EXISTENCE AND CLASSIFICATION OF SINGULAR SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS WITH A GRADIENT TERM

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ABSTRACT. In this paper, we completely classify the behaviour near 0, as well as at ∞ when $\Omega = \mathbb{R}^N$, of all positive solutions of $\Delta u = u^q |\nabla u|^m$ in $\Omega \setminus \{0\}$, where Ω is a domain in $\mathbb{R}^N$ ($N \geq 2$) and $0 \in \Omega$. Here, $q \geq 0$ and $m \in (0, 2)$ satisfy $m + q > 1$. Our classification depends on the position of q relative to the critical exponent $q_* := \frac{N - m(N-1)}{N-2}$ (with $q_* = \infty$ if $N = 2$). We prove the following: If $q < q_*$, then any positive solution u has either (1) a removable singularity at 0, or (2) a *weak singularity* at 0 ($\lim_{|x|\to 0} u(x)/E(x) \in (0, \infty)$, where E denotes the fundamental solution of the Laplacian), or (3) $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$, where ϑ and λ are uniquely determined positive constants (strong singularity). If $q \geq q_*$ (for $N > 2$), then 0 is a removable singularity for all positive solutions. Furthermore, for any positive solution in $\mathbb{R}^N \setminus \{0\}$, we show that it is either constant or has a non-removable singularity at 0 (weak or strong). The latter case is possible only for $q < q_*$, where we use a new iteration technique to prove that all positive solutions are *radial*, non-increasing and converging to *any non-negative* number at ∞. This is in sharp contrast to the case of $m = 0$ and $q > 1$ when all solutions decay to 0. Our classification theorems are accompanied by corresponding existence results in which we emphasise the more difficult case of $m \in (0, 1)$ where new phenomena arise.

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1. Introduction and main results

Let Ω be a domain in $\mathbb{R}^N$ with $N \geq 2$. We assume that $0 \in \Omega$ and set $\Omega^* := \Omega \setminus \{0\}$. We are concerned with the non-negative solutions of nonlinear elliptic equations such as

$$-\Delta u + u^q |\nabla u|^m = 0 \quad \text{in } \Omega^*. \tag{1.1}$$

Unless otherwise stated, we always assume that $m, q \in \mathbb{R}$ satisfy

$$q \geq 0, \quad 0 < m < 2 \quad \text{and} \quad m + q > 1. \tag{1.2}$$

Our aim is to obtain a full classification of the behaviour near 0 (and also at ∞ if $\Omega = \mathbb{R}^N$) for all positive $C^1(\Omega^*)$-distributional solutions of (1.1), together with corresponding existence results. This study is motivated by a vast literature on the topic of isolated singularities. For instance, see [Brandolini et al. 2013; Brezis and Oswald 1987; Brezis and Véron 1980/81; Cîrstea 2014; Cîrstea and Du 2010; Friedman and Véron 1986; Phuoc and Véron 2012; Serrin 1965; Vázquez and Véron 1980/81; 1985; Véron 1981; 1986; 1996] and their references. As a novelty of this article, we reveal new and distinct features of the profile of solutions of (1.1) near 0 (and at ∞ when $\Omega = \mathbb{R}^N$), arising from the introduction of the gradient factor in the nonlinear term. It can be seen from our proofs that more general problems could be considered. However, to avoid further technicalities, we restrict our attention to (1.1).

In a different, but related direction, we mention that problems similar to (1.1), which include a gradient term, have attracted considerable interest in a variety of contexts. With respect to boundary-blow up problems, equations like (1.1) arise in the study of stochastic control theory (see [Lasry and Lions 1989]). We refer to [Alarcón et al. 2012] for a large list of references in the case when the domain is bounded and to [Felmer et al. 2013] when the domain is unbounded. In relation to viscous Hamilton–Jacobi equations, Bidaut-Véron and Dao [2012; 2013] have studied the parabolic version of (1.1) for $q = 0$. For the large time behaviour of solutions of Dirichlet problems for sub-quadratic viscous Hamilton–Jacobi equations, see [Barles et al. 2010]. We refer to [Brezis and Friedman 1983; Brezis et al. 1986; Oswald 1988] for the analysis of nonlinear parabolic versions of (1.1) with $m = 0$. If $\ell := m/(m+q)$ and $w := \ell^{m/(m-\ell)} u^{1/\ell}$, we rewrite (1.1) as

$$\Delta(w^\ell) = |\nabla w|^m \quad \text{in } \Omega^*, \tag{1.3}$$

where $\ell \in (0, 1]$ and $m \in (\ell, 2)$ from (1.2). The parabolic version of (1.3) has been studied in different exponent ranges in connection with various applications (most frequently describing thermal propagation phenomena in an absorptive medium): The case $\ell < 1$ is usually referred to as *fast* diffusion, whereas $\ell > 1$ corresponds to *slow* diffusion. The fast diffusion case with singular absorption (that is $\ell, m \in (0, 1)$) was analysed by Ferreira and Vazquez [2001] (see their references for the existence, uniqueness, regularity and asymptotic behaviour of solutions corresponding to other ranges of m and ℓ). The parabolic form of equations like (1.3) also features in the study of the porous medium equation; see [Vázquez 1992; 2007] for a general introduction to this area.

We now return to our problem (1.1). A solution of (1.1), which is assumed to be a non-negative $C^1(\Omega^*)$-function at the outset, is understood in the sense of distributions (see Definition 1.4). We observe that by the strong maximum principle (see Lemma 3.3), any solution of (1.1) is either identically zero or positive in Ω^*. The behaviour of solutions of (1.1) near zero is controlled by the fundamental solution of the Laplacian, which is denoted by E, see (1.11). For a positive solution u

of (1.1), the origin is a removable singularity if and only if $\lim_{|x|\to 0} u(x)/E(x) = 0$, see Lemma 3.11. Moreover, if 0 is a non-removable singularity, there exists $\lim_{|x|\to 0} u(x)/E(x) = \Lambda \in (0, \infty]$ and, as in [Véron 1986], we say that u has a *weak* (respectively, *strong*) *singularity at* 0 if $\Lambda \in (0, \infty)$ (respectively, $\Lambda = \infty$). The fundamental solution E, together with the nonlinear part of (1.1), plays a crucial role in the existence of solutions with non-removable singularities at 0. We define

$$q_* := \frac{N - m(N-1)}{N-2} \text{ if } N \geq 3 \text{ and } q_* = \infty \text{ if } N = 2. \tag{1.4}$$

If (1.2) holds, we show that (1.1) admits solutions with weak (or strong) singularities at 0 if and only if $q < q_*$ (or equivalently, $E^q|\nabla E|^m \in L^1(B_r(0))$ for some $r > 0$, where $B_r(0)$ denotes the ball centred at 0 of radius r). For $q < q_*$ and a smooth bounded domain Ω, we prove in Theorem 1.1 that (1.1) has solutions with any possible behaviour near 0 and a Dirichlet condition on $\partial\Omega$:

$$\lim_{|x|\to 0} \frac{u(x)}{E(x)} = \Lambda \text{ and } u = h \text{ on } \partial\Omega. \tag{1.5}$$

Theorem 1.1 (Existence I)**.** *Let* (1.2) *hold and $q < q_*$. Assume that Ω is a bounded domain with C^1 boundary. If $q < q_*$, then for any $\Lambda \in [0, \infty) \cup \{\infty\}$ and every non-negative function $h \in C(\partial\Omega)$, there exists a solution of the problem* (1.1)*,* (1.5)*.*

Theorem 1.1 is valid for $m = 0$ in (1.2) and $q \in (1, q_*)$ when the existence and uniqueness of the solution of (1.1), (1.5) is known (see, for example, [Friedman and Véron 1986] and [Cîrstea and Du 2010, Theorem 1.2], where more general nonlinear elliptic equations are treated).

Since $m > 0$ in our framework, the presence of the gradient factor in the nonlinear term of (1.1) creates additional difficulties especially for $0 < m < 1$, where new phenomena arise. By passing to the limit in approximating problems, we construct in Theorem 1.1 both the maximal and the minimal solution of (1.1), (1.5) (see Remark 4.2).[1] If $m \geq 1$ in Theorem 1.1, then (1.1), (1.5) has a *unique* solution (using Lemma 3.2 and Theorem 1.2(a)). In contrast, in Remark 4.3 we note that for $m \in (0, 1)$ the uniqueness of the solution of (1.1), (1.5) may not necessarily hold.[2] In Section 2, using the Leray–Schauder fixed point theorem, we study separately the existence of radial solutions of (1.1) with $\Omega = B_R(0)$ with $R > 0$ and $m \in (0, 1)$. For such a domain Ω and h a non-negative constant γ, the maximal and the minimal solution of (1.1), (1.5) are both radial (see Remark 4.2). For $m \in (0, 1)$, we show that they do not coincide if $\Lambda = 0$ and $\gamma \in (0, \infty)$: The maximal solution is γ, whereas the minimal solution is provided by Theorem 2.2, which gives a radial solution u such that $u' > 0$ in $(0, R)$ and $u(R) = \gamma$. On the other hand, for any $\Lambda \in (0, \infty)$ and under the necessary assumption $q < q_*$, we construct a radial *non-increasing* solution of (1.1) in $B_R(0) \setminus \{0\}$ satisfying $\lim_{r\to 0^+} u(r)/E(r) = \Lambda \in (0, \infty)$ and a Neumann boundary condition $u'(R) = 0$ (see Theorem 2.1).

Notice that if (1.2) holds and $q < q_*$, then $u_0(x) = \lambda |x|^{-\vartheta}$ is a positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ with a strong singularity at 0, where ϑ and λ are positive constants given by

$$\vartheta := \frac{2-m}{q+m-1} \text{ and } \lambda := \left[\vartheta^{1-m}(\vartheta - N + 2)\right]^{\frac{1}{q+m-1}}. \tag{1.6}$$

In Theorem 1.2, we describe all the different behaviours near 0 of the positive solutions of (1.1).

[1] The proof of Theorem 1.1 relies solely on (1.2) if $\Lambda = 0$ in (1.5).

[2] If $0 < m < 1$, we cannot apply Lemma 3.2. The modified comparison principle in Lemma 3.1 requires the extra condition $|\nabla u_1| + |\nabla u_2| > 0$ in D, which restricts its applicability.

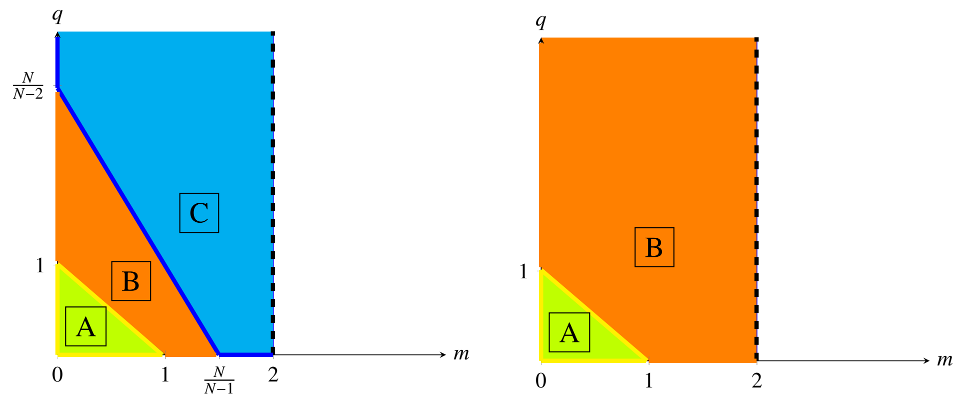

FIGURE 1. The left side (right side) picture pertaining to $N \geq 3$ ($N = 2$) illustrates different classification results established over various ranges of m and q. In region A, the dichotomy result of [Serrin 1965, Theorem 1] is applicable. In this paper, we establish a trichotomy result (removable, weak or strong singularities) in Theorem 1.2(a) for region B, generalising the well-known result of Véron [1981] for $m = 0$ and $q \in (1, \frac{N}{N-2})$ (the existence of weak singularities is also ascertained by Phuoc and Véron [2012] for $q = 0$ and $1 < m < \frac{N}{N-1}$). In region C, we obtain the removability result of Theorem 1.2(b) applicable for $N \geq 3$ (previously known in two cases: $m = 0$ and $q \geq \frac{N}{N-2}$ treated by Brezis and Véron [1980/81]; $q = 0$ and $\frac{N}{N-1} \leq m < 2$ due to Phuoc and Véron [2012]).

Theorem 1.2 (Classification I.). *Let* (1.2) *hold.*

(a) *If $q < q_*$, then any positive solution u of* (1.1) *satisfies exactly one of the following:*

(i) $\lim_{|x|\to 0} u(x) \in (0, \infty)$ *and u can be extended as a continuous solution of* (1.1) *in* $\mathcal{D}'(\Omega)$*, in the sense that* $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ *and*

$$\int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x + \int_\Omega |\nabla u|^m u^q \varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in C^1_c(\Omega). \tag{1.7}$$

(ii) *$u(x)/E(x)$ converges to a positive constant Λ as $|x| \to 0$ and, moreover,*

$$-\Delta u + u^q |\nabla u|^m = \Lambda \delta_0 \quad \text{in } \mathcal{D}'(\Omega), \tag{1.8}$$

where δ_0 denotes the Dirac mass at 0.

(iii) $\lim_{|x|\to 0} |x|^\vartheta u(x) = \lambda$, *where ϑ and λ are as in* (1.6).

(b) *If $q \geq q_*$ for $N \geq 3$, then any positive solution of* (1.1) *satisfies only alternative* (i) *above.*

In Figure 1, we illustrate how our Theorem 1.2 fits into the literature by providing the classification results for the entire eligible range of $m \in [0, 2)$ and $q \in [0, \infty)$, satisfying (1.2) (that is, the regions B and C in Figure 1). We point out that (1.2) is essential for the conclusion of Theorem 1.2 to hold. Indeed, when (1.2) fails such as in region A of Figure 1, then Theorem 1 of Serrin [1965] is applicable so that any positive solution u of (1.1) satisfies exactly one of the following:

(1) The solution u can be defined at 0 and the resulting function is a continuous solution of (1.1) in the whole Ω;
(2) There exists a constant $C > 0$ such that $1/C \leq u(x)/E(x) \leq C$ near $x = 0$.

In Theorem 1.2 we reveal that the behaviour of solutions of (1.1) near 0 for (m, q) in region B is clearly distinct from that corresponding to region C (for $N \geq 3$). In the latter, (1.1) has no solutions with singularities at 0 (see Theorem 1.2(b)). Belonging to the region C, we distinguish the points on the critical line $q = q_* = \frac{N-m(N-1)}{N-2}$, which joins the previously known critical values $\frac{N}{N-2}$ and $\frac{N}{N-1}$ corresponding to $m = 0$ and $q = 0$ in (1.1), respectively. When $N \geq 3$, Theorem 1.2(b) generalises the celebrated removability result of Brezis and Véron [1980/81] for $m = 0$ and $q \geq \frac{N}{N-2}$, as well as the recent one of Phuoc and Véron [2012, Theorem A.2], where the special case $q = 0$ was treated: Any positive $C^2(\overline{\Omega} \setminus \{0\})$-solution of $\Delta u = |\nabla u|^m$ in Ω^* remains bounded and it can be extended as a solution of the same equation in Ω when $\frac{N}{N-1} \leq m < 2$. If, in turn, $1 < m < \frac{N}{N-1}$ and $N \geq 2$, then Phuoc and Véron [2012] ascertain the existence of positive solutions of $\Delta u = |\nabla u|^m$ in Ω^* with a weak singularity at zero. We note that our Theorem 1.2(a) provides a full classification of the behaviour near 0 for all positive solutions of (1.1), corresponding to region B in Figure 1, extending the well-known trichotomy result of Véron [1981] for $m = 0$ and $1 < q < \frac{N}{N-2}$ (see also [Brezis and Oswald 1987] for a different approach).

