

AN ASPHERICAL 5-MANIFOLD WITH PERFECT FUNDAMENTAL GROUP

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ABSTRACT. We construct aspherical closed orientable 5-manifolds with perfect fundamental group. This completes part of our study of PD_n -groups with pro- p completion a pro- p Poincaré duality group of dimension $\leq n - 2$. We also consider the question of whether there are any examples with “dimension drop” 1.

The paper [6] considers the phenomenon of dimension drop on pro- p completion for orientable Poincaré duality groups. Products of aspherical homology spheres, copies of S^1 and copies of the 3-manifold $M(K)$ obtained by 0-framed surgery on a nontrivial prime knot give examples of aspherical closed orientable n -manifolds N such that the pro- p completion of $\pi_1(N)$ is a pro- p Poincaré duality group of dimension r , for all $n \geq r + 2$ and $r \geq 0$, except when $n = 5$ and $r = 0$. (This gap reflects the fact that 5 is not in the additive semigroup generated by 3 and 4, dimensions in which aspherical homology spheres are known.)

We fill this gap in Theorem 1 below. Modifying the construction of [9] gives an aspherical closed 4-manifold with perfect fundamental group and non-trivial second homology. The total space of a suitable S^1 -bundle over this 4-manifold has the required properties. We then apply Theorem 1 to refine the final result of [6]. No such examples with dimension drop $n - r = 1$ are known as yet. The lowest dimension in which there might be such examples is $n = 4$, and we consider this case in the final section. If they exist, products with copies of S^1 would give examples in all higher dimensions.

In this paper all manifolds and PD_n -groups shall be orientable. If G is a group then G' and $G_{[n]}$ shall denote the commutator subgroup and the n th term of the lower central series, respectively. Let $G(\mathbb{Z}) = \bigcap_{\lambda \in \text{Hom}(G, \mathbb{Z})} \text{Ker}(\lambda)$. Then $G/G(\mathbb{Z})$ is the maximal torsion-free abelian quotient of G .

1. AN ASPHERICAL 5-MANIFOLD WITH π_1 PERFECT

Let X be a compact 4-manifold whose boundary components are diffeomorphic to the 3-torus T^3 . A *Dehn filling* of a component Y of ∂X is the adjunction of $T^2 \times D^2$ to X via a diffeomorphism $\partial(T^2 \times D^2) \cong Y$.

If the interior of X has a complete hyperbolic metric then “most” systems of Dehn fillings on some or all of the boundary components give manifolds which admit metrics of non-positive curvature, and the fundamental groups of the cores of the solid tori $T^2 \times D^2$ map injectively to the fundamental group of the filling of X , by the Gromov-Thurston 2π -Theorem. (Here “most” means “excluding finitely many fillings of each boundary component”. See [1].)

Theorem 1. *There are aspherical closed 5-manifolds with perfect fundamental group.*

Proof. Let $M = S^4 \setminus 5T^2$ be the complete hyperbolic 4-manifold with finite volume and five cusps considered in [7] and [9], and let \overline{M} be a compact core, with interior diffeomorphic to M . Then $H_1(\overline{M}; \mathbb{Z}) \cong \mathbb{Z}^5$, $\chi(\overline{M}) = 2$ and the boundary components of \overline{M} are all diffeomorphic to the 3-torus T^3 . There are infinitely many quintuples of Dehn fillings of the components of $\partial\overline{M}$ such that the resulting closed 4-manifold is an aspherical homology 4-sphere [9]. Let \widehat{M} be one such closed 4-manifold, and let $N \subset \widehat{M}$ be the compact 4-manifold obtained by leaving one boundary component of X unfilled. We may assume that the interior of N has a non-positively curved metric, and so N is aspherical. The Mayer-Vietoris sequence for $M = N \cup T^2 \times D^2$ gives an isomorphism

