B}_n$ is one of the following

$$\mathcal{B}_n(t)u := u \qquad \text{Dirichlet boundary condition}$$

$$\mathcal{B}_n(t)u := [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\,\nu_i \qquad \text{Neumann boundary condition.}$$

We wish to show that a sequence of solutions of the above parabolic equations in $\Omega_n \times (0,T]$ converges to the solution of the following limit problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = f(x,t) & \text{in } \Omega \times (0,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{26}$$

However, we will consider the boundary value problems (25) and (26) in the abstract form. As discussed in Section 2, we can write (25) as

$$\begin{cases} u'(t) + A_n(t)u = f_n(t) & \text{for } t \in (0,T] \\ u(0) = u_{0,n}, \end{cases} \tag{27}$$

where $A_n(t) \in \mathscr{L}(V_n, V_n')$ is the operator induced by the bilinear form $a_n(t;,\cdot,\cdot)$. Similarly, we write (26) as

$$\begin{cases} u'(t) + A(t)u = f(t) & \text{for } t \in (0,T] \\ u(0) = u_0. \end{cases} \tag{28}$$

Throughout this section, we denote the variational solution of (27) by u_n and the variational solution of (28) by u. We illustrate an application of Mosco convergence to obtain stability of variational solutions under domain perturbation. The proof is motivated by the techniques presented in [8]. However, we replace the notion of convergence of domains ((3.5) and (3.6) in [8]) by Mosco convergence. It is not difficult to see that the assumption on domains in [8] implies that $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.

4.1. **Dirichlet problems.** When the domain is perturbed, the variational solutions belong to different function spaces. We often extend functions by zero outside the domain. We embed the spaces $H_0^1(\Omega_n)$ into $H^1(D)$ by $v \mapsto \tilde{v}$, where $\tilde{v} = v$ on Ω_n and $\tilde{v} = 0$ on $D\backslash\Omega_n$. Similarly, we may consider the embedding $L^2((0,T), H_0^1(\Omega_n))$ into $L^2((0,T), H^1(D))$ by $w(t) \mapsto \tilde{w}(t)$ for a.e. $t \in (0,T)$. Note that the trivial extension $\tilde{v}$ also acts on $L^2(\Omega_n)$ into $L^2(D)$.

Let us take $V := H^1(D)$, $K_n := H_0^1(\Omega_n)$ and $K := H_0^1(\Omega)$, and consider Mosco convergence of K_n to K. In this case K_n and K are closed and convex subsets of V in the sense of the above embedding. In fact, K_n and K are closed subspace of V. The main application of Theorem 3.4 is to show that the variational solution u_n of (27) converges to the variational solution u of (28) by applying various Mosco conditions.

Theorem 4.1. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T), L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then $\tilde{u}_n$ converges weakly to $\tilde{u}$ in $L^2((0,T), H^1(D))$.*

Proof. Since $f_n \to f$ in $L^2((0,T), L^2(D))$ and $u_{0,n} \to u_0$ in $L^2(D)$, it follows from (9) that $\|u_n\|_{W(0,T,V_n,V_n')}$ is uniformly bounded. Hence $\tilde{u}_n$ is uniformly bounded in $L^2((0,T), H^1(D))$. We can extract a subsequence (denoted again by u_n), such that $\tilde{u}_n \rightharpoonup w$ in $L^2((0,T), H^1(D))$. Mosco condition $(M2')$ (from Theorem 3.4) implies that $w \in L^2((0,T), H_0^1(\Omega))$. It remains to show that $w = u$ in $L^2((0,T), H_0^1(\Omega))$.

Let $\xi \in H_0^1(\Omega)$ and $\phi \in \mathscr{D}([0,T))$. Mosco condition $(M1)$ implies that there exists $\xi_n \in H_0^1(\Omega_n)$ such that $\tilde{\xi}_n \to \tilde{\xi}$ in $H^1(D)$. As u_n is the variational solution of (27), we get from (8) that

$$-\int_0^T (u_n(t)|\xi_n)\phi'(t)\,dt + \int_0^T a_n(t; u_n(t), \xi_n)\phi(t)\,dt$$
$$= (u_{0,n}|\xi_n)\phi(0) + \int_0^T \langle f_n(t), \xi_n\rangle \phi(t)\,dt.$$

By letting $n \to \infty$, we get

$$\begin{aligned} -\int_0^T (w(t)|\xi)\phi'(t)\,dt &+ \int_0^T a(t; w(t), \xi)\phi(t)\,dt \\ &= (u_0|\xi)\phi(0) + \int_0^T \langle f(t), \xi\rangle \phi(t)\,dt. \end{aligned} \tag{29}$$

Hence w is a variational solution of (28). By the uniqueness of solution, we conclude that $w = u$ in $L^2((0,T), H_0^1(\Omega))$ and the whole sequence converges. □

Lemma 4.2. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T), L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then for each $t \in [0,T]$ we have $\tilde{u}_n(t)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$.*

Proof. Since $\tilde{f}_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $\tilde{u}_{0,n}$ is uniformly bounded in $L^2(D)$, we have from (10) that

$$\max_{t\in[0,T]} \|\tilde{u}_n(t)\|_{L^2(D)} \le M,$$

for some $M > 0$. Hence for a subsequence denoted again by $u_n(t)$, there exists $w \in L^2(\Omega)$ such that $\tilde{u}_n(t)_{|\Omega} \rightharpoonup w$ in $L^2(\Omega)$. Let $\xi \in H_0^1(\Omega)$ and $\phi \in \mathscr{D}((0,t])$. Mosco condition $(M1)$ implies that there exists $\xi_n \in H_0^1(\Omega_n)$ such that $\tilde{\xi}_n \to \tilde{\xi}$ in $H^1(D)$. As u_n is the variational solution of (27), we have

$$-\int_0^t (u_n(s)|\xi_n)\phi'(s)\,ds + \int_0^t a_n(s; u_n(s), \xi_n)\phi(s)\,ds$$
$$= -(\tilde{u}_n(t)|\tilde{\xi}_n)_{L^2(D)}\phi(t) + \int_0^t \langle f_n(s), \xi_n\rangle \phi(s)\,ds.$$

Now

$$(\tilde{u}_{0,n})|\tilde{\xi}_n)_{L^2(D)} = (\tilde{u}_{0,n}|\tilde{\xi}_n)_{L^2(\Omega)} + (\tilde{u}_{0,n}|\tilde{\xi}_n)_{L^2(D\setminus\Omega)}.$$

Since $\tilde{\xi}_n \to \tilde{\xi}$ in $L^2(D)$, we have $\tilde{\xi}_n|_\Omega \to \xi$ in $L^2(\Omega)$ and $\tilde{\xi}_n|_{(D\setminus\Omega)} \to 0$ in $L^2(D\setminus\Omega)$. Applying the dominated convergence theorem in the second term above and using the weak convergence of initial condition $u_{0,n}$ in the first term above, we see that

$$(u_{0,n}|\xi_n)_{L^2(\Omega_n)} \to (u_0|\xi)_{L^2(\Omega)}.$$

Hence,

$$
\begin{aligned}
-\int_0^t (u(s)|\xi)\phi'(s)\,ds + \int_0^t a(s;u(s),\xi)\phi(s)\,ds \\
= -(w|\xi)_{L^2(\Omega)}\phi(t) + \int_0^t \langle f(s),\xi\rangle\phi(s)\,ds,
\end{aligned}
\tag{30}
$$

as $n \to \infty$. As u is the variational solution of (28), a similar equation holds with $(w|\xi)_{L^2(\Omega)}$ replaced by $(u(t)|\xi)_{L^2(\Omega)}$. Therefore $(w|\xi)_{L^2(\Omega)} = (u(t)|\xi)_{L^2(\Omega)}$ for all $\xi \in H_0^1(\Omega)$. By the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, $w = u(t)$. Hence, for subsequences $\tilde{u}_n(t)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$. By the uniqueness, the whole sequence $\tilde{u}_n(t)_{|\Omega}$ converges weakly to $u(t)$ in $L^2(\Omega)$. □