Our next goal is to fully understand the profile of all positive solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$, which we show to be *radial*. We stress that the introduction of the gradient factor in the nonlinear term of (1.1) gives rise to new difficulties. In particular, neither the Kelvin transform nor the moving plane method can be applied. To prove radial symmetry, we shall introduce a new iterative method. A key feature that distinguishes our problem from the case $m = 0$ is that any positive solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ admits a limit at ∞, which may be *any* non-negative number. This asymptotic pattern at ∞ is different compared to $m = 0$ in (1.1) when every positive solution of the equation

$$\Delta u = u^q \text{ in } \mathbb{R}^N \setminus \{0\} \quad \text{with } q > 1 \tag{1.9}$$

must decay to 0 at ∞ (see Remark 3.5). Moreover, there are no positive solutions of (1.9) with a removable singularity at 0. For $q > 1$, Brezis [1984] showed that there exists a unique distributional solution ($u \in L^q_{\text{loc}}(\mathbb{R}^N)$) of $\Delta u = |u|^{q-1}u + f$ in $\mathbb{R}^N$ assuming only $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and, moreover, $u \geq 0$ a.e. provided that $f \geq 0$ a.e. in $\mathbb{R}^N$. The existence part of this result has been extended to the p-Laplace operator by Boccardo et al. [1993] (for $q > p - 1 > 0$ and $p > 2 - 1/N$), whereas the question of uniqueness of solutions has been recently investigated by D'Ambrosio et al. [2013].

We recall the profile of all positive solutions of (1.9) (see [Friedman and Véron 1986] for the results corresponding to the p-Laplace operator and $q > p - 1 > 0$):

- If $1 < q < \frac{N}{N-2}$, then *either* $u(x) = \lambda_0 |x|^{-\vartheta_0}$, where λ_0 and ϑ_0 correspond to λ and ϑ in (1.6) with $m = 0$ *or* u is a radial solution with a *weak singularity* at 0 and $\lim_{|x|\to\infty} u(x) = 0$. Moreover, for every $\Lambda \in (0, \infty)$, there exists a *unique* positive radial solution of (1.9) satisfying $\lim_{|x|\to 0} u(x)/E(x) = \Lambda$.
- If $q \geq \frac{N}{N-2}$ for $N \geq 3$, then there are no positive solutions of (1.9).

Compared to (1.9), our Theorem 1.3 reveals a much richer structure of solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$. There exist non-constant positive solutions if and only if $q < q_*$ and in this case, they must be radial, non-increasing and satisfy

$$\lim_{|x|\to 0} \frac{u(x)}{E(x)} = \Lambda \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = \gamma \tag{1.10}$$

with $\Lambda \in (0,\infty]$ and $\gamma \in [0,\infty)$. In addition, all solutions with a *strong singularity* at 0 are given in full by $u(x) = \lambda |x|^{-\vartheta}$ and $u_C(x) = C u_1(C^{1/\vartheta}|x|)$ for $x \in \mathbb{R}^N \setminus \{0\}$. Here, $C > 0$ is arbitrary and u_1 denotes the unique positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ with $\Lambda = \infty$ and $\gamma = 1$ in (1.10). Theorem 1.3 gives a complete classification of all positive solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$.

Theorem 1.3 ($\Omega = \mathbb{R}^N$, Existence and Classification II)**.** *Let* (1.2) *hold and u be any positive solution of* (1.1) *in $\mathbb{R}^N \setminus \{0\}$. The following assertions hold:*

- (i) *If $q < q_*$, then for any $\Lambda \in (0,\infty]$ and any $\gamma \in [0,\infty)$, there exists a unique positive radial solution of* (1.1) *in $\mathbb{R}^N \setminus \{0\}$, subject to* (1.10)*.*
- (ii) *If u is a non-constant solution, then $q < q_*$ and, moreover, u is radial, non-increasing and satisfies* (1.10) *for some $\Lambda \in (0,\infty]$ and $\gamma \in [0,\infty)$. Furthermore, if $\Lambda = \infty$, then $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$, where ϑ and λ are given by* (1.6) *(with $u(x) = \lambda |x|^{-\vartheta}$ if $\gamma = 0$).*
- (iii) *If* 0 *is a removable singularity for u, then u must be constant. In particular, if $q \geq q_*$ and $N \geq 3$, then u is constant.*

Liouville–type theorems for nonlinear elliptic equations have received much attention (in relation to (1.1), we refer to [Farina and Serrin 2011; Filippucci 2009; Li and Li 2012; Mitidieri and Pokhozhaev 2001]). For a broad class of quasilinear elliptic equations with the non-homogeneous term depending strongly on the gradient of the solution, Farina and Serrin [2011] establish that any $C^1(\mathbb{R}^N)$ solution must be constant. Their results apply for solutions unrestricted in sign and, in particular, for the p-Laplace model type equation $\Delta_p u = |u|^{q-1} u |\nabla u|^m$ with $p > 1$, $q > 0$ and $m \geq 0$ under various restrictions on these parameters. With respect to (1.1), if $q > 0$, $0 \leq m < 1$ and $q + m > 1$, then the constant functions are the only non-negative entire solutions of (1.1) (see [Filippucci 2009]). Furthermore, Farina and Serrin [2011] weakened the condition $m < 1$ to $m < \frac{N}{N-1}$. In Theorem 1.3(iii), we further improve this Liouville type result for (1.1) by changing the condition $m < \frac{N}{N-1}$ to $m < 2$ as in (1.2). We give a short and elementary proof of Theorem 1.3(iii), which does not involve the test function method usually employed in the current literature (see Remark 3.14). Our technique relies on local estimates, the comparison principle and the continuous extension at 0 of any solution of (1.1) with a removable singularity at 0 (see Lemma 3.13).

The proof of Theorem 1.3(i) relies on the (radial) maximal solution constructed in Theorem 1.1 for (1.1), (1.5), where $\Omega = B_k(0)$ and $h \equiv \gamma$. For $\Lambda \in (0,\infty)$, we show that as $k \to \infty$, such solution converges to a positive radial solution $u_{\Lambda,\gamma}$ of (1.1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1.10). The existence of the radial solution for $\Lambda = \infty$ is obtained as the limit of $u_{j,\gamma}$ as $j \to \infty$. The uniqueness follows from the comparison principle (Lemma 3.1), based on $\lim_{r\to 0^+} u_1(r)/u_2(r) = 1$ and $\lim_{r\to\infty}(u_1(r) - u_2(r)) = 0$ for any radial solutions u_1, u_2 satisfying (1.10).

The key ingredient in the proof of Theorem 1.3(ii) is Step 1 in Lemma 6.1: *Any positive solution of* (1.1) *in $\mathbb{R}^N \setminus \{0\}$ admits a non-negative limit at* ∞. We prove this fact using a new iterative technique, which we outline here. We take $(x_{n,1})$ with $|x_{n,1}| \nearrow \infty$ and $\lim_{n\to\infty} u(x_{n,1}) = a := \liminf_{|x|\to\infty} u(x)$. Given any sequence (x_n) in $\mathbb{R}^N$ with $|x_n| \nearrow \infty$, we show that for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $u < \limsup_{j\to\infty} u(x_j) + \varepsilon$ in $\overline{B_{|x_n|/2}(x_n)}$ for every $n \geq N_\varepsilon$. Hence, for some $N_1 > 0$, we have $u < a + \varepsilon$ in $\overline{B_{|x_{n,1}|/2}(x_{n,1})}$ for all $n \geq N_1$. Moreover, by choosing $x_{n,2} \in \partial B_{|x_{n,1}|/2}(x_{n,1}) \cap \partial B_{|x_{n,1}|}(0)$, there exists $N_2 > N_1$ such that $u < a + 2\varepsilon$ on $\overline{B_{|x_{n,1}|/2}(x_{n,2})} \cup \overline{B_{|x_{n,1}|/2}(x_{n,1})}$ for all $n \geq N_2$. After a finite number of iterations K (independent of n and ε), we find $N_K > 0$ such that $u < a + K\varepsilon$ on $\partial B_{|x_{n,1}|}(0)$ for all $n \geq N_K$. Since $u(x) \leq \max_{|y|=\delta} u(y)$ for all $|x| \geq \delta > 0$ (see Lemma 3.6), we find that $\limsup_{|x|\to\infty} u(x) \leq a + K\varepsilon$. Letting $\varepsilon \to 0$, we find that there exists $\lim_{|x|\to\infty} u(x) = \gamma \in [0, \infty)$. If u is not a constant solution, then (1.10) holds for some $\Lambda \in (0, \infty]$. For $m \geq 1$, the radial symmetry of u is due to the uniqueness of the solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1.10), and the invariance of this problem under rotation. For $m \in (0, 1)$, we need to think differently (we cannot use Lemma 3.2). For any $\varepsilon > 0$ (and $\varepsilon < \gamma$ if $\gamma > 0$), we construct positive radial solutions u_ε and U_ε of (1.1) in $\mathbb{R}^N \setminus \{0\}$ with the properties: (P1) $u_\varepsilon \leq u \leq U_\varepsilon$ in $\mathbb{R}^N \setminus \{0\}$; (P2) $u_\varepsilon(r)/E(r)$ and $U_\varepsilon(r)/E(r)$ converge to Λ as $r \to 0^+$; (P3) $\lim_{r\to\infty} u_\varepsilon(r) = \max\{\gamma - \varepsilon, 0\}$ and $\lim_{r\to\infty} U_\varepsilon(r) = \gamma + \varepsilon$. As $\varepsilon \to 0$, u_ε increases (U_ε decreases) to a positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1.10). The uniqueness of such a solution and (P1) prove that u is radial.

Notation. Let $B_R(x)$ denote the ball centred at x in $\mathbb{R}^N$ ($N \geq 2$) with radius $R > 0$. When $x = 0$, we simply write B_R instead of $B_R(0)$ and set $B_R^* := B_R \setminus \{0\}$. For abbreviation, we later use B^* in place of B_1^*. By ω_N, we denote the volume of the unit ball in $\mathbb{R}^N$. Let E denote the fundamental solution of the harmonic equation $-\Delta E = \delta_0$ in $\mathbb{R}^N$, namely

$$E(x) = \begin{cases} \dfrac{1}{N(N-2)\,\omega_N} |x|^{2-N} & \text{if } N \geq 3, \\ \dfrac{1}{2\pi} \log(R/|x|) & \text{if } N = 2. \end{cases} \tag{1.11}$$

For a bounded domain Ω of $\mathbb{R}^2$, we let $R > 0$ large so that Ω is included in B_R.

The concept of a solution for (1.1) in an open set D of $\mathbb{R}^N$ is made precise below, where we use $C_c^1(D)$ to denote the set of all functions in $C^1(D)$ with compact support in D.

Definition 1.4. By a *solution* (*sub-solution* or *super-solution*) of $\Delta u = u^q |\nabla u|^m$ in an open set $D \subseteq \mathbb{R}^N$, we mean a *non-negative* function $u \in C^1(D)$ which satisfies

$$\int_D \nabla u \cdot \nabla \varphi \, \mathrm{d}x + \int_D |\nabla u|^m u^q \varphi \, \mathrm{d}x = 0 \quad (\leq 0, \quad \geq 0) \tag{1.12}$$

for every (non-negative) function $\varphi \in C_c^1(D)$.

Outline. We divide the paper into six sections. In Section 2, we study the existence of radial solutions to (1.1) for $m \in (0, 1)$ and $\Omega = B_R$ with $R > 0$. Using the Leray–Schauder fixed point theorem, we prove that: a) There exist radial solutions with a weak singularity at 0 if and only if $q < q_*$ (see Theorem 2.1 and Lemma 2.5); b) For every $\gamma > 0$, there exists a non-constant radial solution with a removable singularity at 0 satisfying $u(R) = \gamma$, assuming only (1.2); see Theorem 2.2. The case $m \in (0, 1)$ deserves special attention since the failure of Lipschitz continuity in the gradient term yields a different version of the comparison principle (Lemma 3.1) compared to Lemma 3.2 pertaining to $m \geq 1$. Besides these comparison principles, Section 3 gives several auxiliary tools to be used later such as *a priori* estimates, a regularity result, and a spherical Harnack inequality. We prove Theorem 1.1 in Section 4 using a suitable perturbation technique. In Section 5 and Section 6, we establish the classification results of Theorem 1.2 and Theorem 1.3, respectively.

2. Existence of radial solutions when $m \in (0,1)$

Here, we assume that $m \in (0,1)$ and study the existence of positive radial solutions of (1.1) with $\Omega = B_R$ for $R > 0$. Without any loss of generality, we let $R = 1$ and consider the problem

$$u''(r) + (N-1)\,u'(r)/r = [u(r)]^q \left|u'(r)\right|^m \quad \text{for every } r \in (0,1). \tag{2.1}$$

In Theorem 2.1, under sharp conditions, we prove that for every $\Lambda \in (0,\infty)$, there exists a positive non-increasing $C^2(0,1]$-solution of (2.1), subject to

$$\lim_{r\to 0^+} \frac{u'(r)}{E'(r)} = \Lambda, \quad u'(1) = 0. \tag{2.2}$$

The first condition in (2.2) yields that $\lim_{r\to 0^+} u(r)/E(r) = \Lambda$, i.e., u has a weak singularity at 0.

Our central result is the following.

Theorem 2.1. *Assume that* $0 < m < 1$ *and* $1 - m < q < q_*$. *Then for every* $\Lambda \in (0,\infty)$*, there exists a positive non-increasing* $C^2(0,1]$*-solution of* (2.1), (2.2).

The proof of Theorem 2.1 is based on the transformation $w(s) = u(r)$ with $s = r^{2-N}$ if $N \geq 3$ and $w(s) = u(r)$ with $s = \ln(e/r)$ if $N = 2$. It is useful to introduce some notation:

$$C_N := \begin{cases} (N-2)^{m-2} & \text{if } N \geq 3 \\ e^{2-m} & \text{if } N = 2 \end{cases} \quad \text{and} \quad g_N(t) := \begin{cases} t^{-(q_*+1)} & \text{if } N \geq 3 \\ e^{(m-2)t} & \text{if } N = 2 \end{cases} \tag{2.3}$$

for all $t \in [1,\infty)$. For the definition of q_*, we refer to (1.4).

We see that u satisfies the differential equation in (2.1) if and only if

$$w''(s) = C_N\, g_N(s)\, [w(s)]^q \left|w'(s)\right|^m \quad \text{for all } s \in (1,\infty), \tag{2.4}$$

where the derivatives here are with respect to s. Moreover, (2.2) is equivalent to

$$\lim_{s\to\infty} w'(s) = \nu, \quad w'(1) = 0, \tag{2.5}$$

where $\Lambda = N(N-2)\,\omega_N \nu$ if $N \geq 3$ and $\Lambda = 2\pi\nu$ if $N = 2$.

In Lemma 2.4, we establish the assertion of Theorem 2.1 by proving that for every $\nu \in (0,\infty)$, there exists a positive non-decreasing $C^2[1,\infty)$ solution of (2.4), (2.5). Moreover, $w'(s) > 0$ for all $s \in (1,\infty)$ if $\nu \in (0,\nu_*]$, where we define

$$\nu_* := \left[(1-m)\, C_N \int_1^\infty t^q g_N(t)\, \mathrm{d}t\right]^{-\frac{1}{q+m-1}}. \tag{2.6}$$

We remark that $\nu_* < \infty$ since $t \longmapsto t^q g_N(t) \in L^1[1,\infty)$.

The proof of Theorem 2.1 is given in Section 2.1 using the Leray–Schauder fixed point theorem. Adapting these ideas, we ascertain in Theorem 2.2 that when $0 < m < 1$ and (1.2) holds, then for every $\gamma > 0$, Eq. (2.1) admits a positive increasing $C^2(0,1]$-solution satisfying $u(1) = \gamma$. If, in turn, $m \geq 1$ in (1.2), then (2.1), subject to $u(1) = \gamma$, has a unique solution with a removable singularity at zero, namely $u \equiv \gamma$.

Theorem 2.2. *Let* $0 < m < 1$ *and* $q > 1-m$. *Then for every* $\gamma > 0$*, there exists a positive increasing* $C^2(0,1]$*-solution of* (2.1)*, subject to* $u(1) = \gamma$.

For the proof of Theorem 2.2, we refer to Section 2.2.

2.1. **Proof of Theorem 2.1.** As mentioned above, Theorem 2.1 is equivalent to Lemma 2.4, whose proof relies essentially on the existence and uniqueness of a positive solution for a corresponding boundary value problem in Lemma 2.3 below.

