Continued fractions of certain Mahler functions

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Abstract

We investigate the continued fraction expansion of the infinite products $g(x) = x^{-1}\prod_{t=0}^{\infty} P(x^{-d^t})$ where polynomials $P(x)$ satisfy $P(0) = 1$ and $\deg(P) < d$. We construct relations between partial quotients of $g(x)$ which can be used to get recurrent formulae for them. We provide that formulae for the cases $d = 2$ and $d = 3$. As an application, we prove that for $P(x) = 1 + ux$ where u is an arbitrary rational number except 0 and 1, and for any integer b with $|b| > 1$ such that $g(b) \neq 0$ the irrationality exponent of $g(b)$ equals two. In the case $d = 3$ we provide a partial analogue of the last result with several collections of polynomials $P(x)$ giving the irrationality exponent of $g(b)$ strictly bigger than two.

1 Introduction

Let $\mathbb{F}$ be a field. Consider the set $\mathbb{F}[[x^{-1}]]$ of Laurent series together with the valuation which is defined as follows: for $f(x) = \sum_{k=-d}^{\infty} c_k x^{-k} \in \mathbb{F}[[x^{-1}]]$ its valuation $||f(x)||$ is the biggest degree d of x having non-zero coefficient c_{-d}. For example, for polynomials $f(x)$ the valuation $||f(x)||$ coincides with their degree. It is well known that in this setting the notion of continued fraction is well defined. In other words, every $f(x) \in \mathbb{F}[[x^{-1}]]$ can be written as

$$f(x) = [a_0(x), a_1(x), a_2(x), \ldots],$$

where $a_i(x)$ are non-zero polynomials of degree at least 1, $i \in \mathbb{Z}_{\geqslant 0}$. We provide some facts about the continued fractions of Laurent series in Section 2 and refer the reader to a nice survey [11] for more details.

It appears that in the case $\mathbb{F} = \mathbb{Q}$ quite often the continued fraction of $f(x) \in \mathbb{Q}[[x^{-1}]]$ can give us the information about the approximational properties of real numbers $f(b)$ for integer values b inside the radius of convergence of $f(x)$. One of the most important such properties is an irrationality exponent. It indicates how well a given irrational number ξ is approximated by rationals and is denoted by $\mu(\xi)$. More precisely, it is the supremum of real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < q^{-\mu}$$

has infinitely many rational solutions p/q. This is one of the most important approximational properties of real numbers indicating how well ξ is approximated by rationals. Note that by the classical Dirichlet theorem we always have $\mu(\xi) \geqslant 2$.

Let $f(x)$ be an infinite product defined as follows

$$f(x) = \prod_{t=0}^{n} P(x^{-d^t}) \qquad (1)$$

where $P \in \mathbb{F}[x]$ is a polynomial and $d \geqslant 2$ is a positive integer. In order that $f(x)$ is correctly defined as a Laurent series, we need an additional condition $P(0) = 1$. An easy check shows that functions $f(x)$ fall into the set of *Mahler functions* which we define as follows: $M(x) \in \mathbb{F}[[x^{-1}]]$ is a Mahler function if it satisfies the equation of the form

$$\sum_{i=0}^{n} P_i(x) M(x^{d^i}) = 0$$

for some integers $n \geqslant 1, d \geqslant 2$, and polynomials $P_0(x), \ldots, P_n(x) \in \mathbb{F}[x]$ with $P_0(x)P_n(x) \neq 0$[1]. For any integer b within the radius of convergence of M the value $M(b)$ is called *Mahler number*.

The question about computing or at least estimating the irrationality exponent of Mahler numbers is currently in the focus of the Diophantine approximation. It was triggered by the work of Bugeaud [4] where he showed that for $b \geqslant 2$ the irrational exponent of the Thue-Morse numbers $f_{TM}(b)$ are equal to 2. Here $f_{TM}(x)$ is the most classical example of Mahler functions and can be defined as follows:

$$f_{TM}(x) := \prod_{t=0}^{\infty} (1 - x^{-2^t}).$$

One of the key ingredients of that paper is the result from [**?**] about non-vanishing of Hankel determinants of $f_{TM}(x)$ (they will be properly defined and discussed in Section 3). Later this approach was further developed and generalised to cover many other Mahler functions, see for example [6, 7, 13]. Finally, Bugeaud, Han, Wen and Yao [5] provided quite a general result where the estimates for $\mu(f(b))$ was given depending on the distribution of non-vanishing Hankel determinants of $f(x)$ (see Theorem BHWY2 in Section 3). The problem with this theorem is that it is usually quite problematic to compute Hankel determinants of $f(x)$ or even to check which of them equal to zero. In [8, 9, 5] the authors used the reduction of $f(x)$ modulo a prime number p to provide local conditions on $f(x)$ which ensure that $\mu(f(b)) = 2$. We present just one example of such results, which appears as Theorem 2.5 in [5].

Theorem BHWY1 *Let $f(x) \in \mathbb{Z}[[x^{-1}]]$ be a power series defined by*

$$f(x) = \prod_{t=0}^{\infty} \frac{C(x^{-3^t})}{D(x^{-3^t})},$$

with $D(x), C(x) \in \mathbb{Z}[x]$ such that $C(0) = D(0) = 1$. Let $b \geqslant 2$ be an integer such that $C(b^{-3^m})D(b^{-3^m}) \neq 0$ for all integer $m \geqslant 0$. If $f(x) \pmod 3$ is not a rational function then the irrationality exponent of $f(b)$ is equal to 2.

Theorem BHWY1 as well as other known results of this kind provide an infinite collections of Mahler functions $f(x)$ such that their values $f(b)$ have irrationality exponent equal two. However, firstly, many series $f(x)$ are not covered by the reduction modulo p approach. Secondly, it can not detect the cases when the irrationality exponent of $f(b)$ is strictly bigger than two.

In Section 3 we show that values of Hankel determinants of $f(x)$ can be derived from the continued fraction of $f(x)$. Therefore in view of Theorem BHWY2, understanding the continued fraction gives us a powerful tool to estimate the irrationality exponents of $f(b)$.

[1] In the literature the notion of Mahler function is often given to $F(x) \in \mathbb{Q}[[x]]$ and not to Laurent series $M(x)$ as in our case. However one can easily convert one notion into another by considering $M(x) = F(x^{-1})$.

The question of computing the continued fraction of certain Mahler function (to the best of authors knowledge) goes back to 1991, when Allouche, Mendès France and van der Poorten [1] showed that all partial quotients of the infinite product

$$f_3(x) := \prod_{t=0}^{\infty}(1 - x^{-3^t})$$

are linear. In [12] the author computed the continued fraction of the solution $f_M(x)$ of the equation

$$f_M(x^2) = xf_m(x) - x^3.$$

Some other papers on this topic are [10, 11]. In particular, in the second of these papers, van der Poorten noted that the continued fraction of the Thue-Morse series

$$f_2(x) := \prod_{t=0}^{\infty}(1 - x^{-2^t})$$

has a regular structure. In [2] the precise formula of the continued fraction of $x^{-1}f_2(x)$ was given. As a consequence of that the authors showed that the Thue-Morse constant $f_2(2)$ is not badly approximable. Later [3] the authors extended their result to the series

$$f_d(x) := \prod_{t=0}^{\infty}(1 - x^{-d^t})$$

for any $d \geqslant 2$. In particular, they show that $f_d(x)$ is badly approximable only for $d = 2$ and $d = 3$ (definition will be given in Section 2) and provide the formula for the continued fraction of $x^{-2}f_3(x)$.

1.1 Main results

In this paper we consider functions $g(x) = x^{-1}f(x)$, where $f(x)$ is given by an infinite product (1). The essential restriction we have to impose on them is $d > ||P(x)||$ because that allows us, given a convergent of $g(x)$, to produce an infinite chain of other convergents of $g(x)$. Under these restrictions we encode each function $g(x)$ by a vector $\mathbf{u} = (u_1, \ldots, u_{d-1}) \in \mathbb{F}^{d-1}$ in the following way:

$$g_{\mathbf{u}}(x) := x^{-1}\prod_{t=0}^{\infty}(1 + u_1x^{-d^t} + u_2x^{-2d^t} + \ldots + u_{d-1}x^{-(d-1)d^t}). \tag{2}$$

The notation $f_{\mathbf{u}}(x)$ is defined in the same way.

We managed to find the relations between the partial quotients of the continued fraction of $g_{\mathbf{u}}(x)$ which can provide the recurrent formulae for them (see Propositions 2 and 3 in Section 6). We then explicitly write down these recurrent formulae in the case $d = 2$ and $d = 3$:

Theorem 1 *Let $u \in \mathbb{F}$. If $g_u(x)$ is badly approximable then its continued fraction is*

$$g_u(x) = \mathop{\mathbf{K}}_{i=1}^{\infty} \frac{\beta_i}{x + \alpha_i}$$

where the coefficients α_i and β_i are computed by the formula

$$\begin{aligned}&\alpha_{2k+1} = -u,\ \alpha_{2k+2} = u;\\ &\beta_1 = 1,\ \beta_2 = u^2 - u,\ \beta_{2k+3} = -\frac{\beta_{k+2}}{\beta_{2k+2}},\ \beta_{2k+4} = \alpha_{k+2} + u^2 - \beta_{2k+3}\end{aligned} \tag{3}$$

for any $k \in \mathbb{Z}_{\geqslant 0}$.

Theorem 2 *Let* $\mathbf{u} = (u, v) \in \mathbb{F}^2$. *If* $g_{\mathbf{u}}(x)$ *is badly approximable then its continued fraction is*

$$g_u(x) = \mathop{\mathbf{K}}_{i=1}^{\infty} \frac{\beta_i}{x + \alpha_i}$$

where the coefficients α_i *and* β_i *are computed by the recurrent formula*

$$\begin{array}{lll} \alpha_1 = -u, & \alpha_2 = \dfrac{u(2v-1-u^2)}{v-u^2}, & \alpha_3 = \dfrac{-u(v-1)}{v-u^2}; \\ \beta_1 = 1, & \beta_2 = u^2 - v, & \beta_3 = \dfrac{u^2+u^4+v^3-3u^2v}{(v-u^2)^2}. \end{array} \tag{4}$$

and

$$\begin{array}{l} \alpha_{3k+4} = -u, \quad \beta_{3k+4} = \dfrac{\beta_{k+2}}{\beta_{3k+3}\beta_{3k+2}}; \\ \beta_{3k+5} = u^2 - v - \beta_{3k+4}, \quad \alpha_{3k+5} = u - \dfrac{\alpha_{k+2} + uv - \alpha_{3k+2}\beta_{3k+4}}{\beta_{3k+5}} \\ \alpha_{3k+6} = u - \alpha_{3k+5}, \quad \beta_{3k+6} = v - \alpha_{3k+5}\alpha_{3k+6}. \end{array} \tag{5}$$

for any $k \in \mathbb{Z}_{\geqslant 0}$.

As one can notice, the complexity of the recurrent formulae grows rapidly with d. Therefore a computer assistance may be needed to provide analogues of Theorems 1 and 2 for larger values of d. These two theorems are proven in Section 6.

We show that formulae in Theorems 1 and 2 can be used to check whether $g_{\mathbf{u}}(x)$ is badly approximable or not:

Theorem 3 *Let* $\mathbb{F} \subset \mathbb{C}$. *The function* $g_u(x)$ *(respectively,* $g_{\mathbf{u}}(x)$*) is badly approximable if and only if none of the parameters* β_n, *computed by formulae* (3) *(respectively* (4) *and* (5)*) vanish. Moreover, if* $\beta_1, \beta_2, \ldots, \beta_n \neq 0$ *then the first* n *partial quotients of* $g_u(x)$ *(respectively* $g_{\mathbf{u}}(x)$*) are linear. And if in addition* $\beta_{n+1} = 0$ *then* $(n+1)$*th partial quotient of* $g_u(x)$ *is not linear.*

Theorem 3 is probably true for any field $\mathbb{F}$, however, as we are mostly interested in $\mathbb{F} = \mathbb{Q}$, we proved it only for subfields of complex numbers and did not make a big effort to generalise the result to an arbitrary field. We prove this result in the beginning of Subsection 7.

