

# Casimir elements and center at the critical level for Takiff algebras

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## Abstract

For every simple Lie algebra  $\mathfrak{g}$  we consider the associated Takiff algebra  $\mathfrak{g}_\ell$  defined as the truncated polynomial current Lie algebra with coefficients in  $\mathfrak{g}$ . We use a matrix presentation of  $\mathfrak{g}_\ell$  to give a uniform construction of algebraically independent generators of the center of the universal enveloping algebra  $U(\mathfrak{g}_\ell)$ . A similar matrix presentation for the affine Kac–Moody algebra  $\widehat{\mathfrak{g}}_\ell$  is then used to prove an analogue of the Feigin–Frenkel theorem describing the center of the corresponding affine vertex algebra at the critical level. The proof relies on an explicit construction of a complete set of Segal–Sugawara vectors for the Lie algebra  $\mathfrak{g}_\ell$ .

## 1 Introduction

For each simple finite-dimensional Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  and any positive integer  $\ell \geq 1$  consider the truncated polynomial Lie algebra  $\mathfrak{g}_\ell$  which is defined as the quotient of  $\mathfrak{g} \otimes \mathbb{C}[v]$  by the ideal  $\mathfrak{g} \otimes \mathbb{C}[v]v^{\ell+1}$ . The Lie algebra  $\mathfrak{g}_\ell$  is also called the *generalized* or  *$\ell$ -th Takiff algebra* following the pioneering work [18], where such algebras were studied in the case  $\ell = 1$ . As shown in that paper, the subalgebra of  $\mathfrak{g}_\ell$ -invariants in the symmetric algebra  $S(\mathfrak{g}_\ell)$  is an algebra of polynomials. This result was extended by Raïs and Tauvel [16] to all values of  $\ell$ . More recently, Macedo and Savage [10] proved its multi-parameter generalization, while Panyushev and Yakimova [15] showed that this generalization remains valid for a wide class of Lie algebras  $\mathfrak{g}$  beyond simple Lie algebras.

The results of Raïs and Tauvel were used by Geoffriau [8] to describe properties of an analogue of the Harish-Chandra homomorphism for the center of the universal enveloping algebra  $U(\mathfrak{g}_\ell)$ . Explicit generators of the center in type  $A$  were given in [11]. In this case the  $\ell$ -th Takiff algebra associated with  $\mathfrak{gl}_n$  coincides with the centralizer of a certain nilpotent element  $e$  in  $\mathfrak{gl}_{n(\ell+1)}$ ; namely,  $e$  is the direct sum of  $n$  Jordan blocks of size  $\ell + 1$ . The construction of central elements was extended by Brown and Brundan [4] to arbitrary nilpotents.

Here we give a uniform explicit construction of algebraically independent generators of the center of  $U(\mathfrak{g}_\ell)$  for all simple Lie algebras  $\mathfrak{g}$  and all  $\ell \geq 1$ .

Then we equip  $\mathfrak{g}_\ell$  with an invariant symmetric bilinear form by extending a standard normalized Killing form on  $\mathfrak{g}$ . The corresponding affine Kac–Moody algebra  $\widehat{\mathfrak{g}}_\ell$  is defined as a central extension of the Lie algebra of Laurent polynomials  $\mathfrak{g}_\ell[t, t^{-1}]$ . The vacuum module over  $\widehat{\mathfrak{g}}_\ell$  is

a vertex algebra whose center  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  is a commutative associative algebra. In the case  $\ell = 0$  the structure of the center  $\mathfrak{z}(\widehat{\mathfrak{g}})$  at the critical level was described by a celebrated theorem of Feigin and Frenkel [6] (see also [7]), which states that  $\mathfrak{z}(\widehat{\mathfrak{g}})$  is an algebra of polynomials in infinitely many variables. We show that this property is shared by the center  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  for all  $\ell \geq 1$ .

Our arguments will rely on matrix presentations of the Lie algebras  $\mathfrak{g}_\ell$  and  $\widehat{\mathfrak{g}}_\ell$ . Such presentations of the classical Lie algebras and the exceptional Lie algebra of type  $G_2$  played a key role in the constructions of generators of the Feigin–Frenkel center  $\mathfrak{z}(\widehat{\mathfrak{g}})$  in [12] and [14]; see also [20] for a different approach. A recent work by Wendlandt [19] provides a significant extension of the matrix techniques by giving a presentation of  $U(\mathfrak{g})$  for any simple Lie algebra  $\mathfrak{g}$  associated with its arbitrary faithful representation. We recall some results from that paper below as they will be needed for our calculations. To show that our central elements are free generators of  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  we use the classical limit and employ the Macedo–Savage theorem [10] in the particular case of ‘double’ Takiff algebras.

An analogue of the Feigin–Frenkel theorem for Takiff algebras in type  $A$  was already proved by Arakawa and Premet [1] as a particular case of a more general theorem describing the centers at the critical level of the affine vertex algebras associated with centralizers of nilpotent elements in simple Lie algebras. Explicit generators of the center in type  $A$  were produced in [13].

As in [1] and [13], our generators of the center  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  can be used to produce generators of quantum shift of argument subalgebras of  $U(\mathfrak{g}_\ell)$ . We expect that under certain regularity conditions they will be ‘quantizations’ of the Mishchenko–Fomenko subalgebras of  $S(\mathfrak{g}_\ell)$  thus yielding a solution of Vinberg’s quantization problem for Takiff algebras.

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## 2 Matrix presentations

We start by recalling some standard tensor product notation. Any  $N \times N$  matrix  $X = [X_{ij}]$  with entries in an associative algebra  $\mathcal{A}$  will be regarded as the element

$$X = \sum_{i,j=1}^N X_{ij} \otimes e_{ij} \in \mathcal{A} \otimes \text{End } \mathbb{C}^N, \quad (2.1)$$

where the  $e_{ij} \in \text{End } \mathbb{C}^N$  denote the standard matrix units. We will need tensor product algebras of the form  $\mathcal{A} \otimes \text{End } (\mathbb{C}^N)^{\otimes m}$ . For any  $a \in \{1, \dots, m\}$  we will denote by  $X_a$  the element (2.1) associated with the  $a$ -th copy of  $\text{End } \mathbb{C}^N$  so that

$$X_a = \sum_{i,j=1}^N X_{ij} \otimes 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \in \mathcal{A} \otimes \text{End } (\mathbb{C}^N)^{\otimes m}.$$

Given any element

$$C = \sum_{i,j,k,l=1}^N c_{ijkl} e_{ij} \otimes e_{kl} \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N,$$

for any two indices  $a, b \in \{1, \dots, m\}$  such that  $a < b$ , we set

$$C_{ab} = \sum_{i,j,k,l=1}^N c_{ijkl} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)} \in \text{End}(\mathbb{C}^N)^{\otimes m}.$$

Sometimes an additional copy of the endomorphism algebra  $\text{End } \mathbb{C}^N$  labelled by 0 will be used so that the notation extends accordingly to that case.

