RATIONALITY OF THE TRIVIAL LATTICE RANK WEIGHTED MOTIVIC HEIGHT ZETA FUNCTION FOR ELLIPTIC SURFACES

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ABSTRACT. Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, set $K = k(t)$, and let $\mathcal{W}_n^{\min}$ be the moduli stack of minimal elliptic curves over K of Faltings height n from the height–moduli framework of [BPS22] applied to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$. For $[E] \in \mathcal{W}_n^{\min}$, let $S \to \mathbb{P}^1_k$ be the associated elliptic surface with section. Motivated by the Shioda–Tate formula, we consider the trivariate motivic height zeta function

$$\mathcal{Z}(u,v;t) := \sum_{n\geq 0}\Big(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\mathrm{rk}(E/K)}\Big)t^n \in K_0(\mathrm{Stck}_k)[u,v][\![t]\!],$$

which refines the height series by weighting each height stratum with the trivial lattice rank $T(S)$ and the Mordell–Weil rank $\mathrm{rk}(E/K)$. We prove rationality for the trivial lattice specialization $Z_{\mathrm{Triv}}(u;t) = \mathcal{Z}(u,1;t)$ by giving an explicit finite Euler product. We conjecture irrationality for the Néron–Severi $Z_{\mathrm{NS}}(w;t) = \mathcal{Z}(w,w;t)$ and the Mordell–Weil $Z_{\mathrm{MW}}(v;t) = \mathcal{Z}(1,v;t)$ specializations.

1. INTRODUCTION

Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, and set $K := k(t)$. An elliptic curve E/K determines a relatively minimal elliptic surface with section

$$f : S \longrightarrow \mathbb{P}^1_k$$

unique up to isomorphism (see [Mir89, SS10] for background on elliptic surfaces).

The arithmetic of E/K is reflected in the geometry of S, and a basic organizing principle is the Shioda–Tate formula [Shi90]

$$\rho(S) = T(S) + \mathrm{rk}(E/K), \tag{1}$$

where $\rho(S) = \mathrm{rk}\,\mathrm{NS}(S_{\bar{k}})$ is the *geometric Picard rank*, $T(S)$ is the *rank of the geometric trivial lattice* generated by the zero section, a fiber class, and the components of reducible fibers not meeting the zero section, and $\mathrm{rk}(E/K)$ is the *Mordell–Weil rank*. For the elliptic surfaces arising from the height moduli considered in this paper (so that $p_g(S) = n-1$), one has the standard bounds.

$$2 \leq \rho(S) \leq 10n, \qquad 2 \leq T(S) \leq 10n, \qquad 0 \leq \mathrm{rk}(E/K) \leq 10n-2, \tag{2}$$

where $\rho(S) \leq 10n = h^{1,1}(S)$ is the Lefschetz bound over $k = \mathbb{C}$ (or in general the Igusa's inequality $\rho(S) \leq b_2(S) = 12n-2$).

In [BPS22], Bejleri–Park–Satriano construct height-moduli stacks of rational points on proper polarized cyclotomic stacks. In the fundamental modular curve example

$$\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6), \qquad \lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1),$$

a minimal elliptic curve over K can be viewed as a rational point of λ–height n on $\overline{\mathcal{M}}_{1,1}$ over K. This yields a separated Deligne–Mumford stack of finite type

$$\mathcal{W}_n^{\min} := \mathcal{W}^{\min}_{n,\mathbb{P}^1_k}\big(\mathcal{P}(4,6), \mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over K of discriminant degree $12n$. Here a K-rational point of $\overline{\mathcal{M}}_{1,1}$ of λ-height n means the stacky height n with respect to the Hodge line bundle λ, in the sense of [ESZB23]. Under the identification $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ one has $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$, and this height agrees with the Faltings height of the corresponding elliptic curve by [BPS22, Cor. 7.6].

Guided by (1), we introduce the following motivic generating series (see [Eke25] for background on the Grothendieck ring of stacks) refining the height generating series in [BPS22, §8] by weighting each height stratum with the *lattice ranks* of the associated relatively minimal elliptic surface.

Definition 1.1. Let k be a perfect field of characteristic $\neq 2,3$, and consider the height moduli stack

$$\mathcal{W}_n^{\min} = \mathcal{W}_{n,\mathbb{P}_k^1}^{\min}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over $K = k(t)$ of discriminant height $12n$. The *trivariate height zeta function* is

$$\mathcal{Z}(u,v;t) \;:=\; \sum_{n\geq 0}\left(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\operatorname{rk}(E/K)}\right) t^n \;\in\; K_0(\mathrm{Stck}_k)[u,v][\![t]\!],$$

where for each $[E] \in \mathcal{W}_n^{\min}$ we write $S \to \mathbb{P}_k^1$ for the associated minimal elliptic surface, and:

- $T(S)$ is the rank of the trivial lattice of S;
- $\operatorname{rk}(E/K)$ is the Mordell–Weil rank.