For the remaining results we assume that $\mathbb{F} = \mathbb{Q}$. Equipped with the continued fraction of $g_{\mathbf{u}}(x) \in \mathbb{Q}[[x^{-1}]]$ we can compute or at least estimate irrationality exponents of the values $g_{\mathbf{u}}(b)$. The first result we want to provide here is as follows:

Theorem 4 *Let* $b \geqslant 2$ *be integer such that* $g_{\mathbf{u}}(b) \neq 0$. *Then* $\mu(g_{\mathbf{u}}(b)) = 2$ *if and only if* $g_{\mathbf{u}}(x)$ *is badly approximable.*

The "if" part of this theorem is covered in Section 3. As we will see, it is essentially an implication of Theorem BHWY2 from [5]. The "only if" part is considered in Section 4.

In the case when $g_{\mathbf{u}}(x)$ is not badly approximable we provide a non-trivial lower bound for the irrationality exponent of $\mu(g_{\mathbf{u}}(b))$. This result is proven in Section 5.

Theorem 5 *Let* $b \geqslant 2$ *be integer. If* $g_{\mathbf{u}}(x)$ *is not badly approximable and* $g_{\mathbf{u}}(b) \neq 0$ *then the irrationality exponent of* $g_{\mathbf{u}}(b)$ *satisfies*

$$\mu(g_{\mathbf{u}}(b)) \geqslant 2 + \frac{c-1}{n_0},$$

where n_0 *is the smallest positive value such that the* n_0*th convergent* $a_{n_0}(x)$ *is not linear and* $c = ||a_{n_0}(x)||$.

We finish the paper by applying the results from above to compute (or estimate) the irrationality exponents of $g_{\mathbf{u}}(b)$ for all integer values $b \geqslant 2$ and as many vectors $\mathbf{u}$ as possible. We manage to completely cover the case $d = 2$ and $u \in \mathbb{Q}$:

Theorem 6 *The series $g_u(x)$ is badly approximable for any $u \in \mathbb{Q}$ except $u = 1$ and $u = 0$ for which $g_u(x)$ becomes a rational function. In particular, if $u \in \mathbb{Q}\backslash\{0,1\}, b \in \mathbb{Z}$, $|b| > 1$ and $b^{2^t} + u \neq 0$ for any $t \in \mathbb{Z}_{\geqslant 0}$ then the real number $g_u(b)$ has irrationality exponent two.*

However, due to complexity of the formulae (4) and (5) we covered many but not all values of $\mathbf{u} \in \mathbb{Q}^2$ for $d = 3$.

Theorem 7 *The series $g_{(u,0)}(x)$ as well as $g_{(0,v)}(x)$ is badly approximable for all $u, v \in \mathbb{Q}$ except $u = 0$ or $v = 0$ respectively. In particular, if $u \in \mathbb{Q}\backslash\{0\}, b \in \mathbb{Z}, |b| > 1$ then the irrationality exponent of $g_{(u,0)}(b)$ and of $g_{(0,v)}(x)$ is two as soon as $b^{3^t} + u \neq 0$ and $b^{2\cdot 3^t} + v \neq 0$ for any $t \in \mathbb{Z}_{\geqslant 0}$.*

Theorem 8 *Let $\mathbf{u} = (u, v) \in \mathbb{R}^2$ satisfy the following conditions:*

(C1) $u^2 \geqslant 6$*;*

(C2) $v \geqslant \max\{3u^2 - 1, 2u^2 + 8\}$.

Then the series $g_{\mathbf{u}}(x)$ is badly approximable. In particular, if $\mathbf{u} \in \mathbb{Q}^2$, $b \in \mathbb{Z}$ satisfies $|b| > 1$ and $b^{2\cdot 3^t} + ub^{3^t} + v \neq 0$ for any $t \in \mathbb{Z}_{\geqslant 0}$, the irrationality exponent of $g_{\mathbf{u}}(x)$ is two.

In the proof of Theorem 8 we sometimes make quite rough estimates, therefore with no doubts the conditions (C1) and (C2) can be made weaker. By this theorem we want to demonstrate that the knowledge of the continued fraction of $g_{\mathbf{u}}(x)$ can produce global conditions on $\mathbf{u}$ for the series to be badly approximable, on top of the local conditions, as in Theorem BHWY1.

Finally, by investigating the equations $\beta_n = 0$ for small values of n we get several series of vectors $\mathbf{u} \in \mathbb{Q}^2$ such that $g_{\mathbf{u}}(x)$ is not badly approximable and therefore non-trivial lower bounds for $g_{\mathbf{u}}(b)$ apply:

Theorem 9 *The functions $g_{\mathbf{u}}(x)$ are not badly approximable for the following vectors $\mathbf{u} \in \mathbb{Q}^2$:*

1. $\mathbf{u} = (\pm u, u^2)$*. Then for any $u \in \mathbb{Q}$ and any $b \in \mathbb{Z}$ with $|b| > 1$ we have $\mu(g_{\mathbf{u}}(b)) \geqslant 5/2$, as soon as $g_{\mathbf{u}}(x) \notin \mathbb{Q}(x)$ and $g_{\mathbf{u}}(b) \neq 0$.*
2. $\mathbf{u} = (\pm s^3, -s^2(s^2 + 1))$*. Then for any $s \in \mathbb{Q}$ and any $b \in \mathbb{Z}$ under the same conditions as before we have $\mu(g_{\mathbf{u}}(b)) \geqslant 3$.*
3. $\mathbf{u} = (2, 1)$*. Then for any $b \in \mathbb{Z}$ with $|b| > 1$ we have $\mu(g_{\mathbf{u}}(b)) \geqslant 13/5$.*

We wrote a computer program which computed the the first 30 partial quotients for all integer values u, v in the range $|u|, |v| < 1000$ and it did not find any other values $\mathbf{u} = (u, v)$, rather than those mentioned above, for which $g_{\mathbf{u}}(x)$ is not badly approximable. This suggests that the following statement may take place:

Conjecture A *The only values $\mathbf{u} \in \mathbb{Z}^2$ such that $g_{\mathbf{u}}(x)$ is not badly approximable are as follows:*

$$(\pm u, u^2),\ (\pm s^3, -s^2(s^2+1))\ \text{ and }\ (\pm 2, 1),$$

where $s \in \mathbb{Z}$.

By observing that $g_{(2,1)}(x) = (g_{1,0}(x))^2$ we get a notable corollary from Theorems 7 and 9: it provides a family of Mahler numbers ξ such that $\mu(\xi) = 2$, but $\mu(\xi^2) \geqslant 13/5$. Namely, one can take $\xi = g_{1,0}(b)$ for any integer b with $|b| > 1$.

Remark. For any integer b, Mahler numbers $g_{\mathbf{u}}(b)$ and $f_{\mathbf{u}}(b)$ are rationally dependent. Therefore they share the same irrationality exponent and Theorems 4 – 9 also provide the information about the irrationality exponents of perhaps "nicer" Mahler numbers $f_{\mathbf{u}}(x)$. Also, as explained in in Section 2, Theorems 3, 6 – 9 give us an insight whether the function $f_{\mathbf{u}}(x)$ is badly approximable or not. However its continued fraction definitely differs from what is provided in the first two theorems.

2 Continued fractions and continuants

Continued fractions of Laurent series share many properties of the classical continued fractions in real numbers. For example, it is known that, as for the standard case, the convergents $p_n(x)/q_n(x) = [a_0(x); a_1(x), \ldots, a_n(x)]$ of $f(x)$ are the best rational approximants of $f(x)$. Furthermore, we have even stronger version of Legendre theorem:

Theorem L *Let $f(x) \in \mathbb{F}[[x^{-1}]]$. Then $p(x)/q(x) \in \mathbb{F}(x)$ in a reduced form is a convergent of $f(x)$ if and only if*

$$\left|\left| f(x) - \frac{p(x)}{q(x)} \right|\right| < -2||q(x)||.$$

The proof of this and other unproven facts from this section can be found in [11].

As we already mentioned, every series $f(x) \in \mathbb{F}[[x^{-1}]]$ allows an expansion into a continued fraction. We will use the following notation:

$$f(x) := [a_0(x); a_1(x), a_2(x), \ldots] = a_0(x) + \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{1}{a_n(x)},$$

where $a_i(z) \in \mathbb{F}[z], i \in \mathbb{N}$. The convergents $p_n(x)/q_n(x)$ of $f(x)$ can be computed by the following formulae

$$\begin{aligned} &p_{-1}(x) = 1,\ p_0(x) = a_0(x),\ p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x);\\ &q_{-1}(x) = 0,\ q_0(x) = 1,\ q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x). \end{aligned} \tag{6}$$

However, unlike the classical setup of real numbers, where the numerators and the denominators p_n and q_n are defined uniquely, $p_n(x)$ and $q_n(x)$ are only unique up to multiplication by a non-zero constant. We can make them unique by putting an additional condition, that $q_n(x)$ must be monic. It is not difficult to see that (6) do not usually give monic polynomials. However we can adjust these formulae a bit to meet the required condition:

$$\begin{aligned} &\hat{p}_{-1}(x) = 1,\ \ \hat{p}_0(x) = a_0(x), \quad \hat{p}_{n+1}(x) = \hat{a}_{n+1}(x)\hat{p}_n(x) + \beta_{n+1}\hat{p}_{n-1}(x);\\ &\hat{q}_{-1}(x) = 0,\ \ \hat{q}_0(x) = 1, \qquad\quad \hat{q}_{n+1}(x) = \hat{a}_{n+1}(x)\hat{q}_n(x) + \beta_{n+1}\hat{q}_{n-1}(x). \end{aligned} \tag{7}$$

where we define, with ρ_n denoting the leading coefficient of $q_n(x)$:

$$\hat{a}_{n+1}(x) := \frac{a_{n+1}(x) \cdot \rho_n}{\rho_{n+1}} \quad \text{and} \quad \beta_0 = \beta_1 = 1,\ \beta_{n+1} = \frac{\rho_{n-1}}{\rho_{n+1}}.$$

One can easily check from (7) that $\hat{a}_n(x)$ are always monic. The formula for β_n suggests that $\beta_m \neq 0$.

It is not difficult to check that from the sequence of monic polynomials $\hat{a}_n(x)$ together with the sequence of non-zero elements β_n one can uniquely restore the initial continued fraction $[a_0(x), a_1(x), \ldots]$. Indeed, we have $a_n(x) = \rho_n \hat{a}_n(x)$ and ρ_n can be derived from the formula $\beta_{n+1} = \rho_{n-1}/\rho_{n+1}$ and initial values $\rho_0 = \rho_1 = 1$. In other words, any Laurent series has a modified continued fraction of the form

$$f(x) = \hat{a}_0(x) + \frac{\beta_1}{\hat{a}_1(x) + \frac{\beta_2}{\hat{a}_2(x)+\cdots}} =: \hat{a}_0(x) + \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{\beta_n}{\hat{a}_n(x)}, \tag{8}$$

where $\hat{a}_n(x) \in \mathbb{F}[x]$ are monic and $\beta_n \in \mathbb{F}$ are non-zero. And vise versa: any sequence of monic $\hat{a}_n(x)$ and non-zero values β_n defines a modified continued fraction for some $f(x)$.