Remark 4.3. In fact, we only require that $\tilde{f}_n|_\Omega \rightharpoonup f$ weakly in $L^2((0,T),L^2(\Omega))$ and $\tilde{u}_{0,n}|_\Omega \rightharpoonup u_0$ weakly in $L^2(\Omega)$ to obtain the conclusion of Theorem 4.1 and Lemma 4.2 as done in [8].

Next we show the strong convergence of solutions. The assumptions on strong convergence of initial values $u_{0,n}$ and inhomogeneous data f_n are required in the proof below (see also Remark 4.7 below).

Theorem 4.4. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T),L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then $\tilde{u}_n$ converges strongly to $\tilde{u}$ in $L^2((0,T),H^1(D))$.*

Proof. We have $\tilde{u}_n$ converges weakly to $\tilde{u}$ in $L^2((0,T),H^1(D))$ from Theorem 4.1. Mosco condition $(M1')$ (from Theorem 3.4) implies that there exists $w_n \in L^2((0,T),H_0^1(\Omega_n))$ such that $\tilde{w}_n \to \tilde{u}$ in $L^2((0,T),H^1(D))$. For $t \in [0,T]$, we consider

$$
d_n(t) = \frac{1}{2}\|\tilde{u}_n(t) - \tilde{u}(t)\|^2_{L^2(D)} + \alpha\int_0^t \|\tilde{u}_n(s) - \tilde{w}_n(s)\|^2_{H^1(D)}ds. \tag{31}
$$

By (24) (with $\lambda = 0$), we have

$$
\begin{aligned}
d_n(t) \le \frac{1}{2}\|\tilde{u}_n(t)\|^2_{L^2(D)} &+ \int_0^t a_n(s;u_n(s),u_n(s))\,ds \\
&+ \frac{1}{2}\|\tilde{u}(t)\|^2_{L^2(D)} + \int_0^t a_n(s;w_n(s),w_n(s))\,ds \\
&- (\tilde{u}_n(t)|\tilde{u}(t))_{L^2(D)} - \int_0^t a_n(s;u_n(s),w_n(s))\,ds \\
&- \int_0^t a_n(s;w_n(s),u_n(s))\,ds,
\end{aligned}
\tag{32}
$$

for all $n \in \mathbb{N}$. It can be easily seen from the weak convergence of $\tilde{u}_n$ and the strong convergence of $\tilde{w}_n$ to $\tilde{u}$ in $L^2((0,T),H^1(D))$ that

$$
\begin{aligned}
&\lim_{n\to\infty}\Big[\int_0^t a_n(s;u_n(s),w_n(s))\,ds + \int_0^t a_n(s;w_n(s),u_n(s))\,ds\Big] \\
&= 2\int_0^t a(s;u(s),u(s))\,ds,
\end{aligned}
\tag{33}
$$

and

$$\lim_{n\to\infty}\int_0^t a_n(s;w_n(s),w_n(s))\,ds=\int_0^t a(s;u(s),u(s))\,ds. \tag{34}$$

Also, by lemma 4.2, we have

$$\lim_{n\to\infty}(\tilde u_n(t)|\tilde u(t))_{L^2(D)}=\lim_{n\to\infty}(\tilde u_n(t)_{|\Omega}|u(t))_{L^2(\Omega)}=\|u(t)\|^2_{L^2(\Omega)}. \tag{35}$$

Finally, as u_n is the variational solution of (27) we get from (4) that

$$\begin{aligned}\frac12\|u_n(t)\|^2_{L^2(\Omega_n)}&+\int_0^t a_n(s;u_n(s),u_n(s))\,ds\\&=\frac12\|u_n(0)\|^2_{L^2(\Omega_n)}+\int_0^t\langle f_n(s),u_n(s)\rangle\,ds.\end{aligned}$$

By the assumption that $\tilde u_{0,n}\to\tilde u_0$ strongly in $L^2(D)$ and $\tilde f_n\to\tilde f$ strongly in $L^2((0,T),L^2(D))$, we get

$$\begin{aligned}\lim_{n\to\infty}&\Big[\frac12\|u_n(t)\|^2_{L^2(\Omega_n)}+\int_0^t a_n(s;u_n(s),u_n(s))\,ds\Big]\\&=\frac12\|u(0)\|^2_{L^2(\Omega)}+\int_0^t\langle f(s),u(s)\rangle\,ds\\&=\frac12\|u(t)\|^2_{L^2(\Omega)}+\int_0^t a(s;u(s),u(s))\,ds.\end{aligned}\tag{36}$$

Hence, it follows from (32) – (36) that $d_n(t)\to0$ for all $t\in[0,T]$. This shows pointwise convergence of $\tilde u_n(t)$ to $\tilde u(t)$ in $L^2(D)$. Moreover, by taking $t=T$ we get

$$\begin{aligned}\int_0^T&\|\tilde u_n(s)-\tilde u(s)\|^2_{H^1(D)}\,ds\\&\le\int_0^T\|\tilde u_n(s)-\tilde w_n(s)\|^2_{H^1(D)}\,ds+\int_0^T\|\tilde w_n(s)-\tilde u(s)\|^2_{H^1(D)}\,ds\\&\to0,\end{aligned}$$

as $n\to\infty$. This proves the strong convergence $\tilde u_n\to\tilde u$ in $L^2((0,T),H^1(D))$. □

In the next theorem we prove convergence of solutions in a stronger norm. We show that Mosco convergence is sufficient for uniform convergence of solutions in $L^2(D)$ with respect to $t\in[0,T]$. We require the following result on Mosco convergence of $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$ [9, Proposition 6.3].

Lemma 4.5. *The following statements are equivalent.*

1. *Mosco condition* $(M1)$*: for every* $w\in H^1_0(\Omega)$*, there exists a sequence* $w_n\in H^1_0(\Omega_n)$ *such that* $\tilde w_n\to\tilde w$ *in* $H^1(D)$*.*
2. $\operatorname{cap}(K\cap\Omega_n^c)\to0$ *as* $n\to\infty$ *for all compact set* $K\subset\Omega$*.*

Theorem 4.6. *Suppose* $\tilde f_n\to\tilde f$ *in* $L^2((0,T),L^2(D))$ *and* $\tilde u_{0,n}\to\tilde u_0$ *in* $L^2(D)$*. If* $H^1_0(\Omega_n)$ *converges to* $H^1_0(\Omega)$ *in the sense of Mosco, then* $\tilde u_n$ *converges strongly to* $\tilde u$ *in* $C([0,T],L^2(D))$*.*

Proof. We notice from the proof of Theorem 4.4 that

$$\int_0^t \|\tilde{u}_n(s) - \tilde{w}_n(s)\|^2_{H^1(D)}\, ds \to 0$$

uniformly with respect to $t \in [0, T]$. Indeed, by (31)

$$\int_0^t \|\tilde{u}_n(s) - \tilde{w}_n(s)\|^2_{H^1(D)}\, ds \le \alpha^{-1} d_n(T)$$

for all $n \in \mathbb{N}$ and for all $t \in [0, T]$. Moreover, it is clear that (33), (34) and (36) hold uniformly on $[0, T]$. It remains to show uniform convergence of (35).