The following specializations are the associated *bivariate height zeta functions*:

$$Z_{\mathrm{Triv}}(u;t) := \mathcal{Z}(u,1;t), \tag{3}$$

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \tag{4}$$

$$Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t). \tag{5}$$

Remark 1.2. Setting $u = v = 1$ forgets the lattice-rank grading and specializes to the *univariate motivic height zeta function* $Z_{\bar{\lambda}}(t) = \mathcal{Z}(1,1;t) \in K_0(\mathrm{Stck}_k)[\![t]\!]$ and likewise to its inertial refinement $\mathcal{I}Z_{\bar{\lambda}}(t)$ which encodes the totality of rational points on $\overline{\mathcal{M}}_{1,1}$ over $K = k(t)$. Theorem [BPS22, Thm. 8.9] shows that both series are in fact rational in t, i.e. lie in $K_0(\mathrm{Stck}_k)(t)$, and gives explicit formulas.

In this paper we focus on $Z_{\mathrm{Triv}}(u;t)$. The key point is that the trivial lattice is governed by *local bad reduction*: its rank is determined by the geometric Kodaira fiber configuration of $\pi_{\bar{k}}\colon S_{\bar{k}} \to \mathbb{P}_{\bar{k}}^1$. Writing $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) := \operatorname{rk}(\mathrm{Triv}(S))$, we have the following explicit formula.

Lemma 1.3. *Let $\pi\colon S \to \mathbb{P}_k^1$ be a relatively minimal elliptic surface with section, and let $\mathfrak{f}$ be the multiset of singular fibers of $\pi_{\bar{k}}\colon S_{\bar{k}} \to \mathbb{P}_{\bar{k}}^1$. If m_v denotes the number of irreducible components of the fiber at v, then*

$$T(S) \;=\; 2 + \sum_{v\in\mathfrak{f}}(m_v - 1).$$

Definition 1.4. Fix $n \geq 1$. For a geometric fiber configuration $\mathfrak{f}$, let $\mathcal{W}_n^{\min,(\mathfrak{f})} \subset \mathcal{W}_n^{\min}$ denote the locus parametrizing those $[E] \in \mathcal{W}_n^{\min}$ whose associated surface $S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ has singular fiber configuration $\mathfrak{f}$.

Definition 1.5. Fix $n \geq 0$. By Proposition 2.1, $\mathcal{W}_n^{\min}$ admits a *finite* constructible stratification by Kodaira data, and $T(S)$ is constant on each stratum. For $n \geq 1$ and each T with $2 \leq T \leq 10n$, let

$$\mathcal{W}_n^{\min}(T) \subset \mathcal{W}_n^{\min}$$

be the finite union of those Kodaira strata on which $T(S) = T$ (hence a finite union of locally closed substacks). For $n = 0$, set $\mathcal{W}_0^{\min} := \mathcal{W}_0^{\min}(2)$.

The trivial–lattice–rank–weighted motivic height zeta function is

$$Z_{\mathrm{Triv}}(u;t) := \sum_{n\geq 0} \sum_{T \geq 2} u^T \, \{\mathcal{W}_n^{\min}(T)\} \, t^n \ \in \ K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

We prove that $Z_{\mathrm{Triv}}(u;t)$ is rational after inverting $\mathbb{L}$, and we give an explicit finite Euler product (Theorem 2.8).

Theorem 1.6. *Let k be a perfect field with* $\mathrm{char}(k) \neq 2,3$ *and put $s = t^{1/12}$. Then*

$$Z_{\mathrm{Triv}}(u;t) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s).$$

Moreover, $Z_{\mathrm{Triv}}(u;t)$ admits an explicit finite Euler product in s.

The proof is a motivic local-to-global factorization argument [CLL16], implemented on the twisted-map stratification of the height moduli $\mathcal{W}_n^{\min}$ via the evaluation morphisms [GP06]. Throughout we use the Bejleri–Park–Satriano correspondence [BPS22, Thm. 3.3] between rational points, minimal weighted linear series, and twisted morphisms; in particular, local reduction conditions are encoded by representable twisted morphisms to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$, yielding a moduli-theoretic Tate's algorithm compatible with the minimal model program [BPS22, Thm. 7.12]. The power structure on $K_0(\mathrm{Stck}_k)$ packages unordered collections of local packets into Euler factors, and the finiteness of packet shapes yields a finite Euler product after inverting $\mathbb{L}$ [GZLMH13]. The only unbounded discrete parameter is the cusp contact order in the two families I_k and I_k^*, which is collapsed by geometric resummation. Finally, specializing $x_\alpha = u^{m(\alpha)-1}$ (for $\alpha \in \mathcal{A}_{\mathrm{nc}}$) together with the cusp substitutions produces the Euler product expression for $Z_{\mathrm{Triv}}(u;t)$ as an element of $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s)$ with $s^{12} = t$.

Remark 1.7. Replacing $\mathcal{W}_n^{\min}(T)$ by its inertia stack (see [HP23, §2] for background on the inertia stack $\mathcal{I}(\mathcal{X})$ of an algebraic stack $\mathcal{X}$) gives

$$\mathcal{I}Z_{\mathrm{Triv}}(u;t) := \sum_{n\geq 0} \sum_{T \geq 2} u^T \left\{\mathcal{I}\mathcal{W}_n^{\min}(T)\right\} t^n \in K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

After inverting $\mathbb{L}$, the same argument yields a finite Euler product for $\mathcal{I}Z_{\mathrm{Triv}}(u;t)$.