In the paper we will use the modified formulae (7) for computing convergents and also for convenience we will not write hats above the the variables p_n, q_n and a_n.

For our function $g(x) = x^{-1}f(x)$ where $f(x)$ is defined by (1), we have $x^{-1} + u_1 x^{-2} + \ldots$, where u_1 is the coefficient coming from $P(x) = 1 + u_1 x + \ldots$. Therefore the first values of the convergents for $g(x)$ are computed as follows:

$$\begin{array}{lll} p_{-1}(x) = 1, & p_0(x) = 0, & p_1(x) = 1; \\ q_{-1}(x) = 0, & q_0(x) = 1, & q_1(x) = x - u_1 = a_1(x). \end{array}$$

Notice that in this case formulae (7) give both monic $p_n(x)$ and $q_n(x)$.

Equations (7) can be written in terms of the generalised continuants as follows. Given two sequences $\overline{a} = (a_n(x))_{n\in\mathbb{N}}$ and $\overline{\beta} = (\beta_n)_{n\in\mathbb{N}}$ and $k \leqslant l$ we write

$$\begin{array}{ll} \overline{a}_{k,l} := (a_k(x), a_{k+1}(x), \ldots, a_l(x)), & \overline{a}_k := \overline{a}_{1,k}; \\ \overline{\beta}_{k,l} := (\beta_k, \beta_{k+1}, \ldots, \beta_l), & \overline{\beta}_k := \overline{\beta}_{1,k}. \end{array}$$

Now generalised continuants $K_n^0(\overline{a}_n, \overline{\beta}_n) \in \mathbb{F}[x]$ and $K_n^1(\overline{a}_n, \overline{\beta}_n) \in \mathbb{F}[x]$ are defined as follows: $K_{-1}^0() = 0$, $K_{-1}^1() = 1$; $K_0^0() = 1$; $K_0^1() = 0$ and for $n \in \mathbb{N}$ both $K_n^0(\overline{a}_n, \overline{\beta}_n)$ and $K_n^1(\overline{a}_n, \overline{\beta}_n)$ satisfy the same recurrent equation:

$$K_n(\overline{a}_n, \overline{\beta}_n) = a_n(x) K_{n-1}(\overline{a}_{n-1}, \overline{\beta}_{n-1}) + \beta_n K_{n-2}(\overline{a}_{n-2}, \overline{\beta}_{n-2}). \tag{9}$$

One can easily check that the $K_n^0(\overline{a}_n, \overline{\beta}_n)$ is always monic while the leading coefficient of $K_n^1(\overline{a}_n, \overline{\beta}_n)$ equals β_1 for any $n \geqslant 1$. We can also check that the degrees of the continuants satisfy

$$||K_n^0(\overline{a}_n, \overline{\beta}_n)|| = \sum_{k=1}^{n} ||a_k(x)||, \quad ||K_n^1(\overline{a}_n, \overline{\beta}_n)|| = \sum_{k=2}^{n} ||a_k(x)||. \tag{10}$$

The enumerator and the denominator of the n'th convergent of $g(x)$ can be written as $p_n(x) = K_n^1(\overline{a}_n, \overline{\beta}_n)$ and $q_n(x) = K_n^0(\overline{a}_n, \overline{\beta}_n)$. Moreover, these polynomials are linked together by the following relations:

Lemma 1 *For $n, m \in \mathbb{Z}_{\geqslant 0}$ we have*

$$p_{n+m}(x) = K_m^0(\overline{a}_{n+1,n+m}, \overline{\beta}_{n+1,n+m}) p_n(x) + K_m^1(\overline{a}_{n+1,n+m}, \overline{\beta}_{n+1,n+m}) p_{n-1}(x). \tag{11}$$

The same relation is true for the polynomials $q_n(x)$ too.

Formula (11) can be checked by applying (7) and by using induction on m.

We will need to quantify the inequality from Theorem L.

Definition 1 *Let $p(x)/q(x) \in \mathbb{F}(x)$ be a rational function and $u(x)$ be a Laurent series. We say that an integer $c > 0$ is* the rate of approximation of $u(x)$ by $p(x)/q(x)$ *if*

$$||u(x) - p(x)/q(x)|| = -2||q(x)|| - c$$

It is known (see [11, displayed equation before Proposition 1]) that the convergent $p_n(x)/q_n(x)$ approximates $u(x)$ with the rate $||a_{n+1}(x)||$.

Definition 2 *We say that $f(x) \in \mathbb{F}((x^{-1}))$ is* badly approximable *if the valuation (i.e.degree) of every its partial quotient is bounded from above by an absolute constant. Otherwise we say that $f(x)$ is well approximable.*

An equivalent formulation of this definition is: $f(x)$ is badly approximable if for any $n \in \mathbb{N}$ the rate of approximation of f by its nth convergent is bounded from above by an absolute constant.

We end up this section with Lemma which shows that the following two statements are equivalent: $f_{\mathbf{u}}(x)$ is badly approximable and $g_{\mathbf{u}}(x)$ is badly approximable. Its proof can be found in [3, Proposition 1].

Lemma 2 *Let $f(x) \in \mathbb{Q}[[x^{-1}]]$, $a(x), b(x) \in \mathbb{Q}[x]\backslash 0$. Then $f(x)$ is badly approximable if and only if $g(x) = \frac{a(x)}{b(x)} f(x)$ is badly approximable.*

3 Relation with Hankel continued fractions

As we mentioned in Introduction, the more popular approach to compute irrationality exponents of Mahler numbers uses Hankel determinants and Hankel continued fractions rather than the classical ones. For example, they can be found in the works of Han [8, 9]. For the power series $f(x) \in \mathbb{F}[[x]]$ they are defined as follows

$$f(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+2}}{1+u_2(x)x - \cfrac{v_2 x^{k_1+k_2+2}}{1+u_3(x)x - \ldots}}} \qquad (12)$$

where $v_i \neq 0$ are constants, k_i are nonnegative integers and $u_i(x)$ are polynomials of degree $\deg(u_i) \leqslant k_{i-1}$. For convenience we will use the following shorter notation instead of (12):

$$f(x) = \mathop{\mathbf{K}}_{i=1}^{\infty} \frac{v_{i-1}^* x^{k_{i-1}+k_{i-2}+2}}{1 + u_i(x)x}.$$

where $v_0^* = v_0$ and $v_i^* = -v_i$ for $i \geqslant 1$.

In particular, the following result was established in [9]:

Theorem H1 *Each Hankel continued fraction defines a power series and conversely, for each power series $f(x)$ there exists unique Hankel continued fraction of $f(x)$.*

If we consider the Laurent series $x^{-1} f(x^{-1})$ then the Hankel continued fraction transforms to the standard continued fraction in the space of Laurent series. Indeed, one can easily check that

$$x^{-1} f(x^{-1}) = \mathop{\mathbf{K}}_{i=1}^{\infty} \frac{v_{i-1}}{x^{k_{i-1}+1} + x^{k_{i-1}} u_i(x^{-1})}.$$

Notice that if we set $a_i(x) = x^{k_{i-1}+1} + x^{k_{i-1}}u_i(x^{-1})$ and $\beta_i = v_{i-1}$ then we get the same notation as in (8). Since $a_i(x)$ is surely a polynomial, this gives a one-to-one correspondence between Hankel continued fractions for $f(x)$ and standard continued fractions for Laurent series of $x^{-1}f(x^{-1})$. In particular, this observation together with the standard fact that continued fractions for $x^{-1}f(x)$ are uniquely defined, gives another proof of Theorem H1.

By applying (6) one can get that the degree of the denominator $q_n(x)$ of nth convergent of $x^{-1}f(x^{-1})$ can be computed as follows:

$$||q_n(x)|| = s_n = k_0 + k_1 + \ldots + k_{n-1} + n. \tag{13}$$

Han used Hankel continued fractions of $f(x)$ to extract some information about the Hankel determinants of $f(x)$ which prove to be a powerful tool for computing the irrationality exponent of numbers $f(b^{-1})$ where b is a positive integer such that b^{-1} is inside the radius of convergence of $f(x)$. For $f(x) = \sum_{n=0}^{\infty} c_n x^n$ Hankel determinants are defined as follows

$$H_n(f) := \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix}.$$

The following result from [9] gives the relation between Hankel continued fraction and Hankel determinants:

Theorem H2 *Let $f(x)$ be a power series such that its Hankel continued fraction is given by (12). Then, for all integers $i \geqslant 0$ all non-vanishing Hankel determinants are given by*

$$H_{s_i}(f) = (-1)^{\epsilon} v_0^{s_i} v_1^{s_i - s_1} \cdots v_{i-1}^{s_i - s_{i-1}}, \tag{14}$$

where $\epsilon = \sum_{j=0}^{i-1} k_j(k_j+1)/2$ and $s_i = k_0 + k_1 + \cdots + k_{i-1} + i$.

By combining this theorem with (13) we get

Corollary 1 *n'th Hankel determinant of $f(x)$ does not vanish if and only if there exists a convergent $p(x)/q(x)$ of $x^{-1}f(x^{-1})$ such that $p(x)$ and $q(x)$ are coprime and $||q(x)|| = n$.*

Another straightforward corollary of the Theorem, applied to continued fractions of $x^{-1}f(x^{-1})$ is as follows:

Corollary 2 *If the continued fraction of $x^{-1}f(x^{-1})$ is badly approximable then there exists an increasing sequence $(s_i)_{i\geqslant 0}$ of positive integers such that $H_{s_i}(f) \neq 0$ for all $i \in \mathbb{Z}_{\geqslant 0}$ and $s_{i+1} - s_i$ is bounded from above by an absolute constant dependent only on $f(x)$.*

Guo, Wu and Wen [7] discovered a relation between the sequence s_i and the irrationality exponent of $f(b^{-1})$ for Mahler functions $f(x)$. Their result was significantly improved and corrected by Bugeaud, Han, Wen and Yao [5].

Theorem BHWY2 *Let $d \geqslant 2$ be an integer and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converge inside the unit disk. Suppose that there exist integer polynomials $A(x), B(x), C(x), D(x)$ with $B(0)D(0) \neq 0$ such that*

$$f(x) = \frac{A(x)}{B(x)} + \frac{C(x)}{D(x)} f(x^d). \tag{15}$$

Let $b \geqslant 2$ be an integer such that $B(b^{-d^n})C(b^{-d^n})D(b^{-d^n}) \neq 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$. If there exists an increasing sequence $(s_i)_{i\geqslant 0}$ of positive integers such that $H_{s_i}(f) \neq 0$ for all $i \in \mathbb{Z}_{\geqslant 0}$ and $\limsup_{i\to\infty} \frac{s_{i+1}}{s_i} = \rho$, then $f(1/b)$ is transcendental and

$$\mu(f(1/b)) \leqslant (1+\rho)\min\{\rho^2, d\}.$$

We apply this theorem to the series

$$\hat{f}_{\mathbf{u}}(x) := f_{\mathbf{u}}(x^{-1}) = \prod_{t=0}^{\infty} P(x^{d^t}).$$

Note that it satisfies the equation $\hat{f}_{\mathbf{u}}(x) = \frac{1}{P(x)}\hat{f}_{\mathbf{u}}(x^d)$ which is of the form (15). The condition $B(b^{-d^n})C(b^{-d^n})D(b^{-d^n}) \neq 0$ in this setting basically means that $g_{\mathbf{u}}(x) \neq 0$. Finally, an application of Corollary 2 asserts that if $g_{\mathbf{u}}(x)$ is badly approximable and $g_{\mathbf{u}}(b) \neq 0$ then $\mu(g_{\mathbf{u}}(b)) = 2$. This finishes the proof of "if" part of Theorem 4.