$$H_1(T^3; \mathbb{Z}) \cong H_1(N; \mathbb{Z}) \oplus H_1(T^2; \mathbb{Z}).$$

Let $\{x, y, z\}$ be a basis for $H_1(T^3; \mathbb{Z})$ compatible with this splitting. Thus x represents a generator of $H_1(N; \mathbb{Z})$ and maps to 0 in the second summand, while $\{y, z\}$ has image 0 in $H_1(N; \mathbb{Z})$ but generates the second summand. Since the subgroup generated by $\{y, z\}$ maps injectively to $\pi_1(\widehat{M})$ [1], the inclusion of ∂N into N is π_1 -injective. Let ϕ be the automorphism of $\partial N = T^3$ which swaps the generators x and y , and let $P = N \cup_{\phi} N$. Then P is aspherical and $\chi(P) = 2\chi(N) = 4$. A Mayer-Vietoris calculation gives $H_1(P; \mathbb{Z}) = 0$, and so $\pi = \pi_1(P)$ is perfect and $H_2(P; \mathbb{Z}) \cong \mathbb{Z}^2$.

Let e generate a direct summand of $H^2(\pi; \mathbb{Z}) = H^2(P; \mathbb{Z})$, and let E be the total space of the S^1 -bundle over P with Euler class e . Then E is an aspherical 5-manifold, and $G = \pi_1(E)$ is the central extension of π by \mathbb{Z} corresponding to $e \in H^2(\pi; \mathbb{Z})$. The Gysin sequence for the bundle (with coefficients in \mathbb{F}_p) has a subsequence

$$0 \rightarrow H^1(E; \mathbb{F}_p) \rightarrow H^0(P; \mathbb{F}_p) \rightarrow H^2(P; \mathbb{F}_p) \rightarrow H^2(E; \mathbb{F}_p) \rightarrow \dots$$

in which the *mod-p* reduction of e generates the image of $H^0(P; \mathbb{F}_p)$. Since e is indivisible this image is nonzero, for all primes p . Therefore $H^1(G; \mathbb{F}_p) = H^1(E; \mathbb{F}_p) = 0$, for all p , and so G is perfect. \square

(From the algebraic point of view, G is a quotient of the universal central extension of π , which is perfect.)

Is there an aspherical 5-dimensional homology sphere? If there is an aspherical 4-manifold X with the integral homology of $\mathbb{C}\mathbb{P}^2$ then the total space of the S^1 -bundle with Euler class a generator of $H^2(X; \mathbb{Z})$ would be such an example. The fake projective planes of [8] are aspherical and have the rational homology of $\mathbb{C}\mathbb{P}^2$, but they have nonzero first homology. The total spaces of the S^1 -bundles over such fake projective planes and with Euler class of infinite order are aspherical rational homology 5-spheres.

2. PRO- p COMPLETIONS AND DIMENSION DROP ≥ 2

The simplest construction of examples of orientable PD_n groups with pro- p completion a pro- p Poincaré duality group of lower dimension uses the fact that finite p -groups are nilpotent. Thus if G is a group with $G' = [G, G']$ the abelianization homomorphism induces isomorphisms on pro- p completions, for all primes p . Hence products of perfect groups with free abelian groups have pro- p completion a free abelian pro- p group. In this section we shall use Theorem 1 to remove a minor constraint on the final result of [6], which excluded a family of such examples.

Theorem 2. *For each $r \geq 0$ and $n \geq \max\{r+2, 3\}$ there is an aspherical closed n -manifold with fundamental group π such that $\pi/\pi' \cong \mathbb{Z}^r$ and $\pi' = \pi''$.*

Proof. Let Σ be an aspherical homology 3-sphere (such as the Brieskorn 3-manifold $\Sigma(2, 3, 7)$) and let P and E be as in Theorem 1. Taking suitable products of copies of Σ , P , E and S^1 with each other realizes all the possibilities with $n \geq r + 3$, for all $r \geq 0$.

Let $M = M(K)$ be the 3-manifold obtained by 0-framed surgery on a nontrivial prime knot K with Alexander polynomial $\Delta(K) = 1$ (such as the Kinoshita-Terasaka knot 11_{n42}). Then M is aspherical, since K is nontrivial [2], and if $\mu = \pi_1(M)$ then $\mu/\mu' \cong \mathbb{Z}$ and μ' is perfect, since $\Delta(K) = 1$. Hence products $M \times (S^1)^{r-1}$ give examples with $n = r + 2$, for all $r \geq 1$. \square

In particular, the dimension hypotheses in Theorem 6.3 of [6] may be simplified, so that it now asserts:

Let $m \geq 3$ and $r \geq 0$. Then there is an aspherical closed $(m+r)$ -manifold M with fundamental group $G = K \times \mathbb{Z}^r$, where $K = K'$. If $m \neq 4$ we may assume that $\chi(M) = 0$, and if $r > 0$ this must be so.