Fix $s \in [0, T]$. For $\epsilon > 0$ arbitrary, we choose a compact set $K \subset \Omega$ such that $\|u(s)\|_{L^2(\Omega\setminus K)} \le \epsilon/2$. Since $u \in C([0, T], L^2(\Omega))$, there exists $\eta > 0$ only depending on ϵ such that $\|u(t) - u(s)\|_{L^2(\Omega)} \le \epsilon/2$ for all $t \in (s - \eta, s + \eta) \cap [0, T]$. It follows that

$$\|u(t)\|_{L^2(\Omega\setminus K)} \le \|u(t) - u(s)\|_{L^2(\Omega)} + \|u(s)\|_{L^2(\Omega\setminus K)} \le \epsilon/2 + \epsilon/2 = \epsilon, \tag{37}$$

for all $t \in (s - \eta, s + \eta) \cap [0, T]$. We next choose a cut-off function $\phi \in C_0^\infty(\Omega)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on K. Since $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, we have from Lemma 4.5 that $\operatorname{cap}(\operatorname{supp}(\phi) \cap \Omega_n^c) \to 0$. By definition of capacity, there exists a sequence $\xi_n \in C_0^\infty(\Omega)$ such that $0 \le \xi_n \le 1$, $\xi_n = 1$ on a neighborhood of $\operatorname{supp}(\phi) \cap \Omega_n^c$ and $\|\xi_n\|_{H^1(\mathbb{R}^N)} \le \operatorname{cap}(\operatorname{supp}(\phi) \cap \Omega_n^c) + 1/n$. Define $\phi_n := (1 - \xi_n)\phi$. We have that $\phi_n \in C_0^\infty(\Omega_n)$ and $\phi_n \to \phi$ in $L^2(D)$. Consider

$$\begin{aligned}
&\left|(\tilde{u}_n(t) - \tilde{u}(t)|\tilde{u}(t))_{L^2(D)}\right| \\
&\le \left|(\tilde{u}_n(t)|\phi_n\tilde{u}(t))_{L^2(D)} - (\tilde{u}(t)|\phi\tilde{u}(t))_{L^2(D)}\right| \\
&\quad + \left|(\tilde{u}_n(t)|(1-\phi_n)\tilde{u}(t))_{L^2(D)} - (\tilde{u}(t)|(1-\phi)\tilde{u}(t))_{L^2(D)}\right| \\
&\le \left|(\tilde{u}_n(t)|\phi_n\tilde{u}(t))_{L^2(D)} - (\tilde{u}(t)|\phi\tilde{u}(t))_{L^2(D)}\right| \\
&\quad + \left|(\tilde{u}_n(t) - \tilde{u}(t)|(1-\phi_n)\tilde{u}(t))_{L^2(D)}\right| + \left|(\tilde{u}(t)|(\phi - \phi_n)\tilde{u}(t))_{L^2(D)}\right|.
\end{aligned} \tag{38}$$

We prove that each term on the right of (38) is uniformly small for $t \in (s - \eta, s + \eta) \cap [0, T]$ if n is sufficiently large. For the first term, applying integration by parts formula (4) and the definition of variational solutions, we obtain

$$\begin{aligned}
&(\tilde{u}_n(t)|\phi_n\tilde{u}(t))_{L^2(D)} \\
&= (u_{0,n}|\phi_n u_0)_{L^2(D)} + \int_0^t \langle u_n'(s), \phi_n\tilde{u}(s)\rangle\, ds + \int_0^t \langle u'(s), \phi_n\tilde{u}_n(s)\rangle\, ds \\
&= (u_{0,n}|\phi_n u_0)_{L^2(D)} + \int_0^t \langle f_n(s), \phi_n\tilde{u}(s)\rangle\, ds - \int_0^t a_n(s; u_n(s), \phi_n\tilde{u}(s))\, ds \\
&\quad + \int_0^t \langle f(s), \phi_n\tilde{u}_n(s)\rangle\, ds - \int_0^t a(s; u(s), \phi_n\tilde{u}_n(s))\, ds.
\end{aligned} \tag{39}$$

It can be easily verified using Dominated Convergence Theorem that $\phi_n\tilde{u} \to \phi\tilde{u}$ in $L^2((0,T),L^2(D))$. Moreover,

$$\begin{aligned}&\int_0^T \|\phi_n\tilde{u}_n(t)-\phi\tilde{u}(t)\|^2_{L^2(D)}\,dt\\&\leq \int_0^T \|\phi_n\tilde{u}(t)-\phi\tilde{u}(t)\|^2_{L^2(D)}\,dt+\int_0^T \|\phi_n\tilde{u}_n(t)-\phi_n\tilde{u}(t)\|^2_{L^2(D)}\,dt\\&\leq \int_0^T \|\phi_n\tilde{u}(t)-\phi\tilde{u}(t)\|^2_{L^2(D)}\,dt+\|\phi_n\|^2_\infty\int_0^T \|\tilde{u}_n(t)-\tilde{u}(t)\|^2_{L^2(D)}\,dt.\end{aligned}$$

It follows also that $\phi_n\tilde{u}_n \to \phi\tilde{u}$ in $L^2((0,T),L^2(D))$. Taking into consideration that $\tilde{u}_{0,n} \to \tilde{u}_0$ and $\tilde{f}_n \to \tilde{f}$, we conclude form (39) that

$$(\tilde{u}_n(t)|\phi_n\tilde{u}(t))_{L^2(D)} \to (\tilde{u}(t)|\phi\tilde{u}(t))_{L^2(D)} \tag{40}$$

uniformly with respect to $t\in[0,T]$. For the last term on the right of (38), applying a similar argument as above, we write

$$(\tilde{u}(t)|\phi_n\tilde{u}(t))_{L^2(D)} = (u_0|\phi_n u_0)_{L^2(D)} + 2\int_0^t \langle u'(s),\phi_n\tilde{u}(s)\rangle\,ds.$$