Remark. The condition $s_{i+1} - s_i \leqslant C$ is much stronger than $\limsup_{i\to\infty} \frac{s_{i+1}}{s_i} = 1$, thus the natural question arises: can we say anything better about the approximational properties of $g_{\mathbf{u}}(b)$ in the case $g_{\mathbf{u}}(x)$ is badly approximable? For example, can we show that

$$\left| f\left(\frac{1}{b}\right) - \frac{p}{q} \right| \leqslant \frac{1}{q^2 \cdot \delta(q)} \tag{16}$$

for some $\delta(q)$ which grows slower than any power function q^{ϵ}? It appears that the proof in [5] can not be easily improved to give us anything like (16).

4 Information about $g(x)$

Recall that we are focused in the following function written as a Laurent series:

$$g_{\mathbf{u}}(x) = x^{-1}f_{\mathbf{u}}(x) = x^{-1}\prod_{t=1}^{\infty} P(x^{-d^t}),$$

where $P(x) \in \mathbb{F}[x]$, $\deg(P) \leqslant \dim(\mathbf{u}) = d-1$ and $P(0) = 1$. By substituting x^d into the formula instead of x we get the functional relation

$$P^*(x)g_{\mathbf{u}}(x^d) = g_{\mathbf{u}}(x), \tag{17}$$

where

$$P^*(x) = x^{d-1}P(x^{-1}) = x^{d-1} + \sum_{n=1}^{d-1} u_n x^{d-n-1}. \tag{18}$$

Lemma 3 *If $p(x)/q(x)$ is a convergent of $g_{\mathbf{u}}(x)$ with the rate at least c then $P^*(x)p(x^d)/q(x^d)$ is also a convergent of $g_{\mathbf{u}}(x)$ with the rate at least $cd-d+1$.*

Proof. We have

$$\left|\left| g_{\mathbf{u}}(x) - \frac{p(x)}{q(x)} \right|\right| \leqslant -2||q(x)|| - c.$$

By substituting x^d instead of x and applying the functional relation we get

$$\left|\left| \frac{g_{\mathbf{u}}(x)}{P^*(x)} - \frac{p(x^d)}{q(x^d)} \right|\right| \leqslant -2d||q(x)|| - cd.$$

Multiply both sides of this equation by $P^*(x)$ and finally get

$$\left|\left|g_{\mathbf{u}}(x) - \frac{P^*(x)p(x^d)}{q(x^d)}\right|\right| \leqslant -2||q(x^d)|| - cd + d - 1.$$

⊠

Remark. This lemma shows the importance of the condition that $||P^*(x)|| < d$ and in turn of the condition $||P(x)|| < d$. In this case, any convergent $p(x)/q(x)$ with the rate of approximation $c \geqslant 1$ allows us to construct an infinite series of other convergents. Otherwise one needs the value of c to be big enough, so that $cd - ||P^*(x)|| > 0$. However we can not guarantee that there exists a convergent of $g_{\mathbf{u}}(x)$ with the rate of approximation strictly bigger than one.

By applying Lemma 3 several times we get the following

Corollary 3 *Let $k \in \mathbb{N}$. If $p(x)/q(x)$ is a convergent of $g_{\mathbf{u}}(x)$ with the rate of approximation at least c then*

$$\prod_{t=0}^{k-1} P^*(x^{d^t}) \frac{p(x^{d^k})}{q(x^{d^k})}$$

is also a convergent of $g_{\mathbf{u}}(x)$ with the rate of approximation at least $(c-1)d^k + 1$.

Lemma 3 provides the following very nice criterium for badly approximable series $g_{\mathbf{u}}(x)$.

Proposition 1 *The series $g_{\mathbf{u}}(x)$ is badly approximable if and only if all partial quotients $a_n(x)$, $n \in \mathbb{N}$ of its continued fraction are linear.*

PROOF. The "if" part of the lemma is straightforward. Let's show the other part. Assume that $||a_n(x)|| \geqslant 2$ for some $n \in \mathbb{N}$. Then the rate of approximation of convergent $r_1(x) := p_n(x)/q_n(x)$ is $c = c_1 \geqslant 2$. Then by Lemma 3 there exists another convergent $r_2(x)$ of $g(x)$ with the rate of approximation $c_2 \geqslant c_1 d - d + 1 > c_1$. We use Lemma 3 iteratively for convergents $r_2(x), r_3(x), ..., r_m(x)$ to construct a new convergent $r_{m+1}(x)$ with the rate of approximation $c_{m+1} > c_m$. Hence $g_{\mathbf{u}}(x)$ has a series of convergents with unbounded rate of approximation which in turn implies that $g_{\mathbf{u}}(x)$ is well approximable.

⊠

With help of the Proposition 1 we show that the "only if" part of Theorem 4 follows from Theorem 5. Indeed, assume that $g_{\mathbf{u}}(x)$ is not badly approximable. Then, by Proposition 1, there exists a partial quotient $a_{n_0}(x)$ of degree $c > 1$. Then Theorem 5 implies that, as soon as $g_{\mathbf{u}}(b) \neq 0$, $\mu(g_{\mathbf{u}}(b)) \geqslant 2 + (c-1)/n_0 > 2$.

In the rest of this section we look at the coordinates $u_1, \dots, u_{d-1}$ of $\mathbf{u}$ as independent variables. Then the Hankel determinant $H_n(f)$ of the series $f_{\mathbf{u}}(x^{-1})$ is a polynomial over them, i.e. $H_n(f) \in \mathbb{F}[u_1, u_2, \dots, u_{d-1}]$.

Lemma 4 *Let $\operatorname{char}\mathbb{F} = 0$ and d be a prime number. Then for any $n \in \mathbb{N}$, the polynomial $H_n(f)$ is not identically zero.*

PROOF. To check this lemma we need to provide just one value of $\mathbf{u}$ (or respectively one polynomial $P(x)$) such that the series $g_{\mathbf{u}}(x)$ is badly approximable. That would imply by Proposition 1 that all partial quotients of $g_{\mathbf{u}}(x)$ are linear and finally Theorem H2 implies that values of $H_n(f)$ for all $n \in \mathbb{N}$ are non-zero, and therefore it is not zero identically.

We use the technique which was firstly introduced by Han in [8]. If $d = 2$ then we know from [2] that $g_{-1}(x)$ is badly approximable. Let $d = p$ be an odd prime number. Then take

$$P(x) = (x+1)^{\frac{p-1}{2}}.$$

Consider the power series

$$\tilde{f}(x) = f(x^{-1}) = \prod_{t=0}^{\infty} P(x^{d^t}).$$

It satisfies the functional relation $\tilde{f}(x) = P(x)\tilde{f}(x^p)$. Consider this equation over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. It becomes

$$\tilde{f}(x) = (x+1)^{\frac{p-1}{2}}\tilde{f}(x)^p \quad \text{or} \quad \left(\tilde{f}(x)^2(x+1)\right)^{\frac{p-1}{2}} = 1,$$

as $f(x) \not\equiv 0 \pmod{p}$. Therefore the series $\tilde{f}(x)$ is a solution of one of the equations $\tilde{f}(x)^2(x+1) = a$ where a is some quadratic residue over $\mathbb{F}_p$. Definitely, $\tilde{f}(x)$ can not be rational, therefore by [9, Theorem 1.1] its Hankel continued fraction is ultimately periodic which in turn yields that the sequence of non-zero values $H_n(f)$ over $\mathbb{F}_p$ is also ultimately periodic. Going back to $\mathbb{Q}$, the Hankel continued fraction of $\tilde{f}$ is badly approximable and hence $g(x) = x^{-1}\tilde{f}(x^{-1})$ is badly approximable.

⊠

Lemma 4 only covers the case of prime d. Almost certainly the same result should be true for any integer $d \geqslant 2$. It would be interesting to see the proof of that statement. The author can extend this lemma to integer powers of prime numbers, however the other cases remain open.

We emphasize that the remaining results of this section are for $\mathbb{F} = \mathbb{C}$ or for the subfields of $\mathbb{C}$.

Lemma 5 *Let d be prime. For any $\mathbf{u} = (u_1, u_2, \ldots, u_{d-1}) \in \mathbb{C}^{d-1}$ there exists a sequence of vectors $\mathbf{v}_i \in \mathbb{C}^{d-1}$ such that $\mathbf{v}_i \to \mathbf{u}$ as $i \to \infty$ and all the series $g_{\mathbf{v}_i}(x)$ are badly approximable.*

Proof. Let $\mathbf{v} \in \mathbb{C}^{d-1}$. From Proposition 1 and Theorem H2 we know that $g_{\mathbf{v}}(x)$ is badly approximable if and only if all Hankel determinants $H_n(f_{\mathbf{v}})$ are not zero. Lemma 4 implies that for every n the equation $H_n(f_{\mathbf{v}}) = 0$ is true for $\mathbf{v}$ on a variety $\mathcal{V}_n$ of zero Lebesgue measure. Whence, $g_{\mathbf{v}}(x)$ is not badly approximable if and only if $\mathbf{v}$ belongs to countably many varieties which total measure is zero:

$$\mathbf{v} \in \mathbb{R}^{d-1} / \bigcup_{n=1}^{\infty} \mathcal{V}_n.$$

Take an arbitrary vector $\mathbf{u} \in \mathbb{R}^{d-1}$. For any $i \in \mathbb{N}$ the set

$$S_i := B(\mathbf{u}, 1/i) / \bigcup_{n=1}^{\infty} \mathcal{V}_n$$

is non-empty, where $B(\mathbf{u}, r)$ is the ball in $\mathbb{R}^{d-1}$ with the center in $\mathbf{u}$ and the radius r. Take any point $\mathbf{v}_i \in S_i$. By the construction $g_{\mathbf{v}_i}(x)$ is badly approximable and also $\mathbf{v}_i \to \mathbf{u}$ as $i \to \infty$. Hence the Lemma.

⊠

As we discussed before in Section 2, for each series $g_{\mathbf{u}}(x)$ we associate partial quotients $a_n(x)$ and parameters β_n where $n \in \mathbb{Z}_{\geqslant 0}$. By Proposition 1, for badly approximable $g_{\mathbf{u}}(x)$

all polynomials $a_n(x)$ can be written as $a_n(x) = x + \alpha_n$. Therefore we have a sequence of parameters β_n and α_n which are uniquely defined by a badly approximable $g_{\mathbf{u}}(x)$. It in turn is defined by $\mathbf{u} \in \mathbb{C}^{d-1}$, hence we can look at α_n and β_n as maps:

$$\alpha_n(\mathbf{u}) \;:\; \mathbb{C}^{d-1}/\bigcup_{i=1}^{n} \mathcal{V}_i \to \mathbb{C}; \quad \beta_n(\mathbf{u}) \;:\; \mathbb{C}^{d-1}/\bigcup_{i=1}^{n} \mathcal{V}_n \to \mathbb{C};$$

Lemma 6 *For each $n \in \mathbb{Z}_{\geqslant 0}$, the maps $\alpha_n(\mathbf{u})$ and $\beta_n(\mathbf{u})$ are continuous.*

PROOF. Firstly note that each coefficient c_n in the formula

$$g_{\mathbf{u}}(x) = \sum_{n=1}^{\infty} c_n x^{-n}$$

is a continuous function of $\mathbf{u}$: $c_n(\mathbf{u})$.