This is best possible, as no PD_1 - or PD_2 -group is perfect, and no perfect PD_4 -group H has $\chi(H) = 0$.

3. DIMENSION DROP ≤ 1 ?

Can we extend Theorem 2 to give examples of PD_n -groups π with $\pi/\pi' \cong \mathbb{Z}^{n-1}$ and $\pi_{[2]} = \pi_{[3]}$, realizing dimension drop $n - r = 1$ on all pro- p completions? In this section we shall weaken some of these conditions, by considering maps to other PD_{n-1} groups and requiring only that pro- p completion be well-behaved for *some* primes p . (This still suits the purposes of [6].)

There are clearly no such examples with $n = 2$. The next lemma rules out any with $\pi/\pi' \cong \mathbb{Z}^2$ and π' perfect when $n = 3$.

Lemma 3. *Let π be a PD_3 -group such that $\pi/\pi' \cong \mathbb{Z}^r$ and $\pi' = \pi''$. Then $r = 0, 1$ or 3 .*

Proof. We may assume that $r > 1$. The augmentation π -module \mathbb{Z} has a finitely generated projective resolution C_* of length 3. Let $\Lambda = \mathbb{Z}[\pi/\pi']$, and let $D_* = \Lambda \otimes_{\pi} C_*$ and $D^* = \text{Hom}_{\Lambda}(D_{3-*}, \Lambda)$. Then $H_1(D_*) = 0$, since π' is perfect, $H_2(D_*) \cong H^1(D^*) \cong \text{Ext}_{\Lambda}^1(\mathbb{Z}, \Lambda) = 0$, since $r > 1$, and $H_3(D_*) \cong H^0(D^*) \cong \text{Hom}_{\Lambda}(\mathbb{Z}, \Lambda) = 0$, since $r > 0$. Therefore D_* is a finitely generated free resolution of the augmentation π/π' -module. Since D_* has length 3, $r = \text{c.d. } \pi/\pi' \leq 3$. \square

The values $r = 0, 1$ and 3 may be realized by the fundamental groups of $\Sigma(2, 3, 7)$, $M(11_{n42})$ and the 3-torus $(S^1)^3$, respectively.

We shall focus on the first undecided case, $n = 4$, but shall reformulate our question in a slightly weaker form. A finitely generated nilpotent group ν of Hirsch length h has a maximal finite normal subgroup $T(\nu)$, with quotient a PD_h -group. Moreover, $\nu/T(\nu)$ has nilpotency class $< h$, and is residually a finite p -group for all p , by Theorem 4 of Chapter 1 of [10]. Thus the pro- p completion of ν is a pro- p Poincaré duality group for all p prime to the order of $T(\nu)$.

If $G_{[k]}/G_{[k+1]}$ is finite, of exponent e , say, then so are all subsequent subquotients of the lower central series, by Proposition 11 of Chapter 1 of [10]. Thus if G is a PD_4 -group such that $G/G_{[3]}$ has Hirsch length 3 and $G_{[3]}/G_{[4]}$ is finite then, setting $\nu = G/G_{[3]}$, the canonical projection to $\nu/T(\nu)$ induces isomorphisms on pro- p completions, for almost all primes p . Taking products of one such group with copies of \mathbb{Z} would give similar examples with dimension drop 1 in all higher dimensions.

We consider first the case when the quotient PD_3 -group is abelian.

Theorem 4. *Let G be a PD_4 -group. Then there is an epimorphism from G to \mathbb{Z}^3 which induces isomorphisms on pro- p completions, for*

almost all p , if and only if $\beta_1(G) = 3$ and the homomorphism from $\wedge^2 H^1(G; \mathbb{Z})$ to $H^2(G; \mathbb{Z})$ induced by cup product is injective.

If these conditions hold then $\chi(G) \geq 2$ and $G(\mathbb{Z})$ is not FP_2 .