We conclude that

$$(\tilde{u}(t)|\phi_n\tilde{u}(t))_{L^2(D)} \to (\tilde{u}(t)|\phi\tilde{u}(t))_{L^2(D)} \tag{41}$$

uniformly with respect to $t\in[0,T]$. Finally, for the second term on the right of (38), we notice that $0\leq 1-\phi_n\leq 1$ on Ω and $1-\phi_n = 1-(1-\xi_n)\phi = \xi_n$ on K. Moreover, using (10) and the assumption that $\tilde{u}_{0,n}\to\tilde{u}_0$ and $\tilde{f}_n\to\tilde{f}$, there exists a constant $M_0>0$ such that

$$\|\tilde{u}_n(t)\|_{L^2(D)}, \|\tilde{u}(t)\|_{L^2(D)} \leq M_0, \tag{42}$$

for all $t\in[0,T]$. Hence, by Cauchy-Schwarz inequality and (37),

$$\begin{aligned}\left|(\tilde{u}_n(t)-\tilde{u}(t)|(1-\phi_n)\tilde{u}(t))_{L^2(D)}\right| &\leq \|\tilde{u}_n(t)-\tilde{u}(t)\|_{L^2(D)}\|(1-\phi_n)\tilde{u}(t)\|_{L^2(D)}\\&\leq 2M_0\left(\|u(t)\|^2_{L^2(\Omega\setminus K)}+\|\xi_n u(t)\|_{L^2(K)}\right)\\&\leq 2M_0\left(\epsilon+\|\xi_n u(t)\|_{L^2(K)}\right),\end{aligned}\tag{43}$$

for all $t\in(s-\eta,s+\eta)\cap[0,T]$ and for all $n\in\mathbb{N}$. Since $\xi_n\to 0$ in $L^2(D)$, a standard argument using Dominated Convergence Theorem implies that $\xi_n u(s)\to 0$ in $L^2(\Omega)$. Hence, there exists $N_{s,\epsilon}\in\mathbb{N}$ such that $\|\xi_n u(s)\|_{L^2(\Omega)}\leq \epsilon/2$ for all $n\geq N_{s,\epsilon}$. Therefore,

$$\begin{aligned}\|\xi_n u(t)\|_{L^2(K)} &\leq \|\xi_n u(s)\|_{L^2(K)} + \|\xi_n u(t)-\xi_n u(s)\|_{L^2(K)}\\&\leq \|\xi_n u(s)\|_{L^2(K)} + \|u(t)-u(s)\|_{L^2(K)}\\&\leq \epsilon/2+\epsilon/2=\epsilon,\end{aligned}$$

for all $t\in(s-\eta,s+\eta)\cap[0,T]$ and for all $n\geq N_{s,\epsilon}$. It follows from (43) that

$$\left|(\tilde{u}_n(t)-\tilde{u}(t)|(1-\phi_n)\tilde{u}(t))_{L^2(D)}\right| \leq 2M_0(\epsilon+\epsilon)=4M_0\epsilon, \tag{44}$$

for all $t \in (s-\eta, s+\eta) \cap [0,T]$ and for all $n \geq N_{s,\epsilon}$. Therefore, by (38), (40), (41), and (44), we conclude that there exist $\tilde{N}_{s,\epsilon} \in \mathbb{N}$ and a positive constant C such that

$$\left|(\tilde{u}_n(t) - \tilde{u}(t)|\tilde{u}(t))_{L^2(D)}\right| \leq C\epsilon,$$

for all $t \in (s-\eta, s+\eta) \cap [0,T]$ and for all $n \geq \tilde{N}_{s,\epsilon}$.

Finally, as $[0,T]$ is a compact interval and η only depends on ϵ, it follows that $(\tilde{u}_n(t)|\tilde{u}(t))_{L^2(D)} \to (\tilde{u}(t)|\tilde{u}(t))_{L^2(D)}$ uniformly with respect to $t \in [0,T]$. □

Remark 4.7. In fact, we can conclude that $\tilde{u}_n \to \tilde{u}$ in $L^2((0,T), L^2(D))$ directly from the weak convergence of solutions in Theorem 4.1 and the compactness result in [8, Lemma 2.1]. This means we only require that $\tilde{f}_n|_\Omega \rightharpoonup f$ weakly in $L^2((0,T), L^2(\Omega))$ and $\tilde{u}_{0,n}|_\Omega \rightharpoonup u_0$ weakly in $L^2(\Omega)$. Under the same assumptions, we can restate convergence result in Theorem 4.6 as $\tilde{u}_n \to \tilde{u}$ in $C([\delta,T], L^2(D))$ for all $\delta \in (0,T]$, as appeared in [8]. The reason we impose stronger assumptions on the initial data and the inhomogeneous terms is to avoid using [8, Lemma 2.1], which is not applicable to Neumann problems, and illustrate a technique that can be applied to both boundary conditions.

It is known that stability under domain perturbation of solution of elliptic equations subject to Dirichlet boundary condition can be obtained from Mosco convergence of $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$ [9]. Hence we can use the same criterion on Ω_n and Ω to conclude the stability of solutions of parabolic equations. In particular, the conditions on domains given in [9, Theorem 7.5] implies convergence of solutions of non-autonomous parabolic equations (25) subject to Dirichlet boundary condition under domain perturbation.

4.2. **Neumann problems.** It is more complicated for Neumann problems because the trivial extension by zero outside the domain of a function u_n in $H^1(\Omega_n)$ does not belong to $H^1(D)$. In addition, as we do not assume any smoothness of domains, there is no smooth extension operator from $H^1(\Omega_n)$ to $H^1(D)$. In order to study the limit of $u_n \in H^1(\Omega_n)$ when the domain is perturbed, we embed the space $H^1(\Omega_n)$ into the following space

$$H^1(\Omega_n) \hookrightarrow L^2(D) \times L^2(D, \mathbb{R}^N)$$

by

$$v_n \mapsto (\tilde{v}_n, \tilde{\nabla} v_n),$$

where $\tilde{v}_n(x) = v(x)$ if $x \in \Omega_n$ and $\tilde{v}_n(x) = 0$ if $x \in D \backslash \Omega_n$. Similarly, $\tilde{\nabla} v_n(x) = \nabla v(x)$ if $x \in \Omega_n$ and $\tilde{\nabla} v_n(x) = 0$ if $x \in D \backslash \Omega_n$. Note that $\tilde{\nabla} v_n$ is not the gradient of $\tilde{v}_n$ in the sense of distribution. By a similar embedding for $H^1(\Omega)$, we can consider Mosco convergence of

$$K_n := \{(\tilde{v}_n, \tilde{\nabla} v_n) \in L^2(D) \times L^2(D, \mathbb{R}^N) \mid v_n \in H^1(\Omega_n)\}$$

to

$$K := \{(\tilde{v}, \tilde{\nabla} v) \in L^2(D) \times L^2(D, \mathbb{R}^N) \mid v \in H^1(\Omega)\}$$

in $V := L^2(D) \times L^2(D, \mathbb{R}^N)$. In this case, K_n and K are closed subspace of V. For simplicity, we use the term $H^1(\Omega_n)$ converges in the sense of Mosco to $H^1(\Omega)$ for K_n and K above.

When dealing with parabolic equations, we regard the space $L^2((0,T), V)$ as $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N))$ via the isomorphism between them. Hence

$$\begin{aligned} L^2((0,T), K_n) &\equiv \{(\tilde{w}_n, \tilde{\nabla} w_n) \mid w_n \in L^2((0,T), H^1(\Omega_n)\} \\ &\subset L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N)), \end{aligned}$$

and

$$\begin{aligned} L^2((0,T), K) &\equiv \{(\tilde{w}, \tilde{\nabla} w) \mid w \in L^2((0,T), H^1(\Omega)\} \\ &\subset L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N)). \end{aligned}$$

As in the case of Dirichlet problem, we apply various Mosco conditions from Theorem 3.4 to prove that the variational solution u_n converges to the variational solution u.