2. Rationality of the trivial lattice rank weighted case

Throughout, let k be a perfect field with $\mathrm{char}(k) \neq 2,3$, set $K = k(t)$, and let $\pi\colon S \to \mathbb{P}^1_k$ be the relatively minimal elliptic surface with section associated to E/K. Write $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) = \mathrm{rk}\,\mathrm{Triv}(S)$.

Proposition 2.1. *Fix $n \geq 1$. The discriminant-degree constraint $\sum_v e(F_v) = 12n$ implies that only finitely many geometric fiber configurations $\mathfrak{f}$ occur among surfaces parametrized by $\mathcal{W}_n^{\min}$. Consequently,*

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a finite constructible stratification. Moreover, the trivial-lattice rank $T(S)$ is constant on each stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$.

Proof. Fix $n \geq 1$ and let $S \to \mathbb{P}^1_k$ be a surface parametrized by $\mathcal{W}_n^{\min}$. For any relatively minimal elliptic surface with section one has $\sum_{v\in\mathbb{P}^1_{\bar{k}}} e(F_v) = e(S_{\bar{k}})$ and in our height-n locus this total Euler number equals $12n$ (equivalently, the discriminant has degree $12n$). For each singular fiber F_v, the Kodaira–Néron classification [Kod63, N64́] gives the types $\mathrm{I}_k, \mathrm{I}_k^*$ $(k \geq 1)$ and $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_0^*, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*$. Their Euler numbers satisfy $e(\mathrm{I}_k) = k$, $e(\mathrm{I}_k^*) = k+6$ while the remaining types have Euler number $e(F_v) \in \{2,3,4,6,8,9,10\}$ (see [Her91, Table 1]). Since $\sum_v e(F_v) = 12n$, the integers k occurring in fibers of type I_k and I_k^* are bounded in terms of n. Hence there are only finitely many multisets of Kodaira symbols (equivalently, fiber configurations $\mathfrak{f}$) whose Euler numbers sum to $12n$. Therefore only finitely many configurations occur, and $\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$ is a finite stratification by locally closed substacks as in [BPS22, Prop. 7.8]. Finally, on a fixed stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ the multiset $\mathfrak{f}$ (hence the integers m_v) is constant, so Lemma 1.3 implies that $T(S) = 2 + \sum_{v\in\mathfrak{f}}(m_v - 1)$ is constant on that stratum. ■

A multivariate height series. We briefly recall the local indexing used in the twisted-maps description of height moduli. By [BPS22, Thm. 5.1] the height-n moduli stack $\mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$ on a proper polarized cyclotomic stack $\mathcal{X}$ with polarizing line bundle $\mathcal{L}$ admits a finite stratification by locally closed substacks indexed by admissible local conditions and degrees: there is a finite disjoint union of morphisms

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_{\Gamma} \longrightarrow \mathcal{M}_{n,C}(\mathcal{X},\mathcal{L}),$$

where $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})$ is the moduli stack of representable twisted morphisms of stable height d to $(\mathcal{X},\mathcal{L})$ with and local twisting conditions

$$\Gamma = \big(\{r_1,a_1\},\ldots,\{r_s,a_s\}\big),$$

recording the stabilizer orders r_i and the corresponding characters a_i at the stacky marked points of the source root stack. The indices (Γ,d) range over those satisfying the height decomposition formula

$$n = d + \sum_{i=1}^{s} \frac{a_i}{r_i}.$$

Here $S_\Gamma \subset S_s$ is the subgroup permuting stacky marked points of the same local type.

Definition 2.2. For the Euler-product argument it is useful to distinguish *packet shapes* from *elementary local patterns*. Let $\mathcal{IP}(4,6)$ be the cyclotomic inertia stack.

(1) Packet shapes. Let J denote the finite set of packet shapes occurring in the Tate-algorithm stratification via twisted maps (see [BPS22, §7]); concretely one may take

$$J = \left\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j \in \{0, 1728\}),\ \mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\right\},$$

where $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$ are the two cusp *shapes* over $j = \infty$.

(2) Elementary local patterns. Let $\mathcal{A}$ denote the set of elementary local patterns used to index evaluation conditions, i.e. the inertia components in which the evaluation maps land. Away from the cusp $j = \infty$, the inertia label determines the Kodaira symbol, so the non-cusp patterns form a finite set

$$\mathcal{A}_{\mathrm{nc}} = \left\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j \in \{0, 1728\})\right\}.$$

At the cusp $j = \infty$, the inertia label records only the cusp shape ($\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$); the additional integer $k \geq 1$ (contact order with the boundary, equivalently the pole order of j) is part of the admissible local data on a twisted-maps chart and is treated as a multiplicity parameter within the cusp shape. Accordingly we set

$$\mathcal{A} := \mathcal{A}_{\mathrm{nc}} \sqcup \{\mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\}.$$

For $\alpha \in \mathcal{A}_{\mathrm{nc}}$, let $m(\alpha) \in \mathbb{Z}_{\geq 1}$ be the number of irreducible components of the corresponding Kodaira fiber, so that $m(\alpha) - 1$ is its contribution to the trivial lattice. For the cusp shapes $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$, the component number depends on the contact order $k \geq 1$ (of the corresponding I_k or I_k^* fiber); this k–dependence will be incorporated later by geometric resummation (Proposition 2.7). In summary, J indexes the *packet shapes* (basic chart types) that become Euler factors under the power structure on $K_0(\mathrm{Stck}_k)$, whereas $\mathcal{A}$ indexes the evaluation labels, i.e. exactly what inertia can see; in particular, over $j = \infty$ inertia distinguishes only the two cusp shapes and not the contact order k.