Secondly, one can easily check that the n'th convergent $p_n(x)/q_n(x)$ of badly approximable $g_{\mathbf{u}}(x)$ is uniquely defined by the first $2n+2$ terms of the series $g_{\mathbf{u}}(x)$. Moreover, if $q_n(x)$ is monic then $q_n(x)$ and $p_n(x)$ is a continuous map from the coefficients $c_1, \ldots, c_{2n}$ to $\mathbb{C}[x]$. Indeed, if $q_n(x) = \sum_{i=0}^{n-1} a_i x^i + x^n$ then the coefficients can be derived from the system

$$\sum_{i=0}^{n-1} c_{i+k} a_i + c_{n+k} = 0$$

for each k between one and n. The matrix of this system is basically n'th Hankel matrix which is invertible, because $H_n(f_{\mathbf{u}}) \neq 0$.

Finally, all terms α_i and β_i are continuous maps from $q_n(x)$ and $p_n(x)$ to $\mathbb{C}$. The last statement follows from the equation

$$\frac{p_n(x)}{q_n(x)} = \mathop{\mathbf{K}}_{i=1}^{n} \frac{\beta_{i+1}}{(x+\alpha_i)}.$$

⊠

Lemma 7 *Let $m \in \mathbb{N}$ and $(\mathbf{u}_i)_{i \in \mathbb{N}}$ be the sequence of vectors in $\mathbb{C}^{d-1}$ with $\lim_{i\to\infty} \mathbf{u}_i = \mathbf{u}$ such that the first m partial quotients of $g_{\mathbf{u}_i}(x)$ are linear. Assume that for any $1 \leqslant n \leqslant m$ there exist positive constants c_n and C_n such that*

$$\lim_{i\to\infty} |\alpha_n(\mathbf{u}_i)| < C_n \quad \text{and} \quad c_n < \lim_{i\to\infty} |\beta_n(\mathbf{u}_i)| < C_n. \tag{19}$$

Then the first m partial quotients of $g_{\mathbf{u}}(x)$ are also linear with coefficients

$$\alpha_n(\mathbf{u}) = \lim_{i\to\infty} \alpha_n(\mathbf{u}_i), \quad \beta_n(\mathbf{u}) = \lim_{i\to\infty} \beta_n(\mathbf{u}_i), \quad 1 \leqslant n \leqslant m.$$

The straightforward corollary of this lemma is that if $g_{\mathbf{u}_i}(x)$ are all badly approximable and (19) is satisfied for all $n \in \mathbb{N}$ then the limiting series $g_{\mathbf{u}}(x)$ is also badly approximable.

PROOF. Let

$$\frac{p_{n,\mathbf{u}_i}(x)}{q_{n,\mathbf{u}_i}(x)}$$

be the n'th convergent of $g_{\mathbf{u}_i}(x)$. Then we have

$$\left|\left| g_{\mathbf{u}_i}(x) - \frac{p_{n,\mathbf{u}_i}(x)}{q_{n,\mathbf{u}_i}(x)} \right|\right| \leqslant -2||q_{n,\mathbf{u}_i}(x)|| - 1$$

Since $\alpha_n(\mathbf{u})$ and $\beta_n(\mathbf{u})$ are continuous, the limits $\alpha_n = \lim_{i\to\infty}\alpha_n(\mathbf{u}_i)$ and $\beta_n = \lim_{i\to\infty}\beta_n(\mathbf{u}_i)$ exist. From (19) we have that $\alpha_n \leqslant C_n$ and $0 < c_n \leqslant \beta_n \leqslant C_n$. By continuity we also have

$$\frac{p_{n,\mathbf{u}_i}(x)}{q_{n,\mathbf{u}_i}(x)} = \mathop{\mathbf{K}}_{n=1}^{m} \frac{\beta_n(\mathbf{u}_i)}{x+\alpha_n(\mathbf{u}_i)} \quad \to \quad \mathop{\mathbf{K}}_{n=0}^{m} \frac{\beta_n}{x+\alpha_n} = \frac{p_n(x)}{q_n(x)}.$$

Then again by continuity we have that the first $2||q_n(x)||+1$ terms of $g_{\mathbf{u}_i}(x)$ tend to the corresponding terms of $g_{\mathbf{u}}(x)$. Therefore

$$\left|\left|g_{\mathbf{u}}(x) - \frac{p_n(x)}{q_n(x)}\right|\right| \leqslant -2||q_n(x)|| - 1,$$

which in turn implies that $p_n(x)/q_n(x)$ are convergents of $g_{\mathbf{u}}(x)$.

⊠

5 Irrationality exponents of $g_{\mathbf{u}}(b)$ for well approximable series

Throughout this section we assume that $g(x)$ is not badly approximable. Proposition 1 asserts that in this case there exists $n \in \mathbb{N}$ such that the n-th convergent $p_n(x)/q_n(x)$ has rate of approximation $c \geqslant 2$. Then we can provide a lower and upper bounds for $\mu(b)$ which depend on the smallest value of n with this property.

Proof of Theorem 5. It is sufficient for any $\epsilon > 0$ to provide an infinite sequence of rational numbers a_k/b_k such that

$$\left|g_{\mathbf{u}}(b) - \frac{a_k}{b_k}\right| < \frac{\gamma}{b_k^{2+(c-1)n_0^{-1}-\epsilon}}.$$

By construction of n_0 we have that $||q_n(x)|| = n$ for all $n \leqslant n_0$ because all partial quotients of $g_{\mathbf{u}}(x)$ are linear for $n \leqslant n_0$. Without loss of generality we may assume that both $p_{n_0}(x)$ and $q_{n_0}(x)$ have integer coefficients. Indeed, otherwise we just multiply both $p_{n_0}(x)$ and $q_{n_0}(x)$ by the least common multiple of the denominators of all the coefficients of both polynomials. We can also write $P^*(x)$ as $D^{-1}\tilde{P}(x)$ where $\tilde{P}(x) \in \mathbb{Z}[x]$ and $D \in \mathbb{Z}$.

Consider the following function

$$F(x) := g_{\mathbf{u}}(x) - \frac{p_{n_0}(x)}{q_{n_0}(x)}.$$

It can be written as an infinite series and moreover, since $p_{n_0}(x)/q_{n_0}(x)$ is a convergent of $g_{\mathbf{u}}(x)$ with rate of approximation c, we have

$$F(x) = \sum_{n=2n_0+c}^{\infty} c_n x^{-n} \quad c_n \in \mathbb{R}.$$

We know that $F(x)$ converges absolutely for all $|x| > 1$ and therefore for all $|x| \geqslant 2$ we have

$$|x^{2n_0+c}F(x)| = \left|\sum_{n=0}^{\infty} c_{n+2n_0+c}x^{-n}\right| \leqslant \sum_{n=0}^{\infty} |c_{n+2n_0+c}|2^{-n} =: \gamma_1. \tag{20}$$

In other words there exists an absolute constant γ_1 such that for all $|x| \geqslant 2$ we have $|F(x)| \leqslant \gamma_1 x^{-2n_0-c}$.

Now apply the functional equation (17) k times to get

$$F(x^{d^k})\prod_{t=0}^{k-1}P^*(x^{d^t}) = g_{\mathbf{u}}(x) - \frac{p_{n_0}(x^{d^k})\prod_{t=0}^{k-1}P^*(x^{d^t})}{q_{n_0}(x^{d^k})} = g_{\mathbf{u}}(x) - \frac{p_{n_0}(x^{d^k})\prod_{t=0}^{k-1}\tilde{P}(x^{d^t})}{D^k q_{n_0}(x^{d^k})}. \quad (21)$$

We set $a_k := p_{n_0}(b^{d^k})\prod_{t=0}^{k-1}\tilde{P}(b^{d^t})$ and $b_k := D^k q_{n_0}(b^{d^k})$. By construction, they are both integer. Moreover, one can check that

$$\lim_{k\to\infty}\frac{q_{n_0}(b^{d^k})}{b^{n_0 d^k}} = \gamma_2,$$

where γ_2 is the leading coefficient of $q_{n_0}(x)$. Therefore for large enough k we have $|b_k| \leqslant 2|\gamma_2|D^k b^{n_0 d^k}$.

Now we use inequality (20) for $F(x)$ and (21) to estimate $|g_{\mathbf{u}}(b) - a_k/b_k|$:

$$\left|g_{\mathbf{u}}(b) - \frac{a_k}{b_k}\right| \leqslant \frac{\gamma_1\prod_{t=0}^{k-1}P^*(b^{d^t})}{b^{(2n_0+c)d^k}} = \frac{\gamma_1\prod_{t=0}^{k-1}P(b^{-d^t})}{b^{(2n_0+c-1)d^k+1}}.$$

Since $\prod_{t=0}^{\infty}P(x^{-d^t})$ converges absolutely for all $|x|>1$, there exists a uniform upper bound γ_3 such that for all $|b|\geqslant 2$ we have $|\prod_{t=0}^{\infty}P(b^{-d^t})| \leqslant \gamma_3$. Next, by solving the equation

$$(D^k b^{n_0 d^k})^x = b^{(2n_0+c-1)d^k}$$

we get

$$x = x(k) = \left(2 + \frac{c-1}{n_0}\right)\left(1 + \frac{k\log_b D}{n_0 d^k}\right)^{-1}$$

As k tends to infinity, $x(k)$ tends to $2+(c-1)/n_0$. Therefore for any $\epsilon>0$ we can find $k(\epsilon)$ large enough so that for any $k>k(\epsilon)$, $x(k) > 2+(c-1)/n_0-\epsilon$ and therefore

$$\left|g_{\mathbf{u}}(b) - \frac{a_k}{b_k}\right| < \frac{(2\gamma_2)^{2+(c-1)/n_0}\gamma_1\gamma_3}{b}\cdot b_k^{-2-(c-1)/n_0+\epsilon}.$$

⊠

6 Recurrent formulae for continued fractions of $g_{\mathbf{u}}(x)$

In this section we construct the continued fraction of the series $g_{\mathbf{u}}(x)$. Throughout the whole section we assume that $g_{\mathbf{u}}(x)$ is badly approximable. Then, by Proposition 1, its continued fraction is determined by the terms α_n and β_n where the partial quotients $a_n(x) = x+\alpha_n$ and the parameters β_n are satisfy the recurrent formulae (7).

Proposition 2 *Let $g_{\mathbf{u}}(x)$ be badly approximable. Then for any $k\in\mathbb{Z}_{\geqslant 0}$ one has*

$$K_d^1(\overline{a}_{dk+1,d(k+1)}, \overline{\beta}_{dk+1,d(k+1)}) = \beta_{dk+1}P^*(x), \quad (22)$$

where $P^*(x)$ is given in (18).

Proof.

Let $p_k(x)/q_k(x)$ be kth convergent of $g_{\mathbf{u}}(x)$. Proposition 1 asserts that $||q_k(x)|| = k$. We know from Lemma 3 that $P^*(x)p_k(x^d)/q_k(x^d)$ is another convergent of $g(x)$. We can assume that $P^*(x)p_k(x)$ and $q_k(x)$ are coprime. Indeed otherwise one can cancel their common divisor from the fraction $P^*(x)p_k(x^d)/q_k(x^d)$ and it's rate of convergence will become bigger than one, which contradicts to Proposition 1. Therefore we get that the fraction $P^*(x)p_k(x^d)/q_k(x^d)$ is in fact dk'th convergent of $g_{\mathbf{u}}(x)$. By following this arguments for each $k \in \mathbb{N}$ we get that

$$p_{dk}(x) \equiv 0 \pmod{P^*(x)}.$$

Consider the equation (11) from Lemma 1 with $n = dk$ and $m = d$ modulo $P^*(x)$:

$$0 \equiv p_{d(k+1)}(x) \equiv K_d^1(\overline{a}_{dk+1,d(k+1)}, \overline{\beta}_{dk+1,d(k+1)})p_{dk-1}(x) \pmod{P^*(x)}.$$

The observation $\gcd(p_{dk-1}(x), p_{dk}(x)) = \gcd(p_{dk-1}(x), P^*(x)) = 1$ implies that

$$K_d^1(\overline{a}_{dk+1,d(k+1)}, \overline{\beta}_{dk+1,d(k+1)}) \equiv 0 \pmod{P^*(x)}.$$

By (10) the degree of the left hand side coincides with those of $P^*(x)$. Then comparing the leading coefficients of the polynomials in the congruence finishes the proof.