Lemma 2.3. *Assume that* $0 < m < 1$ *and* $1 - m < q < q_*$*. Then for any fixed integer* $j \geq 2$ *and every* $\nu \in (0, \nu_*]$*, there exists a unique positive* $C^2[1, j]$*-solution of the problem*

$$\begin{cases} (2.4) & \text{for every } s \in (1, j), \\ w'(s) > 0 & \text{for every } s \in (1, j], \\ w'(1) = 0, & w'(j) = \nu. \end{cases} \tag{2.7}$$

Proof. We first establish the uniqueness of a positive $C^2[1, j]$-solution of (2.7), followed by the proof of the existence of such a solution.

Uniqueness. Suppose that $w_{1,j}$ and $w_{2,j}$ are two positive $C^2[1, j]$-solutions of (2.7). For any $\varepsilon > 0$, we define $P_{j,\varepsilon}(s) = w_{1,j}(s) - (1 + \varepsilon)\, w_{2,j}$ for all $s \in [1, j]$. For abbreviation, we write P_ε instead of $P_{j,\varepsilon}$ since j is fixed. It suffices to show that for every $\varepsilon > 0$, we have $P_\varepsilon \leq 0$ on $[1, j]$. Indeed, by letting $\varepsilon \to 0$ and interchanging $w_{1,j}$ and $w_{2,j}$, we find that $w_{1,j} = w_{2,j}$ in $[1, j]$. Suppose by contradiction that there exists $s_0 \in [1, j]$ such that $P_\varepsilon(s_0) = \max_{s\in[1,j]} P_\varepsilon(s) > 0$. We show that we arrive at a contradiction by analysing three cases:

Case 1. Let $s_0 = j$, that is $P_\varepsilon(j) = \max_{s\in[1,j]} P_\varepsilon(s)$.

From $P'_\varepsilon(j) = -\varepsilon\nu$, we have $P'_\varepsilon < 0$ on $(j - \delta, j)$ if $\delta > 0$ is small. This is a contradiction.

Case 2. Let $s_0 = 1$.

It follows that $P_\varepsilon(s) > 0$ for every $s \in [1, 1 + \delta]$ provided that $\delta > 0$ is small enough. Since $w_{1,j}$ and $w_{2,j}$ satisfy (2.7), for every $s \in (1, 1 + \delta)$, we obtain that

$$\frac{\left|w'_{1,j}(s)\right|^{1-m}}{\left|w'_{2,j}(s)\right|^{1-m}} = \frac{\int_1^s g_N(t) \left[w_{1,j}(t)\right]^q \,\mathrm{d}t}{\int_1^s g_N(t) \left[w_{2,j}(t)\right]^q \,\mathrm{d}t} > (1 + \varepsilon)^q. \tag{2.8}$$

Since $m + q > 1$, we get that $P'_\varepsilon > 0$ on $(1, 1 + \delta)$, which contradicts $P_\varepsilon(1) = \max_{s\in[1,j]} P_\varepsilon(s)$.

Case 3. Let $s_0 \in (1, j)$.

Using (2.7), $P_\varepsilon(s_0) > 0$, $P'_\varepsilon(s_0) = 0$ and $P''_\varepsilon(s_0) \leq 0$, we arrive at a contradiction since

$$\begin{aligned} 0 \geq \frac{w''_{1,j}(s_0) - (1 + \varepsilon)\, w''_{2,j}(s_0)}{C_N\, g_N(s_0) \left[w'_{2,j}(s_0)\right]^m} &= (1 + \varepsilon)^m \left[w_{1,j}(s_0)\right]^q - (1 + \varepsilon) \left[w_{2,j}(s_0)\right]^q \\ &> \left[w_{2,j}(s_0)\right]^q \left[(1 + \varepsilon)^{m+q} - (1 + \varepsilon)\right] > 0. \end{aligned} \tag{2.9}$$

This completes the proof of uniqueness.

Existence. We apply the Leray–Schauder fixed point theorem (see [Gilbarg and Trudinger 2001, Theorem 11.6]) to a suitable homotopy that we construct below.

Step 1. *Construction of the homotopy.*

Let $\mathcal{B}$ denote the Banach space of $C^1[1,j]$-functions with the usual $C^1[1,j]$-norm. Let $\nu \in (0,\nu_*]$, where ν_* is given by (2.6). We define $f_\nu(x) := \frac{1}{2}(\nu + |x| - |x-\nu|)$ for all $x \in \mathbb{R}$, that is

$$f_\nu(x) := \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le \nu, \\ \nu & \text{if } x \ge \nu. \end{cases} \tag{2.10}$$

Since ν is fixed, we will henceforth drop the index ν in f_ν. Let $w \in \mathcal{B}$ be arbitrary. We introduce the function $k = k_w : [0,\infty) \to \mathbb{R}$ as follows

$$k_w(\mu) := \int_1^j g_N(t) \left(\mu + \int_1^t f(w'(\xi))\, \mathrm{d}\xi \right)^q \mathrm{d}t \quad \text{for every } \mu \in [0,\infty). \tag{2.11}$$

We see that for any $w \in \mathcal{B}$, there exists a unique $\mu = \mu_w > 0$ such that

$$k_w(\mu_w) = \frac{\nu^{1-m}}{(1-m)\, C_N}. \tag{2.12}$$

Indeed, $\mu \longmapsto k_w(\mu)$ is increasing and the right-hand side of (2.12) is larger than $k_w(0)$. Using that $\nu \in (0,\nu_*]$ and by a simple calculation, we obtain that $\nu < \mu_w \le \hat{\nu}$, where $\hat{\nu}$ is given by

$$\hat{\nu} := \left(\frac{\nu^{1-m}}{(1-m)\, C_N \int_1^2 g_N(t)\, \mathrm{d}t} \right)^{\frac{1}{q}}.$$

We now define $h_w : [1,j] \to \mathbb{R}$ by

$$h_w(t) := \int_1^t g_N(\tau) \left(\mu_w + \int_1^\tau f(w'(\xi))\, \mathrm{d}\xi \right)^q \mathrm{d}\tau \quad \text{for all } t \in [1,j]. \tag{2.13}$$

In particular, we have $h_w(j) = k_w(\mu_w)$. We prescribe our homotopy $H : \mathcal{B} \times [0,1] \to \mathcal{B}$ as follows

$$H[w,\sigma](s) = \sigma \left(\mu_w + \int_1^s [(1-m)\, C_N\, h_w(t)]^{\frac{1}{1-m}}\, \mathrm{d}t \right) \qquad \text{for all } s \in [1,j], \tag{2.14}$$

where $w \in \mathcal{B}$ and $\sigma \in [0,1]$ are arbitrary.

Step 2. *We claim that H is a compact operator from $\mathcal{B} \times [0,1]$ to $\mathcal{B}$.*

We first show that $H : \mathcal{B} \times [0,1] \to \mathcal{B}$ is continuous, i.e., if $(w_n, \sigma_n) \in \mathcal{B} \times [0,1]$ such that $w_n \to w$ in $\mathcal{B}$ and $\sigma_n \to \sigma$ as $n \to \infty$, then $H[w_n,\sigma_n] \to H[w,\sigma]$ in $\mathcal{B}$. Since f in (2.10) is a continuous function, we have $f(w_n') \to f(w')$ as $n \to \infty$. From (2.13) and (2.14), it is enough to check that $\lim_{n\to\infty} \mu_{w_n} = \mu_w$. Suppose by contradiction that for a subsequence of w_n, relabelled w_n, we have $\lim_{n\to\infty} \mu_{w_n} = \tilde{\mu} \neq \mu_w$. Since $\mu_{w_n} \in (\nu, \hat{\nu}]$, we must have $\tilde{\mu} \in [\nu, \hat{\nu}]$. From (2.12) and the continuity of f, we have that

$$\frac{\nu^{1-m}}{(1-m)\, C_N} = k_{w_n}(\mu_{w_n}) \to k_w(\tilde{\mu}) \quad \text{as } n \to \infty.$$

But k_w is injective and thus $\tilde{\mu} = \mu_w$, which is a contradiction. This proves that $\lim_{n\to\infty} \mu_{w_n} = \mu_w$.

To see that H is compact, let $(w_n, \sigma_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}\times[0,1]$ and define $H_n(s) := H[w_n, \sigma_n](s)$ for all $s \in [1, j]$. We have $H_n \in C^2[1, j]$. We infer that $(H_n)_{n\in\mathbb{N}}$ is both uniformly bounded and equicontinuous in $\mathcal{B}$ since from (2.12), we find that

$$\|H_n\|_{L^\infty(1,j)} \le j\hat{\nu}, \quad \left\|H_n'\right\|_{L^\infty(1,j)} \le \nu, \quad \left\|H_n''\right\|_{L^\infty(1,j)} \le (j\hat{\nu})^q\, \nu^m \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Hence, the Arzelà–Ascoli Theorem implies that $H : \mathcal{B} \times [0, 1] \to \mathcal{B}$ is compact.

Step 3. *The existence of a positive $C^2[1, j]$-solution of (2.7) completed.*

By the first and second inequalities in (2.15), we have that $\|w\|_{C^1[1,j]}$ is bounded for all $(w, \sigma) \in \mathcal{B} \times [0, 1]$ satisfying $w = H[w, \sigma]$. From (2.14), we have $H[w, 0] = 0$ for all $w \in \mathcal{B}$. Therefore, the Leray–Schauder fixed point theorem implies the existence of $w_j \in \mathcal{B} = C^1[1, j]$ such that $H[w_j, 1] = w_j$. Thus, $\mu_{w_j} = w_j(1)$ and w_j satisfies

$$w_j(s) = w_j(1) + \int_1^s \left[(1-m)\, C_N\, h_{w_j}(t)\right]^{\frac{1}{1-m}} \mathrm{d}t \quad \text{for all } s \in [1, j]. \tag{2.16}$$

This gives that $w_j \in C^2[1, j]$. Using (2.12) and (2.13), we find that $w_j'(1) = 0$ and $w_j'(j) = \nu$. By twice differentiating (2.16), we get that

$$w_j'(s) = \left[(1-m)\, C_N\, h_{w_j}(s)\right]^{\frac{1}{1-m}}, \quad w_j''(s) = C_N \left|w_j'(s)\right|^m h_{w_j}'(s) > 0 \quad \text{for all } s \in (1, j). \tag{2.17}$$

It follows that $0 < w_j'(s) \le \nu$ for all $s \in (1, j]$ so that $f(w_j') = w_j'$ in $[1, j]$. Then, we have

$$h_{w_j}(s) = \int_1^s g_N(\tau) \left[w_j(\tau)\right]^q \mathrm{d}\tau, \quad h_{w_j}'(s) = g_N(s) \left[w_j(s)\right]^q \quad \text{for all } s \in (1, j). \tag{2.18}$$

From (2.17) and (2.18), we conclude that w_j is a positive $C^2[1, j]$-solution of (2.7). □

Lemma 2.4. *If $0 < m < 1$ and $1 - m < q < q_*$, then for every positive constant ν, there exists a positive $C^2[1,\infty)$-solution of the problem (2.4), (2.5).*

Proof. We divide the proof into two cases.

Case 1. *Let $\nu \in (0, \nu_*]$, where ν_* is given by (2.6).*

For each integer $j \ge 2$, let w_j denote the unique positive $C^2[1, j]$-solution of (2.7).
Fix $s \in [1,\infty)$ and denote $j_s := \lceil s\rceil$, where $\lceil\cdot\rceil$ stands for the ceiling function.

Claim 1: *The function $j \longmapsto w_j(s)$ is non-increasing for $j \ge j_s$.*

Indeed, for every $\varepsilon > 0$ and $j \ge j_s$, we prove that $P_{j,\varepsilon} \le 0$ on $[1, j]$, where we define $P_{j,\varepsilon}(t) := w_{j+1}(t) - (1+\varepsilon)\, w_j(t)$ for all $t \in [1, j]$. Fix $\varepsilon > 0$. Assume by contradiction that there exists $t_0 \in [1, j]$ such that $P_{j,\varepsilon}(t_0) = \max_{t\in[1,j]} P_{j,\varepsilon}(t) > 0$. By the same argument as in the uniqueness proof of Lemma 2.3, we derive a contradiction when $t_0 = 1$ or $t_0 \in (1, j)$. Suppose now that $t_0 = j$. Since $w_{j+1}''(t) > 0$ for all $t \in (1, j)$ and $w_{j+1}'(j+1) = \nu = w_j'(j)$, it follows that $P_{j,\varepsilon}'(j) < 0$. Thus, $P_{j,\varepsilon}'(t) < 0$ for all $t \in (j-\delta, j)$ if $\delta > 0$ is small enough. This contradicts $P_{j,\varepsilon}(j) = \max_{t\in[1,j]} P_{j,\varepsilon}(t)$, which proves that $P_{j,\varepsilon}(t) \le 0$ for all $t \in [1, j]$. Letting $t = s$ and $\varepsilon \to 0$, we conclude Claim 1.

By Lemma 2.3, we have $w_j(s) \ge w_j(1) > \nu$ for all $s \in [1, j]$. Using Claim 1, for every $s \in [1,\infty)$, we can define $w_\infty(s) := \lim_{j\to\infty} w_j(s)$. We thus have $w_\infty \ge \nu$ on $[1,\infty)$.

Claim 2: *The function* w_∞ *is a positive* $C^2[1,\infty)$*-solution of* (2.4)*,* (2.5)*.*

Let K be an arbitrary compact subset of $[1,\infty)$. We show that

$$w_j \to w_\infty \quad \text{uniformly in } K. \tag{2.19}$$

Let $j_K = j(K)$ be a large positive integer such that $K \subseteq [1, j]$ for all $j \geq j_K$. By Claim 1, we have $w_j \geq w_{j+1}$ in K for every $j \geq j_K$. Moreover, since $w_j \in C(K)$ and $0 \leq w_j' \leq \nu$ in K for all $j \geq j_K$, we obtain (2.19). In particular, $w_\infty \in C[1,\infty)$. From Lemma 2.3, w_j satisfies (2.16) with h_{w_j} given by (2.18). Using (2.19), we can let $j \to \infty$ in (2.16) to obtain that

$$w_\infty(s) = w_\infty(1) + \int_1^s \left[(1-m)\, C_N \int_1^t g_N(\tau)\, [w_\infty(\tau)]^q \, d\tau\right]^{\frac{1}{1-m}} dt \quad \text{for all } s \in (1,\infty). \tag{2.20}$$

Thus, $w_\infty \in C^2[1,\infty)$ satisfies (2.4) and $w_\infty'(1) = 0$.

It remains to prove that $\lim_{s\to\infty} w_\infty'(s) = \nu$. By using (2.20), we find that

$$w_\infty'(s) = \left[(1-m)\, C_N \int_1^s g_N(t)\, [w_\infty(t)]^q \, dt\right]^{\frac{1}{1-m}} \quad \text{for every } s \in (1,\infty). \tag{2.21}$$

On the other hand, from (2.12) and (2.18), we have

$$\int_1^j g_N(t) \left[w_j(t)\right]^q dt = h_{w_j}(j) = k_{w_j}(\mu_{w_j}) = \frac{\nu^{1-m}}{(1-m)\, C_N} \quad \text{for every } j \geq 2. \tag{2.22}$$

Since $w_j'(t) \leq \nu$ for all $t \in [1, j]$, we find that

$$w_j(t) \leq \nu t + w_j(1) - \nu \quad \text{for all } t \in [1, j].$$

Recall that $\nu < w_j(1) \leq w_2(1)$ for all $j \geq 2$. Consequently, we obtain that

$$g_N(t) \left[w_j(t)\right]^q \leq g_N(t) \left[\nu t + w_j(1) - \nu\right]^q \leq [w_2(1)]^q\, t^q g_N(t) \quad \text{for all } t \in [1, j] \text{ and } j \geq 2.$$

For every $t \in [1,\infty)$, it holds $g_N(t)\left[w_j(t)\right]^q \to g_N(t)\, [w_\infty(t)]^q$ as $j \to \infty$. Thus, we can let $j \to \infty$ in (2.22) and use Lebesgue's Dominated Convergence Theorem to find that

$$\int_1^\infty g_N(t)\, [w_\infty(t)]^q \, dt = \frac{\nu^{1-m}}{(1-m)\, C_N}. \tag{2.23}$$

From (2.21) and (2.23), we conclude that $\lim_{s\to\infty} w_\infty'(s) = \nu$, proving Lemma 2.4 in Case 1.