For any  $a \in \{1, \dots, m\}$  the *partial trace*  $\text{tr}_a$  will be understood as the linear map

$$\text{tr}_a : \text{End}(\mathbb{C}^N)^{\otimes m} \rightarrow \text{End}(\mathbb{C}^N)^{\otimes(m-1)}$$

which acts as the usual trace map on the  $a$ -th copy of  $\text{End } \mathbb{C}^N$  and is the identity map on all the remaining copies. Similarly, the *partial transposition*  $t_a$  is the linear map on  $\text{End}(\mathbb{C}^N)^{\otimes m}$  which acts as the usual transposition  $t : e_{ij} \mapsto e_{ji}$  on the  $a$ -th copy of  $\text{End } \mathbb{C}^N$  and is the identity map on all the remaining copies.

For a given simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  introduce a symmetric invariant bilinear form  $\langle \cdot, \cdot \rangle$  as the normalized Killing form

$$\langle X, Y \rangle = \frac{1}{2h^\vee} \text{tr}(\text{ad } X \text{ ad } Y), \quad (2.2)$$

where  $h^\vee$  is the dual Coxeter number for  $\mathfrak{g}$ . Fix a basis  $J^1, \dots, J^d$  of  $\mathfrak{g}$  and let  $J_1, \dots, J_d$  be the basis dual to  $J^1, \dots, J^d$  with respect to the form (2.2). Define Casimir elements by

$$\bar{\Omega} = \sum_{i=1}^d J_i \otimes J^i \in \text{U}(\mathfrak{g}) \otimes \text{U}(\mathfrak{g}) \quad \text{and} \quad \omega = \sum_{i=1}^d J_i J^i \in \text{U}(\mathfrak{g}). \quad (2.3)$$

With the chosen normalization of the Killing form, the eigenvalue of  $\omega$  in the adjoint representation equals  $2h^\vee$  which coincides with the value of the parameter  $c_{\mathfrak{g}}$  in [19]. Now let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a faithful representation of  $\mathfrak{g}$  of dimension  $\dim V = N$ . Identify the vector space  $V$  with  $\mathbb{C}^N$  by choosing a basis and set

$$\Omega = (\rho \otimes \rho)(\bar{\Omega}) \in \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N.$$

Furthermore, define the matrix

$$F = \sum_{i,j=1}^N F_{ij} \otimes e_{ij} \in \text{U}(\mathfrak{g}) \otimes \text{End } \mathbb{C}^N, \quad (2.4)$$

by setting  $F = -(1 \otimes \rho)(\bar{\Omega})$ .

The following presentation of  $\text{U}(\mathfrak{g})$  is due to Wendlandt [19, Proposition 4.4].

**Proposition 2.1.** *The algebra  $\text{U}(\mathfrak{g})$  is generated by the elements  $F_{ij}$  with  $1 \leq i, j \leq N$  subject only to the relations*

$$F_1 F_2 - F_2 F_1 = \Omega F_2 - F_2 \Omega \quad (2.5)$$

in  $\text{U}(\mathfrak{g}) \otimes \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$ , and

$$F^2 - \left( (F^t)^2 \right)^t = h^\vee F. \quad (2.6)$$

It is easy to see that  $[F_1 + F_2, \Omega] = 0$  so that relation (2.5) can be written in the equivalent form

$$F_1 F_2 - F_2 F_1 = F_1 \Omega - \Omega F_1. \quad (2.7)$$

*Remark 2.2.* If  $\rho$  is the vector representation for the classical types then relation (2.6) in types  $B, C$  and  $D$  is equivalent to the skew-symmetry condition  $F + F' = 0$  with respect to the bilinear form defining the orthogonal or symplectic Lie algebra. In type  $A$  the relation poses no extra conditions on the generator matrix.

One consequence of relation (2.5) is the following well-known property of the powers of the generator matrix; cf. [12, Proposition 4.2.1].

**Corollary 2.3.** *All elements  $\text{tr } F^m$  with  $m \geq 0$  belong to the center of  $U(\mathfrak{g})$ .*

It is also known that by taking  $\rho$  to be the lowest-dimension representation of  $\mathfrak{g}$ , one can choose algebraically independent generators of the center of  $U(\mathfrak{g})$  among the Casimir elements  $\text{tr } F^m$  (with the exception of type  $D$ , where a Pfaffian-type element  $\text{Pf } F$  has to be added). The required values of  $m$  coincide with the degrees of basic invariants of the symmetric algebra  $S(\mathfrak{g})$  as given in Table 1.

Type of $\mathfrak{g}$	Degrees of generators
$A_n$	$2, 3, \dots, n + 1$
$B_n$	$2, 4, \dots, 2n$
$C_n$	$2, 4, \dots, 2n$
$D_n$	$2, 4, \dots, 2n - 2, n$
$E_6$	$2, 5, 6, 8, 9, 12$
$E_7$	$2, 6, 8, 10, 12, 14, 18$
$E_8$	$2, 8, 12, 14, 18, 20, 24, 30$
$F_4$	$2, 6, 8, 12$
$G_2$	$2, 6$

Table 1: Degrees of basic invariants

More precisely, the following holds.

**Corollary 2.4.** *Except for type  $D_n$ , the elements  $\text{tr } F^m$  with  $m$  running over the values specified in Table 1 are algebraically independent generators of the center of the algebra  $U(\mathfrak{g})$ .*

*In type  $D_n$  the elements  $\text{tr } F^m$  with  $m = 2, 4, \dots, 2n - 2$  and  $\text{Pf } F$  are algebraically independent generators of the center of the algebra  $U(\mathfrak{g})$ .*

Corollary 2.4 is a classical result for types  $A, B, C, D$ , but it appears to be less known for the exceptional types. In those cases, to prove that the elements  $\text{tr } F^m$  are algebraically independent generators, one only needs to verify that their top degree components are basic  $\mathfrak{g}$ -invariants of the symmetric algebra  $S(\mathfrak{g})$ . The latter property goes back to Kuin [9], with some cases previously

considered by Coxeter [5] ( $E_6$ ) and Takeuchi [17] ( $F_4$ ). A direct claim about these Casimir elements was made in [2], [3].

To give more details for the exceptional types, recall that the Theorem in [9, Sec. 2.2] reads as follows. Suppose that  $\Lambda_1, \dots, \Lambda_N$  are the weights of the lowest-dimension representation  $(V, \rho)$  of  $\mathfrak{g}$ . The weights are identified with elements of the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  via the form (2.2). The Theorem states that the power sums

$$P_m = \sum_{a=1}^N \Lambda_a^m \quad (2.8)$$

with  $m$  running over the respective degrees in Table 1, are algebraically independent generators of the subalgebra of  $W$ -invariants  $S(\mathfrak{h})^W$  in  $S(\mathfrak{h})$ , where  $W$  denotes the Weyl group of  $\mathfrak{g}$ .