⊠

Polynomial equation (22) gives us $d-1$ relations between various values α_n and β_n for each $k \in \mathbb{Z}_{\geqslant 0}$. We just need to compare the corresponding coefficients of the polynomials from both sides of the equation. However they are still not enough to provide the recurrent formula for all values $\alpha_{dk+1}, \ldots, \alpha_{d(k+1)}, \beta_{dk+1}, \ldots, \beta_{d(k+1)}$. More relations can be derived from the following:

Proposition 3 *Let $g_{\mathbf{u}}(x)$ be badly approximable. Then for any $k \in \mathbb{N}$ one has*

$$K_{2d}^1(\overline{a}_{dk+1,d(k+2)}, \overline{\beta}_{dk+1,d(k+2)}) = \beta_{dk+1}(x^d + \alpha_{k+2})P^*(x) \tag{23}$$

and

$$K_{2d}^0(\overline{a}_{dk+1,d(k+2)}, \overline{\beta}_{dk+1,d(k+2)}) = \beta_{k+2} + (x^d + \alpha_{k+2})K_d^0(\overline{a}_{kd+1,k(d+1)}, \overline{\beta}_{kd+1,k(d+1)}). \tag{24}$$

Proof.

For convenience we will use the following notation throughout the proof: $K_{2d}^0(x) := K_{2d}^0(\overline{a}_{dk+1,d(k+2)}, \overline{\beta}_{dk+1,d(k+2)})$, $K_d^0(x) := K_d^0(\overline{a}_{kd+1,d(k+1)}, \overline{\beta}_{kd+1,d(k+1)})$. The notions $K_{2d}^1(x)$ and $K_d^1(x)$ are defined by analogy.

We provide two different relations between $q_{kd}(x), q_{(k+1)d}(x)$ and $q_{(k+2)d}(x)$. The first one comes from the fact that for each $m \in \mathbb{N}$, $q_{md}(x) = q_m(x^d)$, which was shown in the proof of Proposition 2. Therefore the application of (7) gives us

$$q_{(k+2)d}(x) = (x^d + \alpha_{k+2})q_{(k+1)d}(x) + \beta_{k+2}q_{kd}(x). \tag{25}$$

On the other hand (11) implies

$$\begin{aligned} q_{(k+1)d}(x) = {} & K_d^0(x)q_{kd}(x) + K_d^1(x)q_{kd-1}(x) \\ \text{[by Proposition 2]} = {} & K_d^0(x)q_{kd}(x) + \beta_{kd+1}P^*(x)q_{kd-1}(x). \end{aligned}$$

From this formula we can write $q_{kd-1}(x)$ in terms of $q_{kd}(x)$ and $q_{(k+1)d}(x)$.

$$q_{kd-1}(x) = \frac{q_{(k+1)d}(x) - K_d^0(x)q_{kd}(x)}{\beta_{kd+1}P^*(x)}.$$

Next, (11) also gives us

$$\begin{aligned} q_{(k+2)d}(x) = & \quad K_{2d}^0(x)q_{kd}(x) + K_{2d}^1(x)q_{kd-1}(x) \\ = & \quad \left(K_{2d}^0(x) - \frac{K_{2d}^1(x)K_d^0(x)}{\beta_{kd+1}P^*(x)}\right) q_{kd}(x) + \frac{K_{2d}^1(x)}{\beta_{kd+1}P^*(x)} q_{(k+1)d}(x). \end{aligned}$$

Combining the last formula with (25) gives

$$\left(\beta_{k+2} + \frac{K_{2d}^1(x)K_d^0(x)}{\beta_{kd+1}P^*(x)} - K_{2d}^0(x)\right) q_{kd}(x) = \left(\frac{K_{2d}^1(x)}{\beta_{kd+1}P^*(x)} - x^d - \alpha_{k+2}\right) q_{(k+1)d}(x). \quad (26)$$

Adapting the formula (11) to K_{2d}^1 gives

$$K_{2d}^1(x) = K_d^0(\overline{a}_{kd+d+1,(k+2)d}, \overline{\beta}_{kd+d+1,(k+2)d})K_d^1(x) + K_d^1(\overline{a}_{kd+d+1,(k+2)d}, \overline{\beta}_{kd+d+1,(k+2)d})K_{d-1}^1(x) \quad (27)$$

By Proposition 2 we get that $K_d^1(x) = \beta_{kd+1}P^*(x)$ and $K_d^1(\overline{a}_{kd+d+1,(k+2)d}, \overline{\beta}_{kd+d+1,(k+2)d}) = \beta_{kd+d+1}P^*(x)$. This straightforwardly implies that the expressions on the left and right hand sides of (26) are in fact polynomials. Moreover, since the leading coefficient of $K_{2d}^1(x)$ is β_{kd+1} we have that the degree of the polynomial

$$D(x) := \frac{K_{2d}^1(x)}{\beta_{kd+1}P^*(x)} - x^d - \alpha_{k+2}$$

is at most $d-1$.

Two polynomials $q_{kd}(x) = q_k(x^d)$ and $q_{(k+1)d}(x) = q_{k+1}(x^d)$ are coprime. Therefore $D(x)$ should be a multiple of $q_{kd}(x)$. However for $k \geqslant 1$ its degree $||q_{kd}(x)||$ is strictly bigger than $||D(x)||$ which is only possible when $D(x) = 0$. This immediately gives the formula (23). Finally, (24) can be achieved by equating the right hand side of (26) to zero.

⊠

6.1 Recurrent formulae for small d

Relations from Proposition 2 and 3 appear to be enough to provide the recurrent formulae for the values α_n and β_n. We demonstrate that by constructing the recurrent formulae for small values of d.

The case $d = 2$. We have

$$\mathbf{u} = u, \quad g_u(x) = \prod_{t=0}^{\infty}(1 + ux^{-d^t}) \quad \text{and} \quad P^*(x) = x + u.$$

Proof of Theorem 1. By Proposition 2 we have that for any $k \geqslant 0$,

$$K_2^1(\overline{a}_{2k+1,2k+2}, \overline{\beta}_{2k+1,2k+2}) = \beta_{2k+1}(x + \alpha_{2k+2}) = \beta_{2k+1}(x + u).$$

Since we assumed that $g_u(x)$ is badly approximable, $\beta_{2k+1} \neq 0$ and the formula straightforwardly implies that $\alpha_{2k+2} = u$ for any $k \in \mathbb{Z}_{\geqslant 0}$.

Then we apply Proposition 3. From (23) for any $k \in \mathbb{N}$ we have

$$\begin{aligned} & ((x + \alpha_{2k+4})(x + \alpha_{2k+3})(x + \alpha_{2k+2}) + (x + \alpha_{2k+4})\beta_{2k+3} + (x + \alpha_{2k+2})\beta_{2k+4})\beta_{2k+1} \\ = \; & \beta_{2k+1}(x^2 + \alpha_{k+2})(x + u). \end{aligned}$$

We already know that $\alpha_{2k+2} = \alpha_{2k+4} = u$. Then comparing the coefficients for x^2, x and 1 give

$$\alpha_{2k+3} = -u; \quad \beta_{2k+3} + \beta_{2k+4} = \alpha_{k+2} + u^2.$$

Finally, look at equation (24) modulo $K_2^0(x) := K_2^0(\overline{a}_{2k+1,2k+2}, \overline{\beta}_{2k+1,2k+2})$:

$$K_4^0(\overline{a}_{2k+1,2k+4}, \overline{\beta}_{2k+1,2k+4}) \equiv (x+\alpha_{2k+1})(x+\alpha_{2k+4})\beta_{2k+3} = (x+\alpha_{2k+1})(x+\alpha_{2k+2})\beta_{2k+3}. \quad (28)$$

The right hand side of (24) is congruent to β_{k+2} modulo $K_2^0(x)$. We also have,

$$K_2^0(x) = (x + \alpha_{2k+2})(x + \alpha_{2k+1}) + \beta_{2k+2}$$

and therefore the last expression in (28) is congruent to $-\beta_{2k+2}\beta_{2k+3}$. Hence this provides the following relation between β's:

$$\beta_{2k+2}\beta_{2k+3} = -\beta_{k+2}.$$

We collect all the data together and get the recurrent formulae which allow us to confirm formulae (3) for α_n and β_n starting from $n = 5$: for any $k \geqslant 1$,

$$\begin{array}{l} \alpha_{2k+2} = u; \quad \alpha_{2k+3} = -u; \\ \beta_{2k+3} = -\frac{\beta_{k+2}}{\beta_{2k+2}}, \quad \beta_{2k+4} = \alpha_{k+2} + u^2 - \beta_{2k+3}. \end{array}$$

To finish the proof we need to find the values $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_4$. By direct computation one can easily check that the first convergent of $g_u(x)$ is $(x-u)^{-1}$. That together with Lemma 3 gives us

$$\frac{p_1(x)}{q_1(x)} = \frac{1}{x-u}; \quad \frac{p_2(x)}{q_2(x)} = \frac{x+u}{x^2-u}, \quad \frac{p_4(x)}{q_4(x)} = \frac{(x+u)(x^2+u)}{x^4-u}.$$

We find the denominator $q_3(x) = x^3 + ax^2 + bx + c$ of the third convergent by noticing that

$$g_u(x) = x^{-1} + ux^{-2} + ux^{-3} + u^2x^{-4} + ux^{-5} + u^2x^{-6} + \ldots$$

and that the coefficients for x^{-1}, x^{-2} and x^{-3} of the expression $g_u(x)(x^3 + ax^2 + bx + c)$ are all zeroes. That gives us the system of linear equations in a, b, c with solutions $a = -u, b = -u-1, c = u(u+1)$. That finally gives us

$$\frac{p_3(x)}{q_3(x)} = \frac{x^2 - u^2 - 1}{(x-u)(x^2-u-1)}$$

These convergents give us the initial values:

$$\alpha_1 = -u, \ \alpha_2 = u, \ \alpha_3 = -u, \ \alpha_4 = u;$$

$$\beta_1 = 1, \ \beta_2 = u^2 - u, \ \beta_3 = -1, \ \beta_4 = u^2 + u + 1.$$

Now we have all the relations from (3).