Theorem 4.8. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T), L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then $\tilde{u}_n$ converges weakly to $\tilde{u}$ in $L^2((0,T), L^2(D))$ and $\tilde{\nabla} u_n$ converges weakly to $\tilde{\nabla} u$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$.*

Proof. By a similar argument as in the proof of Theorem 4.1, we have the uniform boundedness of $(\tilde{u}_n, \tilde{\nabla} u_n)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N))$. We can extract a subsequence (denoted again by u_n), such that $\tilde{u}_n \rightharpoonup w$ in $L^2((0,T), L^2(D))$ and $\tilde{\nabla} u_n \rightharpoonup (v_1, \dots, v_N)$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$. Mosco condition $(M2')$ (from Theorem 3.4) implies that $w \in L^2((0,T), H^1(\Omega))$.

To show that $w = u$, we let $\xi \in H^1(\Omega)$ and $\phi \in \mathscr{D}([0,T))$ and then use Mosco convergence of $H^1(\Omega_n)$ to $H^1(\Omega)$. In the same way as the proof of Theorem 4.1, we get (29) holds for all $\xi \in H^1(\Omega)$ and all $\phi \in \mathscr{D}([0,T))$. Hence by the uniqueness of solution, $w = u$ in $L^2((0,T), H^1(\Omega))$ and the whole sequence converges. □

Lemma 4.9. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T), L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then for each $t \in [0,T]$ we have $\tilde{u}_n(t)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$.*

Proof. We use the same argument as in the proof of Lemma 4.2 with Mosco convergence of $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$ replaced by Mosco convegence of $H^1(\Omega_n)$ to $H^1(\Omega)$ and the fact that $H^1(\Omega)$ is also dense in $L^2(\Omega)$. □

Remark 4.10. As remarked in the case of Dirichlet problems, we only require that $\tilde{f}_n|_\Omega \rightharpoonup f$ weakly in $L^2((0,T), L^2(\Omega))$ and $\tilde{u}_{0,n}|_\Omega \rightharpoonup u_0$ weakly in $L^2(\Omega)$ to obtain the conclusion of Theorem 4.8 and Lemma 4.9.

We next show the strong convergence.

Theorem 4.11. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T), L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then $\tilde{u}_n$ converges strongly to $\tilde{u}$ in $L^2((0,T), L^2(D))$ and $\tilde{\nabla} u_n$ converges strongly to $\tilde{\nabla} u$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$.*

Proof. The proof is similar to the one in Theorem 4.4. We show some details here for the sake of completeness. By Theorem 4.8, $(\tilde{u}_n, \tilde{\nabla} u_n)$ converges weakly to $(\tilde{u}, \tilde{\nabla} u)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D,\mathbb{R}^2))$. Since $u \in L^2((0,T), H^1(\Omega))$, Mosco condition $(M1')$ (from Theorem 3.4) implies that there exists $w_n \in L^2((0,T), H^1(\Omega_n))$ such that $\tilde{w}_n \to \tilde{u}$ in $L^2((0,T), L^2(D))$ and $\tilde{\nabla} w_n \to \tilde{\nabla} u$ in $L^2((0,T), L^2(D,\mathbb{R}^N))$. For $t \in [0,T]$, we consider

$$\begin{aligned} d_n(t) &= \frac{1}{2}\|\tilde{u}_n(t) - \tilde{u}(t)\|^2_{L^2(D)} + \alpha \int_0^t \|u_n(s) - w_n(s)\|^2_{L^2(\Omega_n)}\, ds \\ &\quad + \alpha \int_0^t \|\nabla u_n(s) - \nabla w_n(s)\|^2_{L^2(\Omega_n,\mathbb{R}^N)}\, ds \\ &= \frac{1}{2}\|\tilde{u}_n(t) - \tilde{u}(t)\|^2_{L^2(D)} + \alpha \int_0^t \|u_n(s) - w_n(s)\|^2_{H^1(\Omega_n)}\, ds. \end{aligned} \tag{45}$$

By (24) (with $\lambda = 0$), we can show that d_n satisfies (32) for all $n \in \mathbb{N}$. It can be easily seen from the weak convergence of $(\tilde{u}_n, \tilde{\nabla} u_n)$ and the strong convergence of $(\tilde{w}_n, \tilde{\nabla} w_n)$ to $(\tilde{u}, \tilde{\nabla} u)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D,\mathbb{R}^N))$ that (33) and (34) also hold. By using Lemma 4.9 instead of Lemma 4.2, we obtain (35). Finally, (36) is also valid for Neumann problem. Hence, $d_n(t) \to 0$ for all $t \in [0,T]$. This shows pointwise convergence of $\tilde{u}_n(t)$ to $\tilde{u}(t)$ in $L^2(D)$. Moreover, by taking $t = T$ we get

$$\begin{aligned} &\int_0^T \|\tilde{u}_n(s) - \tilde{u}(s)\|^2_{L^2(D)}\, ds \\ &\quad \le \int_0^T \|\tilde{u}_n(s) - \tilde{w}_n(s)\|^2_{L^2(D)}\, ds + \int_0^T \|\tilde{w}_n(s) - \tilde{u}(s)\|^2_{L^2(D)}\, ds \\ &\quad \to 0, \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \|\tilde{\nabla} u_n(s) - \tilde{\nabla} u(s)\|^2_{L^2(D,\mathbb{R}^N)}\, ds \\ &\quad \le \int_0^T \|\tilde{\nabla} u_n(s) - \tilde{\nabla} w_n(s)\|^2_{L^2(D,\mathbb{R}^N)}\, ds + \int_0^T \|\tilde{\nabla} w_n(s) - \tilde{\nabla} u(s)\|^2_{L^2(D,\mathbb{R}^N)}\, ds \\ &\quad \to 0. \end{aligned}$$

This proves the strong convergence $\tilde{u}_n \to \tilde{u}$ in $L^2((0,T), L^2(D))$ and $\tilde{\nabla} u_n \to \tilde{\nabla} u$ in $L^2((0,T), L^2(D,\mathbb{R}^N))$. □

Recall that we embed the space $K = H^1(\Omega)$ in $V = L^2(D) \times L^2(D,\mathbb{R}^N)$. If v is a function in $W((0,T), H^1(\Omega), H^1(\Omega)')$, then $v' \in L^2((0,T), H^1(\Omega)')$. It is not always true that we can embed $v'(t) \in V' = L^2(D) \times L^2(D,\mathbb{R}^N)$ a.e. $t \in (0,T)$ and claim that $v \in W((0,T), V, V') \cap L^2((0,T), K)$. However, a similar argument as in the proof of Theorem 3.4 $(i) \Rightarrow (iii)$ for Mosco condition (M1") gives the following result.

Lemma 4.12. *Suppose that $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, If $w \in C^\infty([0,T],H^1(\Omega))$ then there exists $w_n \in C^\infty([0,T],H^1(\Omega_n))$ such that $\tilde{w}_n$ converges to $\tilde{w}$ in $C^\infty([0,T],L^2(D))$.*

Proof. We note that Proposition 3.9 gives uniform convergence of the approximation sequence in $V = L^2(D) \times L^2(D,\mathbb{R}^N)$. The proof follows the same arguments as in the proof of Theorem 3.4 $(i) \Rightarrow (iii)$. The only difference is that we assume here $w \in C^\infty([0,T],H^1(\Omega))$. Hence the stretched function $w_\delta = w \circ S_\delta^{-1}$ belongs to $C^\infty([-\delta,T+\delta],H^1(\Omega))$. We point out that, by using uniform continuity of the k-th order derivative $w^{(k)}$ on $[0,T]$, the restriction of w_δ on $[0,T]$ converges to w in $C^\infty([0,T],H^1(\Omega))$. This gives the required convergence in $C^\infty([0,T],L^2(D))$. □

Using the above lemma, we show in the next theorem that the solution u_n of (27) indeed converges uniformly with respect to $t \in [0,T]$.