Choose a special form of the Casimir element  $\bar{\Omega}$  in (2.3) by taking a basis  $J^1, \dots, J^d$  of  $\mathfrak{g}$  such that  $J^1, \dots, J^n$  form a basis of the Cartan subalgebra  $\mathfrak{h}$ , while the remaining  $J^i$ 's are root vectors. Then the vectors  $J_1, \dots, J_n$  of the dual basis will also belong to  $\mathfrak{h}$ . Now take a weight basis of  $V$  with basis vectors of weights  $\Lambda_1, \dots, \Lambda_N$  and consider the entries  $F_{ij}$  of the matrix  $F$  defined in (2.4) as elements of the symmetric algebra  $S(\mathfrak{g})$ . A nonzero contribution to the image of the element  $\text{tr } F^m \in S(\mathfrak{g})^{\mathfrak{g}}$  under the Chevalley projection  $S(\mathfrak{g})^{\mathfrak{g}} \rightarrow S(\mathfrak{h})^W$  will only come from its part  $\text{tr } \bar{F}^m$ , where

$$\bar{F} = - \sum_{i=1}^n J_i \otimes \rho(J^i).$$

By our choice of parameters,  $\rho(J^i)$  is a diagonal matrix of the form

$$\rho(J^i) = \sum_{a=1}^N \langle \Lambda_a, J^i \rangle e_{aa}.$$

Hence, the Chevalley image of  $\text{tr } F^m$  equals

$$\sum_{a=1}^N \left( - \sum_{i=1}^n \langle \Lambda_a, J^i \rangle J_i \right)^m = (-1)^m \sum_{a=1}^N \Lambda_a^m = (-1)^m P_m$$

which thus coincides with the element (2.8), up to a sign.

### 3 Casimir elements for Takiff algebras

We will use the presentation of  $\mathfrak{g}$  associated with an arbitrary faithful representation  $\rho$ , as given in Proposition 2.1. Introduce elements of the Takiff algebra  $\mathfrak{g}_\ell$  as defined in the Introduction, by  $F_{ij}^{(r)} = F_{ij} v^r$  for  $r = 0, 1, \dots, \ell$ . We combine them into the respective matrices

$$F^{(r)} = \sum_{i,j=1}^N F_{ij}^{(r)} \otimes e_{ij} \in U(\mathfrak{g}_\ell) \otimes \text{End } \mathbb{C}^N.$$

For a variable  $u$  set

$$F(u) = F^{(0)} + F^{(1)}u + \dots + F^{(\ell)}u^\ell.$$

The traces of powers of this matrix are polynomials in  $u$  of the form

$$\mathrm{tr} F(u)^m = \sum_r \theta_m^{(r)} u^r, \quad \theta_m^{(r)} \in \mathbb{U}(\mathfrak{g}_\ell).$$

**Proposition 3.1.** *All elements  $\theta_m^{(r)}$  with  $m \geq 1$  and  $r = m\ell, m\ell - 1, \dots, m\ell - \ell$  belong to the center of the algebra  $\mathbb{U}(\mathfrak{g}_\ell)$ .*

*Proof.* Since  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ , the Lie algebra  $\mathfrak{g}_\ell$  is generated by the elements  $F_{ij}^{(0)}$  and  $F_{ij}^{(1)}$  with  $1 \leq i, j \leq N$ . Hence, it is sufficient to verify that all commutators

$$\left[ F_{ij}^{(0)}, \mathrm{tr} F(u)^m \right] \quad \text{and} \quad \left[ F_{ij}^{(1)}, \mathrm{tr} F(u)^m \right]$$

are polynomials in  $u$  whose degrees are less than  $m\ell - \ell$ . We will use the matrix notation of Section 2 and consider the algebra  $\mathbb{U}(\mathfrak{g}_\ell) \otimes \mathrm{End} \mathbb{C}^N \otimes \mathrm{End} \mathbb{C}^N$  with the tensor factors  $\mathrm{End} \mathbb{C}^N$  labelled by 0 and 1. Relation (2.5) implies

$$\begin{aligned} \left[ F_0^{(0)}, F(u)_1^m \right] &= \sum_{i=1}^m F(u)_1^{i-1} \left[ F_0^{(0)}, F(u)_1 \right] F(u)_1^{m-i} \\ &= \sum_{i=1}^m F(u)_1^{i-1} \left[ \Omega_{01}, F(u)_1 \right] F(u)_1^{m-i} = \left[ \Omega_{01}, F(u)_1^m \right]. \end{aligned}$$

Therefore, taking the partial trace  $\mathrm{tr}_1$  on both sides and using its cyclic property we can conclude that  $\left[ F_0^{(0)}, \mathrm{tr}_1 F(u)_1^m \right] = 0$ . We also have

$$u \left[ F_0^{(1)}, F(u)_1 \right] = \left[ \Omega_{01}, F(u)_1 - F_1^{(0)} \right].$$

Hence, a similar calculation gives

$$u \left[ F_0^{(1)}, \mathrm{tr}_1 F(u)_1^m \right] = - \sum_{i=1}^m \mathrm{tr}_1 F(u)_1^{i-1} \left[ \Omega_{01}, F_1^{(0)} \right] F(u)_1^{m-i}.$$

However, the degree of the polynomial in  $u$  on the right hand side is  $m\ell - \ell$  so that the commutator  $\left[ F_0^{(1)}, \mathrm{tr}_1 F(u)_1^m \right]$  is a polynomial of degree less than  $m\ell - \ell$ , as required.  $\square$

*Remark 3.2.* Proposition 3.1 and its proof obviously extend to the reductive Lie algebra  $\mathfrak{gl}_n$ . In the case where  $\rho$  is the vector representation of  $\mathfrak{gl}_n$ , these Casimir elements are closely related to one of the families produced in [11].  $\square$