Proof. The homomorphism from $\wedge^2 H^1(G; \mathbb{Z})$ to $H^2(G; \mathbb{Z})$ induced by cup product is a monomorphism if and only if $G_{[2]}/G_{[3]}$ is finite [4].

Suppose that there is such an epimorphism. Since G/G' and $G_{[2]}/G_{[3]}$ are finitely generated, it follows easily that $\beta_1(G) = 3$ and $G_{[2]}/G_{[3]}$ is finite. Thus the conditions in the first assertion are necessary.

If they hold then $G_{[2]}/G_{[3]}$ is finite, and the rational lower central series for G terminates at $G(\mathbb{Z})$ [4]. Therefore $G(\mathbb{Z})/G_{[3]}$ is the torsion subgroup of $G/G_{[3]}$. Thus if p is prime to the order of $G(\mathbb{Z})/G_{[3]}$ then the canonical epimorphism from G to $G/G(\mathbb{Z}) \cong \mathbb{Z}^3$ induces an isomorphism of pro- p completions.

The image of cup product from $\wedge^2 H^1(G; \mathbb{Z})$ to $H^2(G; \mathbb{Z})$ has rank 3, and must be self-annihilating, since $\wedge^4(\mathbb{Z}^3) = 0$. Hence $\beta_2(G) \geq 6$, by the non-singularity of Poincaré duality, and so $\chi(G) \geq 2$.

If $G(\mathbb{Z})$ were FP_2 then it would be a PD_1 -group, and so FP , by Theorem 1.19 of [3]. But then $\chi(G) = 0$, since χ is multiplicative in exact sequences of groups of type FP . \square

The conditions in this lemma are detected by de Rham cohomology, which suggests that we should perhaps seek examples among the fundamental groups of smooth manifolds with metrics of negative curvature. Are there any such groups? Since $\chi(G) \neq 0$, no such group is solvable or a semidirect product $H \rtimes \mathbb{Z}$ with H of type FP .

There are parallel criteria in the nilpotent case.

Theorem 5. *Let G be a PD_4 -group. Then there is an epimorphism from G to a nonabelian nilpotent PD_3 -group which induces isomorphisms on pro- p completions, for almost all p , if and only if $\beta_1(G) = 2$, cup product from $\wedge^2 H^1(G; \mathbb{Z})$ to $H^2(G; \mathbb{Z})$ is 0, and $G_{[3]}/G_{[4]}$ is finite.*

If these conditions hold then $\chi(G) \geq 0$.

Proof. The conditions are clearly necessary. Suppose that they hold. Then $G/G(\mathbb{Z}) \cong \mathbb{Z}^2$. The homomorphism from the free group $F(2)$ to G determined by elements of G representing a basis for this quotient induces a monomorphism from $F(2)/F(2)_{[3]}$ to $G/G_{[3]}$ with image of finite index, by the cup-product condition [4]. Thus $G(\mathbb{Z})/G_{[3]}$ is nilpotent and virtually \mathbb{Z} . Let T be the preimage in G of the torsion subgroup of $G(\mathbb{Z})/G_{[3]}$. This is characteristic in G , and G/T is a non-abelian extension of \mathbb{Z}^2 by \mathbb{Z} . Hence it is a nilpotent PD_3 -group. If $G_{[3]}/G_{[4]}$ is finite, then the quotient epimorphism to G/T induces isomorphisms on pro- p completions, for almost all p .

If these conditions hold then the natural map from $H_2(G; \mathbb{Q})$ to $H_2(G/G_{[3]}; \mathbb{Q}) \cong \mathbb{Q}^2$ is an epimorphism, by the 5-term exact sequence of low degree for the homology of G as an extension of $G/G_{[3]}$ by $G_{[3]}$. Hence $\beta_2(G) \geq 2$, and so $\chi(G) \geq 0$. \square

Are there any such groups? Theorem 1.19 of [3] again implies that T cannot be FP_2 . If $\chi(G) = 0$ then T cannot even be finitely generated, by Corollary 6.1 of [5] (used twice). For otherwise T would be \mathbb{Z} , so G would be nilpotent, and $G_{[3]}/G_{[4]}$ would be infinite. (If Γ is a lattice in the nilpotent Lie group Nil^4 then the first two conditions of Lemma 5 hold, and $\chi(\Gamma) = 0$, but $\Gamma_{[3]} \cong \mathbb{Z}$ and $\Gamma_{[4]} = 1$.) Is there such a group with T free of infinite rank?