Case 2. *Let* $\nu > \nu_*$*, where* ν_* *is defined by* (2.6)*.*

From Case 1, there exists a positive $C^2[1,\infty)$-solution w_* of (2.4), (2.5) corresponding to $\nu = \nu_*$. If $N \geq 3$, then we denote $r_* := (\nu/\nu_*)^{\frac{m+q-1}{q_*-q}} \in (1,\infty)$ and define $w : [1,\infty) \to (0,\infty)$ by

$$w(s) = \begin{cases} r_*^{\frac{m+q_*-1}{m+q-1}}\, w_*(s/r_*) & \text{for } r_* \leq s < \infty, \\ r_*^{\frac{m+q_*-1}{m+q-1}}\, w_*(1) & \text{for } 1 \leq s \leq r_*. \end{cases} \tag{2.24}$$

If $N = 2$, we let $r_* := 1 + \frac{q+m-1}{2-m} \ln(\nu/\nu_*) \in (1, \infty)$ and define $w : [1, \infty) \to (0, \infty)$ as follows

$$w(s) = \begin{cases} \dfrac{\nu}{\nu_*} w_* (s + 1 - r_*) & \text{for } r_* \le s < \infty, \\ \dfrac{\nu}{\nu_*} w_*(1) & \text{for } 1 \le s \le r_*. \end{cases} \tag{2.25}$$

It is a simple exercise to check that w is a positive $C^2[1, \infty)$-solution of (2.4), (2.5). □

Lemma 2.5. *Let* (1.2) *hold. If* (2.1) *has a solution with a weak singularity at* 0*, then* $q < q_*$.

Remark 2.6. Theorem 1.2(b) shows that $q < q_*$ is a necessary condition for the existence of solutions of (1.1) with a non-removable singularity at 0 (see Section 5 for its proof).

Proof. We need only consider the non-trivial case $N \ge 3$. Suppose that $u \in C^2(0, 1)$ is a positive solution of (2.1) such that $\lim_{r\to 0^+} u(r)/r^{2-N} =: \eta$ for some $\eta \in (0, \infty)$. Then, u satisfies

$$\frac{d}{dr} \left(r^{N-1} u'(r) \right) = r^{N-1} [u(r)]^q |u'(r)|^m \ge 0 \quad \text{for all } r \in (0, 1). \tag{2.26}$$

Hence, $r \longmapsto r^{N-1}u'(r)$ is non-decreasing on $(0, 1)$ so that it admits a limit as $r \to 0^+$. By L'Hôpital's rule,we obtain that

$$(0, \infty) \ni \eta = \lim_{r\to 0^+} r^{N-2}u(r) = -(N - 2)^{-1} \lim_{r\to 0^+} r^{N-1}u'(r). \tag{2.27}$$

By integrating (2.26) over $(\varepsilon, 1/2)$ for arbitrarily small $\varepsilon > 0$ and letting $\varepsilon \to 0^+$, we find that

$$2^{1-N}u'(1/2) + (N - 2)\,\eta = \int_0^{1/2} r^{N-1}[u(r)]^q|u'(r)|^m \, dr < \infty. \tag{2.28}$$

We use $A(r) \sim B(r)$ as $r \to 0^+$ to mean that $\lim_{r\to 0^+} A(r)/B(r) = 1$. By using (2.27), we have that

$$r^{N-1}[u(r)]^q|u'(r)|^m \sim (N - 2)^m \eta^{q+m} r^{(N-1)(1-m)-q\,(N-2)} \quad \text{as } r \to 0^+.$$

This, jointly with (2.28), leads to $N - m\,(N - 1) > q\,(N - 2)$, which proves that $q < q_*$. □

2.2. **Proof of Theorem 2.2.** In view of the preliminary discussion in Section 2, Theorem 2.2 is equivalent to the following.

Lemma 2.7. *Let* $0 < m < 1$ *and* $m + q > 1$*. For any* $\gamma \in (0, \infty)$*, there exists a positive decreasing* $C^2[1, \infty)$*-solution of* (2.4)*, subject to* $w(1) = \gamma$ *and* $\lim_{s\to\infty} w(s) > 0$.

Proof. We divide the proof into three steps and proceed similarly to Lemmas 2.3 and 2.4.

Step 1: *For every integer* $j \ge 2$*, there exists a unique positive* $C^2[1, j]$*-solution* w_j *of*

$$\begin{cases} (2.4) \quad \text{for every } s \in (1, j) \\ w'(s) < 0 \quad \text{for every } s \in (1, j), \\ w(1) = \gamma, \quad w'(j) = 0. \end{cases} \tag{2.29}$$

To show uniqueness, we follow an argument similar to the uniqueness proof of Lemma 2.3 in Case 3. Keeping the same notation, we see that Case 2 there (that is, $\max_{s\in[1,j]} P_\varepsilon(s) = P_\varepsilon(1) > 0$) cannot happen due to $w(1) = \gamma$ in (2.29). Finally, in Case 1 (i.e., $s_0 = j$), we have $P_\varepsilon > 0$ on

$[j-\delta, j]$ for $\delta > 0$ small enough, which implies (2.8) for all $s \in (j-\delta, j)$. Since $w'(s) < 0$ on $(1, j)$, it follows that $P'_\varepsilon < 0$ on $(j-\delta, j)$, which is a contradiction with $\max_{s\in[1,j]} P_\varepsilon(s) = P_\varepsilon(j)$.

Next, we show existence via the Leray–Schauder fixed point theorem. Let $\mathcal{B}$ denote the Banach space of $C^1[1, j]$-functions with the usual $C^1[1, j]$ norm. Let $\hat{f}(x) := \frac{1}{2}(\gamma + |x| - |x-\gamma|)$ for all $x \in \mathbb{R}$. We prescribe the homotopy $\hat{H} : \mathcal{B} \times [0, 1] \to \mathcal{B}$ as follows

$$\hat{H}[w,\sigma](s) = \sigma\left(\gamma - \int_1^s \left[C_N\,(1-m)\int_\tau^j g_N(t)\left(\hat{f}(w(t))\right)^q\,\mathrm{d}t\right]^{\frac{1}{1-m}}\mathrm{d}\tau\right)\quad\text{for all } s \in [1, j], \tag{2.30}$$

where $w \in \mathcal{B}$ and $\sigma \in [0, 1]$ are arbitrary. We show that $\hat{H}$ is a compact operator from $\mathcal{B} \times [0, 1]$ to $\mathcal{B}$ as in Step 2 in the existence proof of Lemma 2.3. We use that

$$\begin{aligned} &\left\|\hat{H}\right\|_{L^\infty(1,j)} \le \gamma, \quad \left\|\hat{H}'\right\|_{L^\infty(1,j)} \le \left[C_N\,(1-m)\,\gamma^q\int_1^\infty g_N(t)\,\mathrm{d}t\right]^{\frac{1}{1-m}}, \\ &\left\|\hat{H}''\right\|_{L^\infty(1,j)} \le g_N(1)\left[C_N\,(1-m)^m\,\gamma^q\left(\int_1^\infty g_N(t)\,\mathrm{d}t\right)^m\right]^{\frac{1}{1-m}}. \end{aligned} \tag{2.31}$$

Hence, $\|w\|_{C^1[1,j]}$ is bounded for all $(w, \sigma) \in \mathcal{B} \times [0, 1]$ satisfying $w = \hat{H}[w, \sigma]$. From (2.30), we have $\hat{H}[w, 0] = 0$ for all $w \in \mathcal{B}$. Therefore, by the Leray–Schauder fixed point theorem, there exists $w_j \in \mathcal{B} = C^1[1, j]$ such that $\hat{H}[w_j, 1] = w_j$. Thus, $w_j(1) = \gamma$, $w'_j(j) = 0$ and w_j satisfies

$$w_j(s) = \gamma - \int_1^s \left[C_N\,(1-m)\int_\tau^j g_N(t)\left(\hat{f}(w_j(t))\right)^q\,\mathrm{d}t\right]^{\frac{1}{1-m}}\mathrm{d}\tau \quad\text{for all } s \in [1, j]. \tag{2.32}$$

Clearly, $w'_j \le 0$ in $[1, j]$ so that $w(s) \le w(1) = \gamma$ in $[1, j]$.

To conclude Step 1, it remains to show that $w_j(s) > 0$ for all $s \in [1, j]$.

Claim 1: *If there exists $\hat{s} \in (1, j]$ such that $w_j(\hat{s}) = 0$, then $w_j = 0$ on $[\hat{s}, j]$.*

Indeed, since $w'_j \le 0$ in $[1, j]$, it follows that $w_j(s) \le 0$ in $[\hat{s}, j]$ and thus $\hat{f}(w_j(t)) = 0$ for all $t \in [\hat{s}, j]$. In particular, using (2.32), we find that

$$w_j(\hat{s}) - w_j(\xi) = \int_{\hat{s}}^{\xi} \left[C_N\,(1-m)\int_\tau^j g_N(t)\left(\hat{f}(w_j(t))\right)^q\,\mathrm{d}t\right]^{\frac{1}{1-m}}\mathrm{d}\tau = 0 \quad\text{for all } \xi \in [\hat{s}, j].$$

Claim 2: *We have $w_j > 0$ in $[1, j]$.*

If we suppose the contrary, then $\hat{s} \in (1, j]$, where we define $\hat{s} = \inf\left\{\xi \in (1, j] :\ w_j(\xi) = 0\right\}$. Then, $w_j > 0$ on $[1, \hat{s})$ and $w_j = 0$ on $[\hat{s}, j]$. For any $\varepsilon \in (0, \gamma)$ small, there exists $\tilde{s} \in (1, \hat{s})$ such that $w_j(\tilde{s}) = \varepsilon$. Thus, by the mean value theorem, we have $-w'_j(\bar{s}) = \varepsilon/(\hat{s} - \tilde{s})$ for some $\bar{s} \in (\tilde{s}, \hat{s})$. Since $w_j = 0$ in $[\hat{s}, j]$ and $w_j \le \varepsilon$ on $[\tilde{s}, \hat{s}]$, by differentiating (2.32), we find that

$$\frac{\varepsilon}{\hat{s} - \tilde{s}} = -w'_j(\bar{s}) = \left[C_N\,(1-m)\int_{\bar{s}}^{\hat{s}} g_N(t)\,\left(w_j(t)\right)^q\,\mathrm{d}t\right]^{\frac{1}{1-m}} \le \left[C_N\,(1-m)\,g_N(1)\,(\hat{s} - \tilde{s})\,\varepsilon^q\right]^{\frac{1}{1-m}}.$$

This yields that $\varepsilon \geq \left[(j-1)^{2-m} C_N\,(1-m)\,g_N(1)\right]^{-1/(q+m-1)}$. This is a contradiction since $\varepsilon > 0$ can be made arbitrarily small. This proves Claim 2, completing the proof of Step 1.

To complete the proof of Lemma 2.7, we proceed as in Case 1 of Lemma 2.4.

Step 2: *For each fixed $s \in [1,\infty)$, the function $j \mapsto w_j(s)$ is non-increasing whenever $j \geq \lceil s \rceil$.*

It suffices to prove that $P_{j,\varepsilon} \leq 0$ in $[1, j]$ for every $\varepsilon > 0$, where $P_{j,\varepsilon}(t) := w_{j+1}(t) - (1+\varepsilon)\,w_j(t)$ for all $t \in [1, j]$. Assuming the contrary, we have $\max_{t\in[1,j]} P_{j,\varepsilon}(t) = P_{j,\varepsilon}(s_0) > 0$ for some $s_0 \in [1, j]$. We get a contradiction similarly to the proof of uniqueness of solutions to (2.29).

This shows that for each $s \in [1,\infty)$, we may define $w_\infty(s) := \lim_{j\to\infty} w_j(s)$.

Step 3: *The function w_∞ is a positive decreasing $C^2[1,\infty)$-solution of* (2.4)*, satisfying $w_\infty(1) = \gamma$ and $\lim_{s\to\infty} w_\infty(s) > 0$.*

The proof can be completed in the same way as Claim 2 in the proof of Lemma 2.4. We deduce that $w_j \to w_\infty$ uniformly in arbitrary compact sets of $[1,\infty)$. Hence w_∞ satisfies

$$w_\infty(s) = \gamma - \int_1^s \left[C_N\,(1-m)\int_\tau^\infty g_N(t)\,(w_\infty(t))^q\;\mathrm{d}t\right]^{\frac{1}{1-m}} \mathrm{d}\tau \quad \text{for all } s \in [1,\infty). \tag{2.33}$$

It follows that $w_\infty(1) = \gamma$ and $\lim_{s\to\infty} w'_\infty(s) = 0$. The fact that w_∞ is positive in $[1,\infty)$ follows as in Claim 2 of Step 1 above. We thus skip the details.

Finally, we show that $\lim_{s\to\infty} w_\infty(s) > 0$ by adjusting the proof of the positivity of w_∞. Suppose by contradiction that $\lim_{s\to\infty} w_\infty(s) = 0$. For any small $\varepsilon_1 > 0$, there exists $s_1 > 1$ large such that $w_\infty(s_1) = \varepsilon_1$. For any small $\varepsilon_2 \in (0, \gamma - \varepsilon_1)$, chosen independently of ε_1, there exists $\delta \in (0,1)$ such that $w_\infty(s_1 - \delta) = \varepsilon_1 + \varepsilon_2$. By the mean value theorem, we have $-w'_\infty(s_2) = \varepsilon_2/\delta$ for some $s_2 \in (s_1 - \delta, s_1)$. Since $w_\infty \leq \varepsilon_1 + \varepsilon_2$ in $[s_2,\infty)$, by differentiating (2.33), we find that

$$\varepsilon_2 \leq -w'_\infty(s_2) \leq \hat{C}^{\frac{1}{1-m}} (\varepsilon_1 + \varepsilon_2)^{\frac{q}{1-m}}, \quad \text{where } \hat{C} := C_N\,(1-m)\int_1^\infty g_N(t)\,\mathrm{d}t. \tag{2.34}$$

By taking $\varepsilon_1 \to 0$, we would get $\varepsilon_2 \geq \hat{C}^{-1/(q+m-1)}$. This is a contradiction since ε_1 and ε_2 can be chosen arbitrarily small. This finishes the proof of Lemma 2.7. □

3. Auxiliary Tools

We start with two comparison principles to be used often in the paper.

Lemma 3.1 (Comparison principle, see Theorem 10.1 in [Pucci and Serrin 2004])**.** *Let D be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let $\hat{B}(x,z,\boldsymbol{\xi}) : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous in $D \times \mathbb{R} \times \mathbb{R}^N$ and continuously differentiable with respect to $\boldsymbol{\xi}$ for $|\boldsymbol{\xi}| > 0$ in $\mathbb{R}^N$. Assume that $\hat{B}(x,z,\boldsymbol{\xi})$ is non-decreasing in z for fixed $(x,\boldsymbol{\xi}) \in D \times \mathbb{R}^N$. Let u_1 and u_2 be non-negative $C^1(\overline{D})$ (distributional) solutions of*

$$\begin{cases} \Delta u_1 - \hat{B}(x, u_1, \nabla u_1) \geq 0 & \text{in } D, \\ \Delta u_2 - \hat{B}(x, u_2, \nabla u_2) \leq 0 & \text{in } D. \end{cases} \tag{3.1}$$

Suppose that $|\nabla u_1| + |\nabla u_2| > 0$ in D. If $u_1 \leq u_2$ on ∂D, then $u_1 \leq u_2$ in D.

The following result given in [Pucci and Serrin 2007] is a version of Theorem 10.7(i) in [Gilbarg and Trudinger 2001] with the significant exception that $\hat{B}(x,z,\boldsymbol{\xi})$ is allowed to be singular at $\boldsymbol{\xi}=0$ and that the class $C^1(D)$ is weakened to $W^{1,\infty}_{\text{loc}}(D)$.