⊠

The case $\mathbf{d = 3}$. We have

$$\mathbf{u} = (u, v), \quad g_{\mathbf{u}}(x) = \prod_{t=0}^{\infty}(1 + ux^{-3^t} + vx^{-2\cdot 3^t}) \quad \text{and} \quad P^*(x) = x^2 + ux + v.$$

Proof of Theorem 2. We proceed in a similar way as for the case $d = 2$. Proposition 2 gives us that for any integer $k \geqslant 0$,

$$(x + \alpha_{3k+2})(x + \alpha_{3k+3}) + \beta_{3k+3} = x^2 + ux + v \tag{29}$$

This immediately implies some relations between the coefficients:

$$\alpha_{3k+2} + \alpha_{3k+3} = u, \quad \alpha_{3k+2}\alpha_{3k+3} + \beta_{3k+3} = v. \tag{30}$$

Next, we apply the equation (23), where we use (27) to compute $K^1_{2d}(x)$. For $k \geqslant 1$ we get

$$\begin{aligned}&\beta_{3k+1}(x^2+ux+v)((x + \alpha_{3k+4})(x^2+ux+v) + (x+\alpha_{3k+6})\beta_{3k+5} + (x+\alpha_{3k+2})\beta_{3k+4})\\ &= \beta_{3k+1}(x^2 + ux + v)(x^3 + \alpha_{k+2}).\end{aligned}$$

Comparing the coefficients then gives

$$\begin{aligned}&\alpha_{3k+4} + u = 0, \quad u\alpha_{3k+4} + v + \beta_{3k+5} + \beta_{3k+4} = 0,\\ &\alpha_{3k+4}v + \alpha_{3k+6}\beta_{3k+5} + \alpha_{3k+2}\beta_{3k+4} = \alpha_{k+2}.\end{aligned} \tag{31}$$

Finally, as before, apply the equation (24) modulo $K^0_3(x)$:

$$K^0_3(x) = K^0_3(\overline{a}_{3k+1,3k+3}, \overline{\beta}_{3k+1,3k+3}) = (x + \alpha_{3k+1})(x^2 + ux + v) + (x + \alpha_{3k+3})\beta_{3k+2}.$$

We get

$$\begin{aligned}\beta_{k+2} \equiv\ & \beta_4((x + \alpha_{3k+5})(x + \alpha_{3k+6}) + \beta_{3k+6})((x + \alpha_{3k+2})(x + \alpha_{3k+1}) + \beta_{3k+2})\\ \stackrel{(29)}{\equiv}\ & \beta_{3k+4}P^*(x)((x + \alpha_{3k+2})(x + \alpha_{3k+1}) + \beta_{3k+2})\\ \equiv\ & \beta_{3k+4}((x + \alpha_{3k+2})K^0_3(x) - (x + \alpha_{3k+3})(x + \alpha_{3k+2})\beta_{3k+2} + P^*(x)\beta_{3k+2})\\ \stackrel{(29)}{\equiv}\ & \beta_{3k+2}\beta_{3k+3}\beta_{3k+4}\end{aligned}$$

or $\beta_{k+2} = \beta_{3k+2}\beta_{3k+3}\beta_{3k+4}$. Combining this formula with (30) and (31) we finally get recurrent formulae for all values α_n and β_n satisfy (5) starting from $n = 7$:

$$\begin{aligned}&\alpha_{3k+4} = -u, \quad \beta_{3k+4} = \frac{\beta_{k+2}}{\beta_{3k+3}\beta_{3k+2}};\\ &\beta_{3k+5} = u^2 - v - \beta_{3k+4}, \quad \alpha_{3k+5} = u - \frac{\alpha_{k+2} + uv - \alpha_{3k+2}\beta_{3k+4}}{\beta_{3k+5}}\\ &\alpha_{3k+6} = u - \alpha_{3k+5}, \quad \beta_{3k+6} = v - \alpha_{3k+5}\alpha_{3k+6}.\end{aligned}$$

Note that since (30) is also true for $k = 0$, α_6 and β_6 can also be computed by (5). Therefore it remains to compute $\alpha_1, \ldots, \alpha_5$ and $\beta_1, \ldots \beta_5$. We do that straight from calculating the first five convergents of $g_{\mathbf{u}}(x)$. To save the space we will only provide their denominators.

$$q_1(x) = (x - u), \quad q_2(x) = x^2 + \frac{u(v-1)}{v - u^2}x + \frac{u^2 - v^2}{v - u^2}, \quad q_3(x) = x^3 - u$$

This allows us to get the values:

$$\begin{aligned}&\alpha_1 = -u, \alpha_2 = \frac{u(2v - 1 - u^2)}{v - u^2}, \alpha_3 = \frac{-u(v-1)}{v - u^2};\\ &\beta_1 = 1, \beta_2 = u^2 - v, \beta_3 = \frac{u^2 + u^4 + v^3 - 3u^2v}{(v - u^2)^2}.\end{aligned}$$

Finally we use Mathematica to compute $\alpha_4, \alpha_5, \beta_4, \beta_5$ and to confirm that they satisfy the recurrent equations (5) with $k = 0$. This gives us all relations from (4) and (5).

⊠

7 Badly approximable series $g_{\mathbf{u}}(x)$ for small d.

Theorems 1 and 2 are only valid for badly approximable series $g_{\mathbf{u}}(x)$. In this section we try to answer the question: for what values $\mathbf{u}$ the series $g_{\mathbf{u}}(x)$ is in fact badly approximable? Then for such series $g_{\mathbf{u}}(x)$ all machinery of the previous paragraph can be used.

Proof of Theorem 3. Assume that the first n terms α_k and β_k satisfy

$$|\alpha_k| < \infty,\ 0 < |\beta_k| < \infty; \quad 1 \leqslant k \leqslant n. \tag{32}$$

Then by Lemma 5 there exists a sequence of vectors $\mathbf{u}_i$ such that $\mathbf{u}_i \to \mathbf{u}$ and $g_{\mathbf{u}_i}(x)$ are badly approximable. their parameters $\alpha_k(\mathbf{u}_i)$ and $\beta_k(\mathbf{u}_i)$ are computed by formulae (3) for $d = 2$ and by (4) and (5) for $d = 3$. Therefore by continuity the coefficients $\alpha_k(\mathbf{u}_i)$ and $\beta_k(\mathbf{u}_i)$ tend to α_k and β_k respectively. Moreover, for each k there exists $i_0 = i_0(k)$ and $c_k = |b_k|/2, C_k = \max\{2|\alpha_k|, 2|\beta_k|\}$ such that for $i > i_0$ we have $\alpha_k(\mathbf{u}_i) < C_k$ and $c_k < \beta_k(\mathbf{u}_i) < C_k$. Finally, Lemma 1 confirms that α_k and β_k with $1 \leqslant k \leqslant n$ are indeed the coefficients of the continued fraction of $g_{\mathbf{u}}(x)$.

Note that all division in formulae (3), (4) and (5) for α_k and β_k are by some values of β_m with $m \leqslant k$. Therefore, as soon as $\beta_1, \ldots, \beta_n$ do not vanish, the condition (32) is automatically satisfied. Moreover, in this case we also can not have $|\beta_{n+1}| = \infty$.

Finally, assume that $\beta_{n+1} = 0$. If $(n+1)$th partial quotient of $g_{\mathbf{u}}(x)$ is linear than by Lemma 6 the sequence $\beta_{n+1}(\mathbf{u}_i)$ should tend to $\beta_{n+1}(\mathbf{u})$. Therefore $\beta_{n+1}(\mathbf{u}) = 0$ which is impossible, because all values of β_k in a continued fraction for $g_{\mathbf{u}}(x)$ must be non-zero. Hence we have a contradiction and the $(n+1)$th partial quotient in $g_{\mathbf{u}}(x)$ is not zero.

⊠

The case $d = 2$.

Proof of Theorem 6. By Theorem 3, $g_{\mathbf{u}}(x)$ is well approximable if and only if one of the values β_n vanishes. From (3) there are two obvious values $u = 0$ and $u = 1$ when $\beta_2 = 0$. They in fact produce rational functions: $g_0(x) = x^{-1}, g_1(x) = (x-1)^{-1}$. On the other hand the values of β_3 and β_4 do not equal to zero for any rational (and in fact any real) values of u.

Lemma 8 *For any $n \geqslant 3$ the value $\beta_n(u)$ can be written as*

$$\beta_n(u) = \frac{e_n(u)}{d_n(u)},$$

where $e_n(u), d_n(u) \in \mathbb{Z}[x]$ and the leading and constant coefficients of both $e_n(u), d_n(u)$ equal ± 1.

Proof. It can be easily checked by induction. It is true for $n = 3$ and $n = 4$. We assume that the statement is true for all values $\beta_3(u), \ldots \beta_{2k+2}(u)$ and prove it for $\beta_{2k+3}(u)$ and $\beta_{2k+4}(u)$. In addition we will check the following condition: $||e_{2k}(u)|| = ||d_{2k}(u)|| + 2$, $||e_{2k+1}(u)|| \leqslant ||d_{2k+1}(u)||$.

By (3) we have

$$\beta_{2k+3}(u) = \frac{-d_{2k+2}(u)e_{k+2}(u)}{e_{2k+2}(u)d_{k+2}(u)}.$$

Its enumerator and the denominator clearly satisfy the conditions of the lemma together with

$$\begin{aligned} ||e_{2k+3}(u)|| &= ||d_{2k+2}(u)e_{k+2}(u)|| = ||e_{2k+2}(u)|| + ||e_{k+2}(u)|| - 2 \\ &\leqslant ||e_{2k+2}(u)d_{k+2}(u)|| = ||d_{2k+3}(u)||. \end{aligned}$$

For $\beta_{2k+4}(u)$ we have

$$\beta_{2k+4}(u) = \alpha_{k+2}(u) + u^2 - \frac{e_{2k+3}(u)}{d_{2k+3}(u)} = \frac{(u^2 \pm u)d_{2k+3}(u) - e_{2k+3}(u)}{d_{2k+3}(u)}.$$

The leading coefficient of the enumerator on the right hand side comes from $u^2 d_{2k+3}(u)$ and the constant coefficient comes from $-e_{2k+3}(u)$. Both of them by inductional hypothesis are plus or minus one. Since $||e_{2k+3}(u)|| \leqslant ||d_{2k+3}(u)||$, we have

$$||e_{2k+4}(u)|| = ||(u^2 \pm u)d_{2k+3}(u) - e_{2k+3}(u)|| = ||u^2 d_{2k+3}(u)|| = 2 + ||d_{2k+3}(u)|| = ||d_{2k+4}(u)||.$$

This completes the induction.

⊠

The obvious corollary from Lemma 8 is that if $u \in \mathbb{Q}$ and $g_n(u) = 0$ for some $n \geqslant 3$ then u is either plus or minus one. Indeed these are the only possible rational roots of the equation $e_n(u) = 0$. On the other hand it was shown in [3] that $g_{-1}(x)$ is badly approximable. Theorem 6 is proven.

Remark. There exist real values of u such that $g_u(x)$ is well approximable. For example, one can check that if u is any real root of the equation $u^4 - u - 1 = 0$ then $\beta_6(u) = 0$.

The case $d = 3$.

PROOF OF THEOREM 9. As for $d = 2$ we investigate the case when $\beta_n = 0$ for some n. From (4), the equation $\beta_2 = 0$ gives an infinite collection of vectors $\mathbf{u} = (u, u^2)$ such that the series:

$$g_{(u,u^2)}(x) = x^{-1} \prod_{t=0}^{\infty} (1ux^{-3^t} + u^2 x^{-2\cdot 3^t})$$

is well approximable. Theorem 5 then asserts that for any $u \in \mathbb{Q}$ and integer $b \geqslant 2$, as soon as $g_{(u,u^2)}(x) \notin \mathbb{Q}(x)$ and $g_{(u,u^2)}(b) \neq 0$, we have $\mu(g_{(u,u^2)}(b)) \geqslant 5/2$.