Remark 2.3. When an evaluation condition lands over the cusp $j = \infty$, the corresponding component of the cyclotomic inertia stack $\mathcal{IP}(4,6)$ records only the *cusp shape* (multiplicative $\mathrm{I}_\bullet$ or additive $\mathrm{I}_\bullet^*$); it does *not* record the *multiplicity* $k \geq 1$. Equivalently, inertia detects that j has a pole, but not its pole order. The missing discrete datum is the *contact order with the boundary*. Geometrically, it is visible on the log canonical model obtained by contracting, in each reducible fiber, the components not meeting the zero section.

(1) The multiplicative family I_k. If the fiber at $t \in \mathbb{P}^1$ is of type I_k ($k \geq 1$), then the contraction produces an A_{k-1} surface singularity. Étale locally one has

$$xy = u^k,$$

where u is a local parameter at t. Since an étale neighbourhood of the universal nodal fiber over the cusp $[\infty] \in \overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$ is given by $xy = s$ (with s a parameter at the cusp), the classifying map $\varphi_g : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ satisfies $s = u^k$. Thus φ_g meets the boundary with contact order k, and $\nu(\Delta) = k$ for type I_k.

(2) The additive family I_k^*. If the fiber at t is of type I_k^* ($k \geq 1$), then the contraction produces a D_{k+4} surface singularity. The classifying map still lands at $j = \infty$ with boundary contact order k (so locally $s = u^k$), while the discriminant valuation is shifted by the starred contribution: $\nu(\Delta) = k + 6$ for type I_k^*.

For Z_{Triv} one has

$$m(\mathrm{I}_k) - 1 = k - 1, \qquad m(\mathrm{I}_k^*) - 1 = k + 4,$$

so the trivial-lattice exponent depends linearly on k in each cusp family.

Definition 2.4. Fix an auxiliary variable s with $s^{12} = t$. Introduce variables $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ and define

$$\mathcal{H}(s;\mathbf{x}) := \sum_{n \geq 0} \sum_{\mathfrak{f}} \Bigg(\prod_{v \in \mathfrak{f}} x_{\alpha_v} \Bigg) \{\mathcal{W}_n^{\min,(\mathfrak{f})}\} s^{12n} \in K_0(\mathrm{Stck}_k)[\mathbf{x}][\![s]\!], \tag{6}$$

where for fixed n the inner sum ranges over the finitely many geometric fiber configurations $\mathfrak{f}$ occurring in height n (Proposition 2.1).

For each singular fiber F_v in $\mathfrak{f}$, let $\alpha_v \in \mathcal{A}$ denote the corresponding inertia/evaluation label (Definition 2.2). Away from the cusp $j = \infty$ this label is the Kodaira symbol, while over $j = \infty$ it records only the cusp shape $\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$. The additional contact order $k \geq 1$ at the cusp (Remark 2.3) is part of the twisted-maps chart data and is *not* recorded by the variables x_α.

Remark 2.5. The local conditions defining the strata are imposed via evaluation maps ev_i to $\mathcal{IP}(4,6)$, hence are naturally indexed by connected components of the inertia stack. In particular, the same Kodaira symbol may correspond to distinct inertia components. For example, I_0^* splits into distinct inertia components according to whether $j \in \{0, 1728\}$ or $j \notin \{0, 1728\}$. Accordingly we index local conditions by inertia labels, not by Kodaira symbols alone.

Lemma 2.6. *Let $\mathcal{A}$ be the finite set of elementary local patterns from Definition 2.2, and let $\mathcal{H}(s;\mathbf{x})$ be the multivariate height series defined in* (6)*. After inverting $\mathbb{L}$, the series $\mathcal{H}(s;\mathbf{x})$ is a rational function of s with coefficients in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}]$.*

More precisely, there exist:

- *a finite index set J of packet shapes,*
- *motivic classes $A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$,*
- *integers $c_j \geq 1$, recording the discriminant-degree (equivalently height) increment contributed by one packet of type j,*
- *and exponents $\beta_{j,\alpha} \in \mathbb{Z}_{\geq 0}$ for $\alpha \in \mathcal{A}$, recording how many markings of inertia type α occur in a packet of type j,*

such that

$$\mathcal{H}(s;\mathbf{x})=\prod_{j\in J}\frac{1}{1-A_j\Big(\prod_{\alpha\in\mathcal{A}}x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}}\ \in\ K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](s).$$