Theorem 4.13. *Suppose $\tilde{f}_n \to \tilde{f}$ in $L^2((0,T),L^2(D))$ and $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then $\tilde{u}_n$ converges strongly to $\tilde{u}$ in $C([0,T],L^2(D))$.*

Proof. As in the proof of Theorem 4.6, it requires to show uniform convergence of $(\tilde{u}_n(t)|\tilde{u}(t))_{L^2(D)} \to (\tilde{u}(t)|\tilde{u}(t))_{L^2(D)}$.

Let $\epsilon > 0$ arbitrary. By a similar argument as in the proof Lemma 3.8, we have the density of $C^\infty([0,T],H^1(\Omega))$ in $W((0,T),H^1(\Omega),H^1(\Omega)')$. Since the solution u is in the space $W((0,T),H^1(\Omega),H^1(\Omega)')$, there exists $w \in C^\infty([0,T],H^1(\Omega))$ such that

$$\|w-u\|_{W((0,T),H^1(\Omega),H^1(\Omega)')} \le \varepsilon.$$

As $W((0,T),H^1(\Omega),H^1(\Omega)')$ is continuously embedded in $C([0,T],L^2(\Omega))$, we can indeed choose $w \in C^\infty([0,T],H^1(\Omega))$ such that

$$\|w(t)-u(t)\|_{L^2(\Omega)} \le \varepsilon, \tag{46}$$

for all $t \in [0,T]$. By Lemma 4.12, there exists $w_n \in C^\infty([0,T],H^1(\Omega_n))$ such that $\tilde{w}_n \to \tilde{w}$ in $C^\infty([0,T],L^2(D))$. We can write

$$\begin{aligned} \left|(\tilde{u}_n(t)-\tilde{u}(t)|\tilde{u}(t))_{L^2(D)}\right| \le{} & \left|(\tilde{u}_n(t)|\tilde{w}_n(t))_{L^2(D)} - (\tilde{u}(t)|\tilde{w}(t))_{L^2(D)}\right| \\ & + \left|(\tilde{u}_n(t)-\tilde{u}(t)|\tilde{u}(t)-\tilde{w}(t))_{L^2(D)}\right| \\ & + \left|(\tilde{u}_n(t)|\tilde{w}(t)-\tilde{w}_n(t))_{L^2(D)}\right|, \end{aligned} \tag{47}$$

for all $n \in \mathbb{N}$. Since u_n is a solution of (27),

$$\begin{aligned} (\tilde{u}_n(t)|\tilde{w}_n(t))_{L^2(D)} = (\tilde{u}_{0,n}|\tilde{w}_n(0))_{L^2(D)} + \int_0^t \langle f_n(s), w_n(s)\rangle\, ds \\ + \int_0^t \langle w_n'(s), u_n(s)\rangle\, ds - \int_0^t a_n(s; u_n(s), w_n(s))\rangle\, ds, \end{aligned}$$

for all $n \in \mathbb{N}$. Taking into consideration that $\tilde{u}_{0,n} \to \tilde{u}_0$ and $f_n \to f$, we conclude that

$$(\tilde{u}_n(t)|\tilde{w}_n(t))_{L^2(D)} \to (\tilde{u}(t)|\tilde{w}(t))_{L^2(D)} \tag{48}$$

uniformly with respect to $t \in [0,T]$. Moreover, by (46) and the uniform boundedness of solutions as in (42),

$$\begin{aligned}
&\big|(\tilde{u}_n(t) - \tilde{u}(t)|\tilde{u}(t) - \tilde{w}(t))_{L^2(D)}\big| \\
&\leq \|\tilde{u}_n(t) - \tilde{u}(t)\|_{L^2(D)} \|\tilde{u}(t) - \tilde{w}(t)\|_{L^2(D)} \\
&\leq 2M_0\epsilon,
\end{aligned} \tag{49}$$

for all $t \in [0,T]$ and for all $n \in \mathbb{N}$. Finally, as $\tilde{w}_n \to \tilde{w}$ in $C^\infty([0,T], L^2(D))$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\|\tilde{w}_n(t) - \tilde{w}(t)\|_{L^2(D)} \leq \epsilon,$$

for all $t \in [0,T]$ and for all $n \geq N_\epsilon$. Hence,

$$\begin{aligned}
\big|(\tilde{u}_n(t)|\tilde{w}_n(t) - \tilde{w}(t))_{L^2(D)}\big| &\leq \|\tilde{u}_n(t)\|_{L^2(D)} \|\|\tilde{w}_n(t) - \tilde{w}(t)\|_{L^2(D)} \\
&\leq M_0\epsilon,
\end{aligned} \tag{50}$$

for all $t \in [0,T]$ and for all $n \geq N_\epsilon$. Therefore, by (47) – (50), there exists $\tilde{N}_\epsilon \in \mathbb{N}$ and a positive constant C such that

$$\big|(\tilde{u}_n(t) - \tilde{u}(t)|\tilde{u}(t))_{L^2(D)}\big| \leq C\epsilon,$$

for all $t \in [0,T]$ and for all $n \geq \tilde{N}_\epsilon$. As $\epsilon > 0$ was arbitrary, this proves the required uniform convergence of $(\tilde{u}_n(t)|\tilde{u}(t))_{L^2(D)} \to (\tilde{u}(t)|\tilde{u}(t))_{L^2(D)}$ with respect to $t \in [0,T]$. □

We can use the same criterion on Ω_n and Ω as in Neumann elliptic problems to conclude the stability of solutions of Neumann parabolic equations under domain perturbation. In particular, for domains in two dimensional spaces, the conditions on domains given in [5, Theorem 3.1] implies convergence of solutions of non-autonomous parabolic equations (25) subject to Neumann boundary condition.

Remark 4.14. The assumptions on strong convergence of $\tilde{u}_{0,n}$ and $\tilde{f}_n$ can be weaken if we impose some regularity of the domains. We give an example of domains Ω_n satisfying the *cone condition* (see [1, Section 4.3]) uniformly with respect to $n \in \mathbb{N}$.

Let $N = 2$ and let

$$\begin{aligned}
\Omega &:= \{x \in \mathbb{R}^2 : |x| < 1\} \backslash \{(x_1, 0) : 0 \leq x_1 < 1\}, \\
\Omega_n &:= \{x \in \mathbb{R}^2 : |x| < 1\} \backslash \{(x_1, 0) : \delta_n \leq x_1 < 1\},
\end{aligned}$$

where $\delta_n \searrow 0$. This example is an exterior perturbation of the domain, that is $\Omega \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$. It is easy to see that Ω and Ω_n satisfy the cone condition uniformly with respect to $n \in \mathbb{N}$, but $H^1(\Omega)$ and $H^1(\Omega_n)$ do not have the extension property. Moreover, these domains satisfy the conditions in [5]. Hence, $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco. Note that here we take D to be the open unit disk center at 0 in $\mathbb{R}^2$. In this example, we only need that $\tilde{f}_n|_\Omega \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))$ and $\tilde{u}_{0,n}|_\Omega \rightharpoonup u_0$ in $L^2(\Omega)$ to conclude the convergence of solutions $\tilde{u}_n \to \tilde{u}$ in $C([\delta,T], L^2(D))$ for all $\delta \in (0,T]$. In addition, if $\tilde{u}_{0,n} \to \tilde{u}_0$ in $L^2(D)$, then the assertion holds for $\delta = 0$.