Now suppose that  $\mathfrak{g} = \mathfrak{o}_{2n}$  is the orthogonal Lie algebra (of type  $D_n$ ). We will use its presentation where the elements of  $\mathfrak{g}$  are skew-symmetric  $2n \times 2n$  matrices with the usual matrix commutator. In relation (2.5) we have

$$\Omega = \sum_{i,j=1}^{2n} (e_{ij} \otimes e_{ji} - e_{ij} \otimes e_{ij}),$$

while (2.6) is equivalent to  $F + F^t = 0$ . Define the *Pfaffian* of the matrix  $F(u)$  by the formula

$$\text{Pf } F(u) = \sum_{\sigma} \text{sgn } \sigma \cdot F_{\sigma(1)\sigma(2)}(u) \cdots F_{\sigma(2n-1)\sigma(2n)}(u), \quad (3.1)$$

summed over the elements  $\sigma$  of the subset  $\mathcal{A}_{2n} \subset \mathfrak{S}_{2n}$  of the symmetric group  $\mathfrak{S}_{2n}$  which consists of the permutations with the properties  $\sigma(2k-1) < \sigma(2k)$  for all  $k = 1, \dots, n$  and  $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$ . Introduce the coefficients of the polynomial  $\text{Pf } F(u)$  by

$$\text{Pf } F(u) = \sum_r \pi^{(r)} u^r.$$

**Proposition 3.3.** *All coefficients  $\pi^{(r)}$  with  $r = n\ell, n\ell - 1, \dots, n\ell - \ell$  belong to the center of the algebra  $U(\mathfrak{g}_\ell)$ .*

*Proof.* It is enough to prove that the commutators

$$\left[ F_{ij}^{(0)}, \text{Pf } F(u) \right] \quad \text{and} \quad \left[ F_{ij}^{(1)}, \text{Pf } F(u) \right] \quad (3.2)$$

are polynomials in  $u$  of degree less than  $n\ell - \ell$ . Note that for any permutation  $\pi \in \mathfrak{S}_{2n}$  the mapping  $F_{ij}^{(r)} \mapsto F_{\pi(i)\pi(j)}^{(r)}$  defines an automorphism of the Lie algebra  $\mathfrak{g}_\ell$ . Hence, it is sufficient to verify the required properties of the commutators in (3.2) for  $i = 1$  and  $j = 2$ . This follows by a straightforward calculation with the use of the commutation relations

$$\left[ F_{ij}^{(0)}, F_{kl}(u) \right] = \delta_{kj} F_{il}(u) - \delta_{il} F_{kj}(u) - \delta_{ki} F_{jl}(u) + \delta_{jl} F_{ki}(u)$$

and

$$u \left[ F_{ij}^{(1)}, F_{kl}(u) \right] = \left[ F_{ij}^{(0)}, F_{kl}(u) - F_{kl}^{(0)} \right]$$

and arguing as in the proof of Proposition 3.1.  $\square$

*Remark 3.4.* Analogous expressions for the Pfaffian-type central elements can also be written for the presentations of the orthogonal Lie algebra associated with arbitrary non-degenerate symmetric bilinear forms on  $\mathbb{C}^{2n}$ ; cf. [12, Sec. 8].  $\square$

Return to an arbitrary simple Lie algebra  $\mathfrak{g}$  and suppose now that  $\rho$  is the lowest-dimension representation of  $\mathfrak{g}$ .

**Theorem 3.5.** *Except for type  $D_n$ , the elements  $\theta_m^{(r)}$  with  $m$  running over the values specified in Table 1 and  $r = m\ell, m\ell - 1, \dots, m\ell - \ell$ , are algebraically independent generators of the center of the algebra  $U(\mathfrak{g}_\ell)$ .*

*In type  $D_n$  the elements  $\theta_m^{(r)}$  with  $m = 2, 4, \dots, 2n - 2$  and  $r = m\ell, m\ell - 1, \dots, m\ell - \ell$  together with  $\pi^{(r)}$  with  $r = n\ell, n\ell - 1, \dots, n\ell - \ell$ , are algebraically independent generators of the center of the algebra  $U(\mathfrak{g}_\ell)$ .*

*Proof.* All these elements belong to the center of  $U(\mathfrak{g}_\ell)$  by Propositions 3.1 and 3.3. Their symbols in the symmetric algebra  $S(\mathfrak{g}_\ell)$  are  $\mathfrak{g}_\ell$ -invariants. Moreover, these invariants are associated with basic  $\mathfrak{g}$ -invariants in  $S(\mathfrak{g})$  in the way described in [16, Sec. 3.1]. Therefore, applying [16, Théorème 4.5], we can conclude that the symbols are algebraically independent generators of the subalgebra of  $\mathfrak{g}_\ell$ -invariants in  $S(\mathfrak{g}_\ell)$ . This implies the desired property of the central elements in  $U(\mathfrak{g}_\ell)$ .  $\square$

## 4 Segal–Sugawara vectors

We will identify the Lie algebra  $\mathfrak{g}$  with a subalgebra of  $\mathfrak{g}_\ell$  via the embedding  $F_{ij} \mapsto F_{ij}^{(0)}$ . Extend the form (2.2) defined on this subalgebra to the Lie algebra  $\mathfrak{g}_\ell$  by positing that all elements  $F_{ij}^{(r)}$  with  $r = 1, \dots, \ell$  belong to its kernel. This defines a symmetric invariant bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}_\ell$ . The corresponding affine Kac–Moody algebra  $\widehat{\mathfrak{g}}_\ell$  is the central extension

$$\widehat{\mathfrak{g}}_\ell = \mathfrak{g}_\ell[t, t^{-1}] \oplus \mathbb{C}K,$$

where  $\mathfrak{g}_\ell[t, t^{-1}]$  is the Lie algebra of Laurent polynomials in  $t$  with coefficients in  $\mathfrak{g}_\ell$ . For any  $r \in \mathbb{Z}$  and  $X \in \mathfrak{g}_\ell$  we will write  $X[r] = X t^r$ . The commutation relations of the Lie algebra  $\widehat{\mathfrak{g}}_\ell$  have the form

$$[X[r], Y[s]] = [X, Y][r + s] + r \delta_{r, -s} \langle X, Y \rangle K, \quad X, Y \in \mathfrak{g}_\ell, \quad (4.1)$$

and the element  $K$  is central in  $\widehat{\mathfrak{g}}_\ell$ .

The *vacuum module at the level*  $k \in \mathbb{C}$  over  $\widehat{\mathfrak{g}}_\ell$  is the quotient  $V_k(\mathfrak{g}_\ell) = \mathbb{U}(\widehat{\mathfrak{g}}_\ell)/I$ , where  $I$  is the left ideal of  $\mathbb{U}(\widehat{\mathfrak{g}}_\ell)$  generated by  $\mathfrak{g}_\ell[t]$  and the element  $K - k$ . The Poincaré–Birkhoff–Witt theorem implies that this quotient is isomorphic to the universal enveloping algebra  $\mathbb{U}(t^{-1}\mathfrak{g}_\ell[t^{-1}])$ , as a vector space. The vacuum module is equipped with a vertex algebra structure; see e.g. [7]. We will call the level  $k = -(\ell + 1)h^\vee$  *critical*, as the vacuum module  $V_{\text{cri}}(\mathfrak{g}_\ell)$  at this level turns out to exhibit similar properties to its counterpart for  $\ell = 0$ . We will denote by  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  the *center* of the vertex algebra  $V_{\text{cri}}(\mathfrak{g}_\ell)$  which is defined as the subspace

$$\mathfrak{z}(\widehat{\mathfrak{g}}_\ell) = \{v \in V_{\text{cri}}(\mathfrak{g}_\ell) \mid \mathfrak{g}_\ell[t]v = 0\}.$$