Moreover, for $\alpha\in\{\mathrm{I}_\bullet,\mathrm{I}_\bullet^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp shape *in packet type j; it does not record the contact order $k\geq 1$.*

Proof. By [BPS22, Thm. 5.1], for each n the stack $\mathcal{W}_n^{\min}$ admits a finite locally closed stratification by charts of the form

$$\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)/S_\Gamma,$$

where $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}(\mathcal{P}(4,6),\mathcal{O}(1))$ parametrizes representable twisted morphisms equipped with an *ordered* list of stacky markings and admissible local data. We write the admissible local condition as

$$\Gamma=\big(\Gamma_{\mathrm{nc}},(\mathbf{k}^{\mathrm{I}},\mathbf{k}^{\mathrm{I}^*})\big),\qquad \mathbf{k}^{\mathrm{I}}=(k_1^{\mathrm{I}},\dots,k_{m_{\mathrm{I}}}^{\mathrm{I}}),\ \ \mathbf{k}^{\mathrm{I}^*}=(k_1^{\mathrm{I}^*},\dots,k_{m_{\mathrm{I}^*}}^{\mathrm{I}^*}),$$

where Γ_{nc} records the non-cusp inertia labels and $\mathbf{k}^{\mathrm{I}},\mathbf{k}^{\mathrm{I}^*}$ record the contact orders at cusp markings of shape $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$ (equivalently, one may list the cusp markings as pairs (α_j,k_j) with $\alpha_j\in\{\mathrm{I}_\bullet,\mathrm{I}_\bullet^*\}$ and $k_j\geq 1$). Thus the charts are indexed by full local data, including the individual contact orders k_j.

For each marking there is an evaluation morphism

$$\mathrm{ev}_i:\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1_k}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)\to\mathcal{IP}(4,6),\qquad(\varphi,\sigma_1,\dots,\sigma_s)\longmapsto\varphi(\sigma_i),$$

and prescribing an inertia label $\alpha\in\mathcal{A}$ is equivalent to requiring ev_i to land in the corresponding connected component of $\mathcal{IP}(4,6)$ (cf. [BPS22, Thm. 7.12]). Over the cusp $j=\infty$, the inertia label records only the cusp *shape* $\alpha\in\{\mathrm{I}_\bullet,\mathrm{I}_\bullet^*\}$; the contact orders k_j are additional admissible boundary-contact data on the chart (Remark 2.3). The variables $\mathbf{x}=\{x_\alpha\}_{\alpha\in\mathcal{A}}$ record only the inertia labels, so the series $\mathcal{H}(s;\mathbf{x})$ is obtained from the stratification by forgetting the extra contact-order data.

Passing to the quotient by S_Γ forgets the ordering among markings of the same inertia type. Consequently, the incidence data are governed by unordered configurations of marked points on the coarse curve $\pi(\mathcal{C})\simeq\mathbb{P}^1$, hence by symmetric powers $\mathrm{Sym}^N(\mathbb{P}^1)$. The power structure on $K_0(\mathrm{Stck}_k)$ packages these symmetric powers into Euler factors (after inverting $\mathbb{L}$): each packet shape j contributes a factor

$$\frac{1}{1-A_j\Big(\prod_{\alpha\in\mathcal{A}}x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}},\qquad A_j\in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}],$$

where $c_j\geq 1$ records the increment of $\deg(\Delta)$ contributed by one packet of type j (so the s-grading is integral, with $s^{12}=t$). Multiplying over the finitely many packet shapes yields the claimed product formula.

Finally, for $\alpha\in\{\mathrm{I}_\bullet,\mathrm{I}_\bullet^*\}$ the exponent $\beta_{j,\alpha}$ counts only the number of cusp markings of the given cusp *shape* in packet type j; the individual contact orders k_j are handled separately by geometric resummation. ∎

Proposition 2.7. *Let R be a commutative ring.*

(1) Geometric resummation *Fix $A \in R$. For integers $a, c \geq 1$ and $b, d \geq 0$, one has in $R[\![u,t]\!]$*

$$\sum_{k\geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \cdot \frac{1}{1-u^a t^c}. \tag{7}$$

Moreover, if $k_1, \ldots, k_M \geq 1$ are independent and contribute multiplicatively with the same step (a,c), then

$$\sum_{k_1,\ldots,k_M\geq 1} A \prod_{i=1}^{M} u^{ak_i+b} t^{ck_i+d} = A\left(u^{a+b} t^{c+d}\right)^M \cdot \frac{1}{(1-u^a t^c)^M}. \tag{8}$$

Equivalently, each marking contributes one factor $(1-u^a t^c)^{-1}$, so M such markings contribute the power $(1-u^a t^c)^{-M}$, up to the monomial shift $\left(u^{a+b} t^{c+d}\right)^M$.