Lemma 3.2 (Comparison principle, see Corollary 3.5.2 in [Pucci and Serrin 2007])**.** *Let D be a bounded domain in $\mathbb{R}^N$ with $N\ge 2$. Assume that $\hat{B}(x,z,\boldsymbol{\xi}): D\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $D\times\mathbb{R}\times\mathbb{R}^N$ and is non-decreasing in z for fixed $(x,\boldsymbol{\xi})\in D\times\mathbb{R}^N$. Let u_1 and u_2 be (distribution) solutions in $W^{1,\infty}_{\text{loc}}(D)$ of* (3.1)*. If $u_1\le u_2+M$ on ∂D, where M is a positive constant, then $u_1\le u_2+M$ in D.*

Throughout this section, we understand that (1.2) holds. In Lemma 3.3, we show that the strong maximum principle applies to (1.1) (as a simple consequence of Theorem 2.5.1 in [Pucci and Serrin 2007]). Subsequently, we present several ingredients to be invoked later such as:

(i) *A priori* estimates (Lemma 3.4);
(ii) A regularity result (Lemma 3.8);
(iii) A spherical Harnack-type inequality (Lemma 3.9).

Lemma 3.3 (Strong Maximum Principle)**.** *If u is a solution of* (1.1) *such that $u(x_0)=0$ for some $x_0\in\Omega^*$, then $u\equiv 0$ in Ω^*.*

Proof. Using (1.2), we can easily find p such that $p>\max\{1/q,1\}$ and $mp'>1$, where p' denotes the Hölder conjugate of p, that is $p':=p/(p-1)$. By Young's inequality, we have

$$z^q|\boldsymbol{\xi}|^m\le z^{qp}/p+|\boldsymbol{\xi}|^{mp'}/p'\le z^{qp}/p+|\boldsymbol{\xi}|/p'$$

for all $z\in\mathbb{R}^+$ and $\boldsymbol{\xi}\in\mathbb{R}^N$ satisfying $|\boldsymbol{\xi}|\le 1$. Hence, by applying Theorem 2.5.1 in [Pucci and Serrin 2007], we conclude our claim. □

Lemma 3.4 (*A priori* estimates)**.** *Fix $r_0>0$ such that $\overline{B_{2r_0}}\subset\Omega$. Let u be a positive (sub-)solution of* (1.1)*. Then there exist positive constants $C_1=C_1(m,q)$ and $C_2=C_2(r_0,u)$ such that*

$$u(x)\le C_1|x|^{-\vartheta}+C_2\quad \textit{for every } 0<|x|\le 2r_0, \tag{3.2}$$

where ϑ is given by (1.6)*. In particular, we can take $C_1=\left[\vartheta^{1-m}(\vartheta+1)\right]^{1/(m+q-1)}$ and $C_2=\max_{\partial B_{2r_0}}u$.*

Proof. For any $\delta\in(0,2r_0)$, we define the annulus $A_\delta:=\left\{x\in\mathbb{R}^N:\delta<|x|<2r_0\right\}$. We consider the radial function $F_\delta(x)=C_1(|x|-\delta)^{-\vartheta}+C_2$ on A_δ, where $C_1:=\left[\vartheta^{1-m}(\vartheta+1)\right]^{1/(m+q-1)}$ and $C_2:=\max_{\partial B_{2r_0}}u$. Our choice of C_1 ensures that F_δ is a (radial) super-solution to (1.1) in A_δ, that is

$$F_\delta''(r)+(N-1)F_\delta'(r)/r\le[F_\delta(r)]^q\,|F_\delta'(r)|^m\quad\text{for all }\ \delta<r<r_0. \tag{3.3}$$

Indeed, to prove (3.3), it suffices to show that F_δ satisfies

$$F_\delta''(r)+(N-1)F_\delta'(r)/r\le C_1^{q+m}\vartheta^m(r-\delta)^{-[\vartheta(q+m)+m]}\quad\text{for all }\ \delta<r<2r_0. \tag{3.4}$$

By a simple calculation, we see that (3.4) is equivalent to the following inequality

$$\vartheta^{1-m}\left[\vartheta-N+2+(N-1)\delta/r\right]\le C_1^{m+q-1}\quad\text{for all }\ \delta<r<2r_0. \tag{3.5}$$

Since (3.5) holds for our C_1, we obtain that F_δ is a super-solution to (1.1) in A_δ. We show that

$$u(x) \le F_\delta(|x|) \quad \text{for all } x \in A_\delta. \tag{3.6}$$

Clearly, (3.6) holds for every $x \in \partial A_\delta$. Using that $\nabla F_\delta \neq 0$ in A_δ, we can apply Lemma 3.1 to conclude that (3.6) holds. For any fixed $x \in B^*_{2r_0}$, we have $x \in A_\delta$ for all $\delta \in (0, |x|)$. Hence, by letting $\delta \to 0$ in (3.6), we obtain (3.2). This completes the proof. □

Remark 3.5. The presence of the gradient factor in (1.1) implies that every non-negative constant is a solution of (1.1). Hence, the constant C_2 in (3.2) cannot be discarded nor made independent of u. This is in sharp contrast with the case $m = 0$ in (1.2), when it is known (see [Véron 1981, p. 227] or [Friedman and Véron 1986, Lemma 2.1]) that there exists a positive constant C_1, depending only on N and q, such that every positive solution of $\Delta u = u^q$ in Ω^* with $q > 1$ satisfies

$$u(x) \le C_1 |x|^{-\frac{2}{q-1}} \quad \text{for all } 0 < |x| \le r_0. \tag{3.7}$$

As before, $r_0 > 0$ is such that $\overline{B_{2r_0}} \subset \Omega$. Since C_1 is independent of Ω, from (3.7) we infer that *any* positive solution of (1.9) must decay to 0 at ∞.

Lemma 3.6. *If u is a positive solution of* (1.1) *in $\mathbb{R}^N \setminus \{0\}$, then for every $\delta > 0$, we have*

$$u(x) \le \max_{\partial B_\delta} u \quad \textit{for all } |x| \ge \delta. \tag{3.8}$$

Proof. Let $\delta > 0$ be fixed. For any positive integer k, we define the function

$$F_{k,\delta}(x) := C_1 (k - |x|)^{-\vartheta} + \max_{\partial B_\delta} u \quad \text{for all } \delta < |x| < k, \tag{3.9}$$

where C_1 is as in Lemma 3.4. Since $F_{k,\delta}(x) \to \infty$ as $|x| \nearrow k$, for $\varepsilon > 0$ small enough, we have $u(x) \le F_{k,\delta}(x)$ for all $k - \varepsilon \le |x| < k$. With a calculation similar to Lemma 3.4, we find that $F_{k,\delta} \in C^1(\overline{D})$ is a super-solution of (1.1) in $D := \{x \in \mathbb{R}^N : \ \delta < |x| < k - \varepsilon\}$. Since $|\nabla F_{k,\delta}| \neq 0$ in D and $u \in C^1(\overline{D})$, by the comparison principle in Lemma 3.1, we find that $u \le F_{k,\delta}$ in D, i.e.,

$$u(x) \le C_1 (k - |x|)^{-\vartheta} + \max_{\partial B_\delta} u \quad \text{for all } \delta < |x| < k. \tag{3.10}$$

By letting $k \to \infty$ in (3.10), we obtain (3.8). □

Corollary 3.7. *Any positive ($C^1(\mathbb{R}^N)$) solution of* (1.1) *in $\mathbb{R}^N$ must be constant.*

Proof. Let u be a positive solution of (1.1) in $\mathbb{R}^N$, that is $u \in C^1(\mathbb{R}^N)$ is positive function satisfying (1.1) in $\mathcal{D}'(\mathbb{R}^N)$ (see Definition 1.4). Let $y \in \mathbb{R}^N$ be fixed. For any positive integer k and $\delta \in (0, 1)$, we define the function $F_{k,\delta,y}(x) := F_{k,\delta}(x - y)$ for all $\delta < |x - y| < k$, where $F_{k,\delta}$ is given by (3.9). Following the same line of argument as in Lemma 3.6, we find that

$$u(x) \le F_{k,\delta,y}(x) = C_1 (k - |x - y|)^{-\vartheta} + \max_{|z-y|=\delta} u(z) \quad \text{for all } \delta < |x - y| < k. \tag{3.11}$$

Fix $x \in \mathbb{R}^N \setminus \{y\}$. In (3.11), we let $k \to \infty$ and $\delta = \delta_n \searrow 0$ as $n \to \infty$. Hence, we find that $u(x) \le u(y)$ for all $x \in \mathbb{R}^N$. Since $y \in \mathbb{R}^N$ is arbitrary, we conclude that u is a constant. □

Lemma 3.8 (A regularity result)**.** *Fix $r_0 > 0$ such that $\overline{B_{2r_0}} \subset \Omega$. Let ζ and θ be non-negative constants such that $\theta \le \vartheta$ and $\zeta = 0$ if $\theta = \vartheta$. Let u be a positive solution of* (1.1) *satisfying*

$$u(x) \le g(x) := d_1 |x|^{-\theta} [\ln(1/|x|)]^{\zeta} + d_2 \quad \text{for every } 0 < |x| \le 2r_0, \tag{3.12}$$

where d_1 and d_2 are positive constants. Then there exist constants $C > 0$ and $\alpha \in (0,1)$ such that for any x, x' in $\mathbb{R}^N$ with $0 < |x| \le |x'| < r_0$, it holds

$$|\nabla u(x)| \le C\,\frac{g(x)}{|x|} \quad \text{and } |\nabla u(x) - \nabla u(x')| \le C \frac{g(x)}{|x|^{1+\alpha}} |x - x'|^{\alpha}. \tag{3.13}$$

Proof. We only show the first inequality in (3.13), which can then be used to obtain the second inequality as in [Cîrstea and Du 2010, Lemma 4.1]. Fix $x_0 \in B^*_{r_0}$ and define $v_{x_0} : B_1 \to (0,\infty)$ by

$$v_{x_0}(y) := \frac{u(x_0 + \frac{|x_0|}{2} y)}{g(x_0)} \quad \text{for every } y \in B_1. \tag{3.14}$$

By a simple calculation, we obtain that v_{x_0} satisfies the following equation

$$-\Delta v + \tilde{B}(y, v, \nabla v) = 0 \quad \text{in } B_1, \tag{3.15}$$

where $\tilde{B}(y, v, \nabla v)$ is defined by

$$\tilde{B}(y, v, \nabla v) = 2^{m-2} \left[|x_0|^{\vartheta} g(x_0) \right]^{m+q-1} [v(y)]^q |\nabla v(y)|^m \quad \text{for all } y \in B_1. \tag{3.16}$$

From (3.12) and (3.14), there exists a positive constant A_0, which depends on r_0, such that $v_{x_0}(y) \le A_0$ for all $y \in B_1$. Moreover, using the assumptions on θ and ζ, we infer that there exists a positive constant A_1, depending on r_0, such that $|x_0|^{\vartheta} g(x_0) \le A_1$ for all $0 < |x_0| < r_0$. Hence, using that $m \in (0,2)$, we find a positive constant A_2, depending on r_0, but independent of x_0 such that

$$|\tilde{B}(y, v, \boldsymbol{\xi})| \le A_2 (1 + |\boldsymbol{\xi}|)^2 \quad \text{for all } y \in B_1 \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N. \tag{3.17}$$

Then, by applying Theorem 1 in [Tolksdorf 1984], we obtain a constant A_3, which depends on N and A_2, but is independent of x_0, such that $|\nabla v_{x_0}(0)| \le A_3$. Since this is true for every $x_0 \in B^*_{r_0}$, we readily deduce the first inequality of (3.13). □

Lemma 3.9 (A spherical Harnack-type inequality)**.** *Let $r_0 > 0$ be such that $\overline{B_{2r_0}} \subset \Omega$ and u be a positive solution of* (1.1)*. Then there exists a positive constant C_0 depending on r_0 such that*

$$\max_{\partial B_r} u \le C_0 \min_{\partial B_r} u \quad \text{for all } r \in (0, r_0). \tag{3.18}$$

Proof. Fix $x_0 \in B^*_{r_0}$. We define $v_{x_0} : B_1 \to \mathbb{R}$ as in (3.14). By Lemma 3.4, we know that (3.12) holds with $\theta = \vartheta$ and $\zeta = 0$. The proof of Lemma 3.8 shows that v_{x_0} is a solution of (3.15), where $\tilde{B}$ satisfies (3.17). Hence, by the Harnack inequality in [Trudinger 1967, Theorem 1.1], we have

$$\sup_{B_{1/3}} v_{x_0} \le C \inf_{B_{1/3}} v_{x_0}, \quad \text{or, equivalently,} \quad \sup_{B_{\frac{|x_0|}{6}}(x_0)} u \le C \inf_{B_{\frac{|x_0|}{6}}(x_0)} u, \tag{3.19}$$

where C is a positive constant independent of x_0 (but depending on A_2 and thus on r_0). Using (3.19) and a standard covering argument (see, for example, [Friedman and Véron 1986]), we conclude the proof of (3.18) with $C_0 = C^{10}$. □

As a consequence of Lemmas 3.8 and 3.9, we obtain the following.

Corollary 3.10. *Fix $r_0 > 0$ such that $\overline{B_{4r_0}} \subset \Omega$. Let u be a positive solution of* (1.1).

(a) *For any $0 < a < b \le 3/2$, there exists a constant $C_{a,b}$ depending on r_0 such that*

$$\max_{ar\le|x|\le br} u(x) \le C_{a,b} \min_{ar\le|x|\le br} u(x) \quad \text{for every } r \in (0, r_0). \tag{3.20}$$

(b) *There exists a positive constant C depending on r_0 such that*

$$|\nabla u(x)| \le C u(x)/|x| \quad \text{for all } 0 < |x| < r_0. \tag{3.21}$$

Proof. (a) For any $0 < a < b \le 3/2$, we define $\mathcal{D}_{a,b} := \{y \in \mathbb{R}^N : a \le |y| \le b\}$. Since $\mathcal{D}_{a,b}$ is a compact set in $\mathbb{R}^N$, there exists a positive integer $k_{a,b}$ and $y_i \in \mathcal{D}_{a,b}$ with $i = 1, 2, \ldots, k_{a,b}$ such that $\mathcal{D}_{a,b} \subseteq \bigcup_{i=1}^{k_{a,b}} B_{|y_i|/6}(y_i)$. Fix $r \in (0, r_0)$. Letting $x_i = ry_i$ for $i = 1, 2, \ldots, k_{a,b}$, we find that

$$\mathcal{D}_{ar,br} := \left\{x \in \mathbb{R}^N : ar \le |x| \le br\right\} \subseteq \bigcup_{i=1}^{k_{a,b}} B_{\frac{|x_i|}{6}}(x_i).$$

By (3.19), there exists a positive constant $C = C(r_0)$ such that

$$\sup_{B_{\frac{|x_i|}{6}}(x_i)} u(x) \le C \inf_{B_{\frac{|x_i|}{6}}(x_i)} u(x) \quad \text{for all } i = 1, 2, \ldots, k_{a,b}. \tag{3.22}$$

Hence, we obtain (3.20) with $C_{a,b} := C^{k_{a,b}}$.

(b) Fix $x_0 \in B^*_{r_0}$. In the definition of v_{x_0} in (3.14) and also in (3.16), we replace $g(x_0)$ by $u(x_0)$. By (a), the function v_{x_0} is bounded by a positive constant A_0 independent of x_0 since

$$v_{x_0}(y) := \frac{u(x_0 + \frac{|x_0|}{2} y)}{u(x_0)} \le \frac{\max_{|x_0|/2\le|y|\le3|x_0|/2} u(y)}{\min_{|x_0|/2\le|y|\le3|x_0|/2} u(y)} \le A_0 \quad \text{for all } y \in B_1.$$

The proof of (3.21) can now be completed as in Lemma 3.8. □

We give a removability result for (1.1), which will be useful in the proof of Lemma 3.13, as well as to deduce that alternative (i) in Theorem 1.2(a) occurs when $\lim_{|x|\to0} u(x)/E(x) = 0$.

