Applied Probability Trust (6 June 2005)

LETTER TO THE EDITOR: EXTENSION OF DELTHEIL'S STUDY ON RANDOM POINTS IN A CONVEX QUADRILATERAL

Richard Cowan,* *University of Sydney*

S. N. Chiu,** *Hong Kong Baptist University*

Deltheil's 1926 treatise [4] is sometimes cited in the context of Sylvester's famous 4-point problem. The problem, finding the probability p_4 that four uniformly–distributed points within a planar convex body K have a triangular convex hull, has been solved, according to the research literature of the last 40 years, *only* for a few bodies – triangles, ellipses, parallelograms and regular polygons. Whilst Delthiel's name is sometimes linked with these solutions and also with certain extremal issues, it has apparently been forgotten that his work gives a simple expression for p_4 when K is a general convex quadrilateral. In this note we extend Deltheil's result and, as a pleasant side effect, draw attention to his forgotten study.

For n points, the probability p_n equals $\binom{n}{3}\mathbb{E}(A_3^{n-3})/|K|^{n-3}$, where A_n is the area of the convex hull formed by n points, uniformly and independently distributed within a convex quadrilateral (see Effron [5]). We report $\mathbb{E}(A_3^k)$ and $\mathbb{E}(A_n)$ for some small values of n and k, extending work of Deltheil. So we focus on the affine–invariant moments $\mathbb{E}(A_n^k)/|K|^k$. Let K be a convex quadrilateral $ABCD$, whose diagonal AC is cut by the other diagonal BD into two segments of ratio a : 1, with BD in turn being divided in the ratio b : 1. Using a straightforward analysis aided by symbolic calculations, we have derived the

* Postal address: School of Mathematics and Statistics, University of Sydney, NSW, 2006, Australia
** Postal address: Department of Mathematics, Hong Kong Baptist University, Kowloon, Hong Kong

following formulae:

$$\mathbb{E}\Big(\frac{A_3}{|K|}\Big) = \frac{1}{12} - \frac{ab}{9(1+a)^2(1+b)^2}; \qquad \mathbb{E}\Big(\frac{A_3^2}{|K|^2}\Big) = \frac{1}{72} - \frac{ab}{18(1+a)^2(1+b)^2};$$

$$\mathbb{E}\Big(\frac{A_3^3}{|K|^3}\Big) = \frac{31}{9000} - \frac{a\,b\,\left(132\,a\,b + 74\,(a+b)\,(1+a\,b) + 41\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{1500\,(1+a)^4\,(1+b)^4};$$

$$\mathbb{E}\Big(\frac{A_3^4}{|K|^4}\Big) = \frac{1}{900} - \frac{a\,b\,\left(28\,a\,b + 20\,(a+b)\,(1+a\,b) + 13\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{900\,(1+a)^4\,(1+b)^4}.$$

Only $\mathbb{E}(A_3)/|K|$ was found by Deltheil (being expressed by him using a different parametrisation). Our analysis and further discussion can be found in [3].

Our (a,b)-quadrilateral collapses to a triangle when either a or b equals zero and our leading terms agree with known results for triangles, given by Reed [6]. Other special cases are $a = 1$ yielding a (possibly skewed) kite, $a = b$ creating a trapezium and $a = b = 1$, a parallelogram. Our results do not agree with Reed's parallelogram formula. Instead, we agree with a formula of Trott, recently reported by Weisstein [7]:

$$\mathbb{E}\Big(\frac{A_3^k}{|K|^k}\Big)_{\text{parallelogram}} = \frac{3\Big(1 + (k+2)\sum_{r=1}^{k+1} r^{-1}\Big)}{(1+k)\,(2+k)^3\,(3+k)^2\,2^{k-3}}.$$

We have also found some results for $\mathbb{E}(A_n)$ when $n > 3$. Buchta [1] showed that, for general K, $\mathbb{E}(A_4) = 2\mathbb{E}(A_3)$. We supplement this with:

$$\mathbb{E}\Big(\frac{A_5}{|K|}\Big) = \frac{43}{180} - \frac{a\,b\,\left(108\,a\,b + 56\,(a+b)\,(1+a\,b) + 29\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{90\,(1+a)^4\,(1+b)^4};$$

$$\mathbb{E}\Big(\frac{A_6}{|K|}\Big) = \frac{3}{10} - \frac{a\,b\,\left(124\,a\,b + 68\,(a+b)\,(1+a\,b) + 37\,\left(1+a^2\right)\,\left(1+b^2\right)\right)}{90\,(1+a)^4\,(1+b)^4}.$$

Using Buchta's more recent theory [2], in combination with our results, we find that N_5, the number of sides of the convex hull when $n = 5$, takes values $3, 4$ or 5 with probabilities $\frac{5}{36} - \psi$, $\frac{5}{9}$ and $\frac{11}{36} + \psi$ respectively. Here $\psi := 5ab/[9(1+a)^2(1+b)^2]$. It is intriguing that the probability of this convex hull being 4–sided does not depend on a or b.

Our study was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (project no. HKBU200503).

References

[1] BUCHTA, C. (1986). On a conjecture of R. E. Miles about the convex hull of random points. *Monatsh. Math.* **102,** 91–102.

[2] BUCHTA, C. (2005). An identity relating moments of functions of convex hulls. *Discr. Comput. Geom.* **33,** 125–142.

[3] COWAN, R. AND CHIU, S. N. (2005). Technical Report: A solution of Sylvester-like problems for convex quadrilaterals. http://www.maths.usyd.edu.au/u/richardc.

[4] DELTHEIL, R. (1926). *Probabilités Géométriques.* In *Traité du calcul des probabilités et de ses applications,* **2**, ed E. Borel, Gauthier-Villars, Paris.

[5] EFFRON, B. (1965). The convex hull of a random set of points. *Biometrika* **52,** 331–343.

[6] REED, W. J. (1974). Random points in a simplex. *Pacific J. Math.* **54,** 183–198.

[7] WEISSTEIN, E. W. (2005). http://mathworld.wolfram.com/SquareTrianglePicking.html