It follows from the axioms of vertex algebra that  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  is a unital commutative associative algebra which can be regarded as a subalgebra of  $\mathbb{U}(t^{-1}\mathfrak{g}_\ell[t^{-1}])$ . This subalgebra is invariant with respect to the *translation operator*  $T$  which is the derivation of the algebra  $\mathbb{U}(t^{-1}\mathfrak{g}_\ell[t^{-1}])$  whose action on the generators is given by

$$T : X[r] \mapsto -r X[r - 1], \quad X \in \mathfrak{g}_\ell, \quad r < 0. \quad (4.2)$$

Any element of  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  is called a *Segal–Sugawara vector*. By the Feigin–Frenkel theorem [6], [7], in the case  $\ell = 0$  the center  $\mathfrak{z}(\widehat{\mathfrak{g}})$  contains a *complete set of Segal–Sugawara vectors*  $S_1, \dots, S_n$ , which means that the translations  $T^r S_p$  with  $r \geq 0$  and  $p = 1, \dots, n (= \text{rank } \mathfrak{g})$  are algebraically independent generators of the algebra  $\mathfrak{z}(\widehat{\mathfrak{g}})$ .

Our goal is to prove that this property is shared by the algebras  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  for all  $\ell \geq 1$ . Note that this has already been proved in type  $A$  by Arakawa and Premet [1] as a particular case of a more general theorem on affine vertex algebras associated with centralizers of nilpotent elements in  $\mathfrak{g}$ ; see also [13] for an explicit construction of a complete set of Segal–Sugawara vectors.

We begin by producing some families of Segal–Sugawara vectors for  $\mathfrak{g}_\ell$  and then will show how to choose a complete set of such vectors. As in Section 3, we will use the presentation

of  $\mathfrak{g}$  associated with an arbitrary faithful representation  $\rho$ , given in Proposition 2.1. Introduce polynomials in  $u$  of the form

$$\mathcal{F}(u) = F^{(0)}[-1] + F^{(1)}[-1]u + \cdots + F^{(\ell)}[-1]u^\ell,$$

where  $F^{(r)}[p]$  denotes the matrix

$$F^{(r)}[p] = \sum_{i,j=1}^N F_{ij}^{(r)}[p] \otimes e_{ij}, \quad p \in \mathbb{Z}.$$

Define elements  $\Theta_m^{(r)} \in V_{\text{cri}}(\mathfrak{g}_\ell)$  as the coefficients of the polynomial

$$\text{tr } \mathcal{F}(u)^m = \sum_r \Theta_m^{(r)} u^r.$$

**Proposition 4.1.** *Suppose that  $\ell \geq 1$ . Then all coefficients  $\Theta_m^{(r)}$  with the parameters  $m \geq 1$  and  $r = m\ell, m\ell - 1, \dots, m\ell - \ell$  are Segal–Sugawara vectors for  $\mathfrak{g}_\ell$ .*

*Proof.* Since  $F_{ij}^{(0)}[0], F_{ij}^{(0)}[1]$  and  $F_{ij}^{(1)}[0]$  with  $1 \leq i, j \leq N$  are generators of the Lie algebra  $\mathfrak{g}_\ell[t]$ , it is sufficient to verify that they annihilate the elements  $\Theta_m^{(r)}$ . Using the matrix notation as in the proof of Proposition 3.1, we derive from (2.5) that

$$\left[ F^{(0)}[0]_0, \mathcal{F}(u)_1 \right] = \left[ \Omega_{01}, \mathcal{F}(u)_1 \right].$$

Hence,  $F^{(0)}[0]_0 \text{tr}_1 \mathcal{F}(u)_1^m = 0$  in  $V_{\text{cri}}(\mathfrak{g}_\ell)$  which follows in the same way as in the proof of Proposition 3.1. As in that proof, we also have

$$u \left[ F^{(1)}[0]_0, \mathcal{F}(u)_1 \right] = \left[ \Omega_{01}, \mathcal{F}(u)_1 - F^{(0)}[-1]_1 \right],$$

which implies that the degree of the polynomial  $F^{(1)}[0]_0 \text{tr}_1 \mathcal{F}(u)_1^m$  in  $u$  is less than  $m\ell - \ell$ .

Furthermore, as an immediate consequence of (2.7) and (4.1), we obtain

$$\left[ F^{(0)}[r]_0, F^{(0)}[s]_1 \right] = \left[ F^{(0)}[r+s]_0, \Omega_{01} \right] + r \delta_{r,-s} \Omega_{01} K.$$

Hence, for the remaining generators of  $\mathfrak{g}_\ell[t]$  we have

$$\left[ F^{(0)}[1]_0, \mathcal{F}(u)_1 \right] = \left[ \mathcal{F}(0, u)_0, \Omega_{01} \right] + \Omega_{01} K,$$

where we used the notation

$$\mathcal{F}(0, u) = F^{(0)}[0] + F^{(1)}[0]u + \cdots + F^{(\ell)}[0]u^\ell. \quad (4.3)$$

Calculating in the vacuum module we then find

$$F^{(0)}[1]_0 \text{tr}_1 \mathcal{F}(u)_1^m = \sum_{i=1}^m \text{tr}_1 \mathcal{F}(u)_1^{i-1} \left( -\Omega_{01} \mathcal{F}(0, u)_0 + \mathcal{F}(0, u)_0 \Omega_{01} + \Omega_{01} K \right) \mathcal{F}(u)_1^{m-i}.$$

Observe that modulo a polynomial in  $u$  of degree less than  $\ell$ , we can write

$$\left[\mathcal{F}(0, u)_0, \mathcal{F}(u)_1\right] \equiv (\ell + 1) \left[\Omega_{01}, F^{(\ell)}[-1]_1 u^\ell\right] \equiv (\ell + 1) \left[\Omega_{01}, \mathcal{F}(u)_1\right]. \quad (4.4)$$

Therefore,

$$\mathcal{F}(0, u)_0 \mathcal{F}(u)_1^{m-i} \equiv (\ell + 1) \left[\Omega_{01}, \mathcal{F}(u)_1^{m-i}\right] \quad (4.5)$$

modulo a polynomial in  $u$  of degree less than  $(m - i)\ell$ .