Lemma 3.11. *Let u be a positive solution of* (1.1) *with* $\lim_{|x|\to0} u(x)/E(x) = 0$. *Then there exists* $\lim_{|x|\to0} u(x) \in (0, \infty)$ *and, moreover, u can be extended as a continuous solution of* (1.1) *in the whole Ω. If, in addition, $0 < m < 1$, then $u \in C^1(\Omega)$.*

Proof. As in [Cîrstea and Du 2010, Lemma 3.2(ii)], we obtain that $\limsup_{|x|\to0} u(x) < \infty$. We show that (1.7) holds. Indeed, for $\varphi \in C^1_c(\Omega)$ fixed, let $R > 0$ be such that $\operatorname{Supp}\varphi \subset B_R \subset\subset \Omega$. Using the gradient estimates in Lemma 3.8 and $\limsup_{|x|\to0} u(x) < \infty$, we can find positive constants C_1 and C_2 (depending on R), such that

$$|\nabla u|^m u^q \le C_1 |x|^{-m}(u + C_2) \quad \text{for all } 0 < |x| \le R.$$

Since $m < 2$, by [Serrin 1965, Theorem 1], we find that $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ and (1.7) holds.

We next prove that $\lim_{|x|\to0} u(x) > 0$. Fix $r_0 > 0$ small such that $\overline{B_{4r_0}} \subset \Omega$. By using (3.21) in Corollary 3.10, there exists a positive constant C, depending on r_0, such that

$$\Delta u = u^q |\nabla u|^m \le C^m |x|^{-m} u^{m+q} \quad \text{in } B^*_{r_0}. \tag{3.23}$$

For each integer $k > 1/r_0$, let w_k denote the unique positive classical solution of the problem

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B_{r_0} \setminus \overline{B_{\frac{1}{k}}}, \\ w|_{\partial B_{1/k}} = \min\limits_{\partial B_{1/k}} u, \quad w|_{\partial B_{r_0}} = \min\limits_{\partial B_{r_0}} u. \end{cases} \tag{3.24}$$

By uniqueness, w_k must be radially symmetric. Using (3.23) and Lemma 3.2, we infer that

$$w_{k+1}(x) \le w_k(x) \le u(x) \quad \text{for every } \ 1/k \le |x| \le r_0. \tag{3.25}$$

Then, $w_k \to w$ in $C^1_{\rm loc}(B^*_{r_0})$ as $k \to \infty$, where w is a positive radial solution of

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B^*_{r_0}, \\ \lim\limits_{|x|\to 0} w(x)/E(x) = 0 \ \text{ and } \ w|_{\partial B_{r_0}} = \min\limits_{\partial B_{r_0}} u. \end{cases} \tag{3.26}$$

We have $\lim_{|x|\to 0} w(x) > 0$ (see e.g., [Cîrstea 2014, Proposition 3.1(b)] if $N \ge 3$ and [Cîrstea 2014, Proposition 3.4(b)] if $N = 2$). From (3.25), we infer that $w \le u$ in $B^*_{r_0}$ and hence, $\lim_{|x|\to 0} u(x) > 0$.

Finally, we show that $u \in C^1(\Omega)$ when $m \in (0, 1)$. In this case, we can choose $p \in (N, N/m)$. We show that $u \in W^{2,p}_{\rm loc}(B_{r_0})$, where $r_0 > 0$ is small such that $\overline{B_{4r_0}} \subset \Omega$. Since $u \in C^1(\Omega^*)$, we conclude that $u \in C^1(\Omega)$ using the continuous embedding $W^{2,p}(B_r) \subset C^1(B_r)$ for $r > 0$ (see, for example, Corollaries 9.13 and 9.15 in [Brezis 2011] or [Evans 2010, p. 270]).

Observe that $u^q |\nabla u|^m \in L^p(B_{r_0})$. Indeed, using (3.21), there exist constants $c_1, c_2 > 0$ such that

$$\int_{B_{r_0}} |\nabla u|^{mp} dx \le c_1 \int_{B_{r_0}} |x|^{-mp}\, dx \le c_2 r_0^{N-mp} < \infty \text{ since } p < N/m. \tag{3.27}$$

Since $p > N$ and $u \in C(\overline{B_{r_0}})$, by Corollary 9.18 in [Gilbarg and Trudinger 2001, p. 243], there exists a unique solution $v \in W^{2,p}_{\rm loc}(B_{r_0}) \cap C(\overline{B_{r_0}})$ of the problem

$$\begin{cases} \Delta v = u^q |\nabla u|^m & \text{in } B_{r_0}, \\ v = u & \text{on } \partial B_{r_0}. \end{cases} \tag{3.28}$$

(The uniqueness of the solution $v \in W^{2,p}_{\rm loc}(B_{r_0}) \cap C(\overline{B_{r_0}})$ is valid for any $p > 1$.) We have $v \in W^{2,2}(D)$ for any subdomain $D \subset\subset B_{r_0}$ and by Theorem 8.8 in [Gilbarg and Trudinger 2001, p. 183], $u \in W^{2,2}(D)$. By the uniqueness of the solution $v \in W^{2,2}_{\rm loc}(B_{r_0}) \cap C(\overline{B_{r_0}})$ of (3.28), it follows that $u = v$ and thus $u \in W^{2,p}_{\rm loc}(B_{r_0})$. Hence, u is in $C^1(\Omega)$, completing the proof of Lemma 3.11. □

Remark 3.12. If $u \in C^1(\mathbb{R}^N)$ is a positive solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, then by Lemma 3.11, u becomes a positive $C^1(\mathbb{R}^N)$ solution of (1.1) in $\mathbb{R}^N$ (and, by elliptic regularity theory, $u \in C^2(\mathbb{R}^N)$).

We are now ready to prove the first part of the assertion of Theorem 1.3(iii).

Lemma 3.13. *Let $\Omega = \mathbb{R}^N$. If* 0 *is a removable singularity for a positive solution u of* (1.1)*, then u must be constant.*

Proof. Let u be a positive solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ with a removable singularity at 0. By Lemma 3.11, we can extend u as a positive continuous solution of (1.1) in $\mathcal{D}'(\mathbb{R}^N)$. Moreover, using also Lemma 3.6, we find that $\sup_{\mathbb{R}^N} u = u(0) > 0$. We show that

$$u(0) = \limsup_{|y|\to\infty} u(y). \tag{3.29}$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $u(x) \le \limsup_{|y|\to\infty} u(y) + \varepsilon$ for all $|x| \ge R_\varepsilon$. Set $f_\varepsilon(x) = \varepsilon|x|^{2-N}$ if $N \ge 3$ and $f_\varepsilon(x) = (1/R_\varepsilon)\log(R_\varepsilon/|x|)$ if $N = 2$. Clearly, there exists $r_\varepsilon > 0$ small such that $u(x) \le f_\varepsilon(x)$ in $B^*_{r_\varepsilon}$. Fix $z \in \mathbb{R}^N \setminus \{0\}$. Then $0 < |z| < R_\varepsilon$ for every $\epsilon > 0$ small and

$$u(z) \le f_\varepsilon(z) + \limsup_{|y|\to\infty} u(y) + \varepsilon.$$

Letting $\varepsilon \to 0$, we find that $u(0) \le \limsup_{|y|\to\infty} u(y) \le \sup_{\mathbb{R}^N} u = u(0)$. This proves (3.29).

If $u < u(0)$ in $\mathbb{R}^N \setminus \{0\}$, then (3.8) would imply that $u(z) \le \max_{|x|=1} u(x) < u(0)$ for all $|z| \ge 1$, which would contradict (3.29). Thus, there exists $z \in \mathbb{R}^N \setminus \{0\}$ such that $u(z) = u(0)$. Since u is a sub-harmonic function, by the strong maximum principle, we have $u = u(0)$ on $\mathbb{R}^N$. □

Remark 3.14. For $m < 1$, Lemma 3.13 follows from Lemma 3.11, combined with either Corollary 3.7 or [Filippucci 2009, Theorem 2.2], whose proof uses a test function technique in [Mitidieri and Pokhozhaev 2001]. Moreover, if $m < \frac{N}{N-1}$, we regain Lemma 3.13 for the positive $C^1(\mathbb{R}^N)$ solutions of (1.1) using the results in [Farina and Serrin 2011, p. 4422].

4. Proof of Theorem 1.1

Let (1.2) hold and $q < q_*$. Assume that Ω is a bounded domain with C^1 boundary and $h \in C(\partial\Omega)$ is a non-negative function. For any $n \ge 1$, we consider the perturbed problem

$$\Delta u = \frac{u^{q+1}}{\sqrt{u^2 + 1/n}} \frac{|\nabla u|^{m+2}}{|\nabla u|^2 + 1/n} \quad \text{in } \Omega^*. \tag{4.1}$$

Let $\Lambda \in [0, \infty)$. We shall prove the existence of a solution of (1.1), (1.5) based on the following.

Lemma 4.1. *If $\Lambda \in [0, \infty)$, then there is a unique non-negative solution $u_{\Lambda,n}$ of* (4.1)*,* (1.5)*.*

Proof. The uniqueness follows from Lemma 3.2. Indeed, let $\hat{B}$ denote

$$\hat{B}(x, z, \boldsymbol{\xi}) = \hat{B}(z, \boldsymbol{\xi}) := \frac{z|z|^q}{\sqrt{z^2 + 1/n}} \frac{|\boldsymbol{\xi}|^{m+2}}{|\boldsymbol{\xi}|^2 + 1/n} \quad \text{for every } x \in \Omega^*,\ z \in \mathbb{R} \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N.$$

We see that $\hat{B}$ is C^1 with respect to $\boldsymbol{\xi}$ in $\Omega^* \times \mathbb{R} \times \mathbb{R}^N$. By a simple calculation, we obtain that

$$\frac{\partial}{\partial z}\hat{B} = \frac{|\boldsymbol{\xi}|^{m+2}}{|\boldsymbol{\xi}|^2 + 1/n} \frac{|z|^q}{(z^2 + 1/n)^{\frac{3}{2}}} \left[qz^2 + (q+1)/n\right] \ge 0,$$

so that $\hat{B}$ is non-decreasing in z for fixed $(x, \boldsymbol{\xi}) \in \Omega^* \times \mathbb{R}^N$. Let $u_{\Lambda,n}$ and $\hat{u}_{\Lambda,n}$ denote two non-negative solutions of (4.1), (1.5). Fix $\varepsilon > 0$ arbitrary. If $\Lambda = 0$, then $u_{\Lambda,n} \le \varepsilon E + \hat{u}_{\Lambda,n}$ in Ω^*. If $\Lambda \in (0, \infty)$, then $u_{\Lambda,n} \le (1 + \varepsilon)\,\hat{u}_{\Lambda,n}$ in Ω^* using $\lim_{|x|\to 0} u_{\Lambda,n}(x)/\hat{u}_{\Lambda,n}(x) = 1$ and Lemma 3.2. Hence, in both cases, letting $\varepsilon \to 0$, then interchanging $u_{\Lambda,n}$ and $\hat{u}_{\Lambda,n}$, we find that $u_{\Lambda,n} \equiv \hat{u}_{\Lambda,n}$.

The existence of a non-negative solution $u_{\Lambda,n}$ for (4.1), (1.5) is established in two steps.

Step 1: *For any integer $k \geq 2$, let $\mathcal{D}_k := \Omega \setminus \overline{B_{\frac{1}{k}}}$. There exists a unique non-negative solution $u_{n,k} \in C^2(\mathcal{D}_k) \cap C(\overline{\mathcal{D}}_k)$ of the following problem:*

$$\begin{cases} \Delta u = \dfrac{u|u|^q}{\sqrt{u^2 + 1/n}} \dfrac{|\nabla u|^{m+2}}{|\nabla u|^2 + 1/n} & \text{in } \mathcal{D}_k := \Omega \setminus \overline{B_{\frac{1}{k}}}, \\ u = \Lambda E + \max_{\partial\Omega} h \ \text{ on } \partial B_{\frac{1}{k}} \ \text{ and } \ u = h \ \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Moreover $u_{n,k}$ is positive in $\mathcal{D}_k$.

The existence assertion is a consequence of Theorem 15.18 in [Gilbarg and Trudinger 2001]. The conditions of Theorem 14.1 and equation (10.36) in [Gilbarg and Trudinger 2001] can be checked easily. To see that the assumptions of Theorem 15.5 in [Gilbarg and Trudinger 2001] are satisfied, we take $\theta = 1$ in (15.53) and use that $m \in (0, 2)$. The uniqueness and non-negativity of the solution of (4.2) follows from Lemma 3.2. By Lemma 3.3, we obtain that $u_{n,k} > 0$ in $\mathcal{D}_k$. Observe also that $u_{n,k} \geq \min_{\partial\Omega} h$ in $\mathcal{D}_k$.

Step 2: *The limit of $u_{n,k}$ in $C^1_{\text{loc}}(\Omega^*)$ as $k \to \infty$ yields a non-negative solution of* (4.1)*,* (1.5)*.*

Since $\Lambda E + \max_{\partial\Omega} h$ is a super-solution of (4.2), we obtain that

$$0 < u_{n,k+1} \leq u_{n,k} \leq \Lambda E + \max_{\partial\Omega} h \quad \text{in } \mathcal{D}_k. \tag{4.3}$$

Thus, there exists $u_{\Lambda,n}(x) := \lim_{k\to\infty} u_{n,k}(x)$ for all $x \in \Omega^*$ and $u_{n,k} \to u_{\Lambda,n}$ in $C^1_{\text{loc}}(\Omega^*)$ as $k \to \infty$ (see Lemma 3.8), where $u_{\Lambda,n}$ is a non-negative solution of (4.1). We prove that $u_{\Lambda,n}$ satisfies (1.5). From (4.3) and Dini's Theorem, we find that $u_{\Lambda,n} \in C(\overline{\Omega} \setminus \{0\})$ and $u_{\Lambda,n} = h$ on $\partial\Omega$.

If $\Lambda = 0$, then clearly $\lim_{|x|\to 0} u_{\Lambda,n}(x)/E(x) = 0$. If $\Lambda \in (0, \infty)$, then by (4.3), we have $\limsup_{|x|\to 0} u_{\Lambda,n}(x)/E(x) \leq \Lambda$. To end the proof of Step 2, we show that

$$\liminf_{|x|\to 0} \frac{u_{\Lambda,n}(x)}{E(x)} \geq \Lambda. \tag{4.4}$$

Fix $r_0 > 0$ small such that $\overline{B_{4r_0}} \subset \Omega$ and let k be any large integer such that $k > 1/r_0$. By Corollary 3.10(b), there exists a positive constant $C = C(r_0)$ such that

$$\Delta u_{n,k} = \frac{u_{n,k}^{q+1}}{\sqrt{u_{n,k}^2 + 1/n}} \frac{|\nabla u_{n,k}|^{m+2}}{|\nabla u_{n,k}|^2 + 1/n} \leq u_{n,k}^q |\nabla u_{n,k}|^m \leq C^m |x|^{-m} u_{n,k}^{m+q} \quad \text{in } B^*_{r_0}$$

for all $n \geq 1$ and every $k > 1/r_0$. Thus, $u_{n,k}$ is a super-solution of the following problem:

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B_{r_0} \setminus \overline{B_{\frac{1}{k}}}, \\ w = \Lambda E + \max_{\partial\Omega} h \ \text{ on } \partial B_{\frac{1}{k}} \ \text{ and } \ w = 0 \ \text{ on } \partial B_{r_0}. \end{cases} \tag{4.5}$$

On the other hand, (4.5) has a unique positive classical solution w_k. Then, Lemma 3.2 gives that

$$w_k(x) \leq u_{n,k}(x) \quad \text{for every } 1/k \leq |x| \leq r_0. \tag{4.6}$$

By [Cîrstea and Du 2010, Theorem 1.2], $\lim_{k\to\infty} w_k = w$ in $C^1_{\rm loc}(B^*_{r_0})$, where $w > 0$ in $B^*_{r_0}$ satisfies

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} \quad \text{in } B^*_{r_0}, \\ \lim\limits_{|x|\to 0} w(x)/E(x) = \Lambda \ \text{ and } \ w = 0 \ \text{ on } \partial B_{r_0}. \end{cases} \tag{4.7}$$

By letting $k \to \infty$ in (4.6), we obtain that $w \le u_{\Lambda,n}$ in $B^*_{r_0}$, which leads to (4.4). □

Proof of Theorem 1.1 completed. Let $\Lambda \in [0,\infty)$ be arbitrary and $u_{\Lambda,n}$ denote the unique non-negative solution of (4.1), (1.5). By Lemmas 3.2 and 3.3, we obtain that

$$0 < u_{\Lambda,n+1} \le u_{\Lambda,n} \le \Lambda E + \max_{\partial\Omega} h \quad \text{in } \Omega^*. \tag{4.8}$$

Thus, $u_\Lambda(x) := \lim_{n\to\infty} u_{\Lambda,n}(x)$ exists for all $x \in \Omega^*$. By Lemma 3.8, we find that $u_{\Lambda,n} \to u_\Lambda$ in $C^1_{\rm loc}(\Omega^*)$ as $n \to \infty$, where u_Λ is a non-negative solution of (1.1). Moreover, $u_\Lambda > 0$ in Ω^* from Lemma 3.3. As before, $u_\Lambda \in C(\overline{\Omega} \setminus \{0\})$ and $u_\Lambda = h$ on $\partial\Omega$. If $\Lambda = 0$, then $\lim_{|x|\to 0} u_\Lambda(x)/E(x) = 0$. If $\Lambda \in (0,\infty)$, from the proof of Step 2, $w \le u_\Lambda$ in $B^*_{r_0}$, where w is the (unique) positive solution of (4.7). This and (4.8) prove that $\lim_{|x|\to 0} u_\Lambda(x)/E(x) = \Lambda$. Hence, u_Λ is a non-negative solution of (1.1), (1.5) such that $u_\Lambda \ge \min_{\partial\Omega} h$ in Ω^* and $u_\Lambda \in C^{1,\alpha}_{\rm loc}(\Omega^*)$ for some $\alpha \in (0,1)$ (by Lemma 3.8).