(2) Cusp shapes for Z_{Triv} *Assume $\mathrm{char}(k) \neq 2, 3$ and work in $R = K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. Introduce an auxiliary variable s with $t = s^{12}$, so that t^n corresponds to $\deg(\Delta) = 12n$, while s records the integral discriminant degree $\deg(\Delta)$.*

After specializing $x_\beta = u^{m(\beta)-1}$ for $\beta \in \mathcal{A}_{\mathrm{nc}}$, a cusp marking of shape $\mathrm{I}_\bullet$ (resp. $\mathrm{I}_\bullet^*$) with contact order $k \geq 1$ contributes weight $u^{k-1}s^k$ (resp. $u^{k+4}s^{k+6}$), since

$$m(\mathrm{I}_k) - 1 = k-1, \qquad v(\Delta) = k, \qquad m(\mathrm{I}_k^*) - 1 = k+4, \qquad v(\Delta) = k+6$$

(cf. Remark 2.3). Hence summing over $k \geq 1$ at a single cusp marking gives, in $R[\![u,s]\!]$,

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1} s^k = \frac{s}{1-us}, \qquad x_{\mathrm{I}_\bullet^*} = \sum_{k\geq 1} u^{k+4} s^{k+6} = \frac{u^5 s^7}{1-us}. \tag{9}$$

In particular, each cusp marking of either shape contributes one factor $(1-us)^{-1}$ after resummation. Thus a packet type j with $\beta_{j,\mathrm{I}_\bullet}$ markings of shape $\mathrm{I}_\bullet$ and $\beta_{j,\mathrm{I}_\bullet^*}$ markings of shape $\mathrm{I}_\bullet^*$ contributes the cusp factor

$$(1-us)^{-(\beta_{j,\mathrm{I}_\bullet} + \beta_{j,\mathrm{I}_\bullet^*})},$$

together with the monomial shift

$$u^{5\beta_{j,\mathrm{I}_\bullet^*}} s^{\beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}_\bullet^*}}$$

coming from (9).

Proof. For (7), factor out the $k = 1$ term and apply the geometric-series identity:

$$\sum_{k\geq 1} A u^{ak+b} t^{ck+d} = A u^{a+b} t^{c+d} \sum_{k\geq 0} (u^a t^c)^k = A u^{a+b} t^{c+d} \cdot \frac{1}{1-u^a t^c}.$$

Equation (8) follows because the sum over $(k_1, \ldots, k_M)$ factorizes as a product of M copies of (7). Part (2) is (7) with $(a,c) = (1,1)$ applied in $R[\![u,s]\!]$ to the two affine-linear weights $u^{k-1}s^k$ and $u^{k+4}s^{k+6}$, yielding (9) and the stated denominator power. ■

We now prove the Main Theorem.

Theorem 2.8 (Rationality and finite Euler product for Z_{Triv}). *Let k be a perfect field of characteristic $\neq 2,3$, and set $s = t^{1/12}$ (so $t = s^{12}$). Then*

$$Z_{\mathrm{Triv}}(u;t) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s).$$

More precisely, let J, A_j, c_j, and $\beta_{j,\alpha}$ be as in Lemma 2.6. Put

$$\Delta(s) := 1 - us, \qquad b_j := \sum_{\beta\in\mathcal{A}_{\mathrm{nc}}} \beta_{j,\beta}\big(m(\beta)-1\big), \qquad m_j := \beta_{j,\mathrm{I}_\bullet} + \beta_{j,\mathrm{I}_\bullet^*},$$

and define

$$B_j := b_j + 5\beta_{j,\mathrm{I}_\bullet^*}, \qquad C_j := c_j + \beta_{j,\mathrm{I}_\bullet} + 7\beta_{j,\mathrm{I}_\bullet^*}.$$

Then one has the finite Euler product

$$Z_{\mathrm{Triv}}(u;t) = u^2 \cdot \mathbb{L} \cdot \prod_{j\in J} \frac{1}{1 - A_j\, u^{B_j}\, s^{C_j}\, \Delta(s)^{-m_j}}, \qquad (s^{12} = t).$$

Moreover, all dependence on $k \geq 1$ in the cusp families I_k and I_k^* (over $j = \infty$) is absorbed by the single geometric-series denominator $\Delta(s)^{-1} = (1-us)^{-1}$.

Proof. Work in the localized ring $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. By Lemma 2.6 we have

$$\mathcal{H}(s;\mathbf{x}) = \prod_{j\in J} \frac{1}{1 - A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}} \;\in\; K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][\mathbf{x}](s).$$

Trivial-lattice baseline. By Lemma 1.3, for an elliptic surface S with singular fiber configuration $\mathfrak{f}$ one has

$$T(S) = 2 + \sum_{v\in\mathfrak{f}} (m_v - 1).$$

Under the specializations below, the monomial attached to $\mathfrak{f}$ is $u^{\sum_{v\in\mathfrak{f}}(m_v-1)}$, i.e. it records only the fiber contributions. Thus passing from $\mathcal{H}$ to Z_{Triv} introduces the global factor u^2.

Height-zero term. By Definition 1.5 one has $\mathcal{W}_0^{\min} = \mathcal{W}_0^{\min}(2)$, so

$$[t^0]\, Z_{\mathrm{Triv}}(u;t) = u^2\{\mathcal{W}_0^{\min}\}.$$

For $n = 0$ the discriminant degree is 0, hence the corresponding elliptic curve over $K = k(t)$ has everywhere good reduction and is therefore constant. Equivalently, $\mathcal{W}_0^{\min}$ identifies with the moduli stack of smooth elliptic curves,

$$\mathcal{W}_0^{\min} \;\cong\; \mathcal{M}_{1,1}.$$

Therefore $\{\mathcal{W}_0^{\min}\} = \mathbb{L}$, and the required constant term is $u^2 \cdot \mathbb{L}$.