Now use the following general property of the partial transposition  $t_1$ :

$$\mathrm{tr}_1 XY = \mathrm{tr}_1 X^{t_1} Y^{t_1}. \quad (4.6)$$

Taking  $X = \mathcal{F}(u)_1^{i-1} \mathcal{F}(0, u)_0$  and  $Y = \Omega_{01} \mathcal{F}(u)_1^{m-i}$  we obtain

$$\mathrm{tr}_1 \mathcal{F}(u)_1^{i-1} \mathcal{F}(0, u)_0 \Omega_{01} \mathcal{F}(u)_1^{m-i} = \mathrm{tr}_1 \left(\mathcal{F}(u)_1^{i-1}\right)_1^t \mathcal{F}(0, u)_0 \left(\mathcal{F}(u)_1^{m-i}\right)_1^t (\Omega_{01})^{t_1}.$$

Applying the transposition to both sides of (4.5) we get

$$\mathcal{F}(0, u)_0 \left(\mathcal{F}(u)_1^{m-i}\right)_1^t \equiv (\ell + 1) \left[\mathcal{F}(u)_1^{m-i}\right)_1^t, (\Omega_{01})^{t_1}\right].$$

Thus, bringing the calculations together, we obtain the following relation in  $V_{\mathrm{cri}}(\mathfrak{g}_\ell)$  modulo a polynomial in  $u$  of degree less than  $m\ell - \ell$ :

$$\begin{aligned} F^{(0)}[1]_0 \mathrm{tr}_1 \mathcal{F}(u)_1^m &\equiv K \sum_{i=1}^m \mathrm{tr}_1 \mathcal{F}(u)_1^{i-1} \Omega_{01} \mathcal{F}(u)_1^{m-i} \\ &\quad + (\ell + 1) \sum_{i=1}^m \mathrm{tr}_1 \left( \mathcal{F}(u)_1^{i-1} \Omega_{01} \mathcal{F}(u)_1^{m-i} \Omega_{01} - \mathcal{F}(u)_1^{i-1} \Omega_{01}^2 \mathcal{F}(u)_1^{m-i} \right. \\ &\quad \left. - \left(\mathcal{F}(u)_1^{i-1}\right)_1^t (\Omega_{01})^{t_1} \left(\mathcal{F}(u)_1^{m-i}\right)_1^t (\Omega_{01})^{t_1} + \left(\mathcal{F}(u)_1^{i-1}\right)_1^t \left(\mathcal{F}(u)_1^{m-i}\right)_1^t \left((\Omega_{01})^{t_1}\right)^2 \right). \end{aligned}$$

The application of (4.6) to the terms in the last line brings this expression to the form

$$\begin{aligned} F^{(0)}[1]_0 \mathrm{tr}_1 \mathcal{F}(u)_1^m &\equiv \sum_{i=1}^m \mathrm{tr}_1 \mathcal{F}(u)_1^{i-1} \left( K \Omega_{01} + (\ell + 1) \left( \left( (\Omega_{01})^{t_1} \right)^2 \right)^{t_1} - (\ell + 1) \Omega_{01}^2 \right) \mathcal{F}(u)_1^{m-i} \\ &\quad + (\ell + 1) \sum_{i=1}^m \mathrm{tr}_1 \left( \mathcal{F}(u)_1^{i-1} \Omega_{01} \mathcal{F}(u)_1^{m-i} \Omega_{01} - \Omega_{01} \mathcal{F}(u)_1^{i-1} \Omega_{01} \mathcal{F}(u)_1^{m-i} \right). \end{aligned}$$

Recall that  $K = -(\ell + 1)h^\vee$  at the critical level and so the first sum is zero. This follows from the identity

$$\Omega_{01}^2 - \left( \left( (\Omega_{01})^{t_1} \right)^2 \right)^{t_1} + h^\vee \Omega_{01} = 0$$

which is a consequence of relation (2.6); we just need to apply  $\rho \otimes 1$  to its both sides and note that  $\Omega_{01} = -(\rho \otimes 1)(F)$ .

The second sum is a polynomial in  $u$  of degree at most  $m\ell - \ell$ . It remains to verify that the coefficient of  $u^{m\ell - \ell}$  in the sum is zero. This coefficient equals

$$\sum_{i=1}^m \text{tr}_1 \left( \Phi_1^{i-1} \Omega_{01} \Phi_1^{m-i} \Omega_{01} - \Omega_{01} \Phi_1^{i-1} \Omega_{01} \Phi_1^{m-i} \right), \quad (4.7)$$

where we set  $\Phi = F^{(\ell)}[-1]$  for brevity. In the algebra

$$U(\widehat{\mathfrak{g}}_\ell) \otimes \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$$

with the tensor factors  $\text{End } \mathbb{C}^N$  labelled by 0, 1 and 2 we can write

$$\text{tr}_1 \Omega_{01} \Phi_1^{i-1} \Omega_{01} \Phi_1^{m-i} = \text{tr}_{1,2} \Omega_{01} \Phi_1^{i-1} \Omega_{01} \Phi_2^{m-i} P_{12},$$

where

$$P_{12} = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}$$

is the permutation operator. We have used the relation  $\Phi_2^{m-i} P_{12} = P_{12} \Phi_1^{m-i}$  and observed that  $\text{tr}_2 P_{12} = 1$ . Since  $\ell \geq 1$ , the matrix elements of the matrix  $\Phi$  pairwise commute and so

$$\text{tr}_{1,2} \Omega_{01} \Phi_1^{i-1} \Omega_{01} \Phi_2^{m-i} P_{12} = \text{tr}_{1,2} \Phi_2^{m-i} \Omega_{01} \Phi_1^{i-1} \Omega_{01} P_{12} = \text{tr}_{1,2} \Phi_2^{m-i} P_{12} \Omega_{02} \Phi_2^{i-1} \Omega_{02}$$

which equals

$$\text{tr}_2 \Phi_2^{m-i} \Omega_{02} \Phi_2^{i-1} \Omega_{02} = \text{tr}_1 \Phi_1^{m-i} \Omega_{01} \Phi_1^{i-1} \Omega_{01}.$$

Thus, the expression (4.7) is zero so that  $F^{(0)}[1]_0 \text{tr}_1 \mathcal{F}(u)_1^m$  is a polynomial in  $u$  of degree less than  $m\ell - \ell$ .  $\square$

*Remark 4.2.* Proposition 4.1 does not hold under the assumption  $\ell = 0$ ; see e.g. [12, Secs 7.1, 8.2 and 8.4] for counterexamples in classical types.  $\square$

Consider now the orthogonal Lie algebra  $\mathfrak{g} = \mathfrak{o}_{2n}$  with its presentation used in Section 3. Define elements  $\Pi^{(r)} \in V_{\text{cri}}(\mathfrak{g}_\ell)$  as the coefficients of the *Pfaffian* of the matrix  $\mathcal{F}(u)$

$$\text{Pf } \mathcal{F}(u) = \sum_r \Pi^{(r)} u^r,$$

where  $\text{Pf } \mathcal{F}(u)$  is given by formula (3.1) applied to the matrix  $\mathcal{F}(u)$ .