We now prove Theorem 1.1 for $\Lambda = \infty$. For any $j \ge 1$, let $u_{j,n}$ denote the unique positive solution of (4.1), (1.5) with $\Lambda = j$. By Lemmas 3.2 and 3.4, we find $C_1 > 0$ such that

$$0 < u_{j,n}(x) \le u_{j+1,n}(x) \le C_1 |x|^{-\vartheta} + \max_{\partial\Omega} h \quad \text{for all } x \in \Omega^* \text{ and every } n \ge 2. \tag{4.9}$$

By Lemma 3.8, we have $u_{j,n} \to u_{\infty,n}$ in $C^1_{\rm loc}(\Omega^*)$ as $j \to \infty$, where $u_{\infty,n}$ is a solution of (4.1), (1.5) with $\Lambda = \infty$. If u is any solution of (1.1), (1.5) with $\Lambda = \infty$, then $u \le u_{\infty,n+1} \le u_{\infty,n}$ in Ω^*. (We use Theorem 1.2(a)(iii) for $u_{\infty,n}$.) We set $u_\infty(x) := \lim_{n\to\infty} u_{\infty,n}(x)$ for all $x \in \Omega^*$. Hence, $u_{\infty,n} \to u_\infty$ in $C^1_{\rm loc}(\Omega^*)$ as $n \to \infty$ and u_∞ is the maximal solution of (1.1), (1.5) with $\Lambda = \infty$.

Remark 4.2. For any $\Lambda \in [0,\infty) \cup \{\infty\}$, the solution of (1.1), (1.5) constructed in the proof of Theorem 1.1, say $u_{\Lambda,h}$, is the *maximal* one in the sense that any other (sub-)solution is dominated by it. If $m \ge 1$, then $u_{\Lambda,h}$ is the only solution of (1.1), (1.5) (by Lemma 3.2). If $0 < m < 1$, then we can construct the *minimal* solution of (1.1), (1.5) using a similar perturbation argument. More precisely, for any integer $\xi \ge 1$, we consider the perturbed problem

$$\Delta u = u^q \left(|\nabla u|^2 + 1/\xi\right)^{m/2} \quad \text{in } \Omega^*. \tag{4.10}$$

Under the assumptions of Theorem 1.1, it can be shown that (4.10), subject to (1.5), has a unique non-negative solution $u_{\xi,\Lambda,h}$, which is dominated by any solution of (1.1), (1.5) (using Lemma 3.2 for (4.10)). The existence of $u_{\xi,\Lambda,h}$ is obtained by proving Lemma 4.1 with (4.1) replaced by

$$\Delta u = \frac{u^{q+1}}{\sqrt{u^2 + 1/n}} \left(|\nabla u|^2 + 1/\xi\right)^{m/2} \quad \text{in } \Omega^*. \tag{4.11}$$

The proof can be given as before and thus we skip the details. Moreover, $u_{\xi,\Lambda,h} \le u_{\xi+1,\Lambda,h}$ in Ω^* and $u_{\xi,\Lambda,h}$ converges in $C^1_{\rm loc}(\Omega^*)$ as $\xi \to \infty$ to the minimal solution of (1.1), (1.5). Furthermore, if $\Omega = B_\ell$ for some $\ell > 0$ and h is a non-negative constant, then by construction, both the maximal solution and the minimal solution of (1.1), (1.5) are radial.

Remark 4.3. For $m \in (0,1)$, the uniqueness of the solution of (1.1), (1.5) may not necessarily hold (depending on Ω, h and Λ). Indeed, let $\Lambda \in (0,\infty)$ be arbitrary. Then there exists a non-increasing solution u_1 of (2.1), subject to (2.2), such that $u_1'(r) = 0$ for all $r \in [r_1, 1]$ and $u_1' < 0$ on $(0, r_1)$ for some $r_1 \in (0,1]$ (see Theorem 2.1). If $\Lambda > 0$ is small, then $r_1 = 1$ (see Lemma 2.3) and, moreover, u_1 is the unique positive solution of (1.1), (1.5) with $\Omega = B_1$ and $h \equiv u_1(r_1)$ (by Lemma 3.1).

By Theorem 2.2, there exists a positive, radial and increasing solution u_2 of (1.1) in $B^*_{r_1}$, subject to $u|_{\partial B_{r_1}} = u_1(r_1)$. Let $C := \frac{u_2(0)}{u_1(r_1)} \in (0,1)$ and $r_2 := r_1 C^{-1/\vartheta}$. We define $u_3 : (0, r_1 + r_2] \to (0,\infty)$ by

$$u_3(r) := \begin{cases} C u_1(C^{\frac{1}{\vartheta}} r) & \text{for } r \in (0, r_2), \\ u_2(r - r_2) & \text{for } r \in [r_2, r_1 + r_2]. \end{cases}$$

We observe that (1.1) in $B^*_{r_1+r_2}$, subject to $u|_{\partial B_{r_1+r_2}} = u_1(r_1)$ and $\lim_{|x|\to 0} u(x)/E(x) = \Lambda C^{1+\frac{2-N}{\vartheta}}$ has at least two distinct positive solutions: u_3 and the maximal solution, say u_4, as constructed in the proof of Theorem 1.1. We have $u_3 \neq u_4$ since $u_3'(r_2) = 0$ and $u_3 < u_1(r_1) \leq u_4$ on $[r_2, r_1 + r_2)$.

5. Proof of Theorem 1.2

Let (1.2) hold. We first assume that $q < q_*$ and prove the claim of Theorem 1.2(a). Let u be any positive solution of (1.1). We denote $\Lambda := \limsup_{|x|\to 0} u(x)/E(x)$ and analyse three cases: I) $\Lambda = 0$; II) $\Lambda \in (0,\infty)$ and III) $\Lambda = \infty$. In Case I), the claim follows from Lemma 3.11.

Case II). *Let* $\Lambda \in (0,\infty)$.

One can show the assertion of (ii) in Theorem 1.2(a) using an argument similar to [Friedman and Véron 1986, Theorem 1.1] and [Cîrstea and Du 2010, Theorem 5.1(b)]. We sketch the main ideas. Let $r_0 > 0$ be such that $\overline{B_{4r_0}} \subset \Omega$. For any $r \in (0, r_0)$ fixed, we define the function

$$V_{(r)}(\xi) := u(r\xi)/E(r) \quad \text{for all } \xi \in \mathbb{R}^N \text{ with } 0 < |\xi| < r_0/r.$$

We see that $V_{(r)}(\xi)$ satisfies the following equation

$$\Delta V_{(r)}(\xi) = r^{2-m}\,[E(r)]^{q+m-1}\,[V_{(r)}(\xi)]^q\,|\nabla V_{(r)}(\xi)|^m \quad \text{for } 0 < |\xi| < r_0/r. \tag{5.1}$$

We prove that $\lim_{|x|\to 0} u(x)/E(x) = \Lambda$ by showing that for every $\xi \in \mathbb{R}^N \setminus \{0\}$, it holds

$$\lim_{r\to 0^+} V_{(r)}(\xi) = G(\xi), \quad \text{where } G(\xi) := \begin{cases} \Lambda\,|\xi|^{2-N} & \text{if } N \geq 3, \\ \Lambda & \text{if } N = 2. \end{cases} \tag{5.2}$$

For any $\xi \in \mathbb{R}^N \setminus \{0\}$, we define $W(\xi)$ as follows

$$W(\xi) := \begin{cases} |\xi|^{2-N} & \text{if } N \geq 3, \\ 1 + \ln(1/\min\{|\xi|, 1\}) & \text{if } N = 2. \end{cases}$$

Then by Lemma 3.8, there exist positive constants C_1, C and $\alpha \in (0,1)$ such that

$$0 < V_{(r)}(\xi) \leq C_1 W(\xi), \;\; |\nabla V_{(r)}(\xi)| \leq C\,\frac{W(\xi)}{|\xi|} \text{ and } |\nabla V_{(r)}(\xi) - V_{(r)}(\xi')| \leq C\,\frac{|\xi - \xi'|^\alpha}{|\xi|^{1+\alpha}} W(\xi) \tag{5.3}$$

for every $\xi, \xi' \in \mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$. From the assumptions of Theorem 1.2, we infer that $\lim_{r\to 0^+} r^{2-m}[E(r)]^{q+m-1} = 0$. Thus, from (5.1) and (5.3), we find that for any sequence $\overline{r}_n$ decreasing to zero, there exists a subsequence r_n such that

$$V_{(r_n)} \to V \text{ in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{and} \quad \Delta V = 0 \text{ in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}). \tag{5.4}$$

We set $\tilde{\Lambda}(r) := \sup_{|x|=r} u(x)/E(x)$ for $0 < r < r_0$. Then $\lim_{r\to 0^+} \tilde{\Lambda}(r) = \Lambda$ and there exists ξ_{r_n} on the $(N-1)$-dimensional sphere S^{N-1} in $\mathbb{R}^N$ such that $\tilde{\Lambda}(r_n) = u(r_n\xi_{r_n})/E(r_n)$. Passing to a subsequence, relabelled r_n, we have $\xi_{r_n} \to \xi_0$ as $n \to \infty$. We observe that

$$\frac{V_{(r_n)}(\xi)}{\tilde{\Lambda}(r_n|\xi|)} \le \frac{E(r_n\,|\xi|)}{E(r_n)} \quad \text{for any } 0 < |\xi| < \frac{r_0}{r_n} \tag{5.5}$$

with equality for $\xi = \xi_{r_n}$. Therefore, by letting $n \to \infty$ in (5.5) and using (5.4), we obtain that $V \le G$ in $\mathbb{R}^N \setminus \{0\}$ with $V(\xi_0) = G(\xi_0)$. Hence, $V = G$ in $\mathbb{R}^N \setminus \{0\}$. For $N \ge 3$, we also find that

$$\lim_{n\to\infty} \frac{(\nabla u)(r_n\xi)}{r_n^{1-N}} = -\frac{\Lambda}{N\omega_N}\,|\xi|^{-N}\xi \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.6}$$

Since $\{\bar{r}_n\}$ is an arbitrary sequence decreasing to 0, we conclude (5.2). Moreover, it holds

$$\lim_{|x|\to 0} \frac{x \cdot \nabla u(x)}{|x|^{2-N}} = -\frac{\Lambda}{N\omega_N} \quad \text{and} \quad \lim_{|x|\to 0} \frac{|\nabla u(x)|}{|x|^{1-N}} = \frac{\Lambda}{N\omega_N}. \tag{5.7}$$

For $N \ge 3$, the claim of (5.7) follows easily from (5.6). For $N = 2$, one can follow the proof of Theorem 1.1 in [Friedman and Véron 1986] corresponding there to $p = N$ to obtain that $\lim_{r\to 0^+} r\,(\nabla u)\,(r\xi) = \Lambda \nabla E(\xi)$ for $\xi \in \mathbb{R}^N \setminus \{0\}$, which for $|\xi| = 1$ gives (5.7).

To obtain (1.8), we use (5.7) and similar ideas in the proof of (5.1) in [Cîrstea and Du 2010].

Case III). *Let* $\Lambda = \infty$.

Using a contradiction argument based on Lemma 3.9 and the same argument as in [Brandolini et al. 2013, Corollary 4] (or [Cîrstea 2014, Corollary 4.5]), we find that $\lim_{|x|\to 0} u(x)/E(x) = \infty$. We next conclude the proof of Theorem 1.2(a) by showing that $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$.

Lemma 5.1. *Assume that* (1.2) *holds and* $q < q_*$. *Then any positive solution of* (1.1) *with a strong singularity at* 0 *satisfies* $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$, *where* ϑ *and* λ *are given by* (1.6).

Proof. We divide the proof into two steps.

Step 1. *We show that* $\liminf_{|x|\to 0} |x|^{\vartheta} u(x) > 0$.

Fix $r_0 > 0$ such that $\overline{B_{4r_0}} \subset \Omega$ and let C be a positive constant as in Corollary 3.10(b). Let k be a large integer such that $k > 1/r_0$. Consider the problem

$$\begin{cases} \Delta z = C^m |x|^{-m} z^{m+q} & \text{in } B^*_{r_0}, \\ z|_{\partial B_{r_0}} = \min\limits_{\partial B_{r_0}} u. \end{cases} \tag{5.8}$$

Using (1.2) and $q < q_*$, we obtain a unique positive solution $z_k \in C^1(B^*_{r_0})$ of (5.8) satisfying $\lim_{|x|\to 0} z_k(x)/E(x) = k$ (by [Cîrstea and Du 2010, Theorem 1.2]). Since $\lim_{|x|\to 0} u(x)/E(x) = \infty$, by (3.23) and Lemma 3.2, we find that $0 < z_k \le z_{k+1} \le u$ in $B^*_{r_0}$. We have $\lim_{k\to\infty} z_k = z_\infty$ in

$C^1_{\rm loc}(B^*_{r_0})$ and z_∞ is a positive solution of (5.8) with $\lim_{|x|\to 0} z_\infty(x)/E(x) = \infty$ (see [Cîrstea and Du 2010, p. 197]). From $z_\infty \le u$ in $B^*_{r_0}$ and $\lim_{|x|\to 0} |x|^\vartheta z_\infty(x) > 0$ (see Theorem 1.1 in [Cîrstea and Du 2010]), we conclude Step 1.

Step 2. *We have* $\lim_{|x|\to 0} |x|^\vartheta u(x) = \lambda$*, where* λ *and* ϑ *are given by* (1.6).

We use a perturbation technique as introduced in [Cîrstea and Du 2010] to construct a one-parameter family of sub-super-solutions for (1.1). Fix $\varepsilon \in (0, \vartheta - N + 2)$. Observe that if $N \ge 3$, then $q < q_*$ gives that $\vartheta > N - 2$. We define $\lambda_{\pm\varepsilon} > 0$ and $U_{\pm\varepsilon} : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ as follows

$$U_{\pm\varepsilon}(x) = \lambda_{\pm\varepsilon}|x|^{-(\vartheta\pm\varepsilon)} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \quad \text{where } \lambda_\pm := \left[(\vartheta \pm \varepsilon)^{1-m} (\vartheta - N + 2 \pm \varepsilon)\right]^{\frac{1}{q+m-1}}. \tag{5.9}$$

Clearly, we see that $\lambda_{\pm\varepsilon} \to \lambda$ as $\varepsilon \to 0$. By a direct computation, we find that

$$\Delta U_\varepsilon - U_\varepsilon^q |\nabla U_\varepsilon|^m \le 0 \le \Delta U_{-\varepsilon} - U_{-\varepsilon}^q |\nabla U_{-\varepsilon}|^m \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.10}$$

From Step 1, we obtain that $\lim_{|x|\to 0} u(x)/U_{-\varepsilon}(x) = \infty$. On the other hand, by the *a priori* estimates in Lemma 3.4, we have that $\lim_{|x|\to 0} u(x)/U_\varepsilon(x) = 0$. Since $\nabla U_{\pm\varepsilon} \ne 0$ in $\mathbb{R}^N \setminus \{0\}$, by (5.10) and the comparison principle in Lemma 3.1, we deduce that

$$u(x) \le U_\varepsilon(x) + \max_{\partial B_{r_0}} u \quad \text{and} \quad u(x) + \lambda r_0^{-\vartheta} \ge U_{-\varepsilon}(x) \quad \text{for all } 0 < |x| \le r_0, \tag{5.11}$$

where $r_0 \in (0, 1)$ is chosen such that $\overline{B_{r_0}} \subset \Omega$. Letting $\varepsilon \to 0$ in (5.11), we find that

$$\lambda \left(|x|^{-\vartheta} - r_0^{-\vartheta}\right) \le u(x) \le \lambda |x|^{-\vartheta} + \max_{\partial B_{r_0}} u \quad \text{for all } x \in B^*_{r_0}.$$

This concludes the proof of Step 2.