Non-cusp specialization. For $\beta \in \mathcal{A}_{\mathrm{nc}}$ specialize $x_\beta = u^{m(\beta)-1}$. Then the contribution of the non-cusp markings in packet type j becomes the monomial u^{b_j}.

Cusp resummation. Over the cusp $j = \infty$, the contact order $k \geq 1$ varies. In the discriminant-degree grading (with $t = s^{12}$), the two cusp families contribute weights $u^{k-1}s^k$ and $u^{k+4}s^{k+6}$, hence

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1}s^k = \frac{s}{1-us}, \qquad x_{\mathrm{I}_\bullet^*} = \sum_{k\geq 1} u^{k+4}s^{k+6} = \frac{u^5 s^7}{1-us}.$$

Substituting these into the Euler product for $\mathcal{H}$ yields, for each packet type j, the denominator factor $\Delta(s)^{-m_j}$ and the numerator shifts encoded in B_j and C_j, and therefore gives the displayed finite Euler product for $Z_{\mathrm{Triv}}(u;t)$ in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}][u](s)$. ■

Remark 2.9. Assume $\mathrm{char}(k) \neq 2,3$. For each Kodaira type Θ and $n \geq 1$, let

$$\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}$$

be the moduli stack of minimal elliptic fibrations over $\mathbb{P}^1_k$ of discriminant degree $12n$ having exactly one singular fiber of type Θ over a varying degree–one point and semistable everywhere else.

The one–fiber classes $\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}$ carry a universal dependence on the height n coming from the $10n$–dimensional space of Weierstrass coefficients (equivalently, from the spaces of sections of degrees $4n$ and $6n$ in the weighted presentation $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$). In particular, after dividing by the $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$–factor,[1] the remaining motivic class grows as $\mathbb{L}^{10n+O(1)}$, uniformly in Θ. Accordingly we normalize packet coefficients by

$$A^{(C)}_{\Theta} := \frac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n+C}},$$

for some fixed integer C (e.g. $C = -18$ as in Table 2.9). The choice of C is immaterial for the Euler product: changing C rescales *every* packet coefficient by the same global power of $\mathbb{L}$ and not change its shape (i.e. not the exponents B_j, C_j, m_j nor the finite set of Euler factors).

Then [BPS22, Thm. 1.6] and [HP19, Cor. 2] determine the following normalized one–fiber classes.

These one–fiber classes should be viewed as *local building blocks* for the packet–shape Euler product in Theorem 2.8. For each non-cusp type $\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$ and the two I_0^* cases, the corresponding local packet shapes contribute k–independent Euler factors

$$(1 - A_j\, u^{b_j} t^{c_j})^{-1}$$

after the specialization $x_\alpha = u^{m(\alpha)-1}$ for $\alpha \in \mathcal{A}_{\mathrm{nc}}$. In this way, c_j records the height increment contributed by one packet of type j, b_j records the corresponding trivial-lattice-rank increment from its non-cusp markings, and A_j records the normalized motivic classes of the packet locus.

For the cusp families I_k and I_k^*, the table gives the one–fiber contribution for each contact order $k \geq 1$. In the packet–shape Euler product for $\mathcal{H}(t;\mathbf{x})$, the exponents $\beta_{j,\mathrm{I}_\bullet}$ and $\beta_{j,\mathrm{I}^*_\bullet}$ record only the number of cusp markings of each cusp *shape* in packet

[1] The *unparameterized* $\mathbb{P}^1_k$ corresponds to taking the $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ quotient; motivically this factors out $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$, thereby treating the base as a smooth conic. See [PS25] for a comprehensive treatment.

Reduction type Θ	(r,a)	m_v-1	$A_\Theta := \dfrac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n-18}}$	Euler factor in $Z_{\mathrm{Triv}}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$k-1$	$\mathbb{L}^{16}$	$\dfrac{1}{1-\mathbb{L}^{16}s\,\Delta(s)^{-1}}$
II $(j=0)$	$(6,1)$	0	$\mathbb{L}^{15}$	$\dfrac{1}{1-\mathbb{L}^{15}s^2}$
III $(j=1728)$	$(4,1)$	1	$\mathbb{L}^{14}$	$\dfrac{1}{1-\mathbb{L}^{14}u\,s^3}$
IV $(j=0)$	$(3,1)$	2	$\mathbb{L}^{13}$	$\dfrac{1}{1-\mathbb{L}^{13}u^2s^4}$
$\mathrm{I}^*_{k\geq 1}$ $(j=\infty)$	$(2,1)$	$k+4$	$\mathbb{L}^{12}-\mathbb{L}^{11}$	$\dfrac{1}{1-(\mathbb{L}^{12}-\mathbb{L}^{11})\,u^5s^7\,\Delta(s)^{-1}}$
I^*_0 $(j\neq 0,1728)$	$(2,1)$	4	$\mathbb{L}^{12}-\mathbb{L}^{11}$	$\dfrac{1}{1-(\mathbb{L}^{12}-\mathbb{L}^{11})\,u^4s^6}$
I^*_0 $(j=0,1728)$	$(2,1)$	4	$\mathbb{L}^{11}$	$\dfrac{1}{1-\mathbb{L}^{11}u^4s^6}$
IV^* $(j=0)$	$(3,2)$	6	$\mathbb{L}^{10}$	$\dfrac{1}{1-\mathbb{L}^{10}u^6s^8}$
III^* $(j=1728)$	$(4,3)$	7	$\mathbb{L}^{9}$	$\dfrac{1}{1-\mathbb{L}^{9}u^7s^9}$
II^* $(j=0)$	$(6,5)$	8	$\mathbb{L}^{8}$	$\dfrac{1}{1-\mathbb{L}^{8}u^8s^{10}}$