To see this, we note from Lemma 4.9 (taking Remark 4.10 into account) that $\tilde{u}_n(t)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$ weakly for all $t \in [0,T]$. Since $u \in L^2((0,T), H^1(\Omega))$, we have $u(t) \in H^1(\Omega)$ for almost everywhere $t \in (0,T)$. Fix now such $t \in (0,T)$. By the continuity of the solutions $u_n \in C([0,T], L^2(\Omega_n))$, for each $n \in \mathbb{N}$ we can choose $\rho_n > 0$ such that

$$\|u_n(s) - u_n(t)\|_{L^2(\Omega_n)} \le \frac{1}{n}$$

for all $s \in (t-\rho_n, t+\rho_n) \cap (0,T)$. As $u_n \in L^2((0,T), H^1(\Omega_n))$ we can choose $t_n \in (t-\rho_n, t+\rho_n) \cap (0,T)$ such that $u_n(t_n) \in H^1(\Omega_n)$ for all $n \in \mathbb{N}$. For these choices of t_n, we have $\|u_n(t_n) - u_n(t)\|_{L^2(\Omega_n)} \to 0$ as $n \to \infty$. It follows that $\tilde{u}_n(t_n)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$ weakly. Since $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$, the restriction $u_n(t_n)_{|\Omega}$ belongs to $H^1(\Omega)$ for all $n \in \mathbb{N}$. Hence it follows from the weak convergence of $\tilde{u}_n(t_n)_{|\Omega} = u_n(t_n)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$ that

$$\int_\Omega \partial_j\big(u_n(t_n)_{|\Omega}\big)\phi dx = -\int_\Omega u_n(t_n)_{|\Omega}\partial_j\phi dx \to -\int_\Omega u(t)\partial_j\phi dx = \int_\Omega \partial_j u(t)\phi dx,$$

for all $\phi \in C_c^\infty(\Omega)$ for $j=1,2$. This means $\nabla u_n(t_n)_{|\Omega} \rightharpoonup \nabla u(t)$ in $L^2(\Omega, \mathbb{R}^2)$. Thus, $u_n(t_n)_{|\Omega}$ is bounded in $H^1(\Omega)$. As Ω is bounded and satisfies the cone condition, we have from the Rellich-Kondrachov theorem that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see [1, Theorem 6.2]). Therefore, $u_n(t_n)_{|\Omega}$ has a subsequence which converges strongly in $L^2(\Omega)$. Since we have a prior knowledge of weak convergence $u_n(t_n)_{|\Omega} \rightharpoonup u(t)$ in $L^2(\Omega)$, we conclude that the whole sequence $u_n(t_n)_{|\Omega} \to u(t)$ in $L^2(\Omega)$ strongly. By the choices of t_n, we conclude that $u_n(t)_{|\Omega} \to u(t)$ in $L^2(\Omega)$ strongly. Since the above argument works for almost everywhere $t \in (0,T)$, we deduce from the dominated convergence theorem that $\tilde{u}_n|_\Omega = u_n|_\Omega \to u$ in $L^2((0,T), L^2(\Omega))$ strongly. As the cutting line is a set of measure zero in $\mathbb{R}^2$, we have $\tilde{u}_n \to \tilde{u}$ in $L^2((0,T), L^2(D))$. By extracting a subsequence (indexed again by n), we can choose δ arbitrarily closed to zero in $(0,T]$ such that $\tilde{u}_n(\delta) \to \tilde{u}(\delta)$ in $L^2(D)$. The required convergence in $C([\delta,T], L^2(D))$ follows from the argument in the proof of Theorem 4.11 and Theorem 4.13 with the integration taken over $[\delta, T]$ instead of $[0,T]$ (see also the proof of [8, Therem 3.5]).

5. Application in Parabolic Variational Inequalities

In the previous section we have seen some applications of Theorem 3.4 when K_n and K are closed subspaces of V. In this section, we consider the case when K_n and K are just closed and convex subsets of V. We show here a similar convergence properties of solutions of parabolic variational inequalities.

Let K_n, K be closed and convex subsets in V. For each $t \in (0,T)$, suppose $a(t;\cdot,\cdot)$ is a continuous bilinear form on V satisfying (5) and (6). For simplicity, we assume that $\lambda = 0$ in (6). We denote by $A(t)$ the linear operator induced by $a(t;\cdot,\cdot)$. Let us consider the following parabolic variational inequalities. Given $u_{0,n} \in K_n$ and $f_n \in L^2((0,T), V')$, we want to find u_n such that for a.e. $t \in (0,T)$,

$u_n(t) \in K_n$ and

$$\begin{cases} \langle u'(t), v - u(t)\rangle + \langle A(t)u(t), v - u(t)\rangle - \langle f_n(t), v - u(t)\rangle \geq 0, \quad \forall v \in K_n \\ \hfill u(0) = u_{0,n}. \end{cases} \tag{51}$$

When K_n, f_n and $u_{0,n}$ converge to K, f and u_0, we wish to obtain convergence results of weak solution of (51) to the following limit inequalities.

$$\begin{cases} \langle u'(t), v - u(t)\rangle + \langle A(t)u(t), v - u(t)\rangle - \langle f(t), v - u(t)\rangle \geq 0, \quad \forall v \in K \\ \hfill u(0) = u_0. \end{cases} \tag{52}$$

Throughout this section, we denote the weak solution of (51) by u_n and the weak solution of (52) by u. The notion of our weak solutions is given in Definition 2.2.

Theorem 5.1. *Suppose $f_n \to f$ in $L^2((0,T),V')$, $u_{0,n} \rightharpoonup u_0$ in V and $u_{0,n} \to u_0$ in H. Then the sequence of weak solutions u_n is bounded in $L^2((0,T),V)$.*

Proof. Let $v \in W((0,T),V,V') \cap L^2((0,T),K)$ be the constant function defined by $v(t) := u_0$ for $t \in [0,T]$. Similarly, $v_n \in W((0,T),V,V') \cap L^2((0,T),K_n)$ defined by $v_n(t) := u_{0,n}$ for $t \in [0,T]$. It follows that $v_n \rightharpoonup v$ in $L^2((0,T),V)$. Since u_n is a weak solution of (51),

$$\begin{aligned} &\int_0^T \langle A(t)u_n(t), u_n(t) - v_n(t)\rangle \, dt \\ &\quad \leq \int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle - \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 \\ &\quad = -\int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^T \langle A(t)u_n(t) - A(t)v_n(t), u_n(t) - v_n(t)\rangle \, dt \\ &\quad \leq \int_0^T \langle A(t)v_n(t), v_n(t) - u_n(t)\rangle - \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt \\ &\quad \leq \|A(t)v_n - f_n\|_{L^2((0,T),V')} \|v_n - u_n\|_{L^2((0,T),V)}. \end{aligned}$$

By the coerciveness of $A(t)$,

$$\alpha\|u_n - v_n\|_{L^2((0,T),V)} \leq \|A(t)v_n - f_n\|_{L^2((0,T),V')}.$$

We conclude from the weak convergences of v_n and f_n that u_n is bounded in $L^2((0,T),V)$. □

Theorem 5.2. *Suppose $f_n \to f$ in $L^2((0,T),V')$, $u_{0,n} \rightharpoonup u_0$ in V and $u_{0,n} \to u_0$ in H. If K_n converges to K in the sense of Mosco, then the sequence of weak solutions u_n converges weakly to u in $L^2((0,T),V)$.*

Proof. By Theorem 5.1, we can extract a subsequence of u_n (denoted again by u_n) such that $u_n \rightharpoonup \kappa$ in $L^2((0,T),V)$. Since $u_n \in L^2((0,T),K_n)$, we apply Mosco condition $(M2')$ (from Theorem 3.4) to deduce that the weak limit $\kappa \in L^2((0,T),K)$.