Proof of Theorem 1.2 completed. It remains to show Theorem 1.2(b), that is if $q \ge q_*$ for $N \ge 3$, then (1.1) has no positive solutions with singularities at 0. Indeed, when $q > q_*$, the *a priori* estimates in Lemma 3.4 give that $\lim_{|x|\to 0} u(x)/E(x) = 0$ for any solution of (1.1), proving the claim. If $q = q_*$, then $\vartheta = N - 2$, where ϑ is given by (1.6). For every $\varepsilon > 0$, we define U_ε as in (5.9) and from the proof of Lemma 5.1, we see that

$$u(x) \le U_\varepsilon(x) + \max_{\partial B_{r_0}} u = \left[(N - 2 + \varepsilon)^{1-m} \varepsilon\right]^{\frac{1}{q+m-1}} |x|^{-(\vartheta+\varepsilon)} + \max_{\partial B_{r_0}} u \quad \text{for all } 0 < |x| \le r_0.$$

By letting $\varepsilon \to 0$, we find that $u(x) \le \max_{\partial B_{r_0}} u$ for every $0 < |x| \le r_0$, that is 0 is a removable singularity for every solution of (1.1). Using Lemma 3.11, we finish the proof. □

6. Proof of Theorem 1.3

In this section, unless otherwise mentioned, we let $\Omega = \mathbb{R}^N$ in (1.1). Let (1.2) hold. If $q \ge q_*$ for $N \ge 3$, then by Theorem 1.2(b), 0 is a removable singularity for all positive solutions of (1.1), which must be constant by Lemma 3.13. The assertion of Theorem 1.3(iii) is thus proved by Lemma 3.13. It remains to prove (i) and (ii) of Theorem 1.3.

(i) Let $q < q_*$. We divide the proof of Theorem 1.3(i) into two steps.

Step 1: Uniqueness.

From (3.8), any positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ is non-increasing. Furthermore, since it satisfies (2.1) for all $r \in (0,\infty)$, we see that it is convex. Hence, any positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$ satisfies only one of the following cases:

- **Case 1.** There exists $r_u > 0$ such that $u'(r) = 0$ for all $r \geq r_u$ and $u' < 0$ on $(0, r_u)$;
- **Case 2.** $u'(r) < 0$ for all $r > 0$.

We remark that Case 1 does happen for $m \in (0,1)$ as it can be seen from Theorem 2.1 (defining $u(r) = u(1)$ for $1 < r < \infty$). Let u_1 and u_2 denote any positive radial solutions of (1.1), (1.10) for some $\Lambda \in (0,\infty]$ and $\gamma \in [0,\infty)$. (If $\gamma = 0$, then u_1 and u_2 are in Case 2.) Notice that $\lim_{r\to\infty} (u_1(r) - u_2(r)) = 0$ and $\lim_{r\to 0^+} u_1(r)/u_2(r) = 1$ (using Theorem 1.2(a) if $\Lambda = \infty$). If either u_1 or u_2 is in Case 2, then the uniqueness follows from Lemma 3.1, which is allowed because $|u_1'| + |u_2'| \neq 0$ in $\mathbb{R}^+$. Indeed, for every $\varepsilon > 0$, we have $u_1(r) \leq (1+\varepsilon)\, u_2(r) + \varepsilon$ for every $r \in (0,\infty)$. Letting $\varepsilon \to 0$, then interchanging u_1 and u_2, we conclude that $u_1 \equiv u_2$. If both u_1 and u_2 are in Case 1, then $u_1 = u_2 = \gamma$ in $(\max(r_{u_1}, r_{u_2}), \infty)$. Using Lemma 3.1 on $(0, \max(r_{u_1}, r_{u_2})]$ as above, we find that $u_1 = u_2$ on $(0,\infty)$. (When $1 \leq m < 2$, the proof of uniqueness of solutions can be made simpler by using Lemma 3.2 instead of Lemma 3.1, since we do not require that $|u_1'| + |u_2'| > 0$.)

Step 2: Existence.

Let $\Lambda \in (0,\infty)$ and $\gamma \in [0,\infty)$ be fixed. For any integer $\ell \geq 2$, we denote by $u_{\Lambda,\gamma,\ell}$ the maximal non-negative solution of (1.1), (1.5) with $h \equiv \gamma$ and $\Omega = B_\ell$ constructed by Theorem 1.1. For brevity, we write u_ℓ instead of $u_{\Lambda,\gamma,\ell}$. Recall the notation $B_\ell^* := B_\ell(0) \setminus \{0\}$. From the proof of Theorem 1.1, $u_{n,\ell} \to u_\ell$ in $C^1_{\text{loc}}(B_\ell^*)$ as $n \to \infty$, where $u_{n,\ell}$ stands here for the unique non-negative solution of (4.1), (1.5) with $h \equiv \gamma$ and $\Omega = B_\ell$. We observe that $u_{n,\ell}$ is radial by the rotation invariance of the operator and the symmetry of the domain and, hence, u_ℓ is radial, too. Since $u_{n,\ell}(r) \geq \gamma$ for all $r \in (0,\ell)$, by Lemma 3.2, we infer that $u_{n,\ell}(r) \leq u_{n,\ell+1}(r)$ for every $r \in (0,\ell)$. Consequently, letting $n \to \infty$ and using also Lemma 3.1, we deduce that

$$\gamma \leq u_\ell(r) \leq u_{\ell+1}(r) \leq \lambda r^{-\vartheta} + \gamma \quad \text{for all } 0 < r < \ell. \tag{6.1}$$

Thus, $u_\ell \to u_{\Lambda,\gamma}$ in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ as $\ell \to \infty$, where $u_{\Lambda,\gamma}$ is a radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$. Letting $\ell \to \infty$ in (6.1), we find that $\lim_{r\to\infty} u_{\Lambda,\gamma}(r) = \gamma$. Since $u_\ell(1) \leq \lambda + \gamma$, by Lemma 3.1, we get that $u_\ell(r) \leq u_{\ell+1}(r) \leq \Lambda E(r) + \lambda + \gamma$ for all $r \in (0,1)$ and $\ell \geq 2$. Since $\lim_{r\to 0^+} u_\ell(r)/E(r) = \Lambda$, we obtain that $\lim_{r\to 0^+} u_{\Lambda,\gamma}(r)/E(r) = \Lambda$. Thus, $u_{\Lambda,\gamma}$ satisfies (1.10).

When $\Lambda = \infty$, we denote by $u_{j,\gamma}$ the radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1.10), where Λ is replaced by an integer $j \geq 2$. The above argument shows that $\gamma \leq u_{j,\gamma}(r) \leq u_{j+1,\gamma}(r) \leq \lambda r^{-\vartheta} + \gamma$ in $(0,\infty)$ so that $u_{j,\gamma} \to u_{\infty,\gamma}$ in $C^1_{\text{loc}}(0,\infty)$, where $u_{\infty,\gamma}$ is a radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, satisfying (1.10) with $\Lambda = \infty$. This concludes the proof of Theorem 1.3(i).

(ii) In view of Theorem 1.2, we need to establish the following result.

Lemma 6.1. *Let* (1.2) *hold. If u is a positive non-constant solution of* (1.1) *in* $\mathbb{R}^N \setminus \{0\}$*, then $q < q_*$ and there exists* $\lim_{|x|\to\infty} u(x) = \gamma$ *in* $[0,\infty)$*. Moreover, u is radially symmetric and non-increasing in* $\mathbb{R}^N \setminus \{0\}$ *such that* $\lim_{r\to 0^+} u(r)/E(r) = \Lambda \in (0,\infty]$.

Proof. Let u be a positive non-constant solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$. Then, we have $q < q_*$ and $\lim_{|x|\to 0} u(x)/E(x) = \Lambda \in (0,\infty]$ by Theorem 1.2 and Lemma 3.13. We proceed in two steps.

Step 1: *There exists* $\lim_{|x|\to\infty} u(x)$ *in* $[0,\infty)$.

From (3.8), we have $\limsup_{|x|\to\infty} u(x) < \infty$.

Claim: *Let* $(x_n)_{n\in\mathbb{N}}$ *be any sequence in* $\mathbb{R}^N$ *satisfying* $|x_n| \nearrow \infty$ *as* $n\to\infty$. *For each* $\varepsilon>0$, *there exists* $N_\varepsilon>0$ *such that* $u(z) < \limsup_{n\to\infty} u(x_n) + \varepsilon$ for all $z \in \overline{B_{\frac{|x_n|}{2}}(x_n)}$ and every $n \geq N_\varepsilon$.

Indeed, by defining $v_n(y) = u(x_n + y)$ for all $y \in B_{2|x_n|/3}$, we observe that v_n satisfies (1.1) in $B_{2|x_n|/3}$. Let C_1 be as in Lemma 3.4. From (3.10), we have for any $n \in \mathbb{N}$ that

$$v_n(y) \leq C_1\,(2|x_n|/3 - |y|)^{-\vartheta} + u(x_n) \leq C_1(|x_n|/6)^{-\vartheta} + u(x_n) \quad \text{for all } y \in \overline{B_{\frac{|x_n|}{2}}}. \tag{6.2}$$

By letting $N_\varepsilon > 0$ large such that $C_1(|x_n|/6)^{-\vartheta} < \varepsilon/2$ and $u(x_n) < \limsup_{n\to\infty} u(x_n) + \varepsilon/2$ for all $n \geq N_\varepsilon$, we conclude the claim.

To finish the proof of Step 1, we fix $\varepsilon > 0$. Let $(x_{n,1})_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N$ with $|x_{n,1}| \nearrow \infty$ and $\lim_{n\to\infty} u(x_{n,1}) = \liminf_{|x|\to\infty} u(x)$. The above claim gives $N_1 = N_1(\varepsilon) > 0$ such that

$$u(z) < \liminf_{|x|\to\infty} u(x) + \varepsilon \quad \text{for all } z \in \overline{B_{\frac{|x_{n,1}|}{2}}(x_{n,1})} \text{ whenever } n \geq N_1. \tag{6.3}$$

We choose $x_{n,2} \in \partial B_{|x_{n,1}|} \cap \partial B_{|x_{n,1}|/2}(x_{n,1})$. Thus, $|x_{n,2}| = |x_{n,1}| \nearrow \infty$ as $n\to\infty$. Since (6.3) holds for $z = x_{n,2}$ and all $n \geq N_1$, by applying the claim again, there exists $N_2 > N_1$ such that

$$u(z) < \liminf_{|x|\to\infty} u(x) + 2\varepsilon \quad \text{for all } z \in \overline{B_{\frac{|x_{n,1}|}{2}}(x_{n,2})} \cup \overline{B_{\frac{|x_{n,1}|}{2}}(x_{n,1})} \text{ and every } n \geq N_2.$$

We can repeat this process a finite number of times, say K, which is independent of n, such that for each $2 \leq i \leq K$, it generates a number N_i greater than N_{i-1} and a sequence $(x_{n,i})_{n\geq N_i}$ with $|x_{n,i}| = |x_{n,1}|$ with the property that $\partial B_{|x_{n,1}|} \subset \bigcup_{i=1}^K B_{|x_{n,1}|/2}(x_{n,i})$ and

$$u(z) < \liminf_{|x|\to\infty} u(x) + K\varepsilon \quad \text{for all } z \in \partial B_{|x_{n,1}|} \text{ and every } n \geq N_K. \tag{6.4}$$

In light of (3.8), we see that (6.4) implies that $u(z) \leq \liminf_{|x|\to\infty} u(x) + K\varepsilon$ for all $|z| \geq |x_{n,1}|$ and all $n \geq N_K$. Consequently, $\limsup_{|x|\to\infty} u(x) \leq \liminf_{|x|\to\infty} u(x) + K\varepsilon$. By taking $\varepsilon \to 0$, we obtain that $\limsup_{|x|\to\infty} u(x) = \liminf_{|x|\to\infty} u(x)$. This completes the proof of Step 1.

Step 2: *Proof of Lemma 6.1 concluded.*

We need only show that u is radial. Since $\lim_{|x|\to\infty} u(x) = \gamma \in [0,\infty)$, we have that u satisfies (1.10) for some $\Lambda \in (0,\infty]$. If $m \geq 1$, then (1.1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1.10), has a unique positive solution (by Lemma 3.2), which must be radial by the invariance of the problem under rotation.

Let us now assume that $m \in (0,1)$. Let $\varepsilon \in (0,\gamma)$ be arbitrary. By Theorem 1.3(i), there exists a unique positive radial solution U_ε of (1.1) in $\mathbb{R}^N \setminus \{0\}$ such that $\lim_{r\to 0^+} U_\varepsilon(r)/E(r) = \Lambda$ and $\lim_{r\to\infty} U_\varepsilon(r) = \gamma + \varepsilon$. From the proof of Theorem 1.3(i) (with γ there replaced by $\gamma+\varepsilon$ and $\ell > 1$ large such that $u(x) \leq \gamma + \varepsilon$ for all $|x| \geq \ell$), we infer that $u \leq U_\varepsilon$ in $\mathbb{R}^N \setminus \{0\}$.

Using Remark 4.2 and the same ideas as in the existence proof of Theorem 1.3(i), for any integer $\xi \geq 1$, we can construct the unique non-negative radial solution $u_{\xi,\Lambda,\varepsilon}$ of

$$\begin{cases} \Delta u = u^q \left(|\nabla u|^2 + 1/\xi\right)^{m/2} & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x|\to 0} u(x)/E(x) = \Lambda, & \lim_{|x|\to\infty} u(x) = \max\{\gamma - \varepsilon, 0\}. \end{cases} \tag{6.5}$$

By Lemma 3.2, we deduce that $u_{\xi,\Lambda,\varepsilon} \leq u_{\xi+1,\Lambda,\varepsilon} \leq u$ in $\mathbb{R}^N \setminus \{0\}$ since $\lim_{|x|\to 0} u_{\xi,\Lambda,\varepsilon}(x)/u(x) = 1$ and $\lim_{|x|\to\infty} \left(u_{\xi,\Lambda,\varepsilon}(x) - u(x)\right)$ is either 0 if $\gamma = 0$ or $-\varepsilon$ if $\gamma > 0$. Thus, by defining $u_\varepsilon(r) := \lim_{\xi\to\infty} u_{\xi,\Lambda,\varepsilon}(r)$ for all $r \in (0,\infty)$, we obtain that u_ε is a positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, satisfying $\lim_{r\to 0^+} u_\varepsilon(r)/E(r) = \Lambda$ and $\lim_{r\to\infty} u_\varepsilon(r) = \max\{\gamma - \varepsilon, 0\}$. Moreover, we have

$$u_{\varepsilon_2} \leq u_{\varepsilon_1} \leq u \leq U_{\varepsilon_1} \leq U_{\varepsilon_2} \quad \text{in } \mathbb{R}^N \setminus \{0\} \text{ for all } 0 < \varepsilon_1 < \varepsilon_2 < \gamma.$$

Letting ε tend to 0, we get that both u_ε and U_ε converge to a positive radial solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1.10). By the uniqueness of such a solution, we conclude that u is radial. □

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