**Proposition 4.3.** *All coefficients  $\Pi^{(r)}$  with  $r = n\ell, n\ell - 1, \dots, n\ell - \ell$  are Segal–Sugawara vectors for  $\mathfrak{g}_\ell$ .*

*Proof.* Given any permutation  $\pi \in \mathfrak{S}_{2n}$ , the mapping

$$F_{ij}^{(r)}[p] \mapsto F_{\pi(i)\pi(j)}^{(r)}[p], \quad K \mapsto K$$

defines an automorphism of the Lie algebra  $\widehat{\mathfrak{g}}_\ell$ . Therefore, it is sufficient to verify that the elements  $F_{12}^{(0)}[0]$ ,  $F_{12}^{(1)}[0]$  and  $F_{12}^{(0)}[1]$  acting in the vacuum module  $V_{\text{cri}}(\mathfrak{g}_\ell)$  annihilate the given

coefficients  $\Pi^{(r)}$ . The argument for the first two elements is a straightforward calculation with the use of the commutation relations involving the entries of the matrix  $\mathcal{F}(u) = [\mathcal{F}_{ij}(u)]$ ,

$$\left[ F_{ij}^{(0)}[0], \mathcal{F}_{kl}(u) \right] = \delta_{kj} \mathcal{F}_{il}(u) - \delta_{il} \mathcal{F}_{kj}(u) - \delta_{ki} \mathcal{F}_{jl}(u) + \delta_{jl} \mathcal{F}_{ki}(u)$$

and

$$u \left[ F_{ij}^{(1)}[0], \mathcal{F}_{kl}(u) \right] = \left[ \mathcal{F}_{ij}^{(0)}, \mathcal{F}_{kl}(u) - F_{kl}^{(0)}[-1] \right];$$

cf. the case  $\ell = 0$  in [12, Proposition 8.1.4]. For the remaining element we use the relations

$$\begin{aligned} \left[ F_{ij}^{(0)}[1], \mathcal{F}_{kl}(u) \right] &= \delta_{kj} \mathcal{F}_{il}(0, u) - \delta_{il} \mathcal{F}_{kj}(0, u) - \delta_{ki} \mathcal{F}_{jl}(0, u) + \delta_{jl} \mathcal{F}_{ki}(0, u) \\ &\quad + K \left( \delta_{kj} \delta_{il} - \delta_{ki} \delta_{jl} \right) \end{aligned}$$

involving the entries of the matrix (4.3). Consider first the summands in formula (3.1) for the matrix  $\mathcal{F}(u)$  with  $\sigma(1) = 1$  and  $\sigma(2) = 2$ . In the vacuum module we have

$$F_{12}^{(0)}[1] \mathcal{F}_{12}(u) \mathcal{F}_{\sigma(3)\sigma(4)}(u) \dots \mathcal{F}_{\sigma(2n-1)\sigma(2n)}(u) = -K \mathcal{F}_{\sigma(3)\sigma(4)}(u) \dots \mathcal{F}_{\sigma(2n-1)\sigma(2n)}(u).$$

Furthermore, let  $\tau \in \mathcal{A}_{2n}$  with  $\tau(2) > 2$ . Then  $\tau(3) = 2$  and  $\tau(4) > 2$  and so

$$F_{12}^{(0)}[1] \mathcal{F}_{1\tau(2)}(u) \mathcal{F}_{2\tau(4)}(u) \dots \mathcal{F}_{\tau(2n-1)\tau(2n)}(u) = -\mathcal{F}_{2\tau(2)}(0, u) \mathcal{F}_{2\tau(4)}(u) \dots \mathcal{F}_{\tau(2n-1)\tau(2n)}(u).$$

Considering this expression modulo a polynomial in  $u$  of degree less than  $n\ell - \ell$ , we may apply relations (4.4) to conclude that the expression coincides with

$$(\ell + 1) \mathcal{F}_{\tau(2)\tau(4)}(u) \dots \mathcal{F}_{\tau(2n-1)\tau(2n)}(u).$$

Suppose now that  $\sigma \in \mathcal{A}_{2n}$  in an element with the fixed values  $\sigma(1) = 1$  and  $\sigma(2) = 2$ , and calculate the coefficient of the monomial

$$\mathcal{F}_{\sigma(3)\sigma(4)}(u) \dots \mathcal{F}_{\sigma(2n-1)\sigma(2n)}(u)$$

in the expansion of  $F_{12}^{(0)}[1] \text{Pf } \mathcal{F}(u)$  modulo a polynomial in  $u$  of degree less than  $n\ell - \ell$ . Essentially the same calculation was already performed in the case  $\ell = 0$  in [12, Proposition 8.1.4] showing that the coefficient equals  $-K - (\ell + 1)(2n - 2)$ , which is zero at the critical level since  $h^\vee = 2n - 2$ .  $\square$

Now let  $\mathfrak{g}$  be an arbitrary simple Lie algebra and suppose that  $\rho$  is the lowest-dimension representation of  $\mathfrak{g}$ . The next theorem shows that a natural extension of the Feigin–Frenkel theorem [6] holds for the Lie algebra  $\mathfrak{g}_\ell$  with  $\ell \geq 1$ ; that is, the center  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$  is an algebra of polynomials.

**Theorem 4.4.** *Except for type  $D_n$ , the elements  $\Theta_m^{(r)}$  with  $m$  running over the values specified in Table 1 and  $r = m\ell, m\ell - 1, \dots, m\ell - \ell$ , form a complete set of Segal–Sugawara vectors for the Lie algebra  $\mathfrak{g}_\ell$ .*

*In type  $D_n$  the elements  $\Theta_m^{(r)}$  with  $m = 2, 4, \dots, 2n - 2$  and  $r = m\ell, m\ell - 1, \dots, m\ell - \ell$  together with  $\Pi^{(r)}$  with  $r = n\ell, n\ell - 1, \dots, n\ell - \ell$ , form a complete set of Segal–Sugawara vectors for the Lie algebra  $\mathfrak{g}_\ell$ .*

*Proof.* All elements are Segal–Sugawara vectors by Propositions 4.1 and 4.3. We need to show that all the shifted vectors  $T^s \Theta_m^{(r)}$  (together with  $T^s \Pi^{(r)}$  in type  $D_n$ ) with  $s \geq 0$  are algebraically independent generators of the algebra  $\mathfrak{z}(\widehat{\mathfrak{g}}_\ell)$ . We will follow the approach which was used for the case  $\ell = 0$ ; cf. [7, Secs 3.3 and 3.4] and [12, Sec. 6.3].