By the uniqueness of weak solution, it suffices to prove that κ satisfies (15) (with u replaced by κ) in the definition of weak solution.

By Mosco condition $(M1')$, there exists $w_n \in L^2((0,T),K_n)$ such that $w_n \to \kappa$ in $L^2((0,T),V)$. Let $v \in W((0,T),V,V') \cap L^2((0,T),K)$. We again apply Theorem 3.4 for Mosco condition $(M1")$ to get a sequence of functions $v \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $v_n \to v$ in $W((0,T),V,V')$. For each $n \in \mathbb{N}$,

$$\begin{aligned}
&\langle A(t)w_n(t), v_n(t) - u_n(t)\rangle \\
&\quad = \langle A(t)u_n(t), v_n(t) - u_n(t)\rangle + \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - u_n(t)\rangle \\
&\quad = \langle A(t)u_n(t), v_n(t) - u_n(t)\rangle + \langle A(t)w_n(t) - A(t)u_n(t), w_n(t) - u_n(t)\rangle \\
&\qquad + \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - w_n(t)\rangle.
\end{aligned}$$

Hence, by definition of weak solution on K_n and coerciveness of $A(t)$,

$$\begin{aligned}
&\int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle + \langle A(t)w_n(t), v_n(t) - u_n(t)\rangle \, dt \\
&\quad - \int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 \\
&= \int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle + \langle A(t)u_n(t), v_n(t) - u_n(t)\rangle \, dt \\
&\quad - \int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 \\
&\quad + \int_0^T \langle A(t)w_n(t) - A(t)u_n(t), w_n(t) - u_n(t)\rangle \, dt \\
&\quad + \int_0^T \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - w_n(t)\rangle \, dt \\
&\geq \int_0^T \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - w_n(t)\rangle \, dt,
\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$,

$$\begin{aligned}
&\int_0^T \langle v'(t), v(t) - \kappa(t)\rangle + \langle A(t)\kappa(t), v(t) - \kappa(t)\rangle \, dt \\
&\quad - \int_0^T \langle f(t), v(t) - \kappa(t)\rangle \, dt + \frac{1}{2}\|v(0) - u_0\|_H^2 \geq 0.
\end{aligned}$$

This implies κ is a weak solution of (52) as required. □

We finally prove strong convergence of solutions.

Theorem 5.3. *Suppose $f_n \to f$ in $L^2((0,T),V')$, $u_{0,n} \rightharpoonup u_0$ in V and $u_{0,n} \to u_0$ in H. If K_n converges to K in the sense of Mosco, then the sequence of weak solutions u_n converges strongly to u in $L^2((0,T),V)$.*

Proof. By the coerciveness of $A(t)$,

$$\liminf_{n\to\infty} \int_0^T \langle A(t)u_n(t) - A(t)u(t), u_n(t) - u(t)\rangle \, dt \geq 0. \tag{53}$$

For each $\epsilon > 0$, we define u_ϵ by

$$\begin{aligned}\epsilon u_\epsilon' + u_\epsilon &= u\\ u_\epsilon(0) &= u_0.\end{aligned}$$

Then $u_\epsilon \in W((0,T),V,V') \cap L^2((0,T),K)$ and $u_\epsilon \to u$ in $L^2((0,T),V)$ as $\epsilon \to 0$ (see in the proof of [15, Theorem 2.3]). For each $\epsilon > 0$, Mosco condition $(M1")$ (from Theorem 3.4) implies that there exists $u_{\epsilon,n} \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $u_{\epsilon,n} \to u_\epsilon$ in $W((0,T),V,V')$ as $n \to \infty$. Since u_n is a weak solution of (51),

$$\begin{aligned}&\int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle\, dt\\ &\le \int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle\, dt - \int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle\, dt\\ &\quad + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 + \int_0^T \langle A(t)u_n(t), v_n(t) - u(t)\rangle\, dt,\end{aligned}$$

for all $v \in W((0,T),V,V') \cap L^2((0,T),K)$. In particular, taking $v_n = u_{\epsilon,n}$,

$$\begin{aligned}&\int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle\, dt\\ &\le \int_0^T \langle u_{\epsilon,n}'(t), u_{\epsilon,n}(t) - u_n(t)\rangle\, dt - \int_0^T \langle f_n(t), u_{\epsilon,n}(t) - u_n(t)\rangle\, dt\\ &\quad + \frac{1}{2}\|u_{\epsilon,n}(0) - u_{0,n}\|_H^2 + \int_0^T \langle A(t)u_n(t), u_{\epsilon,n}(t) - u(t)\rangle\, dt.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle\, dt\\ &\le \int_0^T \langle u_\epsilon'(t), u_\epsilon(t) - u(t)\rangle\, dt - \int_0^T \langle f(t), u_\epsilon(t) - u(t)\rangle\, dt\\ &\quad + \frac{1}{2}\|u_\epsilon(0) - u_0\|_H^2 + \int_0^T \langle A(t)u(t), u_\epsilon(t) - u(t)\rangle\, dt\\ &= -\epsilon \int_0^T \|u_\epsilon'(t)\|_H^2\, dt - \int_0^T \langle f(t), u_\epsilon(t) - u(t)\rangle\, dt\\ &\quad + \int_0^T \langle A(t)u(t), u_\epsilon(t) - u(t)\rangle\, dt\\ &\le -\int_0^T \langle f(t), u_\epsilon(t) - u(t)\rangle\, dt + \int_0^T \langle A(t)u(t), u_\epsilon(t) - u(t)\rangle\, dt.\end{aligned}$$

This is true for any $\epsilon > 0$. Hence, by letting $\epsilon \to 0$,

$$\limsup_{n\to\infty} \int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle\, dt \le 0.$$

On the other hand, the weak convergence of u_n in Theorem 5.2 implies

$$\lim_{n\to 0} \int_0^T \langle A(t)u(t), u_n(t) - u(t)\rangle\, dt = 0.$$

Thus,

$$\limsup_{n\to\infty}\int_0^T \langle A(t)u_n(t) - A(t)u(t), u_n(t) - u(t)\rangle dt \leq 0. \tag{54}$$

It follows from the coerciveness of $A(t)$, (53) and (54) that

$$\alpha\|u_n - u\|^2_{L^2((0,T),V)} \leq \int_0^T \langle A(t)u_n(t) - A(t)u(t), u_n(t) - u(t)\rangle\, dt \to 0,$$

as $n \to \infty$. Therefore, $u_n \to u$ in $L^2((0,T),V)$. □

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