There is one other class of PD_{n-1} -groups whose pro- p completions are well understood and well-behaved. Surface groups are residually finite p -groups, and have pro- p completion a pro- p Poincaré duality group of dimension 2, for all primes p . But once again there are no examples with $n = 3$.

Lemma 6. *Let G be a PD_3 -group and H a PD_2 -group, and let $f : G \rightarrow H$ be a homomorphism such that $H^1(f; \mathbb{Z})$ is an isomorphism. Then f does not induce isomorphisms on pro- p completion for any prime p .*

Proof. Let $r = \beta_1(G)$ and let $g : F(r) \rightarrow G$ be a homomorphism which induces an isomorphism from $F(r)/F(r)' \cong G/G(\mathbb{Z})$. Then fg induces a similar isomorphism. Since *c.d.* $H = 2$ and $H^1(f)$ is an isomorphism, cup product on $\wedge^2 H^1(G; \mathbb{Z})$ is 0, by the nonsingularity of Poincaré duality for G . Therefore the homomorphism from $Hom(G(\mathbb{Z})/[G, G(\mathbb{Z})], \mathbb{Z})$ to $Hom(F_{[2]}/F_{[3]}, \mathbb{Z})$ induced by g is an isomorphism. On the other hand, the corresponding homomorphism induced by fg is not an isomorphism, since cup product on $\wedge^2 H^1(H; \mathbb{Z})$ is non-trivial. In fact, the induced map from $G_{[2]}/G_{[3]}$ to $H_{[2]}/H_{[3]}$ has infinite kernel. (See [4].) Hence f does not induce isomorphisms on pro- p completion for any prime p . \square

Beyond this it becomes more difficult to find suitable PD_{n-1} groups as candidates for quotients.

The possibility of no dimension drop ($n = r$) is realized by the n -torus $(S^1)^n$, for any n . Are there any examples in which the dimension increases on pro- p completion, i.e., with $n < r$? It again follows from Lemma 3 that if $\pi/\pi' \cong \mathbb{Z}^r$ and π' is perfect then $n \geq 4$. Moreover, if there is such a PD_4 -group G then $\chi(G) \geq 2$, by the non-singularity of Poincaré duality.

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REFERENCES

- [1] Anderson, M. Dehn fillings and Einstein metrics in higher dimensions, *J. Diff. Geom.* 73 (2006), 219–261.
- [2] Gabai, D. Foliations and the topology of 3-manifolds. III, *J. Diff. Geom.* 26 (1987), 479–536.
- [3] Hillman, J.A. *Four-manifolds, Geometries and Knots*, Geometry and Topology Monographs, vol. 5, Geometry and Topology Publications (2002 – revision 2007). Current version – see www.sydney.edu.au/math/u/jonh
- [4] Hillman, J.A. The kernel of integral cup product, *J. Austral. Math. Soc.* 43 (1987), 10–15.
- [5] Hillman, J.A. and Kochloukova, D.H. Finiteness conditions and PD_r -group covers of PD_n -complexes, *Math. Z.* 256 (2007), 45–56.
- [6] Hillman, J.A., Kochloukova, D.H. and Lima, I. Pro- p completions of Poincaré duality groups, *Israel J. Math.* (to appear).
- [7] Ivanišić, D. Hyperbolic structure on a complement of tori in the 4-sphere, *Adv. Geom.* 4 (2004), 119–139.
- [8] Prasad, G. and Yeung, S.-K. Fake projective planes, *Invent. Math.* 168 (2007), 321–370.
- [9] Ratcliffe, J. and Tschantz, S. Some examples of aspherical 4-manifolds that are homology 4-spheres, *Topology* 44 (2005), 341–350.
- [10] Segal, D. *Polycyclic Groups*, Cambridge Tracts in Mathematics 82, Cambridge University Press, Cambridge - London - New York - New Rochelle - Melbourne - Sydney (1983).

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