Regard  $V_{\text{cri}}(\mathfrak{g}_\ell)$  as a  $\mathfrak{g}_\ell[t]$ -module obtained by restriction of the action of  $\widehat{\mathfrak{g}}_\ell$  to the subalgebra  $\mathfrak{g}_\ell[t]$ . By identifying the vector space  $V_{\text{cri}}(\mathfrak{g}_\ell)$  with the algebra  $U(t^{-1}\mathfrak{g}_\ell[t^{-1}])$  and using its canonical filtration, equip the associated graded space  $\text{gr } V_{\text{cri}}(\mathfrak{g}_\ell)$  with the structure of a  $\mathfrak{g}_\ell[t]$ -module. As a vector space,  $\text{gr } V_{\text{cri}}(\mathfrak{g}_\ell)$  will be identified with the symmetric algebra  $S(t^{-1}\mathfrak{g}_\ell[t^{-1}])$ . The action of  $\mathfrak{g}_\ell[t]$  on  $S(t^{-1}\mathfrak{g}_\ell[t^{-1}])$  is obtained by extending the adjoint representation of  $\mathfrak{g}_\ell[t]$  on  $\mathfrak{g}_\ell[t, t^{-1}]/\mathfrak{g}_\ell[t] \cong t^{-1}\mathfrak{g}_\ell[t^{-1}]$  to the symmetric algebra.

The symbols  $\overline{\Theta}_m^{(r)}$  of the Segal–Sugawara vectors  $\Theta_m^{(r)}$  (and the symbols  $\overline{\Pi}^{(r)}$  of the Segal–Sugawara vectors  $\Pi^{(r)}$  in type  $D_n$ ) belong to the subalgebra  $S(t^{-1}\mathfrak{g}_\ell[t^{-1}])^{\mathfrak{g}_\ell[t]}$  of  $\mathfrak{g}_\ell[t]$ -invariants in  $S(t^{-1}\mathfrak{g}_\ell[t^{-1}])$ . The translation operator defined in (4.2) induces a derivation of the symmetric algebra which we will also denote by  $T$ . We only need to verify that all the shifts  $T^s \overline{\Theta}_m^{(r)}$  (together with  $T^s \overline{\Pi}^{(r)}$  in type  $D_n$ ) with  $s \geq 0$  are algebraically independent generators of the algebra of invariants.

It is clear from the definition of the vectors  $\Theta_m^{(r)}$  and  $\Pi^{(r)}$ , that the symbols  $\overline{\Theta}_m^{(r)}$  and  $\overline{\Pi}^{(r)}$  coincide with the images of certain elements of the symmetric algebra  $S(\mathfrak{g}_\ell)$  under the embedding  $\mathfrak{g}_\ell \hookrightarrow t^{-1}\mathfrak{g}_\ell[t^{-1}]$  taking  $X$  to  $X[-1]$ . Namely, they are respective images of the symbols  $\overline{\theta}_m^{(r)}$  and  $\overline{\pi}^{(r)}$  of the central elements of  $U(\mathfrak{g}_\ell)$  constructed in Section 3. As we pointed out in the proof of Theorem 3.5, these symbols are algebraically independent generators of the subalgebra of  $\mathfrak{g}_\ell$ -invariants in  $S(\mathfrak{g}_\ell)$  due to the results of [16]. Moreover, as explained in the proof of Theorem 6.3.3 in [12], these symbols can also be regarded as algebraically independent generators of the subalgebra  $(\text{Fun } \mathfrak{g}_\ell)^{\mathfrak{g}_\ell}$  of  $\mathfrak{g}_\ell$ -invariants in the algebra of polynomial functions  $\text{Fun } \mathfrak{g}_\ell$ . They are obtained according to the following general procedure. Suppose that  $Q_1, \dots, Q_n$  are algebraically independent generators of  $(\text{Fun } \mathfrak{g})^{\mathfrak{g}}$ . Writing each generator  $Q_i$  as a polynomial in coordinates  $x_1, \dots, x_d$  with  $d = \dim \mathfrak{g}$ , make the substitutions

$$x_i(z) = \sum_{r=0}^{\infty} x_{ir} z^r, \quad i = 1, \dots, d,$$

for certain variables  $x_{ir}$ , to define polynomials  $Q_{i(r)}$  by the expansions

$$Q_i(x_1(z), \dots, x_d(z)) = \sum_{r=0}^{\infty} Q_{i(r)} z^r.$$

Then for any nonnegative integer  $\ell$  the polynomials  $Q_{i(r)}$  with  $i = 1, \dots, n$  and  $r = 0, \dots, \ell$  are algebraically independent generators of the algebra of invariants  $(\text{Fun } \mathfrak{g}_\ell)^{\mathfrak{g}_\ell}$ .

Now we want to use the same procedure, with  $\mathfrak{g}$  replaced by  $\mathfrak{g}_\ell$ , for the family of algebraically independent generators of the algebra  $(\text{Fun } \mathfrak{g}_\ell)^{\mathfrak{g}_\ell}$  obtained above. To this end, observe that the first part of the proof of Theorem 6.3.3 in [12] concerning the  $\mathfrak{g}[t]$ -invariants in  $S(t^{-1}\mathfrak{g}[t^{-1}])$

applies in the same way to the Lie algebra  $\mathfrak{g}_\ell$  instead of  $\mathfrak{g}$ . Indeed, the only property of  $\mathfrak{g}$  used in that argument was the existence of a non-degenerate invariant symmetric bilinear form. The Lie algebra  $\mathfrak{g}_\ell$  does possess such a form defined by

$$\langle X_0 + X_1v + \cdots + X_\ell v^\ell, Y_0 + Y_1v + \cdots + Y_\ell v^\ell \rangle_\ell = \langle X_0, Y_\ell \rangle + \cdots + \langle X_\ell, Y_0 \rangle.$$

Therefore, combining the two steps, we come to proving the following general claim. Suppose that  $Q_1, \dots, Q_n$  are algebraically independent generators of  $(\text{Fun } \mathfrak{g})^{\mathfrak{g}}$ , whereas  $\ell$  and  $m$  are nonnegative integers. Apply the above procedure first to the elements  $Q_i$  to get invariant polynomial functions  $Q_{i(r)}$  on  $\mathfrak{g}_\ell$  for  $r = 0, \dots, \ell$ , and then apply it to the elements  $Q_{i(r)}$  to get invariant polynomial functions  $Q_{i(r,s)}$  on  $\mathfrak{g}_{\ell,m}$  for  $s = 0, \dots, m$ , where  $\mathfrak{g}_{\ell,m}$  denotes the quotient of  $\mathfrak{g}_\ell[t]$  by the ideal  $t^{m+1}\mathfrak{g}_\ell[t]$ . The claim is that all these elements  $Q_{i(r,s)}$  are algebraically independent generators of the algebra of invariants  $(\text{Fun } \mathfrak{g}_{\ell,m})^{\mathfrak{g}_{\ell,m}}$ . However, this was already proved in [10]; it is a particular case of Theorem 5.4(b) therein.  $\square$

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