type j; the individual contact orders are not part of the inertia label. The infinite k–variation is collapsed by the geometric resummations

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1}t^k = \frac{t}{1-ut}, \qquad x_{\mathrm{I}^*_\bullet} = \sum_{k\geq 1} u^{k+4}t^{k+6} = \frac{u^5t^7}{1-ut},$$

so that each cusp marking contributes one factor $(1-ut)^{-1}$. Consequently, packet type j contributes the cusp factor

$$(1-ut)^{-(\beta_{j,\mathrm{I}_\bullet}+\beta_{j,\mathrm{I}^*_\bullet})},$$

together with the monomial shift $u^{5\beta_{j,\mathrm{I}^*_\bullet}}t^{\beta_{j,\mathrm{I}_\bullet}+7\beta_{j,\mathrm{I}^*_\bullet}}$ coming from the cusp substitutions.

3. Irrationality of the Néron–Severi and Mordell–Weil specializations

The rationality of $Z_{\mathrm{Triv}}(u;t)$ reflects the fact that the trivial lattice rank $T(S)$ is governed by *local* reduction data. Indeed, by Lemma 1.3 it depends only on the multiset of fiber component numbers m_v, hence is constant on each Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, in the finite constructible stratification of Proposition 2.1. This locality is exactly what makes the evaluation-map factorization and the power structure on $K_0(\mathrm{Stck}_k)$ applicable: unordered collections of local packets assemble into a *finite* Euler product.

In contrast, the Mordell–Weil rank $\mathrm{rk}(E/K)$ is not determined by the fiber configuration. Even on a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the rank typically varies, reflecting genuinely *global* constraints rather than local reduction data. Since $T(S)$ is constant on $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the Shioda–Tate formula (1) shows that variation of $\mathrm{rk}(E/K)$ is equivalent to variation of the Néron–Severi rank $\rho(S)$. Thus any refinement of the height series by $\mathrm{rk}(E/K)$, or equivalently by $\rho(S)$, necessarily detects global jump phenomena invisible to the local packet stratification used for Z_{Triv}.

One way to organize this global complexity is via Néron–Severi jump loci. Fix a fiber configuration $\mathfrak{f}$ and write $\mathrm{Triv}^{(\mathfrak{f})} \subset \mathrm{NS}(S_{\bar{k}})$ for the sublattice generated by the zero section, a fiber class, and the components of reducible fibers in the configuration $\mathfrak{f}$. Inside $\mathcal{W}_n^{\min,(\mathfrak{f})}$, imposing that $\mathrm{NS}(S_{\bar{k}})$ contain additional algebraic classes *independent of* $\mathrm{Triv}^{(\mathfrak{f})}$ (equivalently, that $\rho(S)$, hence $\mathrm{rk}(E/K)$, jump) is an algebraic condition. Over $\mathbb{C}$, these conditions are naturally modeled by Noether–Lefschetz (Hodge) loci for the variation of Hodge structure coming from the family of elliptic surfaces over $\mathcal{W}_n^{\min,(\mathfrak{f})}$, and the theorem of Cattani–Deligne–Kaplan [CDK95] shows that such loci are, in general, only a *countable union of closed algebraic subsets*. In particular, unlike the Kodaira stratification at fixed height n (which is finite by Proposition 2.1), refinements by Néron–Severi *lattice type* are not expected to admit a finite constructible stratification.

This suggests that the local-to-global factorization mechanism producing a finite Euler product for Z_{Triv} should structurally fail for the Mordell–Weil and Néron–Severi specializations, and motivates the following conjecture.

Conjecture 3.1. Let $k = \mathbb{C}$ and $K = \mathbb{C}(z)$. The specializations

$$Z_{\mathrm{MW}}(v;t) := \mathcal{Z}(1,v;t), \qquad Z_{\mathrm{NS}}(w;t) := \mathcal{Z}(w,w;t)$$

do not become rational functions of t after inverting $\mathbb{L}$, i.e.

$$Z_{\mathrm{MW}}(v;t) \notin K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][v](t), \qquad Z_{\mathrm{NS}}(w;t) \notin K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}][w](t).$$

In other words, after inverting $\mathbb{L}$ neither bivariate height zeta series is rational in the height variable t.

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