The equation $\beta_3 = 0$ can be written as $u^2 + u^4 + v^3 - 3u^2 v = 0$. It gives an infinite parametrised series of rational solutions: $u = s^3$, $v = -s^2(s^2+1)$ where $s \in \mathbb{Q}$. It has only one intersection with the collection (u, u^2) above, namely when $s = 0$. This solution can be ignored, because $g_{(0,0)}(x) = x^{-1}$ is a rational function. Hence we have another set of well approximable series:

$$q_{(s^3,-s^2(s^2+1))}(x) = x^{-1} \prod_{t=0}^{\infty} (1 + s^3 x^{-3^t} - s^2(s^2+1) x^{-2\cdot 3^t}), \quad s \in \mathbb{Q}\backslash\{0\}.$$

Direct computation shows that the second convergent of $g_{(s^3,-s^2(s^2+1))}(x)$ is

$$\frac{p_2(x)}{q_2(x)} = \frac{x + s(s^2+1)}{x^2 + sx + s^2}$$

and

$$(x^2 + sx + s^2)x^{-1} \prod_{t=0}^{\infty} (1 + s^3 x^{-3^t} - s^2(s^2+1)x^{-2\cdot 3^t}) = -(s^6 + s^4 + s^2)x^{-5} + \ldots$$

If $s \neq 0$ the term $-(s^6 + s^4 + s^2)$ is non-zero and therefore

$$\left|\left| g_{s^3,-s^2(s^2+1)}(x) - \frac{p_2(x)}{q_2(x)} \right|\right| = -7$$

or in other words the rate of approximation of the second convergent is three. Then the application of Theorem 5 tells us that if $g_{s^3,-s^2(s^2+1)}(x)\not\in\mathbb{Q}(x)$ and $g_{s^3,-s^2(s^2+1)}(b)\neq 0$, we have $\mu(g_{s^3,-s^2(s^2+1)}(b))\geqslant 3$ for all $s\in\mathbb{Q}$ and all integer $b\geqslant 2$.

There is at least one less trivial example of well approximable series. One can note that $g_{(2,1)}(x)$ is well approximable by noticing that $\beta_6(2,1)=0$. Direct computation shows that for the fifth convergent of $g_{(2,1)}(x)$ we have $p_5(x)=x^4+x^3+2x^2+4$ and $q_5(x)=x^5-x^4+x^3-x^2+x-1$ and

$$(x^5-x^4+x^3-x^2+x-1)x^{-1}\prod_{t=0}^{\infty}(1+2x^{-3^t}+x^{-2\cdot 3^t})=3x^{-8}+\dots.$$

Therefore the rate of approximation of fifth convergent of $g_{(2,1)}(x)$ is three. Since for any $|b|>1$ the value $g_{(2,1)}(b)$ is non-zero, Theorem 5 implies that $\mu((g_{(1,0)}(b))^2)\geqslant 13/5$.

⊠

Proof of theorem 7. One can notice that $g_{(0,v)}(x)=xg_{(v,0)}(x^2)$ and therefore $g_{(0,v)}(x)$ is badly approximable if and only if so is $g_{(v,0)}(x)$. Therefore without loss of generality we can only assume the case $u=0$.

Let $u=0$. Then formulae (4), (5) and an easy induction give us that $\alpha_k=0$ and $\beta_{3k+3}=v$ for all $k\geqslant 0$. We can write $\beta_k(\mathbf{u})$ as a rational function of v:

$$\beta_k(v)=:\frac{e_k(v)}{d_k(v)}$$

where $e_k(v)$ and $d_k(v)$ are polynomials with integer coefficients.

Lemma 9 *If $u=0$ then for any $k\in\mathbb{N}$ values β_k satisfy the following properties:*

1. $||\beta_{3k+1}(v)||\leqslant -1$ *and* $||\beta_{3k+2}(v)||=||\beta_{3k+3}(v)||=1$;
2. *The leading coefficient of $e_k(v)$ as well as of $d_k(v)$ is either plus or minus one;*
3. *If $|v|\geqslant 3$ then $|\beta_{3k+1}(v)|<1$ and $|v|-1<|\beta_{3k+2}(v)|<|v|+1$.*

Proof. All these items can simultaneously be shown by induction. For $k=1$ one can easily check that:

$$\beta_4(v)=\frac{1}{v};\quad \beta_5(v)=-\frac{v^2+1}{v};\quad \beta_6(v)=v.$$

Also $\beta_{3k+3}=v$ obviously satisfies all the conditions for each $k\in\mathbb{N}$.

Assume that the properties are true for all integer values up to k and prove it for $k+1$. By (5) we have that

$$||\beta_{3k+4}(v)||=||\beta_{k+2}(v)||-||\beta_{3k+2}(v)||-||\beta_{3k+3}(v)||\leqslant -1;$$

$$|\beta_{3k+4}(v)|\leqslant\frac{|v|+1}{|v|(|v|-1)}.$$

The last expression is always less than one for $|v|\geqslant 3$. Next, since we already know that $||\beta_{3k+4}(v)||<0$,

$$||\beta_{3k+5}(v)||=||-v-\beta_{3k+4}(v)||=1.$$

$$|v|-1<|\beta_{3k+5}(v)|=|-v-\beta_{3k+4}|<|v|+1.$$

For Property 2. we have

$$\beta_{3k+4}(v)=\frac{e_{k+2}(v)d_{3k+2}(v)d_{3k+3}(v)}{d_{k+2}(v)e_{3k+2}(v)e_{3k+3}(v)}$$

Therefore the leading coefficient of both $e_{3k+4}(v)$ and $d_{3k+4}(v)$ is ± 1. Finally,

$$\beta_{3k+5}(v)=-v-\beta_{3k+4}(v)=\frac{-vd_{3k+4}(v)-e_{3k+4}(v)}{d_{3k+4}}.$$

Since, as we have shown, the degree of $e_{3k+4}(v)$ is less than that of $d_{3k+4}(v)$, we have that the leading coefficient of $e_{3k+5}(v)$ comes from $-vd_{3k+4}(v)$ and therefore it equals ± 1.

$\boxtimes$

By Theorem 3, $g_{(0,v)}(x)$ is well approximable if and only if v is a root of at least one equation $\beta_n(v)=0$. By Lemma 9 leading coefficients of each $e_n(v)$ are plus or minus one. Therefore all rational roots of $\beta_n(v)=0$ must also be integer.

Assume now that $v\in\mathbb{Z}$. If $v=0$ then we obviously have $g_{(0,0)}(x)=x^{-1}$ which is a rational function. If $v\not\equiv 0 \pmod 3$ then we use Theorem BHWY1 for $\tilde{f}(x)=xg_{(0,v)}(x^{-1})$. We have $C(x)=1+vx^2, D(x)=1$ and the functional equation (17) for $\tilde{f}(x)$ modulo 3 is

$$(1+vx^2)\tilde{(x)}^2=1$$

As $v\neq 0$ over $\mathbb{F}_3$, we get that $\tilde{f}(x)$ is not a rational function, therefore its continued fraction is ultimately periodic. Going back to $\mathbb{Q}$, this means that $\tilde{f}(x)$ is badly approximable and so is $g_{(0,v)}(x)$.

Finally consider the remaining case that $v\in\mathbb{Z}, v\neq 0$ and $v\equiv 0 \pmod 3$. In this case $|v|\geqslant 3$ and we can use property 3 from Lemma 9. It shows that $\beta_{3k+2}(v)\neq 0$. Finally, recurrent formulae (5) confirm that v is not a root of the remaining terms $\beta_{3k+1}(v)$ and $\beta_{3k+3}(v)$. Application of theorem 3 finishes the proof.

$\boxtimes$

Proof of Theorem 8. Without loss of generality we can assume that $u>0$. Indeed, replacing u by $-u$ does not change any of the conditions (C1), (C2) and the property of $g_{\mathbf{u}}(x)$ being badly approximable is invariant under the change of sign of u.

We will prove by induction that for each integer $k\geqslant 0$ the following is satisfied:

$$\begin{array}{l}2u\leqslant\alpha_{3k+2}\leqslant 3u;\quad -2u\leqslant\alpha_{3k+3}\leqslant -u;\\ |\beta_{3k+1}|\leqslant 1;\quad u^2-v-1\leqslant\beta_{3k+2}\leqslant u^2-v+1;\quad v+2u^2\leqslant\beta_{3k+3}\leqslant v+6u^2.\end{array}\tag{33}$$

For the base of induction we check the initial formulae (4): $2u\leqslant\alpha_2\leqslant 3u$ is equivalent to

$$2(v-u^2)\leqslant 2v-1-u^2\leqslant 3(v-u^2).$$

These two inequalities are in turn equivalent to $1\leqslant u^2$ and $v\geqslant 2u^2-1$ which follow from (C1) and (C2). The inequalities $-2u\leqslant\alpha_3\leqslant -u$ follow from the fact that $\alpha_3=u-\alpha_2$. We obviously have $|\beta_1|=1\leqslant 1$ and $\beta_2=u^2-v\in[u^2-v-1,u^2-v+1]$. Finally we check $v+2u^2\leqslant\beta_3\leqslant v+6u^2$. Since $v\geqslant 2u^2+8$, the enumerator $u^2+u^2+v^3-3u^2v$ of β_3 is positive. Therefore the bounds for β_3 are equivalent to

$$(v+2u^2)(v-u^2)^2\leqslant u^2+u^4+v^3-3u^2v\leqslant (v+6u^2)(v-u^2)^2.$$

The first inequality leads to $u^4+u^2+3u^2(u^2-1)v\geqslant 2u^6$ which can easily be verified, provided that $v\geqslant 2u^2+8$. By simplifying the second inequality we get

$$u^2+u^4+11vu^4\leqslant 3u^2+4u^2v^2+6u^6.$$

Since $v \geqslant 3u^2 - 1$, we have $4u^2v^2 \geqslant 4u^2(3u^2-1)v$ and the last inequality follows from

$$u^2 + u^4 + u^2v \leqslant u^4v + 6u^6$$

which is obviously satisfied for all positive v and $|u| \geqslant 1$.

Now we use recurrent formulae (5) to check estimates (33) for $k+1$ assuming that they are satisfied for all integer indices up to k.

$$|\beta_{3k+4}| = \left|\frac{\beta_{k+2}}{\beta_{3k+2}\beta_{3k+3}}\right| \leqslant \frac{v+6u^2}{(v-u^2-1)(v+2u^2)} \overset{(C2)}{\leqslant} \frac{v+6u^2}{(u^2+7)(v+2u^2)}.$$

The right hand side is obviously less than one. Then the inequalities for β_{3k+5} follow automatically from $\beta_{3k+5} = u^2 - v - \beta_{3k+4}$ and $-1 \leqslant \beta_{3k+4} \leqslant 1$. Note that under conditions (C1), (C2), β_{3k+5} is negative.

We have

$$\alpha_{3k+6} = \frac{\alpha_{k+2} + uv - \alpha_{3k+2}\beta_{3k+4}}{\beta_{3k+5}}.$$

Since $\beta_{3k+5} < 0$ inequality $-2u \leqslant \alpha_{3k+6}$ follows from

$$\frac{\alpha_{k+2} + uv - \alpha_{3k+2}\beta_{3k+4}}{v-u^2-1} \leqslant 2u \quad \Leftarrow \quad 3u + uv + 3u \leqslant 2u(v-u^2-1)$$

which is equivalent to $v \geqslant 2u^2 + 8$. Another inequality $\alpha_{3k+6} \leqslant -u$ follows from

$$\frac{\alpha_{k+2} + uv - \alpha_{3k+2}\beta_{3k+4}}{v-u^2+1} \geqslant u \quad \Leftarrow \quad -2u + uv - 3u \geqslant u(v-u^2+1)$$

which is equivalent to $u^2 \geqslant 6$.

The inequalities $2u \leqslant \alpha_{3k+5} \leqslant 3u$ follow from the formula $\alpha_{3k+5} = u - \alpha_{3k+6}$. Finally,

$$v + 2u^2 \leqslant v - \alpha_{3k+5}\alpha_{3k+6} \leqslant v + 6u^2$$

which implies the inequalities for β_{3k+6}. The claim (33) is verified.

Inequalities (33) suggest that β_{3k+2} and β_{3k+3} can not be zeroes. The value β_{3k+1} also can not be zero because from (5) it is a product of non-zero terms. Thus the values of β can never reach zero and $g_{\mathbf{u}}(x)$ is badly approximable.

⊠

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